Symmetry group and group representations associated with the thermodynamic covariance principle

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The main objective of this work [previously appeared in literature, the thermodynamical field theory (TFT)] is to determine the nonlinear closure equations (i.e., the flux-force relations) valid for thermodynamic systems out of Onsager's region. The TFT rests upon the concept of equivalence between thermodynamic systems. More precisely, the equivalent character of two alternative descriptions of a thermodynamic system is ensured if, and only if, the two sets of thermodynamic forces are linked with each other by the so-called *thermodynamic coordinate transformations* (TCT). In this work, we describe the Lie group and the group representations associated to the TCT. The TCT guarantee the validity of the so-called thermodynamic covariance principle (TCP): *The nonlinear closure equations, i.e., the flux-force relations, everywhere and in particular outside the Onsager region, must be covariant under TCT*. In other terms, the fundamental laws of thermodynamics should be manifestly covariant under transformations between the admissible thermodynamic forces, i.e., under TCT. The TCP ensures the validity of the fundamental theorems for systems far from equilibrium. The symmetry properties of a physical system are intimately related to the conservation laws characterizing that system. Noether's theorem gives a precise description of this relation. We derive the conserved (thermodynamic) currents and, as an example of calculation, a system out of equilibrium (tokamak plasmas) where the validity of TCP imposed at the level of the kinetic equations is also analyzed.

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I. INTRODUCTION

In a previous work, a macroscopic theory for closure relations for thermodynamic systems out of Onsager's region has been introduced [1,2]. The most important closure equations are the so-called transport equations, relating the thermodynamic forces with the conjugate dissipative fluxes that produce them. The thermodynamic forces are related to the spatial inhomogeneity and (in general) they are expressed as gradients of the thermodynamic quantities. The study of these relations is the object of nonequilibrium thermodynamics. Indicating with X^μ and J_μ the thermodynamic forces and fluxes, respectively, the flux-force relations read as

$$J_\nu = \tau_{\mu\nu}(X)X^\mu, \tag{1}$$

where $\tau_{\mu\nu}(X)$ are the transport coefficients, and it is clearly put in evidence that the transport coefficients may depend on the thermodynamic forces. We suppose that all quantities involved in Eq. (1) are written in dimensionless form. In this equation, as well as in the sequel, the Einstein summation convention on the repeated indices is understood. Matrix $\tau_{\mu\nu}(X)$ can be decomposed into a sum of two matrices, one symmetric and the other skew symmetric, which we denote with $g_{\mu\nu}(X)$ and $f_{\mu\nu}(X)$, respectively. The second law of thermodynamics requires that $g_{\mu\nu}$ is a positive-definite matrix. Note that, in general, the dimensionless *entropy production*, denoted by σ with $\sigma = \tau_{\mu\nu}(X)X^\mu X^\nu = g_{\mu\nu}(X)X^\mu X^\nu$, may not be a bilinear expression of the thermodynamic forces (since the transport coefficients may depend on the thermodynamic forces). For conciseness, in the sequel we drop the symbol X in $g_{\mu\nu}$ as well as in the skew-symmetric piece of the transport coefficients $f_{\mu\nu}$ being implicitly understood that these matrices may depend on the thermodynamic forces. The aim of the theory in Ref. [1] is to determine the nonlinear flux-force relations, which are valid for thermodynamic systems out of the linear region of the transport processes (commonly referred to as the Onsager region). This task has been accomplished by means of three hypotheses: two constraints (A) and (B), and one assumption (C). In order to establish the vocabulary and notations that shall be used in the sequel of this work, we briefly recall these hypotheses [1].

(A) The thermodynamic principles and the theorems demonstrated for systems far from equilibrium must be satisfied.

(B) The validity of the thermodynamic covariance principle (TCP) must be ensured. The TCP establishes that *the closure relations should be covariant under the thermodynamic coordinate transformations* (TCT). The TCT are the most general thermodynamic force transformations which leave unaltered both the entropy production σ and the Glansdorff-Prigogine dissipative quantity P [for the definition of P, see the forthcoming Eq. (2)]. As we shall see soon, the invariance

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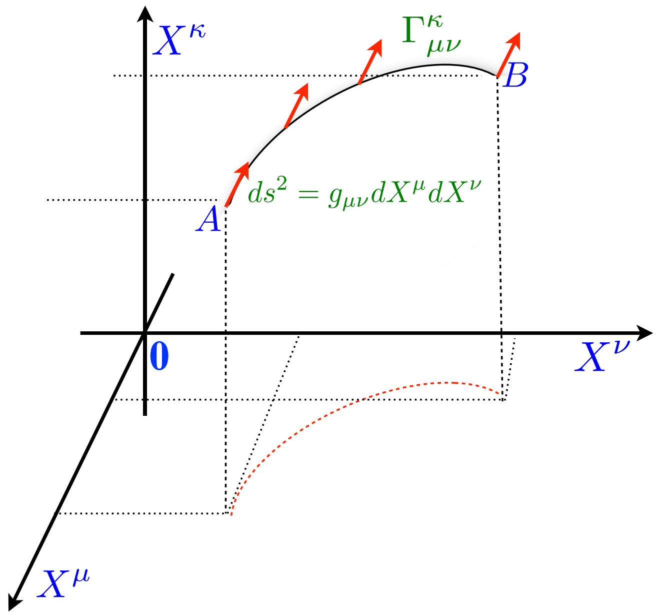

FIG. 1. Thermodynamic space. The space is spanned by the thermodynamic forces. The metric tensor is identified with the symmetric piece $g_{\mu\nu}$ of the transport coefficients, and the expression of the affine connection $\tilde{\Gamma}^\kappa_{\mu\nu}$ is determined by imposing the validity of the universal criterion of evolution. Note that the square of the length element, $ds^2 = \mathbf{d}s \cdot \mathbf{d}s$, is always a non-negative quantity for the second law of thermodynamics.

under TCT is intimately related to the concept of equivalence between thermodynamic systems.

In addition to (A) and (B), the theory rests upon the following assumption:

(C) Close to the steady states, the nonlinear closure equations can be derived by the principle of least action.

This theory, based on (A), (B), and (C), is referred to as the thermodynamical field theory (TFT).

The physical justifications leading to the introduction of the above-mentioned hypotheses can be found in the Appendixes.[1] Here, we shall limit ourselves to provide only a brief description of the mathematical expressions stemmed by (A), (B), and (C) that we shall use in the forthcoming sections.

Constraint (A) allows introducing a thermodynamic space equipped by a metric tensor and the appropriate affine connection. The coordinates of this thermodynamic space are the thermodynamic forces, the metric tensor is identified with the symmetric piece of the transport coefficients (which, for the second law of thermodynamics, is a positive-definite matrix), and the parallel transport of a vector is made by the affine connection constructed in such a way that the one of the most general thermodynamic theorems valid for systems out of equilibrium, i.e., the universal criterion of evolution (UCE) [4,5], is always automatically satisfied (see Fig. 1 and Ref. [1]).

[1]These hypotheses are also more extensively explained in [3].

For easy reference, we enunciate the universal criterion of evolution (UCE) [4,5]: Without using either the Onsager reciprocal relations or the assumption that the phenomenological coefficients (or linear phenomenological laws) are constant, for time-independent boundary conditions, and for conservative systems,[2] when the system relaxes towards a stable steady state, the dissipative quantity P, defined as

$$P \equiv \int_\Omega J_\mu \frac{\partial X^\mu}{\partial t} dv \leqslant 0, \tag{2}$$

is always a negative quantity. The equality is saturated at the steady state.

In Eq. (2), Ω is the spatial volume occupied by the system and dv denotes the spatial volume element, respectively. Hence, quantity $P \equiv \int_\Omega J_\mu \frac{\partial X^\mu}{\partial t} dv$ is a sort of intrinsic quantity of a dissipative system, and it may be referred to as *the Glansdorff-Prigogine dissipative quantity*.[3] As mentioned, the UCE determines the expression of the affine connection. When the transport coefficients are purely symmetric (i.e., when $f_{\mu\nu} = 0$), it is possible to show that the affine connection reads as [1]

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \left\{ {}^{\;\mu}_{\alpha\beta} \right\} + \frac{1}{2\sigma} X^\mu X^\kappa g_{\alpha\beta,\kappa} - \frac{X^\kappa X^\lambda}{2(n+1)\sigma}\left(\delta^\mu_\alpha g_{\beta\kappa,\lambda} + \delta^\mu_\beta g_{\alpha\kappa,\lambda}\right) \tag{3}$$

with

$$\left\{ {}^{\;\mu}_{\alpha\beta} \right\} = \frac{1}{2} g^{\mu\lambda}(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}) \tag{4}$$

denoting the Levi-Civita affine connection. From Eq. (3) we get

$$\Delta\Gamma^\mu_{\alpha\beta} \equiv \tilde{\Gamma}^\mu_{\alpha\beta} - \left\{ {}^{\;\mu}_{\alpha\beta} \right\} = \frac{1}{2\sigma} X^\mu X^\kappa g_{\alpha\beta,\kappa} - \frac{X^\kappa X^\lambda}{2(n+1)\sigma}\left(\delta^\mu_\alpha g_{\beta\kappa,\lambda} + \delta^\mu_\beta g_{\alpha\kappa,\lambda}\right). \tag{5}$$

Equation (5) clearly shows that the thermodynamic affine connection differs, widely, from the Levi-Civita connection.[4] These two affine connections tend to identify with each other only for very large values of the entropy production ($\sigma \gg 1$).[5]

[2]We recall that conservative systems satisfy the equation $\partial_t F_{\mu i} = 0$, with $F_{\mu i}$ denoting the ith component of the external force per unit mass $\mathbf{F}_\mu$ [5].

[3]Notice that Eq. (2) generalizes the so-called minimum entropy production theorem (MEPT), proved by Prigogine in 1947 [6,7], which applies only in the Onsager region. The MEPT reads as follows: When a thermodynamic system near equilibrium relaxes towards a steady state, for time independent boundary conditions and for conservative systems, the following inequality $\int_\Omega \partial_t \sigma \; dv \leqslant 0$ is satisfied during the evolution. Saturation is reached at the steady state.

[4]Note that $\Delta\Gamma^\mu_{\alpha\beta}$ is a mixed third-order tensor under TCT.

[5]As shown in Ref. [1], the geometries of the general relativity (GR) and the thermodynamical field theory (TFT) are widely different. Indeed, in GR the geometry is pseudo-Riemannian, the field is symmetric, and the affine connection is given by the Levi-Civita expression. In addition, the GR rests upon the validity of the general

It is worth noting that the angles $\theta_{\mu\nu} \equiv \arctan(J_\mu/X^\nu)$ and $\alpha_{\mu\nu} \equiv \arctan(\delta J_\mu/\delta X^\nu)$ provide information on the *metric tensor* and the *affine connection* of the *thermodynamic space*, respectively. These angles may be measured experimentally.

Now, let us deal with constraint (B). This constraint refers to the symmetry underneath the thermodynamic covariance principle (TCP) and the related concept of *equivalent thermodynamic systems*. This is the starting point of this work. In Refs [1,8], it is shown that the invariance of the entropy production may not be sufficient to ensure the *equivalent* character of the alternative descriptions (J_μ, X^μ) and (J'_μ, X'^μ). Additional conditions may be necessary (e.g. [4,5,8]). The equivalent character of two alternative descriptions in terms of the thermodynamic forces requires that also the Glansdorff-Prigogine dissipative quantity remains invariant under transformation of the thermodynamic forces $\{X^\mu\} \to \{X'^\mu\}$.[6] This additional restriction must be kept in mind to avoid misinterpretations. Hence, the admissible thermodynamic forces should satisfy the following two conditions:

(1) The entropy production σ should be invariant under transformation of the thermodynamic forces $\{X^\mu\} \to \{X'^\mu\}$.

(2) The Glansdorff-Prigogine dissipative quantity P should also remain invariant under the force transformations $\{X^\mu\} \to \{X'^\mu\}$.

Condition 2 stems from the fact that

(a) The steady state should be transformed into a steady state.

(b) The stable steady state should be transformed into a stable state state, with the same *degree* of stability.

In mathematical terms, this implies that [1][7]

$$\begin{aligned} \sigma &= J_\mu X^\mu = J'_\mu X'^\mu = \sigma', \\ P &= P' \Longrightarrow J_\mu \delta X^\mu = J'_\mu \delta X'^\mu \qquad (t = t'). \end{aligned} \tag{6}$$

Equations (6) are satisfied iff the transformed thermodynamic forces and conjugate fluxes read as [1]

$$X'^\mu = \frac{\partial X'^\mu}{\partial X^\nu} X^\nu, \quad J'_\mu = \frac{\partial X^\nu}{\partial X'^\mu} J_\nu. \tag{7}$$

By direct inspection, it is easy to verify that the general solutions of Eq. (7) are [1]

$$X'^\mu = X^1 F^\mu\left(\frac{X^2}{X^1}, \frac{X^3}{X^2}, \ldots, \frac{X^n}{X^{n-1}}\right) \quad (t = t'), \tag{8}$$

where F^μ are arbitrary functions of variables X^j/X^{j-1} with $j = 2, \ldots, n$. Transformations (8) may be referred to as the thermodynamic coordinate transformations (TCT). Hence, the TCT may be highly nonlinear coordinate transformations but, in the Onsager region, we may or we must require that they have to reduce to

$$X'^\mu = c^\mu_\nu X^\nu \quad (t = t'), \tag{9}$$

where c^μ_ν are constant coefficients (i.e., independent of the thermodynamic forces).

