Glassy dynamics of Brownian particles with velocity-dependent friction

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We consider a two-dimensional model system of Brownian particles in which slow particles are accelerated while fast particles are damped. The motion of the individual particles is described by a Langevin equation with Rayleigh-Helmholtz velocity-dependent friction. In the case of noninteracting particles, the time evolution equations lead to a non-Gaussian velocity distribution. The velocity-dependent friction allows negative values of the friction or energy intakes by slow particles, which we consider active motion, and also causes breaking of the fluctuation dissipation relation. Defining the effective temperature proportional to the second moment of velocity, it is shown that for a constant effective temperature the higher the noise strength, the lower the number of active particles in the system. Using the Mori-Zwanzig formalism and the mode-coupling approximation, the equations of motion for the density autocorrelation function are derived. The equations are solved using the equilibrium structure factors. The integration-through-transients approach is used to derive a relation between the structure factor in the stationary state considering the interacting forces, and the conventional equilibrium static structure factor.

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I. INTRODUCTION

An active particle is defined as a particle which has the ability to absorb energy from its environment or an internal source of energy and dissipate the energy to undertake an out-of-equilibrium motion [1,2]. Different collections of active particles, e.g., biological microswimmers [3,4] or artificial self-propelled particles [5,6], are considered active systems. It has been shown by simulation and experiment that active systems can reach a frozen steady state where single particle fluctuations are arrested [7]. The possibility that an active system undergoes a glass transition is investigated and shown theoretically [8].

Nonequilibrium systems such as sheared colloidal suspensions [9,10] and granular matter [11,12] can undergo a glass transition or melt out of the glassy state. Active microrheology [13,14] is applied to near glass transition colloidal systems to probe the nonequilibrium regimes. For exploring the dynamics of each of the three aforementioned systems, mode-coupling theory [15] has been extended to the far-from-equilibrium situations. In Refs. [9,10], the integration-through-transients (ITT) method is developed and used to obtain the relevant correlation functions from solving the Smoluchowski equation. Farage *et al.* [16] used ITT to calculate the structure factor of an active system using the Smoluchowski operator. Recently an extended mode-coupling scheme was derived by Szamel *et al.* [17] to describe the glassy dynamics of athermal self-propelled particles. Nonequilibrium motion of active particles near the glass transition has been studied using different modeling methods, e.g., considering self-propulsion of a constant speed in the direction of the orientations of the particles and body forces generated by external shear flows [16], assuming an internal driving force [17] or a colored driving and dissipation mechanism [8].

In many cases the motion of biological active particles is confined to a plane [18,19] and numbers of experiments and simulated systems of artificial active particles are prepared in two dimensions [20–22]. It has been shown that charged particles (grains) in plasma can undertake Brownian motion [23]. Dunkel *et al.* [24] studied a two-dimensional layer of charged particles in plasma which is trapped in an external field, numerically. They modeled the charged particles by a Langevin equation with velocity-dependent friction. They suggest that negative (active) friction can be helpful in explaining some effects arising in experiment, such as the higher apparent temperature of the grains in comparison to the surrounding plasma. One of the simple ways to account for an internal propulsion mechanism is introducing a velocity-dependent friction in the Langevin equation [2,25]. The Rayleigh-Helmholtz [26] model of friction considers a nonlinear velocity-dependent friction force $-\gamma(\mathbf{v})\mathbf{v} = \alpha\mathbf{v} - \beta\mathbf{v}^3$. The coefficient $\gamma(\mathbf{v}) = -\alpha + \beta\mathbf{v}^2 = \alpha(-1 + \mathbf{v}^2/\mathbf{v}_0{}^2)$ is similar to the damping coefficient which was used by van der Pol [27] to describe the oscillations in self-sustained oscillators. A self-oscillator transfers a nonperiodic source of energy to a periodic process, which is the functionality various motors have [28]. Badoual *et al.* [29] used the Rayleigh-Helmholtz model to describe the motion of molecular motors. In many other cases the Rayleigh-Helmholtz force has been used to model self-propulsion as a nonequilibrium Brownian motion [2,25,30].

In this paper, we consider a two-dimensional system of N Brownian particles. We model the motion of each particle by the Langevin equation with a Rayleigh-Helmholtz friction. We choose this friction because of its ability to model the pumping of energy to the slow particles, without any rotational or directional dependence. We develop the time evolution operators and, from the corresponding Fokker-Planck equation, we estimate the steady state distributions. The mode-coupling equations for the density correlation functions are then derived to study the dynamical behavior of the system near a glass transition point [31]. To find out about the possible structural changes emerging from the nonequilibrium conditions, we use the ITT formalism.

II. NONLINEAR LANGEVIN EQUATION

To describe the motion of Brownian particles with additional energy input or so-called activity we use the Langevin equation with a velocity-dependent friction [2]:

$$\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i - \gamma(\mathbf{v}_i)\mathbf{p}_i + \boldsymbol{\xi} R_i(t). \tag{1}$$

The rapidly fluctuating force $\boldsymbol{\xi} R_i(t)$, with an ensemble average equal to zero, represents the interaction of the Brownian particle with the solvent molecules. The fluctuation force is a Gaussian white noise [32], which conveys that the fluctuation force values are normally distributed but are uncorrelated in time:

$$\begin{aligned} \langle R_i(t)\rangle &= 0, \\ \langle \boldsymbol{\xi} R_i(t)\boldsymbol{\xi} R_j(t')\rangle &= \xi^2\delta_{ij}\,\delta(t-t'). \end{aligned} \tag{2}$$

In some regions in the phase space, the velocity-dependent friction $\gamma(\mathbf{v}_i)$ allows for negative friction values. When friction is negative, the $-\gamma(\mathbf{v}_i)\mathbf{p}_i$ force pumps additional mechanical energy into the particle rather than dissipating the energy.

III. TIME EVOLUTION OPERATORS

The Liouville equations for a phase variable $A(\boldsymbol{\Gamma}) = A(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N,\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_N)$ and for a nonequilibrium distribution f are defined as [33]

$$\frac{dA(\boldsymbol{\Gamma})}{dt} = i\mathcal{L}A(\boldsymbol{\Gamma}) \tag{3}$$

and

$$\frac{\partial f(\boldsymbol{\Gamma},t)}{\partial t} = -i\mathcal{L}^\dagger f(\boldsymbol{\Gamma},t). \tag{4}$$

In these two equations, $i\mathcal{L}$ and $i\mathcal{L}^\dagger$ are the time evolution operators for phase variables and the distribution function, respectively. Using Eq. (1) we can derive the time evolution operators

$$\begin{aligned} i\mathcal{L} = \dot{\boldsymbol{\Gamma}}\cdot\frac{\partial}{\partial\boldsymbol{\Gamma}} &= \sum_i\left(\frac{\mathbf{p}_i}{m}\cdot\frac{\partial}{\partial\mathbf{r}_i} + \boldsymbol{F}_i\cdot\frac{\partial}{\partial\mathbf{p}_i}\right) \\ &+ \sum_i\left(\boldsymbol{\xi} R_i(t)\cdot\frac{\partial}{\partial\mathbf{p}_i} - \frac{\gamma(\mathbf{v}_i)}{m}\mathbf{p}_i\cdot\frac{\partial}{\partial\mathbf{p}_i}\right) \end{aligned} \tag{5}$$

and

$$\begin{aligned} -i\mathcal{L}^\dagger &= -\dot{\boldsymbol{\Gamma}}\cdot\frac{\partial}{\partial\boldsymbol{\Gamma}} - \left(\frac{\partial}{\partial\boldsymbol{\Gamma}}\cdot\dot{\boldsymbol{\Gamma}}\right) \\ &= \sum_i\left(-\frac{\mathbf{p}_i}{m}\cdot\frac{\partial}{\partial\mathbf{r}_i} - \boldsymbol{F}_i\cdot\frac{\partial}{\partial\mathbf{p}_i}\right) \\ &+ \sum_i\left(-\boldsymbol{\xi} R_i(t)\cdot\frac{\partial}{\partial\mathbf{p}_i} + \frac{\gamma(\mathbf{v}_i)}{m}\mathbf{p}_i\cdot\frac{\partial}{\partial\mathbf{p}_i}\right) \\ &+ \sum_i\left(\frac{1}{m}\frac{\partial\gamma(\mathbf{v}_i)}{\partial\mathbf{p}_i}\cdot\mathbf{p}_i + \frac{\gamma(\mathbf{v}_i)}{m}\right). \end{aligned} \tag{6}$$

The term $\boldsymbol{\xi} R_i(t)\cdot\frac{\partial}{\partial\mathbf{p}_i}$ appears in both time evolution operators $i\mathcal{L}$ and $i\mathcal{L}^\dagger$. Since $\boldsymbol{\xi} R_i(t)$ is a stochastic force, for every realization the time evolution will be different. Thus the variables the operators will operate on do not have a direct time dependence; we take an average over the noise here. We follow the averaging procedure in Ref. [34] (see Appendix A) and assume $m = 1$ for simplicity; therefore,

$$\begin{aligned} i\mathcal{L} &= \sum_i\left(\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{r}_i} + \boldsymbol{F}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}\right) \\ &+ \sum_i\left(-\frac{1}{2}\xi^2\frac{\partial^2}{\partial{\mathbf{v}_i}^2} - \gamma(\mathbf{v}_i)\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}\right), \end{aligned} \tag{7}$$

and

$$\begin{aligned} -i\mathcal{L}^\dagger &= \sum_i\left(-\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{r}_i} - \boldsymbol{F}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}\right) \\ &+ \sum_i\left(\frac{1}{2}\xi^2\frac{\partial^2}{\partial{\mathbf{v}_i}^2} + \gamma(\mathbf{v}_i)\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}\right) \\ &+ \sum_i\left(\frac{\partial\gamma(\mathbf{v}_i)}{\partial\mathbf{v}_i}\cdot\mathbf{v}_i + \gamma(\mathbf{v}_i)\right). \end{aligned} \tag{8}$$