As we shall see more in detail in the next section, the TCT are not, simply, transformations written in a projective form, but they form a nontrivial bundle whose base is the projective space and whose fiber is the space of maps from the projective space to the nonvanishing real numbers. The thermodynamic equivalence principle leads, naturally, to the following (TCP): the nonlinear closure equations, i.e., the flux-force relations, must be covariant under TCT [8]. The essence of the TCP is the following. The equivalent character between two representations is warranted if, and only if, the fundamental thermodynamic equations are covariant under under transformations of the (admissible) thermodynamic forces (the TCT). This is the correct mathematical formalism to ensure the equivalence between two different representations.

Finally, let us discuss assumption (C). According to this assumption, the (nonlinear) closure equations can be derived by the principle of least action. More specifically, assumption (C) states the following [1]: There exists an action which is stationary with respect to arbitrary variation of the transport coefficient and the affine connection. The physical justification of this assumption can be found in the Appendix A. In the framework of the TFT introduced by one of us [1,2], one can find the expression of the thermodynamic action [1]:

$$I = \int \left[R - \left(\Gamma^\lambda_{\alpha\beta} - \tilde{\Gamma}^\lambda_{\alpha\beta}\right) S^{\alpha\beta}_\lambda\right] \sqrt{g}\, d^n X \tag{10}$$

with $d^n X$ denoting an infinitesimal volume element in the space of the thermodynamic forces and g the determinant of $g_{\mu\nu}$ (see [1]).[8] In addition, R is the curvature of the space. In the Appendixes, it is indicated how (10) has been derived.

As shown in [1], in general, action (10) is quite complex. However, in the case of magnetically confined plasmas (which is the case analyzed in this work) the skew-symmetric pieces of the transport coefficients $f_{\mu\nu}$ are zero, and the action simplifies notably because the terms appearing in Eq. (10) reduce to [1]

$$\begin{aligned} R &= R_{\mu\nu} g^{\mu\nu}, \\ R_{\mu\nu} &= \Gamma^\kappa_{\nu\kappa,\mu} - \Gamma^\kappa_{\nu\mu,\kappa} + \Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\kappa\mu} - \Gamma^\kappa_{\nu\mu}\Gamma^\lambda_{\kappa\lambda}, \\ S^{\mu\nu}_\lambda &= \Psi^\nu_{\lambda\alpha} g^{\nu\alpha} + \Psi^\mu_{\lambda\alpha} g^{\mu\alpha} - \frac{1}{2}\Psi^\mu_{\alpha\beta} g^{\alpha\beta}\delta^\nu_\lambda - \frac{1}{2}\Psi^\nu_{\alpha\beta} g^{\alpha\beta}\delta^\mu_\lambda, \\ \Psi^\mu_{\alpha\beta} &= \frac{1}{2\sigma} X^\kappa X^\mu g_{\alpha\beta,\kappa} - \frac{X^\kappa X^\lambda}{2(n+1)\sigma}\left(\delta^\mu_\alpha g_{\beta\kappa,\lambda} + \delta^\mu_\beta g_{\alpha\kappa,\lambda}\right) \end{aligned} \tag{11}$$

covariance principle in the space-time and on the validity of the equivalence principle. In the GR, the universal criterion of evolution is not satisfied. In the TFT, the geometry is non-Riemannian, the field is asymmetric, and the thermodynamic affine connection is given by $\Gamma^\mu_{\alpha\beta} = \tilde{\Gamma}^\mu_{\alpha\beta}$. The TFT rests upon the validity of the (special) covariance principle TCP and on the validity of the universal criterion of evolution. In the TFT, the equivalence principle is not satisfied. For more details, see the annex of [1].

[6] An extensive explanation on this point can be found in the Appendixes.

[7] Notice that we have to use also the invariance $t = t'$. This will avoid certain paradoxes to which Verschaffelt [9] has called attention (cf. also [10]).

[8] Hence, the Lagrangian depends only on the thermodynamic forces. Of course, the thermodynamic forces depend on space and time.

with $g^{\mu\nu}$ denoting the inverse matrix of $g_{\mu\nu}$ and the coma (,) in the subscripts stands for the partial derivative with respect to the thermodynamic forces. The most general expressions for $S_\lambda^{\mu\nu}$, $\Psi^\mu_{\alpha\beta}$, and $\tilde\Gamma^\mu_{\alpha\beta}$ (and, then, for $\Delta\Gamma^\mu_{\alpha\beta}$), valid when $f_{\mu\nu}\neq 0$, can be found in [1]. It can be shown that action (10) is stationary when the *thermodynamic affine connection* $\Gamma^\mu_{\alpha\beta}$ is equal to $\tilde\Gamma^\mu_{\alpha\beta}$, with $\tilde\Gamma^\mu_{\alpha\beta}$ given by Eq. (3) (only when $f_{\mu\nu}=0$). An explanation about the physical meaning of the terms appearing in the thermodynamic action (10) is reported in the Appendixes. By variational methods we get the differential equations for $g_{\mu\nu}$ and $f_{\mu\nu}$. These equations can be found in Ref. [1]. In the Appendixes, we report the approximated differential equation for $g_{\mu\nu}$ obtained in the so-called weak-field approximation (i.e., when the metric tensor is close to Onsager's matrix) and for $\sigma\gg 1$ [see Eqs. (A4) and (A5)].

The aim of this paper is to describe and to study the group of symmetry stemmed by the hypotheses (A), (B), and (C), and in particular by the TCP. This work is constituted by two parts. In the first two sections of the paper we derive theoretical results on the group of symmetry associated to the invariance of the (thermodynamic) Lagrangian under TCT. The last two sections are, instead, devoted to the application of the obtained results to concrete (and quite complex) cases such as transport processes in magnetically confined plasmas. More specifically, we have the following.

(i) *Theory*. In Sec. II, we show that the TCT group (indicated by G^n) is a fiber bundle where the bundle is S_+^{n-1} (the top half of the unit sphere embedded in R^n) and the fiber is the space of maps $S_+^{n-1}\to\mathbb{R}^\times$ (with $\mathbb{R}^\times$ denoting the nonvanishing real numbers). The nature of this bundle is made clear by the algebraic theorem demonstrated in Sec. III. In Sec. III, we demonstrate the following theorem: *The TCT group may be split in a semidirect product of two subgroups where the first one is a normal, Abelian, subgroup*. In the subsections (of Sec. III) we shall see that the TCT group is a noncompact, infinite Lie group. However, there exist nonlinear TCT admitting Lie subalgebras (i.e., subspaces whose the Lie brackets remain in the subspace) associated to finite, compact TCT subgroups. As known to the N generators of a Lie group, there are N conserved Noether's currents.[9] This motivates the presence of the subsequent sections.

(ii) *Applications*. Section IV provides an example of application of the TCT group. Here, we derive the expressions of the two Noether's currents for fully collisional tokamak plasmas (more precisely, for fully collisional, L-mode, JET plasmas). In Sec. V, we can find an example of application of the TCP. We derive the expression of the collisional operator in tokamak plasmas which is, currently, largely used in the numerical simulations. At the end of this section, we apply the TFT formalism, based upon the validity of the TCP, to estimate the heat loss in fully collisional FTU plasmas. We show that the theoretical predictions are in fairly good agreement with experimental data in the expected region of validity. To perform this comparison, the Shafranov shift has also been taken into account. Concluding remarks can be found in Sec. VI. In the Appendixes, we report a brief description of the thermodynamical field theory (TFT), the demonstration of the theorem satisfied by the TCT group, and the details of calculation of Noether's currents, respectively.

[9] Note that in our case the N Noether's currents are *conserved* in the thermodynamic space (i.e., the variables are the thermodynamic forces and not the spatial and time variables).

II. TOPOLOGICAL DESCRIPTION OF THE TCT GROUP

A. Construction

As shown in Ref. [1], the TCT are given by Eqs. (8):

$$X^\mu\to X'^\mu = X^1F^\mu\left(\frac{X^2}{X^1},\frac{X^3}{X^2},\ldots,\frac{X^n}{X^{n-1}}\right), \tag{12}$$

where F^μ are *arbitrary functions* of variables X^j/X^{j-1} with $(j=2,\ldots,n)$. We demand that the n functions F^μ be smooth, so that the TCT preserve equations satisfied by the derivatives of thermodynamic quantities, and also that the transformation be nondegenerate with a smooth inverse, so that the transformed theory contain all of the information of the original theory. The nondegenerate property is also a necessary and sufficient condition for the finiteness of the transformed transport coefficients, even though it implies that the F^μ themselves may sometimes diverge. For example, from the transformation

$$X'^1 = X^2,\quad X'^2 = X^1,$$

one obtains

$$F^1(X^2/X^1) = X^2/X^1,\quad F^2(X^2/X^1) = 1$$

showing that $F^1(X^2/X^1)$ diverges at $X^1=0$, whereas the X'^μ are always finite.

The space of thermodynamic forces, linear combinations of $\{X^1,\ldots,X^n\}$, is the real Euclidean space $\mathbb{R}^n$. On the other hand, the ratios $\{X^\mu/X^{\mu-1}\}$ are coordinates for a different space, the real projective space $\mathbb{RP}^{n-1}$, which is defined to be the quotient of $\mathbb{R}^n$ minus the origin by the scaling map $X^\mu\to\alpha X^\mu$ where α is any nonzero real number. Note that some of the X^μ may vanish; removing the origin simply implies that not all of the X^μ vanish simultaneously. Figure 2 illustrates a space, which is diffeomorphic to the $\mathbb{RP}^{n-1}$. Observe that X'^μ is an

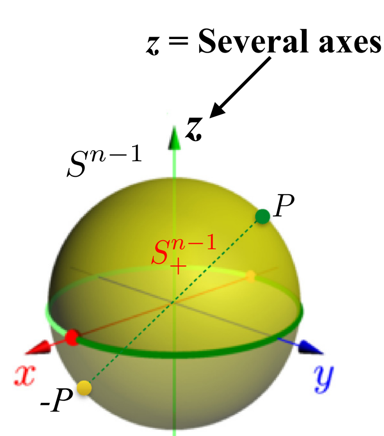

FIG. 2. The projective space $\mathbb{RP}^{n-1}$ is diffeomorphic to S_+^{n-1} made by the upper hemisphere + half equator (without the red and yellow points) + the red point.

arbitrary smooth, degree 1 function of the X's with the property that $X \to X'$ is invertible. This implies that $X'^{\mu}/X'^{\mu-1}$ is an arbitrary degree 0 function with these same properties. Now, $\{X'^{\mu}/X'^{\mu-1}\}$ are again coordinates of $\mathbb{RP}^{n-1}$. The fact that $X'^{\mu}/X'^{\mu-1}$ is degree 0 implies that it is invariant under the transformation $X^{\mu} \to \alpha X^{\mu}$ and so the map $X'^{\mu}/X'^{\mu-1}$ is in fact a map from $\mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$:

$$\mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}: \frac{X^{\mu}}{X^{\mu-1}} \mapsto \frac{X'^{\mu}}{X'^{\mu-1}} = \frac{F^{\mu}\left(\frac{X^2}{X^1}, \frac{X^3}{X^2}, \dots, \frac{X^n}{X^{n-1}}\right)}{F^{\mu-1}\left(\frac{X^2}{X^1}, \frac{X^3}{X^2}, \dots, \frac{X^n}{X^{n-1}}\right)}.$$

So, we have learned that the TCT yields a map from $\mathbb{RP}^{n-1}$ to itself. Furthermore, the invertibility condition implies that this map is invertible and the smooth inverse condition implies that this map is a diffeomorphism. Thus, every TCT defines a diffeomorphism of $\mathbb{RP}^{n-1}$ to itself.

At this point, it is tempting to conclude that the group of TCTs is just the group diff($\mathbb{RP}^{n-1}$) of such diffeomorphisms. However, this is not quite true because the ratios $X'^{\mu}/X'^{\mu-1}$ do not contain all the information in the X'^{μ}. To reconstruct all the X'^{μ} from the ratios, one also needs to know, for example, X'^1 or equivalently the real-valued function $F^1 : \mathbb{RP}^{n-1} \to \mathbb{R}$, which intuitively gives the overall scale dependence of the TCT. Therefore, the group G of TCTs is a product of diff($\mathbb{RP}^{n-1}$) with the multiplicative group of maps from $\mathbb{RP}^{n-1}$ to the nonvanishing reals $\mathbb{R}^{\times}$ where the nonvanishing condition is needed to ensure nondegeneracy.

B. A subtlety

This is the right answer locally. Globally there is one subtlety: we have double counted the map which flips the sign of all of the forces X. It was the element $\alpha = -1$ which we have the quotient when constructing $\mathbb{RP}^{n-1}$ from $\mathbb{R}^n$. More precisely, $\mathbb{RP}^{n-1}$ can be constructed from $\mathbb{R}^n$ minus the origin in two steps: first, the quotient by the maps $X \mapsto \alpha X$ with α positive, yielding the sphere S^{n-1}, and then the quotient by $\alpha = -1$ yielding $\mathbb{RP}^{n-1}$. This second action, whose quotient maps S^{n-1} to $\mathbb{RP}^{n-1}$, has the same action on the X's as the map -1 in $\mathbb{RP}^{n-1} \to \mathbb{R}^{\times}$. How does this double counting affect G?

Given a TCT, one may calculate the ratio $\{X'^{\mu}/X'^{\mu-1}\}$. Since G is the group of TCTs while diff($\mathbb{RP}^{n-1}$) is the group of maps $\{X'^{\mu}/X'^{\mu-1}\}$, there must exist a projection $G \to$ diff($\mathbb{RP}^{n-1}$). The argument above implies that the kernel of this projection is the space of nonvanishing maps from $\mathbb{RP}^{n-1}$ to $\mathbb{R}^{\times}$. Therefore, the group G of TCTs is a bundle whose base is diff($\mathbb{RP}^{n-1}$) and whose fiber is the space of maps $\mathbb{RP}^{n-1} \to \mathbb{R}^{\times}$. Which bundle is it?

When traversing a noncontractible loop in $\mathbb{RP}^{n-1}$, which necessarily lifts in S^{n-1} to a path between two antipodal points, the sign of the $\mathbb{R}^{\times}$ must change. This means that the group of scalings $\mathbb{RP}^{n-1} \to \mathbb{R}^{\times}$ is nontrivial fibered over diff($\mathbb{RP}^{n-1}$) such that, upon traversing the nontrivial cycle once, the sign of $\mathbb{R}^{\times}$ is inverted.

Assembling all these arguments, we arrive at our final result. The group G of TCTs is the nontrivial bundle of the maps $\mathbb{RP}^{n-1} \to \mathbb{R}^{\times}$ over diff($\mathbb{RP}^{n-1}$). There is a simple mathematical formulation for this group G. Let $P : \mathbb{R}^n\backslash\{0\} \to \mathbb{RP}^{n-1}$ be the quotient $X \sim \alpha X$ which defines the real projective space $\mathbb{RP}^{n-1}$. Then, the group G of TCTs is the group of maps $f : \mathbb{RP}^{n-1} \to \mathbb{R}^n\backslash\{0\}$ such that $P \circ f : \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$ is a diffeomorphism. Here, the centered circle ($\circ$) stands for function composition. Note that given a TCT $F : \mathbb{R}^n \to \mathbb{R}^n$ is given by $F(0) = 0$ and away from the origin $F = f \circ P$. This construction is summarized in the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n\backslash\{0\} & \xrightarrow{f\circ P} & \mathbb{R}^n\backslash\{0\} \\ P\big\downarrow & \overset{f}{\nearrow} & \big\downarrow P \\ \mathbb{RP}^{n-1} & \xrightarrow{P\circ f} & \mathbb{RP}^{n-1} \end{array} \tag{13}$$

where the definition of the group G of TCTs is the set of maps f such that the diagram commutes and $P \circ f$ is a diffeomorphism.

C. Examples

The simplest example is the case $n = 1$, where there is only one force X. Now, $\mathbb{RP}^{n-1}$ is just a point. The group of diffeomorphisms of the point is a trivial group, consisting of only the identity element. Any bundle over a point is trivial, so in this case the total space of the bundle is just $\mathbb{R}^{\times}$ itself and so the group of TCTs is the group of maps from the point to $\mathbb{R}^{\times}$ which is just $\mathbb{R}^{\times}$ itself, the multiplicative group of nonvanishing real numbers α. The action of this group on the force X is just multiplication by α. So, there is a one to one correspondence between TCTs and nonzero real numbers α. Therefore, we find that if there is only one thermodynamic force, then the TCTs are linear.

The case $n = 2$ shows the full structure of the group. The projective space $\mathbb{RP}^1$ is a semicircle with both extremes identified, which topologically is just the circle S^1. Therefore, the group of TCTs is locally the product of the group of diffeomorphisms of the circle, which physically describes the mixing between X^1 and X^2, with the group of scalings $S^1 \to \mathbb{R}^{\times}$. Now, a rotation of the (X^1, X^2) plane by 180° is a rotation of $\mathbb{RP}^1$ all the way around and so it acts trivially on $\mathbb{RP}^1$. However, it corresponds to the element -1 of the maps from $\mathbb{RP}^1$ to $\mathbb{R}^{\times}$. So indeed the group G is not simply a product of the groups of scalings and rotations; the scalings are nontrivially fibered over the rotations.

III. ALGEBRAIC DESCRIPTION OF THE TCT GROUP

In this section, we shall provide the algebraic description of the TCT group. In particular, we shall define the TCT group and we enunciate the theorem satisfied by the TCT group. Details related to the demonstration of this theorem can be found in the Appendix.

Let S^{n-1} be the $n-1$ dimensional unit sphere ($\|\mathbf{x}\| = 1$), represented as a C^{∞} differentiable manifold, as a submanifold embedded in $\mathbb{R}^n$. Define the equivalent relation $\mathbb{R}$ as follows: $\mathbf{x}, \mathbf{y} \in S^{n-1}$ are equivalent iff $\mathbf{y} = \pm\mathbf{x}$. Denote by Γ_n^p the subgroup of diff ($\mathbb{RP}^{n-1}$) and let $\mathbf{Y} \in \Gamma_n^p$ iff $\mathbf{Y}(-\mathbf{x}) = -\mathbf{Y}(\mathbf{x})$ where $S^{n-1} \ni \mathbf{x} \to \mathbf{Y}(\mathbf{x}) \in S^{n-1}$. The TCT group, denoted by G^s, is the subgroup of homogeneous diffeomorphisms from diff($\mathbb{R}^n\backslash\{0\}$) i.e., diff($\mathbb{R}^n\backslash\{0\}$) $\ni \mathbf{x} \mapsto \mathbf{Y}_g(\mathbf{x}) \in$ diff($\mathbb{R}^n\backslash\{0\}$). Then, $\mathbf{Y}_g \in G^n$ iff

$$\mathbf{Y}_g(\lambda\mathbf{x}) = \lambda\mathbf{Y}_g(\mathbf{x}), \quad \lambda \in \mathbb{R}, g \in G^n. \tag{14}$$

It is possible to demonstrate that the TCT group G^n may be split in a semidirect product of two subgroups where the first one is a normal, Abelian, subgroup. In particular, let us introduce two subgroups N^n and H^n defined as follows.

Let N^n denote the subset (normal subgroup) of G^n having the form

$$\mathbf{Y}_g(\mathbf{x}) = \mathbf{x} r_g(\mathbf{x}), \quad g \in N^n \subset G^n \tag{15}$$

with $r_g(\mathbf{x})$ denoting a positive $C^\infty(\mathbb{R}^n\backslash\{0\})$ homogeneous function, i.e.,

$$r_g(\lambda\mathbf{x}) = r_g(\mathbf{x}) > 0, \quad \lambda \in \mathbb{R}. \tag{16}$$

Denote by H^n the subgroup of G^n with the properties

$$\|\mathbf{Y}_h(\mathbf{x})\| = \|\mathbf{x}\|, \quad \mathbf{Y}_h(-\mathbf{x}) = -\mathbf{Y}_h(\mathbf{x}) \quad \text{with} \quad h \in H^n. \tag{17}$$

As it is proved in the Appendix, the TCT group is the semidirect product of the Abelian normal subgroup N^n and the subgroup H^n, i.e.,

$$G^n = N^n \rtimes H^n. \tag{18}$$

The irreducible representations of the group G are then related to the irreducible representations of the subgroups H and N. In the previous section, we have shown that G is a bundle whose base is diff $(\mathbb{RP}^{n-1})$. Expression (18) specifies, in more rigorous terms, which bundle it is.

A. Properties of the general element of the TCT group

From Eq. (12), we easily get (no Einstein's convention on the repeated indexes)

$$\begin{aligned} U^\mu_\nu \equiv& \frac{\partial X'^\mu}{\partial X^\nu} = F^\mu \delta^1_\nu + \frac{\partial F^\mu}{\partial Y^\nu}\frac{X^1}{X^{\nu-1}}\left(1-\delta^1_\nu\right) \\ &- \frac{X^1 X^{\nu+1}}{(X^\nu)^2}\frac{\partial F^\mu}{\partial Y^{\nu+1}}\left(1-\delta^n_\nu\right) \quad \text{with} \\ Y^\nu \equiv& \frac{X^\nu}{X^{\nu-1}}, \quad \mu,\nu = 1,\ldots,n. \end{aligned} \tag{19}$$

As shown in [1], matrix U^μ_ν satisfies the important relations

$$X^\nu \frac{\partial U^\mu_\nu}{\partial X^\kappa} = 0, \quad X^\kappa \frac{\partial U^\mu_\nu}{\partial X^\kappa} = 0. \tag{20}$$

Close to the identity, it is useful to write the TCT as

$$\begin{aligned} &U^\mu_\nu = \delta^\mu_\nu + \epsilon^\alpha \delta U^\mu_{\nu(\alpha)}, \quad X'^\mu = X^\mu + \epsilon^\alpha \xi^\mu_{(\alpha)} \quad \text{with} \\ &\delta U^\mu_{\nu(\alpha)} = \omega^\mu_{\nu(\alpha)} + X^\kappa \partial_\nu \omega^\mu_{\kappa(\alpha)}, \quad \xi^\mu_{(\alpha)} = \omega^\mu_{\nu(\alpha)} X^\nu \end{aligned} \tag{21}$$

where ϵ^α are infinitesimal parameter coefficients.

B. Examples

1. Linear TCT

In general, the TCT group G^n is a noncompact, infinite Lie group. However, G^n admits several compact and finite subgroups. Linear transformations of the thermodynamic forces are an important subgroup of the G^n. A significant example is the two-dimensional linear transformation

$$\begin{aligned} &X'^1 = a_1 X^1 + \epsilon_1 a_2 X^2 = X^1 + \epsilon_1 \xi^1, \\ &X'^2 = \epsilon_2 b_1 X^1 + b_2 X^2 = X^2 + \epsilon_2 \xi^2 \quad \text{with} \\ &a_1 - 1 = \epsilon_1 \alpha_1, \quad b_2 - 1 = \epsilon_2 \beta_2 \end{aligned} \tag{22}$$

and

$$\omega^\mu_\nu = \begin{pmatrix} \alpha_1 & a_2 \\ b_1 & \beta_2 \end{pmatrix} \quad \text{and} \quad \xi^\mu = \begin{pmatrix} \alpha_1 X^1 + a_2 X^2 \\ b_1 X^1 + \beta_2 X^2 \end{pmatrix}. \tag{23}$$

The four generators of the group are

$$\begin{aligned} &t_1 = -iX^1\partial_{X^1}, \quad t_2 = -iX^1\partial_{X^2}, \\ &t_3 = -iX^2\partial_{X^1}, \quad t_4 = -iX^2\partial_{X^2}. \end{aligned} \tag{24}$$

The Lie algebra reads as

$$\begin{aligned} &[t_\mu,t_\mu] = 0, \quad [t_\mu,t_\nu] = -[t_\nu,t_\mu], \\ &[t_2,t_4] = -it_2, \quad [t_3,t_4] = it_3, \\ &[t_1,t_2] = -it_2, \quad [t_1,t_3] = it_3, \\ &[t_1,t_4] = 0, \quad [t_2,t_3] = it_4 - it_1. \end{aligned} \tag{25}$$

From the Lie algebra, we may construct the adjoint representations of the generators of the group $T^{(\kappa)}_{\mu\nu}$ through the structure constants

$$T^{(\kappa)}_{\mu\nu} = i f^\nu_{\mu(\kappa)} \quad \text{with} \quad [t_\mu,t_\kappa] = f^\nu_{\mu(\kappa)} t_\nu. \tag{26}$$

We get

$$\begin{aligned} &T^{(1)}_{\mu\nu} = \begin{pmatrix} 0&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&0 \end{pmatrix}, \quad T^{(2)}_{\mu\nu} = \begin{pmatrix} 0&-i&0&0\\0&0&0&0\\i&0&0&-i\\0&i&0&0 \end{pmatrix}, \\ &T^{(3)}_{\mu\nu} = \begin{pmatrix} 0&0&i&0\\-i&0&0&i\\0&0&0&0\\0&0&-i&0 \end{pmatrix}, \quad T^{(4)}_{\mu\nu} = \begin{pmatrix} 0&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&0 \end{pmatrix}. \end{aligned} \tag{27}$$

It is worth mentioning that the previous transformations play an important role in physics, for example, in tokamak plasmas in the fully collisional transport regime (the so-called Pfirsch-Schlüter transport regime) [11,12]. Here, the two thermodynamic forces read as $X^1 = (n_e T_e)^{-1}\nabla_r P$ and $X^2 = -T_e^{-1}\nabla_r T_e$, with n_e, T_e, and P denoting the electron density number, the electron temperature, and the total pressure of the plasma, respectively. In this case, the subgroup of G^n is finite and compact.

Another example of linear TCT, widely used in tokamak plasmas in the weak-collisional transport regime (the so-called banana regime), is provided by the Hinton-Hazeltine transformations [13]. In this case, the TCT read as

$$\begin{aligned} &X'^1 = X^1 - \tfrac{5}{2}X^2 - \tfrac{5}{2}Z^{-1}X^3, \\ &X'^2 = X^2, \\ &X'^3 = X^3, \\ &X'^4 = X^4 \end{aligned} \tag{28}$$

with Z denoting the charge number. In this particular case, $X^1 = -(n_e T_e)^{-1}\nabla_r P$, $X^2 = -T_e^{-1}\nabla_r T_e$,

$X^3 = -T_i^{-1}\nabla_r T_i$, and $X^4 = \langle B^2\rangle^{-1/2}\langle BE_\parallel^A\rangle$, with B and $E_\parallel^A$ denoting the *intensity of the magnetic field* and the *electric field generated by the external coils, parallel to the magnetic field,* respectively. The angular brackets denote the *averaged magnetic surface operation* (see, for example, [11]). In this case, the space of the thermodynamic forces is four dimensional. The TCT group possesses 16 generators, which are similar to the ones given by Eqs. (24), with adjoint representations also similar to Eqs. (27), but the dimensions of the matrices are 16×16. Note that, also in this case, the TCT subgroup is compact and finite.