IV. DISTRIBUTION FUNCTION

Using the time evolution operator $-i\mathcal{L}^\dagger$ in Eq. (8), one can write the time evolution equation (4) for the distribution of one particle,

$$\frac{\partial f}{\partial t} + \mathbf{v}_i\cdot\frac{\partial f}{\partial\mathbf{r}_i} + \boldsymbol{F}_i\cdot\frac{\partial f}{\partial\mathbf{v}_i} = \frac{\partial}{\partial\mathbf{v}_i}\left(\gamma(\mathbf{v}_i)\mathbf{v}_i f + \frac{1}{2}\xi^2\frac{\partial f}{\partial\mathbf{v}_i}\right), \tag{9}$$

which is a Fokker-Planck equation. When friction is velocity dependent, the stationary solution of Eq. (9) is only trivial when neglecting the interaction forces, $F_i = 0$ [2]:

$$f_s(\mathbf{v}) = C\exp\left(-\frac{2}{\xi^2}\int^{\mathbf{v}} d\mathbf{v}'\gamma(\mathbf{v}')\mathbf{v}'\right). \tag{10}$$

When $\gamma(\mathbf{v}_i) = \gamma_0 = \text{const}$, $\xi^2 = 2k_\mathrm{B}T\gamma_0$ according to the fluctuation-dissipation theorem [35]. In the case of velocity-dependent friction, the fluctuation-dissipation relation does not hold, which is consistent with the nonequilibrium situation. We consider a Rayleigh-Helmholtz model of friction,

$$\gamma(\mathbf{v}) = -\alpha + \beta\mathbf{v}^2 = \alpha\left(-1 + \frac{\mathbf{v}^2}{{\mathbf{v}_0}^2}\right) = \beta(\mathbf{v}^2 - {\mathbf{v}_0}^2), \tag{11}$$

where $\alpha/\beta = {\mathbf{v}_0}^2$ and β takes only positive values. When $v < v_o$, the friction is negative and the particles receive energy. On the other hand, when $v > v_0$ the particles are damped due to the positive friction. For simplicity of analytically calculating the distributions, we consider $\beta = 1$, so that $\alpha = {\mathbf{v}_0}^2$ and

$$\gamma(\mathbf{v}) = -\alpha + \mathbf{v}^2. \tag{12}$$

We show in Fig. 1 the regions in the $\alpha, v = |\mathbf{v}|$ plane which leads to Brownian particles being active [energy intake, $\gamma(\mathbf{v}) < 0$] or passive [energy dissipation, $\gamma(\mathbf{v}) > 0$].

Considering that $\gamma(\mathbf{v}) = -\alpha + \mathbf{v}^2$, the stationary velocity distribution in Eq. (10), in terms of $D_v = \xi^2/2$, can be written as

$$f_{\mathrm{SR}}(\mathbf{v}) = C\exp\left[-\frac{1}{D_v}\left(\frac{\mathbf{v}^4}{4} - \alpha\frac{\mathbf{v}^2}{2}\right)\right]. \tag{13}$$

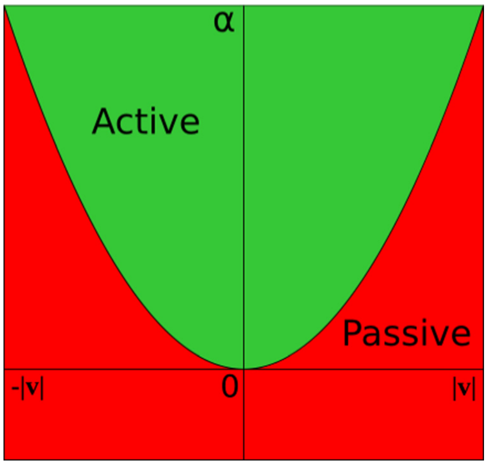

FIG. 1. Distinct regions in the α-v plane which are associated with Brownian particles being active (energy intake) or passive (energy dissipation). The curve $\gamma(\mathbf{v}) = -\alpha + \mathbf{v}^2 = 0$ specifies the boundary of the active region.

In two dimensions where $d\mathbf{v} = 2\pi v\, dv$ [25],

$$\frac{1}{C} = 2\pi \int_0^\infty \exp\left[-\frac{1}{D_v}\left(\frac{\mathbf{v}^4}{4} - \alpha\frac{\mathbf{v}^2}{2}\right)\right] v\, dv$$
$$= \pi\sqrt{\pi D_v}\exp\left(\frac{\alpha^2}{4D_v}\right)\left[1 + \mathrm{erf}\left(\frac{\alpha}{2\sqrt{D_v}}\right)\right]. \quad (14)$$

Figure 2 shows the two-dimensional (2D) normalized distribution $f_{\mathrm{SR}}(\mathbf{v})$ for $\alpha = 1$ and different values of D_v.

The second, fourth, and sixth moment of the velocity in two dimensions can be written as

$$\langle \mathbf{v}^2\rangle = 2\pi \int_0^\infty f_{\mathrm{SR}}(\mathbf{v}) v^2 v\, dv$$
$$= \alpha + 2\sqrt{\frac{D_v}{\pi}}\exp\left(-\frac{\alpha^2}{4D_v}\right)\left[1 + \mathrm{erf}\left(\frac{\alpha}{2\sqrt{D_v}}\right)\right]^{-1}, \quad (15)$$

$$\langle \mathbf{v}^4\rangle = 2D_v + \alpha\langle \mathbf{v}^2\rangle, \quad (16)$$

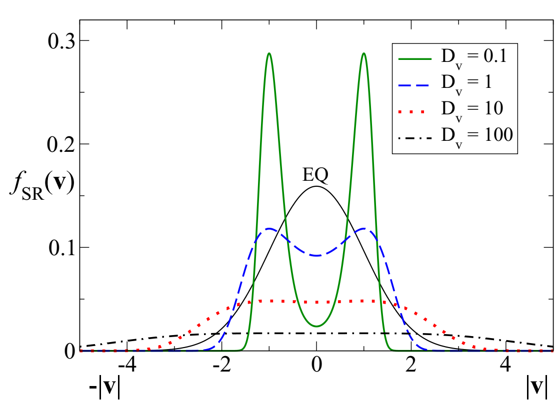

FIG. 2. Stationary velocity distribution for noninteracting Brownian particles shown in Eq. (13) for $\alpha = 1$ and different values of $D_v = \xi^2/2$. The solid black line labeled EQ shows the normalized equilibrium Gaussian distribution $\exp(\alpha v^2/2D_v)/2\pi D_v$ for $\alpha = -1$ and $D_v = k_B T = 1$.

and

$$\langle \mathbf{v}^6\rangle = 2\alpha D_v + (\alpha^2 + 4D_v)\langle \mathbf{v}^2\rangle. \quad (17)$$

These equations are derived in Appendix B where we have also explained the slight difference between $\langle \mathbf{v}^2\rangle$, $\langle \mathbf{v}^4\rangle$ and what was shown in Ref. [25]. Since the velocity distribution is an even function, the odd moments of the velocity are zero in any dimension. The velocity distribution function only contains $\mathbf{v}^2$ terms, thus in two dimensions: $\langle \mathrm{v}_x^2\rangle = \langle \mathrm{v}_y^2\rangle = \langle \mathbf{v}^2\rangle/2$. We define the effective temperature of the system as

$$k_B T_{\mathrm{eff}} = \langle \mathrm{v}_x^2\rangle = \langle \mathrm{v}_y^2\rangle = \frac{\langle \mathbf{v}^2\rangle}{2}. \quad (18)$$

In the case of the normal Langevin equation with constant friction γ_0, the fluctuation-dissipation relation holds and $\xi^2/2\gamma_0 = k_B T = \langle \mathbf{v}^2\rangle/2$, so that there is a linear relation between $\langle \mathbf{v}^2\rangle$ and $\xi^2/2$. But as we can see in Eq. (15), $\langle \mathbf{v}^2\rangle$ and $D_v = \xi^2/2$ have a nonlinear relation. This nonlinearity originates from the velocity-dependent friction.