2. Nonlinear TCT

The linear transformations are an example of (closed) subalgebra of the TCT group. However, it is easy to convince ourselves that nonlinear examples of TCT subalgebras may also be found. Consider, for example, the following TCT:

$$X'^1 = a_1X^1,\quad X'^2 = b_1X^1 + b_2\left(\frac{X^2}{X^1}\right)X^2 \quad \text{with}$$
$$a_1 - 1 = \epsilon_1\alpha_1,\quad b_2 - 1 = \epsilon_2\beta_2. \tag{29}$$

The three generators of the group and their adjoint representations read as, respectively,

$$t_1 = -iX^1\partial_{X^1},\quad t_2 = -iX^2\partial_{X^2},\quad t_3 = -i\left(\frac{X^2}{X^1}\right)X^2\partial_{X^2},$$
$$T^{(1)}_{\mu\nu} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&-i\end{pmatrix},\quad T^{(2)}_{\mu\nu} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&i\end{pmatrix},$$
$$T^{(3)}_{\mu\nu} = \begin{pmatrix}0&0&i\\0&0&-i\\0&0&0\end{pmatrix}. \tag{30}$$

IV. NOETHER'S CURRENT FOR FULLY COLLISIONAL TOKAMAK PLASMAS

As known, through Noether's theorem one can determine the conserved quantities from the observed symmetries of a physical system. In particular, consider the action

$$I = \int \mathcal{L}(\Phi^A,\partial_\mu\Phi^A,X^\mu)\sqrt{g}\,dX^n \tag{31}$$

with ϕ^A denoting the set of differentiable fields defined over all space of the thermodynamic forces and $\mathcal{L}$ the Lagrangian density, respectively. In our case, $\Phi^A = \{g_{\kappa\nu}, f_{\kappa\nu}, \Gamma^\lambda_{\kappa\nu}\}$. Let the action be invariant under certain transformations of the thermodynamic forces coordinates X^μ and the field Φ^A:

$$\begin{aligned} X^\mu &\to X^\mu + \delta X^\mu = X^\mu + \epsilon_\alpha\xi^\mu_{(\alpha)},\\ \Phi^A(X) &\to \Phi^A(X) + \delta\Phi^A(X)\\ &= \Phi^A(X) + \bar\delta\Phi^A(X) + \tilde\delta\Phi^A(X)\\ &= \Phi^A(X) + \epsilon_\alpha\Psi^A_{(\alpha)}(X), \end{aligned} \tag{32}$$

where $\delta\Phi^A$ denotes the *transformation in the field variables*, $\bar\delta\Phi^A$ the *intrinsic changes of the field*, and $\tilde\delta\Phi^A$ the *transformation of the field variables due to the coordinates variation*, respectively. Noether's theorem states that N *currents densities* are conserved, with N equals to the number of generators of the Lie group associated to the TCT [14,15]. In our case, the action remains invariant only under TCT (and not under the field transformations). Hence, the expressions of the N Noether currents j^μ_α reduce to

$$j^\mu_\alpha = \frac{\partial L}{\partial\Phi^A_\mu}\mathcal{L}_{\xi_\alpha}\Phi^A - L\xi^\mu_\alpha \quad \text{with} \tag{33}$$
$$\partial_\mu\left[\sqrt{g}J^\mu_\alpha\right] = 0,\quad J^\mu_\alpha \equiv g^{-1/2}j^\mu_\alpha \quad (\alpha = 1,\ldots,N).$$

Here, $\mathcal{L}_\xi$ denotes the Lie derivatives along the ξ^μ_α vector and $L \equiv \mathcal{L}\sqrt{g}$, respectively.

As an example of application, let us consider the action (10) and the case of tokamak plasmas in fully collisional transport regime, with the TCT given by Eqs. (22)–(25). After (some tedious) calculations, and under the realistic approximation $1/\sigma \ll 1$ valid for tokamak plasmas, we finally get (see Appendixes)

$$\begin{aligned} J^{\mu\lambda}_\nu &= \tfrac{1}{2}g^{\kappa\lambda}A^{\mu\eta}_{\nu\kappa\eta} + \tfrac{1}{2}g^{\beta\lambda}A^{\mu\eta}_{\nu\eta\beta} - g^{\kappa\beta}A^{\mu\lambda}_{\nu\kappa\beta},\\ A^{\mu\eta}_{\nu\kappa\beta} &= X^\mu\Gamma^\eta_{\kappa\beta,\nu} - \Gamma^\mu_{\kappa\beta}\delta^\alpha_\nu + \Gamma^\alpha_{\nu\beta}\delta^\mu_\kappa + \Gamma^\mu_{\nu\kappa}\delta^\mu_\beta. \end{aligned} \tag{34}$$

Note that there are no Noether's currents in the fully collisional transport regimes since in this case all derivatives of the transport coefficients with respect to X^μ (and, hence, $\Gamma^\kappa_{\mu\nu}$ and its derivatives $\Gamma^\kappa_{\mu\nu,\eta}$) are identically equal to zero. These currents appear only in the nonlinear transport regime where the derivatives of the transport coefficients with respect to the thermodynamic forces do not vanish [1]. Even though calculations are more complex, it is possible to show that the above conclusions apply also to the weak collisional (banana) and these plateau transport regimes. Figure 3 shows one component of Noether's current J_1^{11} against the two thermodynamic forces X^1 and X^2. The contour plot of this current is illustrated in Fig. 4 [these graphics have been

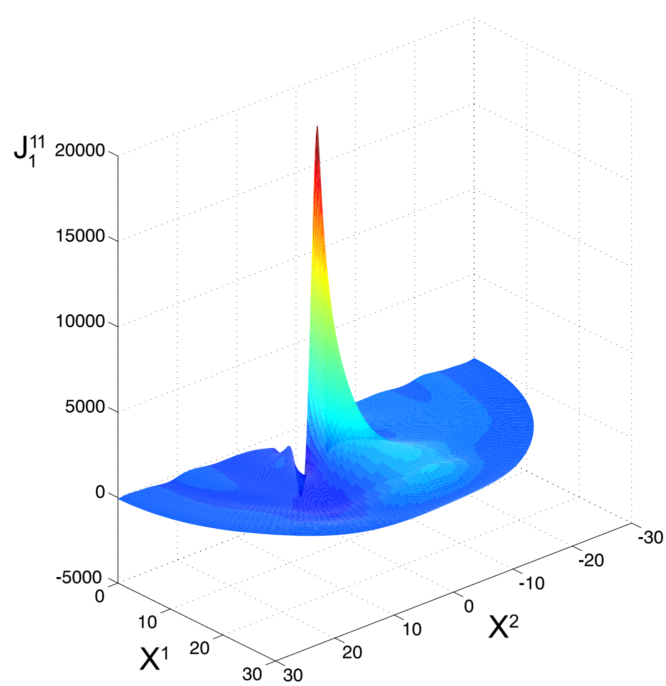

FIG. 3. Component J_1^{11} corresponding to Eq. (33).

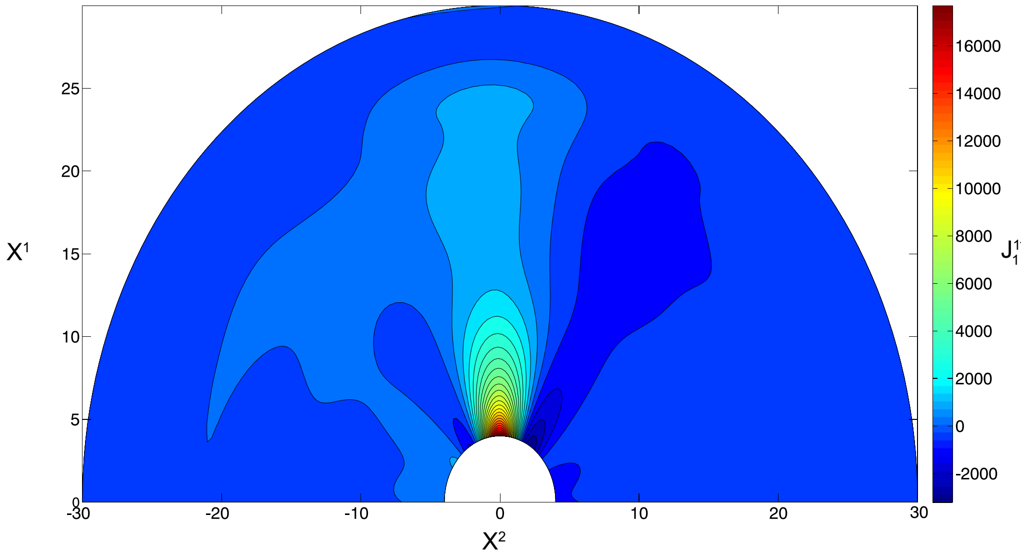

FIG. 4. Contour plot of the J_1^{11} profile.

produced by Peeters, from the Université Libre de Bruxelles (ULB), Brussels (Belgium)].

V. DERIVATION OF THE COLLISIONAL OPERATOR THAT ENSURES, AT THE LOWEST ORDER, THE COVARIANCE UNDER TCT OF THE CLOSURE TRANSPORT RELATIONS

The aim of this section is to derive the expression of the collisional operator for magnetically confined plasmas, which guarantees that the thermodynamic covariance principle (TCP) is satisfied by the closure transport relations (i.e., the flux-force relations). Let us consider a two-component system of charged particles. The statistical state is represented by two reduced distribution functions f^α corresponding to ions i and electrons e [16] (no Einstein's convention on index α):

$$\frac{\partial}{\partial t}f^\alpha(\mathbf{q},\mathbf{v},t) = -\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{q}}f^\alpha(\mathbf{q},\mathbf{v},t) - \frac{e_\alpha}{m_\alpha}\left[\mathbf{E}(\mathbf{q},t)+\frac{1}{c}\mathbf{v}\wedge\mathbf{B}(\mathbf{q})\right]\cdot\frac{\partial}{\partial\mathbf{v}}f^\alpha(\mathbf{q},\mathbf{v},t) + C^\alpha(f,f). \tag{35}$$

Here, $\alpha = e,i$ and c is the speed of light in vacuum. Moreover, e_α and m_α are the *charge* and the *mass* of species α, $\mathbf{q}$ and $\mathbf{v}$ denote the *generalized coordinates* and the *velocity of the particle*, and $\mathbf{E}$ and $\mathbf{B}$ the *electric* and the *magnetic fields*, respectively. Note that the first term on the right-hand side of Eq. (35) represents the *free flow*, the second term corresponds to the *electromagnetic contribution*, and the last term is the contribution due to *collisions*. In many applications in plasmas physics (including those involving the radio-frequency waves), collisions dominate the thermal particles. Therefore, the distribution function can conveniently be expanded about a Maxwellian

$$f^\alpha(\mathbf{x},\mathbf{v},t) = f^\alpha_{eq}(\mathbf{x},\mathbf{v},t)[1+\chi(\mathbf{x},\mathbf{v},t)], \tag{36}$$

where $\mathbf{x}$ denotes the *position of the particle*. In Eq. (36), we have introduced the *reference state* $f^\alpha_{eq}(\mathbf{x},\mathbf{v},t)$, i.e., the local plasma equilibrium (L.P.E.), and the *deviation from the reference state* χ. The local plasma equilibrium is defined in the following way. The electron-electron and ion-ion collisions bring the plasma in a short time to a state of local plasma equilibrium satisfying the equations

$$C^{ee} = C^{ii} = 0 \tag{37}$$

with C^{ee} and C^{ii} denoting the electron-electron and ion-ion collisions, respectively (see below the expression of the collisional operator). The L.P.E. is the solution of Eqs. (37):

$$f^\alpha_{eq}(\mathbf{x},\mathbf{v},t) = (2\pi)^{-3/2}n_\alpha(\mathbf{x},t)\left[\frac{m_\alpha}{T_\alpha(\mathbf{x},t)}\right]^{3/2}\exp(-\mathbf{c}_N\cdot\mathbf{c}_N),$$
$$\mathbf{c}_N \equiv \left(\frac{m_\alpha}{T_\alpha}\right)^{1/2}[\mathbf{v}-\mathbf{u}(\mathbf{x},t)], \tag{38}$$

where $\mathbf{u}^\alpha$, n_α, T_α are the *mean velocity*, the *number density*, and *temperature*, respectively. The deviation χ may be developed in terms of the Hermite polynomials $H^{(m)}_{r_1 r_2\ldots}$:

$$\chi^\alpha(\mathbf{x},\mathbf{c}_N,t) = \sum_{n=0}^{\infty} q^{\alpha(2n)}(\mathbf{x},t)H^{(2n)}(\mathbf{c}_N) + \sum_{n=0}^{\infty} q_r^{\alpha(2n+1)}(\mathbf{x},t)H_r^{(2n+1)}(\mathbf{c}_N) + \sum_{n=0}^{\infty} q_{rs}^{\alpha(2n)}(\mathbf{x},t)H_{rs}^{(2n)}(\mathbf{c}_N) + \ldots \tag{39}$$

with $q^{\alpha(m)}(\mathbf{x},t)$ denoting the Hermitian moments [16]. The Landau collisional operator can be brought into the form [11,17]

$$C^\alpha(f,f) = \sum_{\beta=e,i} C^{\alpha\beta} \quad \text{with}$$

$$C^{\alpha\beta}(f^\alpha(1),f^\beta(2)) = 2\pi e_\alpha^2 e_\beta^2 \ln\Lambda \int d\mathbf{v}_2 \tilde{\partial}_r G_{rs}(\mathbf{v}_1-\mathbf{v}_2) \times \tilde{\partial}_s f^\alpha(1) f^\beta(2) \tag{40}$$

with r,s identifying the components of a vector, and indices (1) and (2) the colliding particles (1) and (2), i.e., $(1) \equiv (\mathbf{x}_1,\mathbf{v}_1,t)$ and $(2) \equiv (\mathbf{x}_2,\mathbf{v}_2,t)$. Here, $\ln\Lambda$ and G_{rs} are the *Coulomb logarithm* (linked to the *Debye length* λ_D) and the *Landau tensor*, respectively, i.e.,

$$\ln\Lambda = \ln\frac{3(T_e+T_i)\lambda_D}{2Ze^2},$$
$$\lambda_D = \left[\frac{4\pi Ze^2(n_eT_e+n_iT_i)}{T_eT_i(1+Z)}\right]^{-1/2}, \tag{41}$$
$$G_{rs}(\mathbf{a}) = \frac{a^2\delta_{rs}-a_ra_s}{a^3}$$

with $\mathbf{a}$ denoting the *relative velocity of two particles*, i.e., $\mathbf{a}\equiv\mathbf{v}_1-\mathbf{v}_2$. The operator $\tilde{\partial}_r$ is defined as follows:

$$\tilde{\partial}_r \equiv m_\alpha^{-1}\partial_{v_{1r}} - m_\beta^{-1}\partial_{v_{2r}}. \tag{42}$$

By inserting Eqs. (36)–(39) into Eq. (35), and by truncating the expansion up to the second order of the (small) *drift parameter* ϵ (defined as the Larmour radius over a macroscopic length), we get the *vector moment equations* [11]

$$\Omega_\alpha\tau_\alpha\epsilon_{rmn}q_m^{\alpha(1)}b_n + \tau_\alpha Q_r^{\alpha(1)} + g_r^{\alpha(1)} + \bar{g}_r^{\alpha(1)} + O(\epsilon^2) = 0,$$
$$\Omega_\alpha\tau_\alpha\epsilon_{rmn}q_m^{\alpha(3)}b_n + \tau_\alpha Q_r^{\alpha(3)} + g_r^{\alpha(3)} + \bar{g}_r^{\alpha(3)} + O(\epsilon^2) = 0,$$
$$\Omega_\alpha\tau_\alpha\epsilon_{rmn}q_m^{\alpha(5)}b_n + \tau_\alpha Q_r^{\alpha(5)} + \bar{g}_r^{\alpha(5)} + O(\epsilon^2) = 0,$$
$$Q_r^{\alpha(m)} = n_\alpha^{-1}\int d\mathbf{v}\, H_r^{(m)}[(m_\alpha/T_\alpha)^{1/2}(\mathbf{v}-\mathbf{u}^\alpha)]C^\alpha, \tag{43}$$

where the Einstein convention is adopted on the repeated indices m and n, but not on the index α. Here, b_n is a unit vector along the magnetic field $\mathbf{B}$, i.e., $b_n\equiv B_n/B$, ϵ_{rmn} is the completely antisymmetric Levi-Civita symbol, Ω_α is the Larmor frequency of species α, and τ_α is the *relaxation time of species* α, respectively. Moreover, $g_r^{\alpha(n)}$, $\bar{g}_r^{\alpha(n)}$, and $Q_r^{\alpha(n)}$ are the *dimensionless source terms related to the thermodynamic forces*, the *additional sources terms in the long mean free path transport regime*, and the *dimensionless friction terms*, respectively (the exact definitions of these quantities may be found in Ref. [11]).

For collision-dominated plasmas (i.e., in absence of turbulence), the entropy production Σ^α of the plasma for species α may be brought into the form [11]

$$\Sigma^\alpha = -\tau_\alpha\sum_{n=0(1)}^{N} q_r^{\alpha(n+1)}Q_r^{\alpha(2n+1)}. \tag{44}$$

The lower limit for n is 0 for the electrons and 1 for the ions. Hence, thanks to this theorem, $Q_r^{\alpha(n)}$ and $q_r^{\alpha(n)}$ are the *thermodynamic forces* and the *thermodynamic fluxes* for magnetic confined plasmas, respectively. Equation (44) tells us that the last equation of Eqs. (43) is the closure equation (flux-forces relation) for tokamak plasmas, derived by kinetic theory. The region where the transport coefficients do not depend on the thermodynamic forces is referred to as *Onsager's region* or the *linear thermodynamic regime*. A well-founded microscopic explanation on the validity of the linear phenomenological laws was developed by Onsager in 1931 [18,19]. Onsager's theory is based on three assumptions: (i) the probability distribution function for the fluctuations of thermodynamic quantities (temperature, pressure, degree of advancement of a chemical reaction, etc.) is a Maxwellian; (ii) fluctuations decay according to a linear law; and (iii) the principle of the detailed balance (or the microscopic reversibility) is satisfied. Out of Onsager's regime, the transport coefficients may depend on the thermodynamic forces. This happens when the above-mentioned assumption (1) and/or assumption (2) are/is not satisfied. Magnetically confined tokamak plasmas are a typical example of thermodynamic systems out of Onsager's region. In this case, even in absence of turbulence, the local distribution functions of species (electrons and ions) deviate from the (local) Maxwellian [see Eq. (36)]. After a short transition time, the plasma remains close to (but, it is not in) a state of local equilibrium (see, for example, [11,12]). The neoclassical theory is a linear transport theory (see, for example, [11]) meaning by this, a theory where the moment equations are coupled to the closure relations (i.e., flux-force relations), which have been linearized with respect to the generalized frictions (see, for example, Ref. [16]). This approximation is clearly in contrast with the fact that the distribution function of the thermodynamic fluctuations is *not* a Maxwellian and it could be a possible cause of disagreement between the theoretical predictions and the experimental profiles [12,20]. However, it is important to mention that it is well accepted that the main reason of this discrepancy is attributed to turbulent phenomena existing in tokamak plasmas. Fluctuations in plasmas can become unstable and therefore amplified, with their nonlinear interaction, successively leading the plasma to a state, which is far away from equilibrium. In this condition, the transport properties are supposed to change significantly and to exhibit qualitative features and properties that could not be explained by collisional transport processes, e.g., size scaling with machine dimensions and nonlocal behaviors that clearly point at turbulence spreading, etc. (see, for example, Ref. [21]). Hence, the truly complete transport theory of plasmas must self-consistently incorporate the instability theory that includes the influence of nonlinear transformations on fluctuations. This global approach is the purpose of the so-called anomalous transport theory (still far from a complete and comprehensive theory). This type of problem is, however, far beyond the scope of this work. Here, more modestly, we deal with plasmas in the collisional-dominated transport regime, characterized by a time scale which is much longer than one involved in the so-called fluctuation-induced turbulence transport.

Our aim is to determine the simplest expression of the collisional operator such that the resulting closure equation satisfies the TCP (without, of course, violating the energy, mass, and momentum conservation laws). Concretely, in mathematical terms, we need to identify an operator able *to kill* the terms that do not satisfy the TCP and, in order not to violate the conservation laws, which commutes with the operator $\tilde{\partial}_r$. The last equation in Eq. (43) will satisfy the TCP if

$$\text{when}\quad C^{\alpha\beta}\to\lambda C^{\alpha\beta}\quad\text{then}\quad Q_r^{\alpha(m)}\to\lambda Q_r^{\alpha(m)} \tag{45}$$

with λ denoting a constant parameter. We introduce now the operator $\mathcal{O}_{\mathrm{TCT}}$ defined as follows:

$$\mathcal{O}_{\mathrm{TCT}} \equiv \left[2-\chi^\alpha(1)\frac{\partial}{\partial\chi^\alpha(1)} - \chi^\beta(2)\frac{\partial}{\partial\chi^\beta(2)}\right]. \tag{46}$$

It is easily checked that this operator possesses the following properties:

$$\mathcal{O}_{\rm TCT}(\chi^\alpha)=\chi^\alpha,\quad \mathcal{O}_{\rm TCT}((\chi^\alpha)^2)=0,$$
$$\mathcal{O}_{\rm TCT}(\chi^\alpha(1)\chi^\beta(2))=0,\quad [\mathcal{O}_{\rm TCT},\tilde{\partial}_r]=0, \tag{47}$$

where the square brackets denote the *Lie brackets*. The last equation in Eq. (43) (the closure equation) satisfies the TCP iff

$$C_{\rm TCT}^{\alpha\beta}=C^{\alpha\beta}[\mathcal{O}_{\rm TCT}(f(1)f(2))] \tag{48}$$

or

$$C_{\rm TCT}^{\alpha\beta}=2\pi e_\alpha^2e_\beta^2\ln\Lambda\int d\mathbf{v}_2\tilde{\partial}_rG_{rs}(\mathbf{v}_1-\mathbf{v}_2)\tilde{\partial}_s$$
$$\times\{f^\alpha(1)_{eq}f^\beta(2)_{eq}[\chi^\alpha(1)+\chi^\beta(2)]\}. \tag{49}$$

Equation (49) can conveniently be written in the form

$$C_{\rm TCT}^{\alpha\beta}=2\pi e_\alpha^2e_\beta^2\ln\Lambda\int d\mathbf{v}_2\tilde{\partial}_rG_{rs}(\mathbf{v}_1-\mathbf{v}_2)$$
$$\times\ \tilde{\partial}_s(f^\alpha(1)f_{eq}^\beta(2))_+\quad\text{with}$$
$$(f^\alpha(1)f_{eq}^\beta(2))_+\equiv f^\alpha(1)f_{eq}^\beta(2)+f_{eq}^\alpha(1)f^\beta(2). \tag{50}$$

Thanks to the last relation in Eqs, (47), we also get

$$\int d\mathbf{v}\,C_{\rm TCT}^\alpha=0\qquad(\alpha=e,i)\qquad\text{number of particles conservation,} \tag{51}$$

$$\sum_\alpha m_\alpha\int d\mathbf{v}\,v_rC_{\rm TCT}^\alpha=0\quad(r=1,2,3)\qquad\text{momentum conservation,} \tag{52}$$

$$\sum_\alpha\frac{1}{2}m_\alpha\int d\mathbf{v}\,v^2C_{\rm TCT}^\alpha=0\qquad\text{energy conservation} \tag{53}$$

with $C_{\rm TCT}^\alpha=\Sigma_{\beta=e,i}C_{\rm TCT}^{\alpha\beta}$. Equation (50) is the linearized collision operator used in existing literature [13,22]. However, it should be noted that in previous literature the quadratic contributions in the distribution functions are ignored without any physical justification. Here, on the contrary, the linearization process of the collisional operator rests upon the validity of a fundamental principle, that is, the thermodynamic covariance principle. Notice that to linearize the collisional operator does not mean that we are in the Onsager regime. As known, this regime is attained by performing two operations: (1) the transport phenomena is evaluated by determining a finite number of Hermitian moments of the distribution functions, and (2) the truncated set of moment equations is linearized in some appropriate way [11,16].

In order to test the validity of the TFT, we have computed a concrete example of heat loss in L-mode, collisional, tokamak plasmas. To perform correctly this calculation, we have solved Eq. (A5), subject to boundary conditions, by taking into account the so-called Shafranov's shift. The Shafranov shift is the outward radial displacement $\Delta(r)$ of the center of the magnetic flux surfaces with the minor radius r of the tokamak, induced by the *plasma pressure* $\beta_{\rm mag}$. This shift compresses the surfaces on the outboard side [23]. In terms of number density, temperature and the intensity of the magnetic field $\beta_{\rm mag}=P/P_{\rm mag}$, with $P=nT$ and $P_{\rm mag}$ given in the forthcoming Eq. (55). Figure 5 depicts the Shafranov shift due to $\beta_{\rm mag}$ versus the minor radius of the tokamak. Equation (A5) has been solved with the boundary conditions obtained by imposing that, for very large values of the thermodynamic forces there are no privileged directions for $h_{\mu\nu}$ in the thermodynamic space [12]. In order to take into account the Shafranov displacement, in the limit of high aspect ratio, the *magnetic configuration* $\mathbf{B}$ is written in the form (normalized to 2π)

$$\mathbf{B}=F\nabla\phi+\nabla\phi\times\nabla\Psi\quad\text{with}\quad F\simeq B_0R_0\quad\text{and}\quad\Psi(r)$$
$$\simeq B_0\int_0^r r'/q(r')\,dr'. \tag{54}$$

Here, ϕ, r, and Ψ are the toroidal angle, the minor radius coordinate, and the poloidal magnetic flux, respectively. B_0 and R_0 denote the intensity of the magnetic field at the magnetic axis of the tokamak and the major radius of the tokamak, respectively. In coordinates (R,ϕ,Z), the Shafranov shift $\Delta(r)$ is estimated to be roughly equal to

$$R=R_0+\Delta(r)+r\cos\theta,\quad Z=r\sin\theta\quad\text{with}$$
$$\Delta(r)\simeq\beta_{\rm mag}r^2/R_0,\quad\beta_{\rm mag}=2\mu_0nT/B_v^2 \tag{55}$$

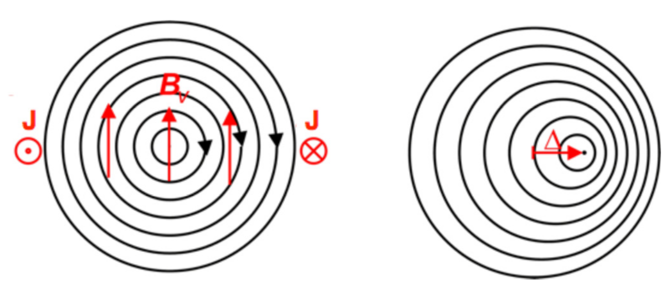

FIG. 5. Shafranov shift. In tokamak plasmas, the plasma pressure leads to an outward shift Δ of the center of the magnetic flux surfaces. J indicates the direction of the electric current that flows inside the plasma. Note that the poloidal magnetic field increases and the magnetic pressure can, then, balance the outward force.