We assume that we can model the distribution of the particles with separating the position and velocity-dependence part. For the Rayleigh-Helmholtz model of friction this leads to

$$f(\{\mathbf{r}_i\},\{\mathbf{v}_i\}) = C\exp\left(-2\frac{U(\{\mathbf{r}_i\})}{\langle \mathbf{v}^2\rangle}\right)$$
$$\times \exp\left[-\frac{1}{D_v}\sum_i\left(\frac{{\mathbf{v}_i}^4}{4} - \alpha\frac{{\mathbf{v}_i}^2}{2}\right)\right]. \quad (19)$$

Using this distribution function in the Fokker-Planck equation and $D_v = \xi^2/2$ we have

$$\frac{\partial f}{\partial t} = \sum_i\left(-\frac{2}{\langle \mathbf{v}^2\rangle}\boldsymbol{F}_i\cdot\mathbf{v}_i - \frac{\alpha}{D_v}\boldsymbol{F}_i\cdot\mathbf{v}_i + \frac{1}{D_v}\mathbf{v}_i^2\boldsymbol{F}_i\cdot\mathbf{v}_i\right)f. \quad (20)$$

Multiplying the nonlinear Langevin equation (1) by $\mathbf{v}_i$ results in

$$\mathbf{v}_i\cdot\frac{d\mathbf{v}_i}{dt} - \mathbf{F}_i\cdot\mathbf{v}_i = -\gamma(\mathbf{v}_i)\mathbf{v}_i^2 + \boldsymbol{\xi}R_i(t)\cdot\mathbf{v}_i, \quad (21)$$

which represents the mechanical energy loss or gain of one particle in the system. For having the same equation in a more general form we use Eqs. (3) and (7) to evaluate the time evolution of the variable $\sum_i \frac{\mathbf{v}_i^2}{2}$:

$$\frac{d}{dt}\sum_i\frac{\mathbf{v}_i^2}{2} = \sum_i \mathbf{v}_i\cdot\frac{d\mathbf{v}_i}{dt} = i\mathcal{L}\sum_i\frac{\mathbf{v}_i^2}{2}$$
$$= \sum_i \mathbf{F}_i\cdot\mathbf{v}_i - \sum_i\gamma(\mathbf{v}_i)\mathbf{v}_i^2 + \sum_i D_v. \quad (22)$$

In an overdamped motion where $d\mathbf{v}_i/dt = 0$ we have

$$\sum_i \mathbf{F}_i\cdot\mathbf{v}_i = \sum_i\gamma(\mathbf{v}_i)\mathbf{v}_i^2 - \sum_i D_v$$
$$= -\sum_i\alpha\mathbf{v}_i^2 + \sum_i\mathbf{v}_i^4 - \sum_i D_v. \quad (23)$$

We bring up that in case we did not have the nonlinear friction and instead we had the Langevin equation with the constant friction γ_0 which models the normal Brownian motion; $\sum_i \mathbf{F}_i \cdot \mathbf{v}_i = \sum_i \gamma_0 \mathbf{v}_i^2 - \sum_i \xi^2/2$ would be equal to zero, according to the fluctuation-dissipation relation $\xi^2 = 2k_B T \gamma_0$. But here because of the nonlinear friction the fluctuation-dissipation relation does not hold.

Replacing Eq. (23) in Eq. (20) leads to

$$\frac{\partial f}{\partial t} = \Lambda f, \tag{24}$$

where

$$\Lambda = \left(\alpha N + \frac{2ND_v}{\langle \mathbf{v}^2\rangle}\right) + \left(\frac{\alpha^2}{D_v} + \frac{2\alpha}{\langle \mathbf{v}^2\rangle} - 1\right)\sum_i \mathbf{v}_i^2 + \left(-\frac{2\alpha}{D_v} - \frac{2}{\langle \mathbf{v}^2\rangle}\right)\sum_i \mathbf{v}_i^4 + \frac{1}{D_v}\sum_i \mathbf{v}_i^6. \tag{25}$$

With help of the ITT formalism, we use Λ in Sec. VIII to write a structural relation between the stationary state at $t \to \infty$ and the equilibrium state.

A. Probability of finding particles with negative friction (active particles)

For every system having a distribution function with a specific value of α and D_v, which follows Eq. (13), the probability of finding particles which have a velocity less than $\sqrt{\alpha}$ is equal to

$$P_{\text{active}} = \int_0^{\sqrt{\alpha}} 2\pi f_{\text{SR}}(\mathbf{v}) v\, dv. \tag{26}$$

The integral can be solved as

$$P_{\text{active}} = 2\pi C \int_0^{\sqrt{\alpha}} \exp\left[-\frac{1}{D_v}\left(\frac{\mathbf{v}^4}{4} - \alpha\frac{\mathbf{v}^2}{2}\right)\right] v\, dv = \frac{\text{erf}\left(\frac{\alpha}{2\sqrt{D_v}}\right)}{1 + \text{erf}\left(\frac{\alpha}{2\sqrt{D_v}}\right)}. \tag{27}$$

Therefore, to compare two systems which have different values of α and D_v, we can use Eq. (27). The larger the P_{active}, the larger the percentage of particles in the system with negative friction. As represented in Fig. 3, for a constant temperature $\langle \mathbf{v}\rangle = 2k_B T_{\text{eff}} = 3$, we choose three pairs of (α, D_v). Using Eq. (27), we can obtain the probability of finding active particles in the systems which are determined by these three pairs. The P_{active} is equal to 0.021, 0.288, and 0.357 for ($\alpha = 0.1$, $D_v = 6.897$), ($\alpha = 1, D_v = 5.315$), and ($\alpha = 2$, $D_v = 3.415$), respectively. The probability that a particle is active is equal to the area under the corresponding $2\pi f_{\text{SR}}(\mathbf{v})v$ curve between zero and $v = \sqrt{\alpha}$; see Fig. 4. For a constant effective temperature, the larger the α is (or the smaller the D_v is), the higher the percentage of active particles in the system.

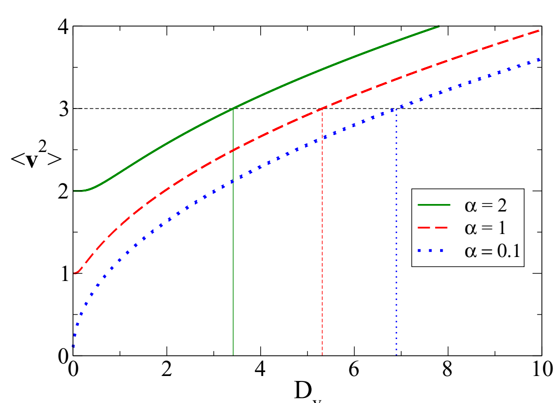

FIG. 3. Second moment of the velocity vs D_v for three different values of α according to Eq. (15). With the α values chosen, $\langle \mathbf{v}^2\rangle = 2k_B T_{\text{eff}} = 3$ leads to three different pairs of $(\alpha, D_v) = (0.1, 6.897)$, (1,5.315), and (2,3.415).

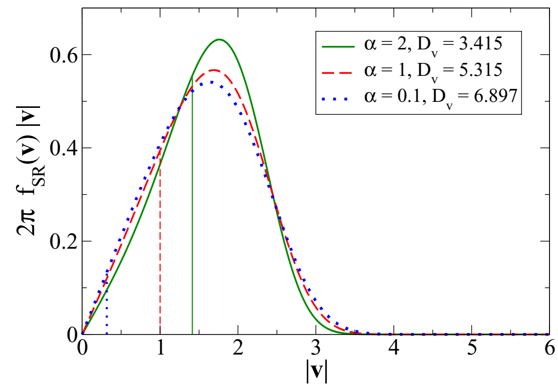

FIG. 4. Stationary velocity distributions for noninteracting Brownian particles shown in Eq. (13), multiplied by $2\pi v$, for different pairs of α and D_v. The (α, D_v) pairs are chosen as in Fig. 3. The value $\sqrt{\alpha}$ is shown with vertical lines having identical line style with every curve. The probability of finding particles with the velocity between zero and $\sqrt{\alpha}$ is equal to the area under the curves in that interval. This area is 0.021, 0.288, and 0.357 for the dotted curve ($\alpha = 0.1$, $D_v = 6.897$), the dashed curve ($\alpha = 1$, $D_v = 5.315$), and the solid curve ($\alpha = 2$, $D_v = 3.415$), respectively. When temperature is constant, with increasing the α, the probability of finding the particles which show activity increases.

B. Definition of the averages

It is useful for later sections to have a consistent definition of the ensemble averages of the product of the phase variables A and $i\mathcal{L}B$:

$$\langle A^* | i\mathcal{L}B\rangle = \int f A^* i\mathcal{L}B\, d\boldsymbol{\Gamma} \tag{28}$$

and

$$\langle -i\mathcal{L}^\dagger A^* | B\rangle = -\int (i\mathcal{L}^\dagger f A^*) B\, d\boldsymbol{\Gamma}. \tag{29}$$

The effect of $i\mathcal{L}^\dagger$ on fA^* can be evaluated as [33]

$$\begin{aligned} i\mathcal{L}^\dagger f A^* &= \dot{\boldsymbol{\Gamma}} \cdot \frac{\partial}{\partial \boldsymbol{\Gamma}}(fA^*) + \left(\frac{\partial}{\partial \boldsymbol{\Gamma}} \cdot \dot{\boldsymbol{\Gamma}}\right) f A^* \\ &= f\dot{\boldsymbol{\Gamma}} \cdot \frac{\partial A^*}{\partial \boldsymbol{\Gamma}} + A^*\dot{\boldsymbol{\Gamma}} \cdot \frac{\partial f}{\partial \boldsymbol{\Gamma}} + A^*\left(\frac{\partial}{\partial \boldsymbol{\Gamma}} \cdot \dot{\boldsymbol{\Gamma}}\right) f \\ &= f i\mathcal{L}A^* + A^* i\mathcal{L}^\dagger f. \end{aligned} \tag{30}$$

The distribution function noted in Eq. (19) is not the stationary solution of the Fokker-Planck equation. Therefore, $i\mathcal{L}^\dagger f$ is nonzero. In that case,

$$i\mathcal{L}^\dagger f A^* = f i\mathcal{L}A^* + A^* i\mathcal{L}^\dagger f = f i\mathcal{L}A^* + A^*\Lambda f, \quad (31)$$

where Λ is noted in Eq. (25). Consequently,

$$\langle -i\mathcal{L}^\dagger A^*|B\rangle = -\int f B i\mathcal{L}A^* d\boldsymbol{\Gamma} - \int A^* B\Lambda f\, d\boldsymbol{\Gamma}. \quad (32)$$