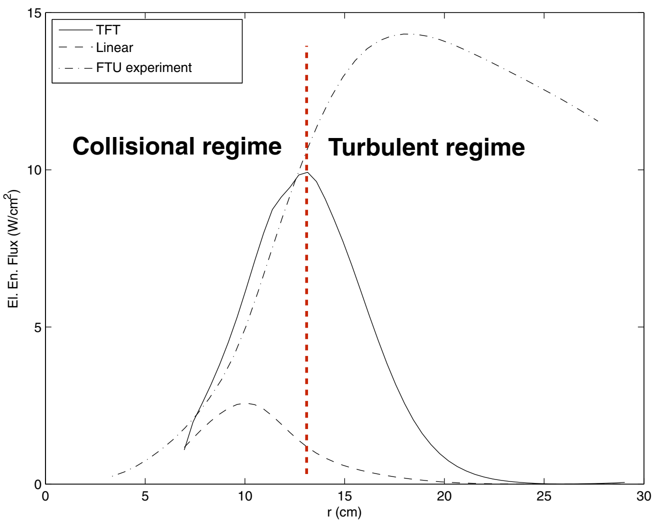

FIG. 6. Electron heat loss in fully collisional FTU plasmas vs the minor radius of the tokamak. The highest dashed line is the experimental profile. These data have been provided by Marinucci from the ENEA C.R.-EUROfusion in Frascati [24]. The bold line is the theoretical profile obtained by the nonlinear theory satisfying the TCP (TFT) and the lowest dashed profile corresponds to the theoretical prediction obtained by Onsager's theory (i.e., by the neoclassical theory).

with θ, μ_0, and B_v denoting the poloidal angle, the magnetic permeability constant, and the poloidal magnetic field, respectively.

Figure 6 shows a comparison between experimental data for fully collisional FTU (Frascati tokamak upgrade) plasmas and the theoretical predictions. In the vertical axis we have the (surface magnetic-averaged) radial electron heat flux, and in the horizontal axis the minor radius of the tokamak. The lowest dashed profile corresponds to the Onsager (neoclassical) theory and the bold line to the nonlinear theory [thermodynamical field theory (TFT)] satisfying the TCP, respectively. The highest profile is the experimental data provided by the ENEA C.R.-EUROfusion [24]. As we can see, the TCP principle is well satisfied in the core of the plasma where plasma is in the collisional transport regime. Towards the edge of the tokamak, transport is dominated by turbulence. We conclude this section by mentioning that close to Onsager's region, i.e., $g_{\mu\nu} \simeq L_{\mu\nu} + h_{\mu\nu}$ (with $L_{\mu\nu}$ and $h_{\mu\nu}$ denoting the Onsager matrix and its perturbation, respectively), *at the leading order in* $h_{\mu\nu}$ we may transform the closure equation for magnetically confined plasmas [i.e., the last equation in Eqs. (43)] in a differential equation, which is covariant under TCT. It is possible to show that the equation to be satisfied by perturbation $h_{\mu\nu}$ is the covariant (under TCT) Laplacian operator constructed with the metric $L_{\mu\nu}$.

VI. CONCLUSIONS AND PERSPECTIVES

We have studied the Lie group associated to the thermodynamic covariance principle (TCP). This principle affirms that the nonlinear closure equations must be covariant under the transformations of thermodynamic forces leaving invariant the entropy production and the Glansdorff-Prigogine dissipative quantity. This class of admissible transformations, referred to as the thermodynamic covariant transformations (TCT), is the most general class of force transformations able to warrant the equivalence between thermodynamic systems. According to the TCP, the Lagrangian of a thermodynamic system should be invariant under TCT. The TCT form a group.

The first part of the work deals with theory. In particular, we have shown that the TCT group is a bundle whose base is diff($\mathbb{RP}^{n-1}$) and the fiber is the space of maps $\mathbb{RP}^{n-1} \to \mathbb{R}^{\times}$. The TCT group may also be split as semidirect product of an Abelian normal subgroup and another subgroup of TCT. The irreducible representations of the TCT group are therefore related to the irreducible representations of these two subgroups.

The second part of the work is devoted to applications. We applied the formalism to magnetically confined plasmas. As an example of calculation, we have derived the Noether's current associated to this TCT invariance for magnetically confined plasmas in fully collisional transport regime. Always in the case of collisional tokamak plasmas, we have derived the expression of the collisional operator able to determine the closure equations satisfying the TCP. We have shown that (contrary to Onsager's theory) the theoretical predictions based on the validity of TCP are in fairly good agreement with experiments in the expected region of validity.

The mathematical study corresponding to the Lie symmetry group associated to this symmetry is under progress. Currently, we are also studying the symmetry-breaking mechanism and the Hamiltonian formulation of problems related to thermodynamic systems out of equilibrium.

It is worth mentioning that the TCP is actually largely used in a wide variety of thermodynamic processes ranging from nonequilibrium chemical reactions to transport processes in tokamak plasmas. As far as we know, the validity of the thermodynamic covariance principle has been verified empirically without exception in physics until now. The influence of nonlinear transformations on fluctuations will be subject of future works.

We close this section by mentioning some perspectives of this work.

(i) Up to now, we have applied the above formalism to thermodynamic systems under the hypothesis of *weak-field approximation* (i.e., the correction to the Onsager metrics is very small) and by assuming that *the dimensionless entropy is much greater than one* (i.e., $\sigma \gg 1$). Of course, these approximations correspond to a limit case.[10] Under these assumptions, by analyzing several cases of thermodynamic systems out of equilibrium, we checked that there is a fairly good agreement between the theoretical predictions of the TFT and experiments (see Sec. V and the examples examined in the works cited in the Bibliography). However, disagreements appear in the region where $\sigma \sim 1$. In particular, in Sec. V we showed that the disagreement appears in the region of the tokamak where the plasma is in the turbulent regime. Incidentally, this corresponds also to the region where $\sigma \sim 1$. Until now, we never explored the solutions of the equations when the system is in this intermediate region. Note that when

[10]Note that the case $\sigma \ll 1$ corresponds to Onsager's regime.

$\sigma \sim 1$, Eqs. (A4) and (A5) lose validity and they have to be replaced by the expressions obtained by solving the equations reported in Ref. [1]. To be clear, the differential equations for getting the nonlinear closure equations may still be solved by adopting the weak-field approximation, but we have to give up to the approximation $\sigma \gg 1$. In addition, when the system is in this intermediate region the *source of the thermodynamic space* $T_{\mu\nu} = -S_\lambda^{\alpha\beta}\frac{\delta\tilde{\Gamma}^\lambda_{\alpha\beta}}{\delta g^{\mu\nu}}$ takes all its importance and its expression can no longer be neglected. In this regard, we would like to recall that the universal criterion of evolution (UCE) is derived by Glansdorff and Prigogine without neglecting, in the hydrodynamic or in the plasma-dynamic equations, the terms leading to turbulence and without assuming the linear phenomenological laws. Briefly, the main result of our analysis is that the UCE holds also for systems in the turbulent regime and comparisons between the TFT predictions and experiments should be carried out in the region $\sigma \sim 1$. The study of transport processes in the intermediate region $\sigma \sim 1$ is under progress.

(ii) Another aspect that should be analyzed more in depth is the *stability of the solutions*. This important problem is treated in Ref. [5] from a thermodynamical point of view and in Ref. [25] by geometrical methods. In [25], the problem has been approached by supposing the validity of the general covariance. However, as mentioned in the Introduction of this paper, this kind of covariance is not always satisfied and the thermodynamic covariance principle (TCP) must be applied. Works on this matter are also under progress.

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APPENDIX A: THERMODYNAMICAL FIELD THEORY IN BRIEF [1]

In this Appendix, we summarize briefly the main results of the thermodynamical field theory (TFT). The following table provides a sketch of the TFT.

More in detail, the TFT is based upon two constraints (A) and (B) and one assumption (C). Let us analyze these hypotheses.

Constraint (A). The second law of thermodynamics and the universal criterion of evolution (UCE) [with the minimum entropy production theorem (MEP) as special case of the UCE] should be respected. Constraint (A) allows to introduce the space of the thermodynamic forces (or, simply, the *thermodynamic space*). In particular,

(i) the second law of thermodynamics allows introducing the metric tensor;

(ii) the UCE determines the expression of the affine connection.

As mentioned in the Introduction, these quantities may be measured experimentally (and estimated theoretically by TFT).

Constraint (B). This constraint refers to the *principle of equivalence* between two thermodynamic systems. Equivalence is ensured by imposing the invariance of both expressions: the dimensionless entropy production σ and the Glansdorff-Prigogine dissipative quantity P. To a principle of equivalence is associated a symmetry group. In our case, to the thermodynamic equivalence is associated the TCT (thermodynamic covariance transformations) group. The concept of equivalent thermodynamic systems was originally introduced by De Donder and Prigogine, and it has been recently deeply investigated and revised in Refs. [1,8]. This statement

	Thermodynamical field theory (TFT)
Hypotheses of the TFT	Two Constraints and one assumption
Constraint (A)	Validity of the laws and the theorems of the thermodynamics of irreversible processes
Constraint (B)	Validity of the thermodynamic covariance principle (TCP)
Assumption (C)	Introduction of the principle of least action
Geometry	Non-Riemannian
Metric	Symmetric piece of the transport coefficients
Affine connection	$\tilde{\Gamma}^\mu_{\alpha\beta}$. If $f_{\mu\nu}=0$, $\tilde{\Gamma}^\mu_{\alpha\beta}$ is given by Eq. (11) [or by Eq. (A4) in the weak-field approximation and for $\sigma \gg 1$] In general, $\tilde{\Gamma}^\mu_{\alpha\beta}$ is given by Eq. (57) reported in Ref. [1]
Nonlinear transport coefficients	Solutions of Eq. (A5) in the weak-field approximation and for $\sigma \gg 1$. In general, they are the solution of the first equation in Eqs. (65) reported in Ref. [1]
Action	$I = \int[R - (\Gamma^\lambda_{\alpha\beta} - \tilde{\Gamma}^\lambda_{\alpha\beta})S_\lambda^{\alpha\beta}]\sqrt{g}\, d^nX$
Source term of the space	$T_{\mu\nu} = -S_\lambda^{\alpha\beta}\frac{\delta\tilde{\Gamma}^\lambda_{\alpha\beta}}{\delta g^{\mu\nu}}$

stems from the Einstein formula linking the *probability of a fluctuation* $\mathcal{W}$ with the *entropy production strength* $\Delta_I S$, associated with the fluctuations from the nonequilibrium steady state. Denoting by ξ_i $(i = 1 \ldots m)$ the m deviations of the thermodynamic quantities from their equilibrium value, Prigogine proposed that the probability distribution of finding a state in which the values ξ_i lie between ξ_i and $\xi_i + d\xi_i$ is given by [7]

$$\mathcal{W} = W_0 \exp[\Delta_{\rm I} S/k_B] \qquad \text{where} \quad \Delta_{\rm I} S = \int_E^F d_{\rm I} s,$$
$$\frac{d_{\rm I} s}{dt} \equiv \int_\Omega \sigma \, dv \tag{A1}$$

with dv denoting the spatial volume element and the integration is over the spatial volume Ω occupied by the system. Here, k_B is the Boltzmann constant and W_0 is a normalization constant that ensures the sum of all probabilities equals one, respectively. E and F indicate the equilibrium state and the state to which a fluctuation has driven the system, respectively. We note that the probability distribution (A1) remains unaltered for flux-force transformations leaving invariant the entropy production. On the basis of the above observations, and other concrete examples analyzed in [6,7], De Donder and Prigogine formulated, for the first time, the concept of *equivalent systems from the thermodynamical point of view*. For De Donder and Prigogine, *thermodynamic systems are thermodynamically equivalent if, under transformation of fluxes and forces, the bilinear form of the entropy production remains unaltered, i.e.,* $\sigma = \sigma'$ [7]. Hence, in classical textbooks on linear and nonlinear irreversible thermodynamics, the concepts of *equivalence of thermodynamic systems* is formulated only in terms of invariance of the entropy production under the thermodynamic force transformations (see, for example, [26,27]).

However, the condition of the invariance of the entropy production is not sufficient to ensure the equivalence character of the two descriptions (J_μ, X^μ) and (J'_μ, X'^μ). Indeed, we can convince ourselves that there exists a large class of transformations such that, even though they leave unaltered the expression of the entropy production, they may lead to misinterpretations reported by Prigogine and Glansdorff [5,7]. In addition, the above De Donder–Prigogine definition is unable to determine, univocally, the most general class of the thermodynamic force transformations able to ensure the equivalence among thermodynamic systems (see [26]). These obstacles may be overcome if we take into account one of the most fundamental and general theorems valid in thermodynamics of irreversible processes: the universal criterion of evolution (UCE) [4,5]. In Refs [1,8], it is shown that to ensure the equivalent character of the two descriptions $\{X^\mu\}$ and $\{X'^\mu\}$, it is not sufficient to require that the entropy production of the system $\sigma = g_{\mu\nu} X^\mu X^\nu$ is invariant under the flux-force transformation, but we should also require that the Glansdorff-Prigogine dissipative quantity remains invariant under transformation of the thermodynamic forces $\{X^\mu\} \to \{X'^\mu\}$. In the Introduction, it is shown that the most general force transformations can be brought into the form [see also Eq. (8)]

$$X'^\mu = X^1 F^\mu \left(\frac{X^2}{X^1}, \frac{X^3}{X^2}, \ldots, \frac{X^n}{X^{n-1}} \right) \quad (t = t'), \tag{A2}$$

where F^μ are arbitrary functions of variables X^j/X^{j-1} with $j = 2, \ldots, n$. In Sec. II, it is shown that the TCT group is a fiber bundle where the bundle is S_+^{n-1} (the top half of the unit sphere embedded in R^n) and the fiber is the space of maps $S_+^{n-1} \to R^x$ (the real numbers, where the real number zero is excluded).