V. MORI-ZWANZIG FORMALISM

We consider two dynamical variables

$$\rho_{\mathbf{q}}(t) = \sum_k \exp(i\mathbf{q}\cdot\mathbf{r}_k(t)) \quad (33)$$

and

$$j_{\mathbf{q}}^L(t) = \sum_k v_k^L \exp(i\mathbf{q}\cdot\mathbf{r}_k(t)), \quad (34)$$

where $\mathbf{q} = (0,0,\mathrm{q})$ and L is the longitudinal direction parallel to $\mathbf{q}$. The inner product of $\rho_{\mathbf{q}}(t=0)$ with itself is $\langle\rho_{\mathbf{q}}^*|\rho_{\mathbf{q}}\rangle = NS_q$. For $j_{\mathbf{q}}^L(t=0)$, knowing that the odd moments of velocity are zero,

$$\langle j_{\mathbf{q}}^{L*}|j_{\mathbf{q}}^L\rangle = N\langle {v_i^L}^2\rangle = \frac{N}{2}\langle\mathbf{v}^2\rangle, \quad (35)$$

where $\langle\mathbf{v}^2\rangle$ follows Eq. (15). Here we have used the fact that the velocity distribution, Eq. (13), depends on the velocity merely through $|\mathbf{v}|$. So the average of the longitudinal component of the velocity is equal to the average of the transverse component and in two dimensions

$$\langle {v^L}^2\rangle = \langle {v^T}^2\rangle = \tfrac{1}{2}\langle\mathbf{v}^2\rangle. \quad (36)$$

In the following we use the Mori-Zwanzig formalism [36], using the following projection operators,

$$\mathcal{P} = A_1\langle A_1^*|\cdots\rangle + A_2\langle A_2^*|\cdots\rangle = \frac{1}{NS_q}\rho_{\mathbf{q}}\langle\rho_{\mathbf{q}}^*|\cdots\rangle + \frac{2}{N\langle\mathbf{v}^2\rangle} j_{\mathbf{q}}^L\langle j_{\mathbf{q}}^{L*}|\cdots\rangle, \quad (37)$$

and $\mathcal{Q} = 1 - \mathcal{P}$, where $\langle A_1^*|A_1\rangle$ and $\langle A_2^*|A_2\rangle = 1$. Then the equation of motion for the correlation function can be written as [15]

$$(z\mathbf{I} + \boldsymbol{\Omega} - \mathbf{M})\mathbf{Y}(z) = -\mathbf{I}, \quad (38)$$

where

$$Y_{nm}(z) = \langle A_n^*|\tilde{A}_m(z)\rangle, \quad (39)$$

$$\Omega_{nm} = \langle A_n^*|\mathcal{L}A_m\rangle, \quad (40)$$

and

$$M_{nm} = \langle A_n^*|\mathcal{L}\mathcal{Q}(z+\mathcal{Q}\mathcal{L}\mathcal{Q})^{-1}\mathcal{Q}\mathcal{L}A_m\rangle. \quad (41)$$

The $\tilde{A}_m(z) = i\int_0^\infty dt\exp(izt)A(t)$ is a Laplace transform of $A_m(t)$. With use of Eq. (7), since $\langle v^L\rangle = 0$, $\Omega_{11} = \frac{1}{NS_q}\langle\rho_{\mathbf{q}}^*|\mathcal{L}\rho_{\mathbf{q}}\rangle = 0$. From Eqs. (28) and (7) we have

$$\Omega_{21} = \frac{1}{iN\sqrt{S_q\langle\mathbf{v}^2\rangle/2}}\langle j_{\mathbf{q}}^{L*}|i\mathcal{L}\rho_{\mathbf{q}}\rangle = \frac{\sqrt{2}}{iN\sqrt{S_q\langle\mathbf{v}^2\rangle}}\int d\boldsymbol{\Gamma} f\sum_k v_k^L\exp(-i\mathbf{q}\cdot\mathbf{r}_k) \times\sum_i \mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{r}_i}\left(\sum_{k'}\exp(i\mathbf{q}\cdot\mathbf{r}_{k'})\right) = \mathrm{q}\sqrt{\frac{\langle\mathbf{v}^2\rangle}{2S_q}}. \quad (42)$$

To evaluate Ω_{12} we note

$$\Omega_{12} = \frac{1}{iN}\sqrt{\frac{2}{S_q\langle\mathbf{v}^2\rangle}}\langle\rho_{\mathbf{q}}^*|i\mathcal{L}j_{\mathbf{q}}^L\rangle = \frac{1}{iN}\sqrt{\frac{2}{S_q\langle\mathbf{v}^2\rangle}}\left[\left\langle\rho_{\mathbf{q}}^*\middle|\sum_i\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{r}_i}j_{\mathbf{q}}^L\right\rangle + \left\langle\rho_{\mathbf{q}}^*\middle|\sum_i\mathbf{F}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}j_{\mathbf{q}}^L\right\rangle - \left\langle\rho_{\mathbf{q}}^*\middle|\sum_i(-\alpha+\mathbf{v}_i^2)\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}j_{\mathbf{q}}^L\right\rangle\right]. \quad (43)$$

The third term inside the brackets contains odd moments of velocity which are zero and

$$\left\langle\rho_{\mathbf{q}}^*\middle|\sum_i\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{r}_i}j_{\mathbf{q}}^L\right\rangle = iq\int d\boldsymbol{\Gamma} f\sum_{i,k}{v_i^L}^2\exp[i\mathbf{q}\cdot(\mathbf{r}_i-\mathbf{r}_k)] = iqN\frac{\langle\mathbf{v}^2\rangle S_q}{2}. \quad (44)$$

Also,

$$\left\langle\rho_{\mathbf{q}}^*\middle|\sum_i\mathbf{F}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}j_{\mathbf{q}}^L\right\rangle = \int d\boldsymbol{\Gamma} f\sum_k\exp(-i\mathbf{q}\cdot\mathbf{r}_k) \times\left(\sum_i\mathbf{F}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}\right)\sum_{k'}v_{k'}^L\exp(i\mathbf{q}\cdot\mathbf{r}_{k'}) = \int d\boldsymbol{\Gamma} f\sum_k\exp(-i\mathbf{q}\cdot\mathbf{r}_k) \times\sum_i F_i^L\exp(i\mathbf{q}\cdot\mathbf{r}_i). \quad (45)$$

We use the method applied in Ref. [14] for a related case, to obtain the average in Eq. (45). According to Eq. (19),

$$\frac{\partial f}{\partial\mathbf{r}_i} = -\frac{2}{\langle\mathbf{v}^2\rangle}\frac{\partial U}{\partial\mathbf{r}_i}f = \frac{2}{\langle\mathbf{v}^2\rangle}\mathbf{F}_i f, \quad (46)$$

and also by means of partial integration

$$\int B\frac{\partial f}{\partial\mathbf{r}_i}d\boldsymbol{\Gamma} = -\int f\frac{\partial B}{\partial\mathbf{r}_i}d\boldsymbol{\Gamma}. \quad (47)$$

Therefore,

$$\int d\boldsymbol{\Gamma} f \sum_k \exp(-i\mathbf{q}\cdot\mathbf{r}_k) \sum_i F_i^L \exp(i\mathbf{q}\cdot\mathbf{r}_i)$$
$$= -\frac{\langle \mathbf{v}^2\rangle}{2} \sum_i \int d\boldsymbol{\Gamma} f \frac{\partial}{\partial r_i^L}\left(\exp(i\mathbf{q}\cdot\mathbf{r}_i) \sum_k \exp(-i\mathbf{q}\cdot\mathbf{r}_k)\right)$$
$$= -iqN\frac{\langle \mathbf{v}^2\rangle}{2}(S_q - 1). \tag{48}$$

Substituting Eq. (48) and (44) into Eq. (43) leads to

$$\Omega_{12} = \Omega_{21} = \mathrm{q}\sqrt{\frac{\langle \mathbf{v}^2\rangle}{2S_q}}. \tag{49}$$

This result is equivalent to the case of usual Brownian motion with constant friction where $\langle \mathbf{v}^2\rangle = 2k_B T$. Ω_{22} describes sound damping and can be evaluated as

$$\Omega_{22} = \frac{2}{iN\langle \mathbf{v}^2\rangle}\langle j_{\mathbf{q}}^{L*}|i\mathcal{L} j_{\mathbf{q}}^L\rangle$$
$$= \frac{2}{iN\langle \mathbf{v}^2\rangle}\int d\boldsymbol{\Gamma} f \sum_k v_k^L \exp(-i\mathbf{q}\cdot\mathbf{r}_k)$$
$$\times\left(-\sum_i(-\alpha + \mathbf{v}_i^2)\mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{v}_i} + \sum_i \mathbf{F}_i\cdot\frac{\partial}{\partial \mathbf{v}_i}\right)$$
$$\times \sum_{k'} v_{k'}^L \exp(i\mathbf{q}\cdot\mathbf{r}_{k'})$$
$$= \frac{1}{i\langle \mathbf{v}^2\rangle}(\alpha\langle \mathbf{v}^2\rangle - \langle \mathbf{v}^4\rangle) + \frac{2}{iN\langle \mathbf{v}^2\rangle}\int d\boldsymbol{\Gamma} f \sum_k v_k^L F_k^L. \tag{50}$$

Recalling from Eq. (16), $\alpha\langle \mathbf{v}^2\rangle - \langle \mathbf{v}^4\rangle = -2D_v = -\xi^2$. Knowing that $\sum_k v_k^L F_k^L = \frac{1}{2}\sum_k \mathbf{v}_k\cdot\mathbf{F}_k$, from Eq. (23) we obtain

$$\int d\boldsymbol{\Gamma} f \sum_k v_k^L F_k^L = \frac{N}{2}\left(-\alpha\langle \mathbf{v}^2\rangle + \langle \mathbf{v}^4\rangle - \frac{\xi^2}{2}\right). \tag{51}$$

Therefore,

$$\Omega_{22} = \frac{iD_v}{\langle \mathbf{v}^2\rangle} = \frac{i\xi^2}{2\langle \mathbf{v}^2\rangle}. \tag{52}$$

Consequently, the existence of a velocity-dependent friction term in the Langevin equation leads to $\langle j_{\mathbf{q}}^{L*}|i\mathcal{L} j_{\mathbf{q}}^L\rangle = iND_v/2$, where the D_v is related to the second and fourth moment of velocity through Eq. (15). However, $\langle \rho_{\mathbf{q}}{}^*|i\mathcal{L}\rho_{\mathbf{q}}\rangle$ is zero, similar to normal Brownian motion, since the odd moments of velocity are zero. The elements of the Ω matrix can be written as

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 & \mathrm{q}\sqrt{\frac{\langle \mathbf{v}^2\rangle}{2S_q}} \\ \mathrm{q}\sqrt{\frac{\langle \mathbf{v}^2\rangle}{2S_q}} & \frac{i\xi^2}{2\langle \mathbf{v}^2\rangle} \end{pmatrix}. \tag{53}$$

In the case of normal Brownian motion (equilibrium case) [37], $\sum_i \mathbf{F}_i\cdot\mathbf{v}_i = 0$ and $\Omega_{22} = i\gamma_0$.

VI. MODE-COUPLING APPROXIMATION

For writing the complete equation of motion, Eq. (38), we still need to know the elements of the memory kernel M_{nm}. We recall from Eq. (42) that $\mathcal{L}A_1 = \mathrm{q}\sqrt{\frac{\langle \mathbf{v}^2\rangle}{2S_q}}A_2$ so $\mathcal{QL}A_1 = 0$ and $M_{11} = M_{21} = 0$. M_{22} can be written as

$$M_{22} = \langle A_2^*|\mathcal{LQ}(z + \mathcal{QLQ})^{-1}\mathcal{QL}A_2\rangle$$
$$= \langle A_2^*|\mathcal{LQ}\exp(it\mathcal{QLQ})\mathcal{QL}A_2\rangle. \tag{54}$$

For separating the remaining fast decaying fluctuations from the slow memory kernel we use the projection operator $\mathcal{P}_M = \sum_{\mathbf{k}<\mathbf{p}} \rho_{\mathbf{k}}\rho_{\mathbf{p}} \frac{\langle \rho_{\mathbf{k}}^*\rho_{\mathbf{p}}^*|\cdots\rangle}{\langle \rho_{\mathbf{k}}^*\rho_{\mathbf{p}}^*|\rho_{\mathbf{k}}\rho_{\mathbf{p}}\rangle}$. By projecting the kernel onto the pair modes of density, the slowly decaying parts of the memory kernel remain which have the longest relaxation times [38]. We also use the first mode-coupling approximation [15] and replace $\exp(it\mathcal{QLQ})$ with $\mathcal{P}_M\exp(i\mathcal{L}t)\mathcal{P}_M$:

$$M_{22} \approx \langle A_2^*|\mathcal{LQP}_M^1\exp(i\mathcal{L}t)\mathcal{P}_M^1\mathcal{QL}A_2\rangle$$
$$= \frac{2}{N\langle \mathbf{v}^2\rangle}\sum_{\mathbf{k}<\mathbf{p},\mathbf{k}'<\mathbf{p}'}\frac{1}{\langle \rho_{\mathbf{k}}^*\rho_{\mathbf{p}}^*|\rho_{\mathbf{k}}\rho_{\mathbf{p}}\rangle\langle \rho_{\mathbf{k}'}^*\rho_{\mathbf{p}'}^*|\rho_{\mathbf{k}'}\rho_{\mathbf{p}'}\rangle}$$
$$\times\langle j_{\mathbf{q}}^{L*}|\mathcal{LQ}\rho_{\mathbf{k}'}\rho_{\mathbf{p}'}\rangle\langle \rho_{\mathbf{k}'}^*\rho_{\mathbf{p}'}^*|\exp(i\mathcal{L}t)\rho_{\mathbf{k}}\rho_{\mathbf{p}}\rangle$$
$$\times\langle \rho_{\mathbf{k}}^*\rho_{\mathbf{p}}^*|\mathcal{QL}j_{\mathbf{q}}^L\rangle. \tag{55}$$

Also according to the factorization ansatz $\langle \rho_{\mathbf{k}}^*\rho_{\mathbf{p}}^*|\rho_{\mathbf{k}}\rho_{\mathbf{p}}\rangle \approx \langle \rho_{\mathbf{k}}^*|\rho_{\mathbf{k}}\rangle\langle \rho_{\mathbf{p}}^*|\rho_{\mathbf{p}}\rangle$ and $\langle \rho_{\mathbf{k}'}^*\rho_{\mathbf{p}'}^*|\exp(i\mathcal{L}t)\rho_{\mathbf{k}}\rho_{\mathbf{p}}\rangle \approx \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{p},\mathbf{p}'} N^2 S_k S_p \phi_{\mathbf{k}}(t)\phi_{\mathbf{p}}(t)$, where $\phi_{\mathbf{k}}(t) = \langle \rho_{\mathbf{k}}^*|\exp(i\mathcal{L}t)\rho_{\mathbf{k}}\rangle/NS_k$. We need to calculate two terms. The first one is

$$\langle j_{\mathbf{q}}^{L*}|\mathcal{LQ}\rho_{\mathbf{k}}\rho_{\mathbf{p}}\rangle = \langle j_{\mathbf{q}}^{L*}|(\mathcal{L}\rho_{\mathbf{k}})\rho_{\mathbf{p}}\rangle + \langle j_{\mathbf{q}}^{L*}|\rho_{\mathbf{k}}(\mathcal{L}\rho_{\mathbf{p}})\rangle$$
$$-\frac{q\langle \mathbf{v}^2\rangle}{2S_q}\langle \rho_{\mathbf{q}}^*|\rho_{\mathbf{k}}\rho_{\mathbf{p}}\rangle$$
$$= \frac{\langle \mathbf{v}^2\rangle}{2}k\langle \rho_{\mathbf{q}-\mathbf{k}}^*|\rho_{\mathbf{p}}\rangle + \frac{\langle \mathbf{v}^2\rangle}{2}p\langle \rho_{\mathbf{q}-\mathbf{p}}^*|\rho_{\mathbf{k}}\rangle$$
$$-\frac{q\langle \mathbf{v}^2\rangle}{2S_q}\langle \rho_{\mathbf{q}}^*|\rho_{\mathbf{k}}\rho_{\mathbf{p}}\rangle$$
$$= N\frac{\langle \mathbf{v}^2\rangle}{2}\delta_{\mathbf{q},\mathbf{k}+\mathbf{p}}(kS_p + pS_k - qS_kS_p), \tag{56}$$

where we used the convolution approximation $\langle \rho_{\mathbf{q}}^*|\rho_{\mathbf{k}}\rho_{\mathbf{p}}\rangle \approx N\delta_{\mathbf{q},\mathbf{k}+\mathbf{p}}S_qS_kS_p$. Above and in the following equations, k and p are the longitudinal components of $\mathbf{k}$ and $\mathbf{p}$, respectively. The second term to calculate is

$$\langle \rho_{\mathbf{k}}^*\rho_{\mathbf{p}}^*|\mathcal{QL}j_{\mathbf{q}}^L\rangle = \frac{1}{i}\langle \rho_{\mathbf{k}}^*\rho_{\mathbf{p}}^*|i\mathcal{L}j_{\mathbf{q}}^L\rangle$$
$$-\langle \rho_{\mathbf{k}}^*\rho_{\mathbf{p}}^*|\rho_{\mathbf{q}}\rangle\frac{1}{NS_q}\langle \rho_{\mathbf{q}}^*|\mathcal{L}j_{\mathbf{q}}^L\rangle. \tag{57}$$

In equilibrium, $\langle \rho_{\mathbf{k}}^*\rho_{\mathbf{p}}^*|i\mathcal{L}j_{\mathbf{q}}^L\rangle = \langle(-i\mathcal{L}^\dagger\rho_{\mathbf{k}}^*)\rho_{\mathbf{p}}^*|j_{\mathbf{q}}^L\rangle + \langle \rho_{\mathbf{k}}^*(-i\mathcal{L}^\dagger\rho_{\mathbf{p}}^*)|j_{\mathbf{q}}^L\rangle$. However, here we need to let the operator $\mathcal{L}$ act on the variable $j^L_{\mathbf{q}}$:

$$\frac{1}{i}\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}|i\mathcal{L}j^L_{\mathbf{q}}\rangle = \frac{1}{i}\left\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}\middle|\sum_i \mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{r}_i}j^L_{\mathbf{q}}\right\rangle + \frac{1}{i}\left\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}\middle|\sum_i \mathbf{F}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}j^L_{\mathbf{q}}\right\rangle - \frac{1}{i}\left\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}\middle|\sum_i(-\alpha+\mathbf{v}_i^2)\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}j^L_{\mathbf{q}}\right\rangle. \quad (58)$$

The third term is zero since the odd moments of velocity are zero. The first term can be written as

$$\frac{1}{i}\left\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}\middle|\sum_i \mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{r}_i}j^L_{\mathbf{q}}\right\rangle = \frac{1}{i}\left\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}\middle|\sum_i\left(\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{r}_i}\right)\times\sum_m v^L_m\exp(i\mathbf{q}\cdot\mathbf{r}_m)\right\rangle = \frac{q\langle\mathbf{v}^2\rangle}{2}\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}|\rho_{\mathbf{q}}\rangle. \quad (59)$$