The thermodynamic equivalence principle leads, naturally, to the following thermodynamic covariance principle (TCP): the nonlinear closure equations, i.e., the flux-force relations, must be covariant under TCT [8]. Note that the TCP is trivially satisfied by the closure equations valid in the Onsager region.

Magnetically confined tokamak plasmas are a typical example of thermodynamic systems, out of Onsager's region, where the equivalence between two different choices of the thermodynamic forces is warranted only if both the entropy production and the Glansdorff-Prigogine dissipative quantity P, defined above, remain unaltered under transformation of these thermodynamic forces [12]. We also mention that the linear version of the TCT is actually widely used for studying transport processes in tokamak plasmas (see, for examples, the papers cited in the book [11] and [13]).

Assumption (C). According to this assumption, there exists an action which is stationary with respect to arbitrary variation of the transport coefficient and the affine connection [1]. We can understand the reason why we may introduce this principle. To this, first let us consider two cases: magnetically confined plasmas and chemical reactions far from equilibrium.

In tokamak plasmas, the thermodynamic forces and the conjugate fluxes are the parallel-generalized frictions and the Hermitian moments, respectively [11]. By using the fluid moment equations, in Ref. [28] it is shown that magnetically confined plasmas tend to relax towards the mechanical equilibrium[11] following the shortest path, traced out in the space of the parallel-generalized frictions.

In chemical reactions out of equilibrium the thermodynamic forces and the conjugate flows are the chemical affinities (over temperature) and the chemical velocities, respectively [7]. By using the law of mass action, in Ref. [29] it is shown that chemical reactions tend to relax towards nonequilibrium steady state following the shortest path, traced out in the space of the chemical reactions (over temperature).

It is quite natural to ask the following: Why should the thermodynamic forces, and not other quantities, obey this law? The answer is because these variables are purely *nonconservative quantities*. As known, the thermodynamic variables can be classified as conservative or nonconservative. A fluctuation of a conservative variable can be *dissipated* only through the boundaries and, due to this severe constraint, its evolutionary trajectory towards the steady state may be very complex in the phase space. On the contrary, a fluctuation of a nonconservative variable can be dissipated freely into the

[11]In tokamak plasmas, the mechanical equilibrium corresponds to a nonequilibrium (generally stable) steady state.

surrounding and its evolutionary trajectory tends to approximate that of the shortest path. As mentioned, in thermodynamics, one of the most dissipative quantities is P. By noticing that the transport coefficients depend only on the thermodynamic forces X^μ, it turns out that also quantity P depends only on the thermodynamic forces. Since the thermodynamic forces are nonconservative variables (and quantity P is defined in the space of the thermodynamic forces), it is not so surprising to have been able to prove that, in the space of the thermodynamic forces, the X^μ tend to follow the shortest path for reaching the (nonequilibrium) steady state. Of course, also the fluxes, conjugate to the thermodynamic forces, have a similar behavior. In fact, fluxes are nonconservative quantities and they are linked to the conjugate thermodynamic forces through the closure relations (1).[12]

The above results lead to the idea of imaging that there are *true* closure equations and that any other transport equations we draw are *false*. This concept allows introducing an action admitting an *extremum* and to enunciate the principle of least action [30].[13] If there is a change in the first order, when we deviate the curve in a certain way, there is a change in the action that is proportional to the deviation. So that if we calculate the action for the false closure equations, we will get a value that is bigger (or lower) than if we calculate the action for the true closure equations. If we have the true path, a curve that differs only a little bit from it will, in the first approximation, make no difference in the action; any difference will be in the second approximation, if we really have an extremum. The only way that it could really be an extremum is that in the first approximation it does not make any change. This leads to enunciate the above-mentioned principle of least action [30]. Following the procedure indicated in the Introduction, we get the expression of the thermodynamic action [see Eq. (10)]:

$$I = \int \left[R - \left(\Gamma^\lambda_{\alpha\beta} - \tilde{\Gamma}^\lambda_{\alpha\beta}\right)S^{\alpha\beta}_\lambda\right]\sqrt{g}\, d^n X. \tag{A3}$$

Action (10) is derived by imposing that [1] (i) it is constructed with the two pieces of the transport coefficients $g_{\mu\nu}$ and $f_{\mu\nu}$, the affine connection $\Gamma^\mu_{\alpha\beta}$, and *only* with the first-order derivatives of these fields; (ii) it is invariant under TCT [this constraint ensures the validity of the TCP, i.e., the closure equations should be covariant under TCT, Constraint (B)]; (iii) it is stationary when $\Gamma^\mu_{\alpha\beta} = \tilde{\Gamma}^\mu_{\alpha\beta}$, i.e., the action is stationary only when the affine connection coincides with the expression able to satisfy (automatically) the UCE; (iv) metric $g_{\mu\nu}$ and the skew-symmetric piece of the transport coefficients $f_{\mu\nu}$ tend to the Onsager matrices as the thermodynamic system approaches equilibrium.

The nonlinear closure equations are derived from a variational principle. These correspond to the differential equations that make the action (10) locally stationary for arbitrary variations with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$, performed independently with each other [1].

Let us now explain the physical meaning of the terms appearing in the action (10). As known, the first piece of Lagrangian, i.e., $R\sqrt{g}$, is due to geometry and it is a generic contribution which appears whenever the curvature of the space is constructed through the Riemann tensor (e.g., space-time, thermodynamic space, etc). Indeed, the vanishing divergence of the tensor derived by the term $R\sqrt{g}$ reflects the geometric unchangeable property which comes from the theorem that the *boundary of a boundary is zero*. As known, this theorem is the geometrical version of the algebraic Bianchi identity.[14] On the other hand, the tensor derived by the term $(\Gamma^\lambda_{\alpha\beta} - \tilde{\Gamma}^\lambda_{\alpha\beta})S^{\alpha\beta}_\lambda\sqrt{g}$ reflects physics. The physical meaning of the tensor, constructed by this piece of Lagrangian, rests upon the Noether current. In our case (i.e., the TFT), this theorem reflects the required symmetry expressing the invariance of the Lagrangian under TCT. The tensor derived by the Noether current is the *source* of the thermodynamic space and, as shown in [1], it vanishes in the Onsager region. Indeed, it is possible to show that the *source term* of the thermodynamic space is the second order thermodynamic tensor $T_{\mu\nu} = -S^{\alpha\beta}_\lambda \frac{\delta\tilde{\Gamma}^\lambda_{\alpha\beta}}{\delta g^{\mu\nu}}$ [1], with $\frac{\delta\tilde{\Gamma}^\lambda_{\alpha\beta}}{\delta g^{\mu\nu}}$ denoting the variation of $\tilde{\Gamma}^\lambda_{\alpha\beta}$ with respect to $g^{\mu\nu}$.

By imposing the stationary of action (10) with respect to small variations of the transport coefficients, we get the nonlinear transport equations [1]. These equations tend to the Onsager transport equations when the system approaches equilibrium. We mention that it is possible to prove that, in the *weak-field approximation*, i.e., when $g_{\mu\nu}(X) \simeq L_{\mu\nu} + h_{\mu\nu}(X)$, with $L_{\mu\nu}$ and $h_{\mu\nu}(X)$ denoting the Onsager transport coefficients matrix and the (weak) perturbation of the Onsager matrix, respectively, and *for very large values of the entropy production* ($\sigma \gg 1$), we have [1]

$$\Gamma^\kappa_{\mu\nu} = \tilde{\Gamma}^\kappa_{\mu\nu} = \tfrac{1}{2}L^{\kappa\eta}(h_{\mu\nu,\nu} + h_{\nu\eta,\mu} - h_{\mu\nu,\eta}) + \text{h.o.t.}, \tag{A4}$$

where h.o.t. stands for higher order terms. Clearly, a transport theory without knowledge of microscopic dynamical laws cannot be developed. Transport theory is only but an aspect of nonequilibrium statistical mechanics, which provides the link between microlevel and macrolevel. This link appears indirectly in the *unperturbed* matrices, i.e., the $L^{\mu\nu}$ (and the $f_0^{\mu\nu}$) coefficients used as an input in the equations. These coefficients, which depend on the specific material under consideration, have to be calculated in the usual way by kinetic theory. The perturbation fields (i.e., the corrections to the Onsager transport coefficients) $h_{\mu\nu}(X)$ depend on the

[12] Note that if the skew-symmetric piece of the transport coefficients $f_{\mu\nu}$ is zero, the thermodynamic forces and the conjugate fluxes can be seen as the contravariant and the covariant components of the same thermodynamic vector, respectively.

[13] To avoid misunderstanding, while it is correct to mention that this postulate affirms the possibility of deriving the nonlinear closure equations by a variational principle, it does not state that the expressions and theorems obtained from the solutions of these equations can also be derived by a variational principle. In particular, the universal criterion of evolution *cannot* be derived by a variational principle.

[14] If we consider an infinitesimal cubical coordinate volume in the space of the thermodynamic forces, when a generic vector $\mathbf{A}^\mu$ is parallel transported around all the six surfaces of the cube, all the edges are traversed twice, once in each direction. These displacements have signs depending upon the direction in which an edge is traversed, so all the displacements add up to zero [31].

thermodynamic forces and for $\sigma \gg 1$ they are solutions of the equations [1]

$$L^{\lambda\kappa}\frac{\partial^2 h_{\mu\nu}}{\partial X^\lambda X^\kappa}+L^{\lambda\kappa}\frac{\partial^2 h_{\lambda\kappa}}{\partial X^\mu X^\nu}-L^{\lambda\kappa}\frac{\partial^2 h_{\lambda\nu}}{\partial X^\kappa X^\mu}-L^{\lambda\kappa}\frac{\partial^2 h_{\lambda\mu}}{\partial X^\kappa X^\nu} = 0+\text{h.o.t.} \tag{A5}$$

For $\sigma \sim 1$, Eqs. (A4) and (A5) lose validity and the correct equations become much more complex. Note that for $\sigma \ll 1$ we enter into the Onsager regime. Equation (A5) should be solved with the appropriate boundary conditions. Concrete examples can be found in Refs. [2,25], where the nonlinear thermoelectric effect and chemical reactions out of the Onsager region are analyzed in detail. In these cases, the boundary conditions are obtained by imposing that, for very large values of the gradient of the inverse of the temperature and of the applied electric field, the electrical and heat fluxes and the chemical flows have no privileged directions in the thermodynamic space [2,25]. As mentioned above, the Onsager matrix $L^{\mu\nu}$ is derived by kinetic theory and introduced, as an input, into Eq. (A5). It is worth mentioning that for the case of chemical reactions, the solution of Eq. (A5), subject to the appropriate boundary conditions, coincides, exactly, with *De Donder's law of mass* [25,26].

APPENDIX B: SPLITTING OF THE TCT GROUP

In this Appendix, we shall prove the validity of Eq. (18). Denote by S^{n-1} the $n-1$ dimensional unit sphere represented as a C^∞ differentiable manifold, which in our case is a submanifold embedded in $\mathbb{R}^n$ of the form

$$\|\mathbf{x}\| = 1. \tag{B1}$$

Here, the function $\mathbb{R}^n \ni \mathbf{x} \mapsto \|\mathbf{x}\| \in \mathbb{R}^+$ is some $C^\infty(\mathbb{R}^n)$ function having also the properties of a norm. For instance,

$$\|\mathbf{x}\| = \left[\sum_{j=1}^{n}(x_j)^{2k}w_j\right]^{\frac{1}{2k}}, \quad w_j > 0,\ k = 1,2,\ldots .$$

Let $\Gamma_n^S = \text{diff}(S^{n-1})$ be the group of diffeomorphisms of S^{n-1} and let $\Gamma_n^p \subset \text{diff}(S^{n-1})$ be the subgroup of Γ_n^S that preserves the equivalence relation $\mathcal{R}$ induced on S^{n-1} by $\mathbf{x},\mathbf{y} \in S^{n-1}$ are equivalent iff $\mathbf{y} = \pm\mathbf{x}$.

Remark 1. The quotient space $S^{n-1}/\mathcal{R}$ is a diffeomorphism with the $n-1$ dimensional projective space $\mathbb{P}^{n-1}$, so Γ_n^p is isomorphic to $\text{diff}(\mathbb{RP}^{n-1})$. For all map $\mathbf{x} \to \mathbf{Y}(\mathbf{x})$ where $\mathbf{Y} \in \Gamma_n^S$ we have $\mathbf{Y} \in \Gamma_n^P$ iff

$$\mathbf{Y}(-\mathbf{x}) = -\mathbf{Y}(\mathbf{x}). \tag{B2}$$

We denote by N^n the Abelian group generated by all $C^\infty(S^{n-1})$ positive functions where the group operation is defined by multiplication, with the additional symmetry property

$$f(\mathbf{x}) \in N^n \quad \text{if} \quad f(-\mathbf{x}) = f(\mathbf{x}).$$

We also denote by $G^n \subset \text{diff}(\mathbb{R}^n\backslash\{0\})$ the TCT group: the subgroup of the group of diffeomorphisms of $\mathbb{R}^n\backslash\{0\}$ has the additional homogeneity property

$$\mathbb{R}^n\backslash\{0\} \ni \mathbf{x} \mapsto \mathbf{Y}_g(\mathbf{x}) \in \mathbb{R}^n\backslash\{0\}, \tag{B3}$$

$$\mathbf{Y}_g(\lambda\mathbf{x}) = \lambda\mathbf{Y}_g(\mathbf{x});\, \lambda \in \mathbb{R}, g \in G^n. \tag{B4}$$

We denote by N^n the subset (normal subgroup, see below) of G^n having the form

$$\mathbf{Y}_g(\mathbf{x}) = \mathbf{x}r_g(\mathbf{x}), \quad g \in N^n \subset G^n \tag{B5}$$

where $r_g(\mathbf{x})$ is a positive $C^\infty(\mathbb{R}^n\backslash\{0\})$ homogeneous function

$$r_g(\lambda\mathbf{x}) = r_g(\mathbf{x}) > 0, \quad \lambda \in \mathbb{R}. \tag{B6}$$

We have the following proposition.

Proposition 2. N^n is a normal Abelian subgroup, and for all $g,g_1,g_2 \in N^n$ we have

$$r_{g_1g_2}(\mathbf{x}) = r_{g_1}(\mathbf{x})r_{g_2}(\mathbf{x}), \tag{B7}$$

$$r_{g^{-1}}(\mathbf{x}) = \frac{1}{r_g(\mathbf{x})}. \tag{B8}$$

Proof. The group properties (B7) and ((B8)) result immediately, by direct calculation from the general definition of the group product in G^n:

$$\mathbf{Y}_{g_1g_2}(\mathbf{x}) := [\mathbf{Y}_{g_1} \circ \mathbf{Y}_{g_2}](\mathbf{x}), \quad g_1,g_2 \in G^n$$

and by the definition (B5). The Abelian character results from Eq. (B7). In order to prove that N^n is a normal subgroup, let $h \in N^n$ and let be $g \in G^n$ an arbitrary element of the TCT group. We have to prove that

$$u := ghg^{-1} \in N^n \tag{B9}$$

or, equivalently, to prove that

$$Y_u(\mathbf{x}) = Y_{ghg^{-1}}(\mathbf{x}) = [Y_g \circ Y_h \circ Y_{g^{-1}}](\mathbf{x}) = \mathbf{x}r(\mathbf{x}), \tag{B10}$$

where

$$Y_h(\mathbf{x}) = \mathbf{x}r_h(\mathbf{x}) \tag{B11}$$

with $r_h(\mathbf{x})$ denoting a positive $C^\infty(\mathbb{R}^n\backslash\{0\})$ homogeneous function. Let us also denote

$$Y_z(\mathbf{x}) = [Y_g \circ Y_h](\mathbf{x}). \tag{B12}$$

From Eqs. (B4) and (B11), we get

$$Y_z(\mathbf{x}) = r_h(\mathbf{x})Y_g(\mathbf{x}) \tag{B13}$$

and from Eqs. (B10), (B13), and (B4) we find

$$\begin{aligned}Y_u(\mathbf{x}) &= [Y_z \circ Y_{g^{-1}}](\mathbf{x}) = [r_hY_g \circ Y_{g^{-1}}](\mathbf{x})\\ &= [r_h \circ Y_{g^{-1}}](\mathbf{x})[Y_g \circ Y_{g^{-1}}](\mathbf{x}) = r[Y_{g^{-1}}(\mathbf{x})]\mathbf{x}.\end{aligned}$$

Observe that $r[Y_{g^{-1}}(\mathbf{x})]$ possesses all the properties required by Eq. (B6), which proves Eq. (B10). ■

Let us now denote by H^n the subgroup of G^n having the properties

$$\|\mathbf{Y}_h(\mathbf{x})\| = \|\mathbf{x}\|, \tag{B14}$$

$$\mathbf{Y}_h(-\mathbf{x}) = -\mathbf{Y}_h(\mathbf{x}), \tag{B15}$$

$$h \in H^n. \tag{B16}$$

Remark 3. Setting in Eq. (B14) $\|\mathbf{x}\| = 1$ and by using Eq. (B15), we note that the diffeomorphism group H^n is isomorphic to the diff($\mathbb{RP}^{n-1}$) where $\mathbb{P}^{n-1}$ is the $n-1$ dimensional projective space since $\mathbb{P}^{n-1}$ can be represented as S^{n-1} with identified antipodal points.

We have the following proposition.

Proposition 4. For all $g \in G^n$ we have the unique representation

$$g = hg_N, \;\; g_N \in N^n, \, h \in H^n, \tag{B17}$$

$$\mathbf{Y}_g(\mathbf{x}) = [\mathbf{Y}_h \circ \mathbf{Y}_{g_N}](\mathbf{x}). \tag{B18}$$

Proof. Existence of the representation: note that by setting

$$\mathbf{Y}_h(\mathbf{x}) = \frac{\mathbf{Y}_g(\mathbf{x})\|\mathbf{x}\|}{\|\mathbf{Y}_g(\mathbf{x})\|}, \tag{B19}$$

$$\mathbf{Y}_{g_N}(\mathbf{x}) = \mathbf{x}r_N(\mathbf{x}), \tag{B20}$$

$$r_N(\mathbf{x}) = \frac{\|\mathbf{Y}_g(\mathbf{x})\|}{\|\mathbf{x}\|}, \tag{B21}$$

Eq. (B18) is verified and $r_N(\mathbf{x})$ has the property (B6). In order to prove uniqueness, we consider that in Eq. (B18) $\mathbf{Y}_h \in N^n$, with properties (B14) and (B15), but otherwise arbitrary, and $\mathbf{Y}_{g_N}(\mathbf{x}) = \mathbf{x}r(\mathbf{x})$ with $r(\mathbf{x})$ an arbitrary smooth, homogeneous function of zero degree. We rewrite Eq. (B18) by using Eq. (B4):

$$\mathbf{Y}_g(\mathbf{x}) = \mathbf{Y}_h[\mathbf{x}r(\mathbf{x})] = r(\mathbf{x})\mathbf{Y}_h(\mathbf{x}). \tag{B22}$$

Since $r(\mathbf{x}) > 0$ we have

$$\|\mathbf{Y}_g(\mathbf{x})\| = |r(\mathbf{x})|\|\mathbf{Y}_{g_N}(\mathbf{x})\| = r(\mathbf{x})\|\mathbf{x}\| \tag{B23}$$

which leads to

$$r(\mathbf{x}) = r_h(\mathbf{x}) = \frac{\|\mathbf{Y}_g(\mathbf{x})\|}{\|\mathbf{x}\|}. \tag{B24}$$

From Eqs. (B24) and (B23) we obtain Eq. (B19), so the proof of uniqueness of the representation (B17). ■

Irrespective to the choice of the norm in the definition of the subgroup H^n, we may easily convince ourselves that they are all equivalent up to a group isomorphism. For easy reference, we recall the semidirect product definition and properties [32–35].

Theorem 5. Let N,H subgroups of the group G, where N is a normal subgroup. Then, the following statements are equivalent: (*a*) $G = NH$ and $N \cap H = \{e\}$. (*b*) For all $g \in G$ there exists a unique representation $g = nh$ with $n \in N$ and $h \in H$. (*c*) For all $g \in G$ there exists a unique representation $g = hn$ with $n \in N$ and $h \in H$. (d) The natural embedding $i : H \to G$, composed with the natural projection $p : G \to G/N$, yields an isomorphism $\psi : H \to G/N$, $\psi = p \circ i$ with inverse $\widehat{\chi} : G/N \to H$. (*e*) There exists a homomorphism $\chi : G \to H$ that is the identity on H and whose kernel is N.

If one of the above properties are verified, G is said to split in a semidirect product of the subgroups H and normal subgroup N. In this case, the representations of the group G are related to the representations of the subgroups H and N.

By using Theorem 5 and Propositions 4 and 2, we finally get the following theorem.

Theorem 6. The TCT group G^n is a semidirect product of the Abelian normal subgroup N^n and the subgroup H^n:

$$G^n = N^n \rtimes H^n. \tag{B25}$$

APPENDIX C: CALCULATION OF THE NOETHER CURRENT

We sketch in some detail the derivation of Eq. (34). From action (10), we have

$$I = \int \left[R - (\Gamma^\lambda_{\alpha\beta} - \tilde{\Gamma}^\lambda_{\alpha\beta})S^{\alpha\beta}_\lambda\right]\sqrt{g}\, d^nX = \int L\, d^nX \tag{C1}$$

with

$$L \equiv \left[R - \left(\Gamma^\lambda_{\alpha\beta} - \tilde{\Gamma}^\lambda_{\alpha\beta}\right)S^{\alpha\beta}_\lambda\right]\sqrt{g}. \tag{C2}$$

For easy reference, we report the expression of the Noether current

$$j^\mu_\alpha = \frac{\partial L}{\partial \Phi^A_\mu}\mathcal{L}_{\xi_\alpha}\Phi^A - L\xi^\mu_\alpha \quad \text{with}$$
$$\Phi^A = \left(g_{\mu\nu}, \Gamma^\kappa_{\mu\nu}\right) \quad (\alpha = 1,\ldots,N). \tag{C3}$$

The first term appearing in Noether's current (C3), i.e., $\partial L/\partial\Phi^A_\mu$, is computed directly from Eqs. (11) and (C2). We get

$$\begin{aligned}
\frac{\partial L}{\partial g_{\kappa\nu,\lambda}} = \Bigg[&\frac{1}{2}g^{\alpha\kappa}S^{\nu\kappa}_\alpha + \frac{1}{2}g^{\alpha\nu}S^{\kappa\lambda}_\alpha - \frac{1}{2}g^{\alpha\lambda}S^{\kappa\nu}_\alpha + \frac{1}{2\sigma}X^\alpha X^\lambda S^{\nu\kappa}_\alpha \\
&- \frac{X^\kappa X^\lambda}{2(n+1)\sigma}S^{\alpha\nu} - \frac{X^\nu X^\lambda}{2(n+1)\sigma}S^{\alpha\kappa} \\
&+ \frac{1}{2\sigma}X^\alpha X^\lambda\left(\Gamma^\kappa_{\alpha\beta}g^{\beta\nu} + \Gamma^\nu_{\alpha\beta}g^{\alpha\kappa}\right) \\
&- \frac{X^\lambda(X^\nu + X^\kappa)}{2(n+1)\sigma}\left(\Gamma^\alpha_{\alpha\beta}g^{\beta\kappa} + \Gamma^\kappa_{\alpha\beta}g^{\alpha\beta}\right) - \frac{1}{2\sigma} \\
&\times\left(X^\beta X^\lambda g^{\kappa\nu} - \frac{X^\nu X^\lambda}{n+1}g^{\beta\kappa} - \frac{X^\kappa X^\lambda}{n+1}g^{\beta\nu}\right)\Gamma^\alpha_{\alpha\beta}\Bigg]\sqrt{g}, \\
\frac{\partial L}{\partial \Gamma^\eta_{\kappa\eta,\lambda}} = &\left(\frac{1}{2}g^{\kappa\lambda}\delta^\nu_\eta + \frac{1}{2}g^{\nu\lambda}\delta^\kappa_\eta - g^{\kappa\nu}\delta^\lambda_\eta\right)\sqrt{g}.
\end{aligned} \tag{C4}$$

The Lie derivatives of the fields $\Phi^A = (g_{\mu\nu}, \Gamma^\mu_{\nu\kappa})$ read as

$$\begin{aligned}
\mathcal{L}_{\delta X_{(\alpha)}\epsilon^\alpha}g_{\mu\nu} &= \left[\partial_\mu\left(\delta X^\lambda_{(\alpha)}\epsilon^\alpha\right)\right]g_{\lambda\nu} + \left[\partial_\nu\left(\delta X^\lambda_{(\alpha)}\epsilon^\alpha\right]g_{\lambda_\mu}\right. \\
&\quad + \delta X^\lambda_{(\alpha)}\epsilon^\alpha g_{\mu\nu,\lambda}, \\
\mathcal{L}_{\delta X_{(\alpha)}\epsilon^\alpha}\Gamma^\mu_{\nu\kappa} &= \left[\partial_\kappa(\delta X^\eta_{(\alpha)}\epsilon^\alpha)\right]\Gamma^\mu_{\nu\eta} + \left[\partial_\nu(\delta X^\eta_{(\alpha)}\epsilon^\alpha)\right]\Gamma^\mu_{\eta\kappa} \\
&\quad - \left[\partial_\beta(\delta X^\mu_{(\alpha)}\epsilon^\alpha)\right]\Gamma^\beta_{\nu\kappa} + \delta X^\lambda_{(\alpha)}\epsilon^\alpha\Gamma^\mu_{\nu\kappa,\lambda} \\
&\quad + \partial^2_{\nu\kappa}\left(\delta X^\mu_{(\alpha)}\epsilon^\alpha\right)
\end{aligned} \tag{C5}$$

with displacement $\delta\mathbf{X}_{(\alpha)}$ coinciding with ξ_α (i.e., $\delta\mathbf{X}_{(\alpha)} \equiv \xi_\alpha$). Note that, in literature, the Lie derivative of the affine connection is referred to as the *pseudo-Lie derivative* due to the presence of the last term in the second equation of Eq. (C5) (i.e., the second derivative of the infinitesimal vector $\delta X^\mu_{(\alpha)}\epsilon^\alpha$). Now, taking into account Eqs. (22)–(25), in the limit of $\sigma \gg 1$, we get Eq. (34).

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