With the help of Eqs. (46) and (47) the second term of Eq. (58) can be evaluated as

$$\frac{1}{i}\left\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}\middle|\sum_i \mathbf{F}_i\cdot\frac{\partial}{\partial\mathbf{v}_i}j^L_{\mathbf{q}}\right\rangle = \frac{1}{i}\left\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}\middle|\sum_i F^L_i\exp(i\mathbf{q}\cdot\mathbf{r}_i)\right\rangle = -\frac{\langle\mathbf{v}^2\rangle}{2i}\sum_i\int d\mathbf{\Gamma}\,f\frac{\partial}{\partial r^L_i}[\exp(i\mathbf{q}\cdot\mathbf{r}_i)\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}]$$
$$= -\frac{\langle\mathbf{v}^2\rangle}{2i}\sum_i\int d\mathbf{\Gamma}\,f\{iq\exp(i\mathbf{q}\cdot\mathbf{r}_i)\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}} - ik\exp[i(\mathbf{q}-\mathbf{k})\cdot\mathbf{r}_i]\rho^*_{\mathbf{p}} - ip\exp[i(\mathbf{q}-\mathbf{p})\cdot\mathbf{r}_i]\rho^*_{\mathbf{k}}\}$$
$$= -\frac{\langle\mathbf{v}^2\rangle}{2}(q\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}|\rho_{\mathbf{q}}\rangle - \delta_{\mathbf{q},\mathbf{k}+\mathbf{p}}NkS_p - \delta_{\mathbf{q},\mathbf{k}+\mathbf{p}}NpS_k). \quad (60)$$

By adding up Eq. (60) and Eq. (59) we have

$$\frac{1}{i}\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}|i\mathcal{L}j^L_{\mathbf{q}}\rangle = N\frac{\langle\mathbf{v}^2\rangle}{2}\delta_{\mathbf{q},\mathbf{k}+\mathbf{p}}(kS_p+pS_k), \quad (61)$$

so

$$\langle\rho^*_{\mathbf{k}}\rho^*_{\mathbf{p}}|\mathcal{Q}\mathcal{L}j^L_{\mathbf{q}}\rangle = N\frac{\langle\mathbf{v}^2\rangle}{2}\delta_{\mathbf{q},\mathbf{k}+\mathbf{p}}(kS_p+pS_k-qS_kS_p). \quad (62)$$

Placing Eqs. (56) and (62) in Eq. (55) leads to

$$M_{22} = \frac{\langle\mathbf{v}^2\rangle}{2N}\sum_{\mathbf{k}<\mathbf{p}}\delta_{\mathbf{q},\mathbf{k}+\mathbf{p}}\left(\frac{kS_p+pS_k-\mathrm{q}S_kS_p}{S_kS_p}\right)^2 \times S_kS_p\phi_{\mathbf{k}}(t)\phi_{\mathbf{p}}(t). \quad (63)$$

Therefore, the expression for the kernel is the same as the mode-coupling theory (MCT) kernel for conventional liquids [15] considering $\langle\mathbf{v}^2\rangle/2 = k_BT_{\mathrm{eff}}$. The effective temperature will drop out by defining

$$m^{mc}_q = \frac{1}{\Omega^2_{12}}M^1_{22} \quad (64)$$

and m^{mc}_q can be written in integral form; in two dimensions [39],

$$m^{\mathrm{MCT}}_q = \int\frac{d^2k}{(2\pi)^2}\frac{\rho S_qS_pS_k}{2q^4}(\mathbf{q}\cdot\mathbf{k}c_k+\mathbf{p}\cdot\mathbf{q}c_p)^2\phi_k(t)\phi_p(t), \quad (65)$$

where $\mathbf{p}=\mathbf{q}-\mathbf{k}$, $\rho c_k = 1-1/S_k$, and ρ is the average density for N particles in an area L^2.

VII. EQUATION OF MOTION FOR THE DENSITY AUTOCORRELATION FUNCTION

The equation of motion following Eqs. (38), (53), and (65) can be written as

$$\partial^2_t\phi_{\mathbf{q}}(t) + \frac{D_v}{\langle\mathbf{v}^2\rangle}\partial_t\phi_{\mathbf{q}}(t) + \Omega^2_q\phi_{\mathbf{q}}(t) + \Omega^2_q\int_0^t\partial_{t'}\phi_{\mathbf{q}}(t)m^{\mathrm{MCT}}_q(t-t')\,dt' = 0, \quad (66)$$

where $\phi_{\mathbf{q}}(t) = \phi_{11}(t)$ and $\Omega^2_q = \Omega^2_{12} = q^2\langle\mathbf{v}^2\rangle/(2S_q)$. For the overdamped case, the equation of motion can be written as

$$\frac{D_v}{\langle\mathbf{v}^2\rangle\Omega^2_q}\partial_t\phi_{\mathbf{q}}(t) + \phi_{\mathbf{q}}(t) + \int_0^t\partial_{t'}\phi_{\mathbf{q}}(t)m^{\mathrm{MCT}}_q(t-t')\,dt' = 0. \quad (67)$$

The equation of motion presented as Eq. (66) contains one more approximation in comparison to the overdamped case in Eq. (67). Seeing that, we have used the property of an overdamped motion conveyed in Eq. (23) to calculate Ω_{22}.

As the kernel m^{MCT}_q obtained here is the same as in the case of normal Brownian motion, the glass transition packing fraction will also not change. But the damping coefficient in both Eq. (66) and Eq. (67) is different from the equilibrium case. The input to the equations of motion is the static structure factor S_q. In the next section, we use the ITT formalism to investigate the possible changes in the structure factor as a result of the nonequilibrium situation. For now, we use the Baus-Colot [39,40] analytical expression for the structure factor of the hard-sphere system in two dimensions (hard disks) to solve the equations of motion. The glass transition happens at the critical packing fraction $\varphi_c = 0.72464$. We have used 500 grid points in the range $q_{\min} = 0.04$ to $q_{\max} = 39.96$ with $\Delta q = 0.08$ to solve the integral equations.

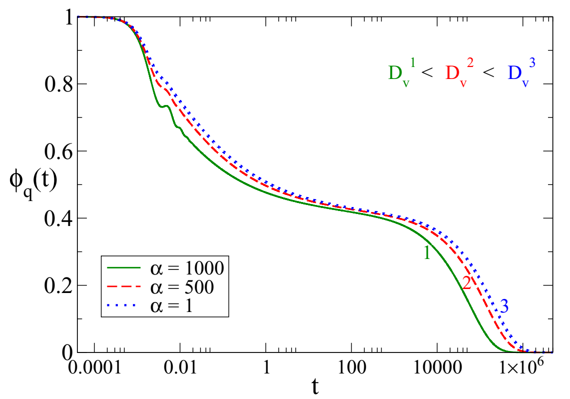

FIG. 5. Density correlation function $\phi_q(t)$ following Eq. (66) for $q = 4.2$ and packing fraction $\varphi = 0.72449$ equivalent to $\varepsilon = (\varphi - \varphi_c)/\varphi_c \simeq 0.0002$, when $\langle \mathbf{v}^2 \rangle = 2k_B T_{\text{eff}} = 1010$; α values presented in the legend and from Eq. (15) $D_v^1 = 88385.66$, $D_v^2 = 501096.48$, and $D_v^3 = 800608.13$. The higher the activity of the system (larger α and smaller D_v) the sooner the correlation function decays.

We choose the temperature $\langle \mathbf{v}^2 \rangle = 2k_B T_{\text{eff}} = 1010$ and we consider three pairs of parameters $(\alpha, D_v) = (1000, 88385.66), (500, 501096.48), (1, 800608.13)$ with the mentioned temperature. We use Eq. (27) to obtain the probability of finding active particles in the system for these three different pairs of parameters. The resulting values are $P_{\text{active}} = 0.0006,\ 0.2767,$ and 0.4956 for $(\alpha, D) = (1, 800608.13)$, $(500, 501096.48)$, and $(1000, 88385.66)$, respectively. In Fig. 5, the solution of Eq. (66) for $\phi_q(t)$ with the packing fraction $\varphi = 0.72449$ in the liquid state and close to transition is presented for the three aforementioned pairs of (α, D_v). The higher the probability of finding active particles in the system, the smaller the time that the correlation function decays to zero. The same behavior is observed for the overdamped case. The solution of Eq. (67), considering the same input, is shown in Fig. 6.

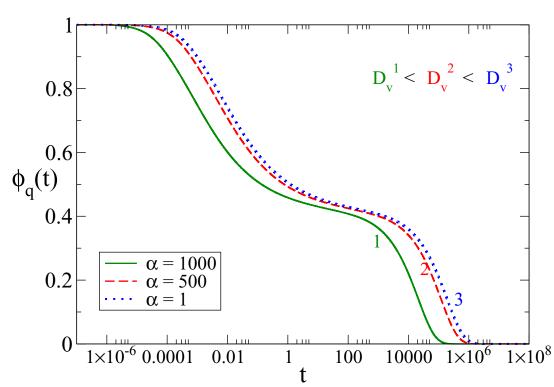

FIG. 6. Density correlation function $\phi_q(t)$ following Eq. (67) for overdamped motion for $q = 4.2$ and packing fraction $\varphi = 0.72449$ equivalent to $\varepsilon = (\varphi - \varphi_c)/\varphi_c \simeq 0.0002$, when $\langle \mathbf{v}^2 \rangle = 2k_B T_{\text{eff}} = 1010$; α values presented in the legend and from Eq. (15) $D_v^1 = 88385.66$, $D_v^2 = 501096.48$, and $D_v^3 = 800608.13$. The higher the activity of the system (larger α and smaller D_v) the sooner the correlation function decays.

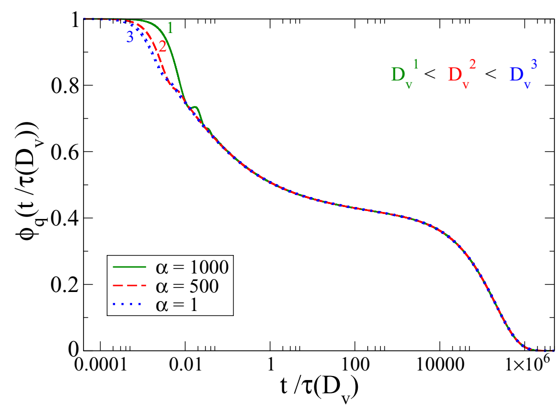

FIG. 7. Scaled density correlation function $\phi_q(\tilde{t})$ according to Eq. (68) for $q = 4.2$ and packing fraction $\varphi = 0.72449$ equivalent to $\varepsilon = (\varphi - \varphi_c)/\varphi_c \simeq 0.0002$, when $\langle \mathbf{v}^2 \rangle = 2k_B T_{\text{eff}} = 1010$. $\tau(D_v) = 1$ for $(\alpha, D_v^3) = (1, 800608.13)$, $\tau(D_v^2) = 0.681$ for $(\alpha, D_v^2) = (500, 501096.48)$, and $\tau(D_v^1) = 0.283$ for $(\alpha, D_v^1) = (1000, 88385.66)$.

Since introducing the velocity-dependent friction does not cause any change in the memory kernel, the activity in the presented model does not affect directly the glass transition packing fraction, which indicates that activity does not melt the glass. However, it can shift the correlation function in a way that, for a constant temperature and below the glass transition packing fraction, the higher the percentage of active particles in the system, the smaller is the time that the correlation function decays to zero. For a better comparison we use the second scaling law (α scaling) [15]. We scale the time in the correlation functions shown in Fig. 5 in a way that all three correlations fall on top of each other in the long time regime. The scaling follows

$$\phi_q(\tilde{t}) = \phi_q\left(\frac{t}{\tau(D_v)}\right), \tag{68}$$

where $\tau(D_v)$ is the scaling time depending on D_v. For the correlation function corresponding to $(\alpha, D_v) = (1000, 88385.66)$, we find $\tau(D_v) = 0.283$; for $(\alpha, D_v) = (500, 501096.48)$, $\tau(D_v) = 0.681$; and for $(\alpha, D_v) = (1, 800608.13)$, the time scale is $\tau(D_v) = 1$. The scaled correlation functions are shown in Fig. 7. Except for the short time dynamics, the correlation functions fall on top of each other. One should have in mind that the scaling time $\tau(D_v)$ will not diverge as a function of D_v, since the glass transition packing fraction is not dependent on activity, and for packing fractions below φ_c, the correlation function will always decay to zero. Since the structure factor is the static input to the equations, small changes in structure factors can change the mode-coupling predictions about the glass transition drastically. In the next section, we study the possible changes in the structure factor.

VIII. INTEGRATION THROUGH TRANSIENTS

If the distribution function f in Eq. (19) was a stationary solution of the Fokker-Planck equation (9), substituting f inside the Fokker-Planck equation would result in $\partial f/\partial t = 0$. But, as mentioned before, f is not a general solution of the Fokker-Planck equation and is only an estimate of the stationary distribution. Replacing f in the Fokker-Planck equation yields $\partial f/\partial t = \Lambda f$, where Λ follows Eq. (25). From Eq. (20) it is seen that f will be a solution of the Fokker-Planck equation under the condition that $\mathbf{F}_i = 0$. In the situation $\mathbf{F}_i \neq 0$ with normal friction, the equilibrium structure factor S_q is justified. We use this fact here and assume when $t < 0$ the interaction forces $\mathbf{F}_i$ are switched off, and at $t = 0$ we switch on the interaction forces. Therefore, we can refer to f as the stationary distribution function when $t < 0$. Using the ITT formalism we are able to evaluate the time dependence of the distribution function as

$$f(\mathbf{\Gamma},t) = \begin{cases} f(\mathbf{\Gamma}), & t \leqslant 0 \\ e^{\Lambda t} f(\mathbf{\Gamma}), & t > 0. \end{cases} \tag{69}$$

Here f follows Eq. (19) and $f(\mathbf{\Gamma},t)$ is the time-dependent distribution function. One can write [10]

$$e^{\Lambda t} = 1 + \int_0^t dt' e^{\Lambda t'} \Lambda; \tag{70}$$

therefore, when $t \to \infty$ according to the ITT formalism [10]

$$\int d\mathbf{\Gamma} f(\mathbf{\Gamma},t) \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} = \int d\mathbf{\Gamma} f(\mathbf{\Gamma}) \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} + \int d\mathbf{\Gamma} \int_0^\infty dt \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} e^{\Lambda t} \Lambda f(\mathbf{\Gamma}) \tag{71}$$

or

$$N S^s_q = N S_q + \int_0^\infty dt \int d\mathbf{\Gamma} \Lambda f(\mathbf{\Gamma}) e^{-\Lambda t} \rho^*_{\mathbf{q}} \rho_{\mathbf{q}}. \tag{72}$$

Here, S^s_q is the structure factor in the stationary state which is reached for $t \to \infty$. We assume that we can replace $-\Lambda$ with $i\mathcal{L}$:

$$e^{-\Lambda t} \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} = e^{i\mathcal{L}t} \rho^*_{\mathbf{q}} \rho_{\mathbf{q}}. \tag{73}$$

Using the projection operator $\mathcal{Q} = 1 - \sum_{\mathbf{q}} \rho_{\mathbf{q}} \langle \rho^*_{\mathbf{q}} | \cdots \rangle / N S_q$, from Eqs. (72) and (73) we arrive at

$$N S^s_q = N S_q + \int_0^\infty dt \langle \Lambda \mathcal{Q} e^{i\mathcal{Q}\mathcal{L}\mathcal{Q}t} \mathcal{Q} \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} \rangle. \tag{74}$$

Using the mode-coupling approximation

$$\langle \Lambda \mathcal{Q}\mathcal{P} e^{i\mathcal{L}t} \mathcal{P}\mathcal{Q} \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} \rangle = \sum_{\mathbf{k}<\mathbf{p}} \frac{\langle \Lambda \mathcal{Q} | \rho_{\mathbf{k}} \rho_{\mathbf{p}} \rangle \langle \rho^*_{\mathbf{k}} \rho^*_{\mathbf{p}} | \exp{(i\mathcal{L}t)} \rho_{\mathbf{k}} \rho_{\mathbf{p}} \rangle \langle \rho^*_{\mathbf{k}} \rho^*_{\mathbf{p}} | \mathcal{Q} \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} \rangle}{\langle \rho^*_{\mathbf{k}} \rho^*_{\mathbf{p}} | \rho_{\mathbf{k}} \rho_{\mathbf{p}} \rangle^2}. \tag{75}$$

From Eq. (25)

$$\langle \Lambda \rangle = \int d\mathbf{\Gamma} f(\mathbf{\Gamma}) \Lambda = \left(3\langle \mathbf{v}^2 \rangle - \frac{\xi^2}{\langle \mathbf{v}^2 \rangle} - \alpha \right) = \left(3\langle \mathbf{v}^2 \rangle - \frac{2D_v}{\langle \mathbf{v}^2 \rangle} - \alpha \right), \tag{76}$$

where $f(\mathbf{\Gamma})$ follows Eq. (19). Also,

$$\langle \Lambda \mathcal{Q} | \rho_{\mathbf{k}} \rho_{\mathbf{p}} \rangle = N \delta_{-\mathbf{k},\mathbf{p}} \langle \Lambda \rangle S_k, \tag{77}$$

and

$$\begin{aligned} \langle \rho^*_{\mathbf{k}} \rho^*_{\mathbf{p}} | \mathcal{Q} \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} \rangle &= \langle \rho^*_{\mathbf{k}} \rho^*_{\mathbf{p}} | \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} \rangle - \sum_{\mathbf{q}} \frac{\langle \rho^*_{\mathbf{k}} \rho^*_{\mathbf{p}} | \rho_{\mathbf{q}} \rangle \langle \rho^*_{\mathbf{q}} | \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} \rangle}{\langle \rho_{\mathbf{q}} | \rho^*_{\mathbf{q}} \rangle} \\ &= \delta_{-\mathbf{k},\mathbf{p}} \delta_{\mathbf{q},\mathbf{k}+\mathbf{p}} N^2 S_k S_q - \sum_{\mathbf{q}} \frac{N^2 \delta_{\mathbf{q},\mathbf{k}+\mathbf{p}} \delta_{\mathbf{q},\mathbf{q}+\mathbf{q}} S_k S_p S_q S^3_q}{N S_q} \\ &= \delta_{-\mathbf{k},\mathbf{p}} \delta_{\mathbf{q},\mathbf{k}+\mathbf{p}} N^2 S_k (1 - S_k). \end{aligned} \tag{78}$$

Substitution of Eqs. (77) and (78) into Eq. (75) results in

$$\langle \Lambda \mathcal{Q}\mathcal{P} e^{i\mathcal{L}t} \mathcal{P}\mathcal{Q} \rho^*_{\mathbf{q}} \rho_{\mathbf{q}} \rangle = \tfrac{1}{2} \langle \Lambda \rangle N (1 - S_k) \phi^2_k(t). \tag{79}$$

Therefore,

$$S^s_q = S_q + \frac{1}{2} \langle \Lambda \rangle (1 - S_q) \int_0^\infty \phi^2_q(t) dt, \tag{80}$$

or finally,

$$S^s_q = S_q + \frac{1}{2} \left(3\langle \mathbf{v}^2 \rangle - \frac{2D_v}{\langle \mathbf{v}^2 \rangle} - \alpha \right) (1 - S_q) \int_0^\infty \phi^2_q(t) dt. \tag{81}$$

This equation is very similar to what Farage *et al.* [16] obtained.

We obtain the correlation function $\phi_q(t)$ from Eq. (67) and substitute it into Eq. (81) to calculate S^s_q. The integral $\int_0^\infty \phi^2_q(t) dt$ becomes infinitely large at the glass transition; therefore, we are able to calculate S^s_q only when we are sufficiently away from the glass transition and inside the liquid state. The other necessity for Eq. (81) to result in a reasonable S^s_q is that the effective temperature should be sufficiently low. In other words, solving Eq. (81) requires that the perturbations are adequately small.

For $\varepsilon = (\varphi_c - \varphi)/\varphi_c \simeq 0.0215$ and $\langle \mathbf{v}^2 \rangle = 2k_B T_{\text{eff}} = 0.1$ we have solved Eq. (67) for three pairs of $(\alpha, D_v) = (0.08, 0.00284)$, $(0.05, 0.004881)$, and $(0.02, 0.006697)$. As we discussed in Sec. IV A, the higher the α (the smaller the D_v), the higher is the percentage of active particles in the system. Therefore, these three pairs correspond to monotonically decreasing fractions of active particles, with all three pairs at the same effective temperature. For having a good comparison we also introduce a fourth pair (α', D'_v) at a smaller effective temperature than the aforementioned three pairs, but the same fraction of active particles as in $(0.05, 0.004881)$. We chose the effective temperature for the fourth term to be $2k_B T_{\text{eff}} = 0.08$. According to Eq. (27), for the (α', D'_v) to have the same P_{active} as $(0.05, 0.004881)$ has, $\alpha'/\sqrt{D'_v}$ must be equal to $0.05/\sqrt{0.004881}$. This together with the condition that $2k_B T_{\text{eff}} = 0.08$ results in $(\alpha', D'_v) = (0.04, 0.003124)$. For solving Eq. (67) we use the Baus-Colot analytical expression for the structure factor S_q of the hard-sphere system in two dimensions [39,40]. For every q value, replacing $\phi_q(t)$ in Eq. (81) and calculating the integral $\int_0^\infty \phi^2_q(t) dt$ results in the S^s_q. We show the S^s_q values around the first peak, in Fig. 8. For the three pairs with the same effective temperature, one can observe that with decreasing α, the peak value of the S^s_q decreases too. This is different from Ref. [16]. Here, we model the activity with velocity-dependent friction which

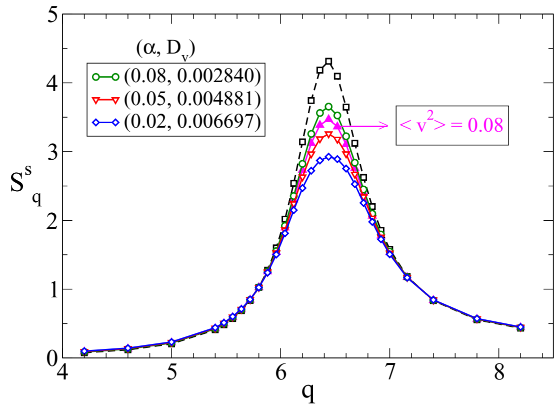

FIG. 8. Structure factor S_q^s for values around the first peak, calculated via Eq. (81), for three pairs of (α, D_v) as indicated in the legends when $\langle \mathbf{v}^2\rangle = 2k_B T_{\text{eff}} = 0.1$ and also for $(\alpha', D_v') = (0.04, 0.003124)$ when $\langle \mathbf{v}^2\rangle = 2k_B T_{\text{eff}} = 0.08$ shown with solid upward triangles. The Baus-Colot equilibrium structure factor S_q is shown with squares. $\varepsilon = (\varphi_c - \varphi)/\varphi_c \simeq 0.0215$.

is isotropic and does not have any rotational or directional dependence. But we are adding an additional constraint to the system. This additional constraint is D_v related to the percentage of active particles in the system. The higher is that percentage (the smaller is the D_v), the more ordered the system becomes and the higher is the peak value of the structure factor. A comparison between the structure factor peak of (0.05,0.004881) and $(\alpha', D_v') = (0.04, 0.003124)$ shows that, as we may expect, although these two curves correspond to the same percentage of activity in the system, since the temperature is lower when $(\alpha', D_v') = (0.04, 0.003124)$ the structure factor peak has larger peak value.

In general, the structure factors S_q^s are less pronounced than the equilibrium Baus-Colot structure factor. In other systems, e.g., colloidal suspensions with short-ranged attractive interactions [41], it has been shown that a decrease in the structure factor peak value yields an increase of the packing fraction for the glass transition according to MCT equations. Therefore, we conclude that the less pronounced peak in the structure factors S_q^s would result in higher transition packing fractions. The change in the structure factor first peak due to activity has been reported before. Ni *et al.* [42] showed by simulation that the structure factor peak value of an active system of self-propelled hard spheres will reduce by increasing activity and the glass transition shifts to higher packing fractions. The same result for the structure factor was obtained earlier in a simulated system of motorized particles [43]. Szamel *et al.* [17] also showed the changes in structure factor and transition point in response to increasing activity although those changes are not monotonic.

IX. CONCLUSION

We analyzed the glassy dynamics of a system in which slow particles are accelerated and fast particles are damped, by means of extending mode-coupling theory to nonequilibrium situations. We have approximated the distribution function by the solution of the Fokker-Planck equation for a noninteracting system. In that case, the activity does not affect the glass transition directly in the memory kernel as in the case for granular matter [11,12]. However, in the present system activity leads to a modification of the static structure factor as shown above by employing the ITT formalism together with a factorization approximation; cf. Fig. 8. In general the structure factor peak values for the considered active systems are smaller than the equilibrium Baus-Colot structure factor peak value. Hence, one expects a shift of the glass transition packing fractions in the active systems towards higher values in comparison to the equilibrium case. Such a trend was observed in the numerical simulation results [42] for a related active system for both the glass transition density as well as the variation of the static structure factor with activity, lending support to the *a priori* uncontrolled approximations used in the MCT and ITT calculations.

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APPENDIX A: NOISE TERMS

As mentioned in Sec. III, both time evolution operators $i\mathcal{L}$ and $i\mathcal{L}^\dagger$ contain the term $\boldsymbol{\xi} R_i(t)\cdot\frac{\partial}{\partial \mathbf{p}_i}$. Since $\boldsymbol{\xi} R_i(t)$ is a stochastic force, the time evolution would be different for every realization. Therefore, we take an average over the noise. Here we review the calculation of these averages in detail following Ref. [34]. We assume $\frac{dB(\Gamma(t))}{dt} = i\mathcal{L}_1 B(\Gamma(t)) = -\boldsymbol{\xi} R_i(t)\cdot\frac{\partial}{\partial \mathbf{p}_i} B(\Gamma(t))$; therefore,

$$B(t+\Delta t) - B(t) = \int_t^{t+\Delta t} i\mathcal{L}_1 B(t_1) dt_1. \qquad \text{(A1)}$$

We substitute B from Eq. (A1) into itself and drop B from both sides of the equation; $-\boldsymbol{\xi} R_i(t)\cdot\frac{\partial}{\partial \mathbf{p}_i}$ is equal to

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_t^{t+\Delta t} -\boldsymbol{\xi} R_i(t_1)\cdot\frac{\partial}{\partial \mathbf{p}_i} dt_1 + \int_t^{t+\Delta t}\int_t^{t_1}\left(\boldsymbol{\xi} R_i(t_1)\cdot\frac{\partial}{\partial \mathbf{p}_i}\right)\left(\boldsymbol{\xi} R_i(t_2)\cdot\frac{\partial}{\partial \mathbf{p}_i}\right) dt_1 dt_2\right]. \qquad \text{(A2)}$$

Since the time scale of $R_i(t)$ is much shorter than the phase variables, we can choose Δt long enough that we can replace the terms inside the integrals by their averages:

$$-\boldsymbol{\xi} R_i(t)\cdot\frac{\partial}{\partial \mathbf{p}_i} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_t^{t+\Delta t} -\langle \boldsymbol{\xi} R_i(t_1)\cdot\frac{\partial}{\partial \mathbf{p}_i}\rangle dt_1 + \int_t^{t+\Delta t}\int_t^{t_1}\left\langle\left(\boldsymbol{\xi} R_i(t_1)\cdot\frac{\partial}{\partial \mathbf{p}_i}\right)\left(\boldsymbol{\xi} R_i(t_2)\cdot\frac{\partial}{\partial \mathbf{p}_i}\right)\right\rangle dt_1 dt_2\right]. \qquad \text{(A3)}$$

According to Eq. (2) the first part of the right-hand side of Eq. (A3) is zero and

$$-\boldsymbol{\xi} R_i(t)\cdot\frac{\partial}{\partial \mathbf{p}_i}=\lim_{\Delta t\to 0}\frac{\xi^2}{\Delta t}\int_t^{t+\Delta t}\int_t^{t_1}\langle R_i(t_1)R_i(t_2)\rangle\frac{\partial^2}{\partial {p_i}^2}dt_1dt_2=\lim_{\Delta t\to 0}\frac{\xi^2}{2\Delta t}\int_t^{t+\Delta t}\frac{\partial^2}{\partial {p_i}^2}dt_1=\frac{1}{2}\xi^2\frac{\partial^2}{\partial {p_i}^2},\tag{A4}$$

where we have used the property of the Dirac delta $\int_t^{t_1}\delta(t_1-t_2)dt_2=1/2$ where $t<t_2<t_1$.

APPENDIX B: VELOCITY INTEGRALS

Here we calculate the integrals in Eqs. (14)–(17) as

$$\frac{1}{C}=2\pi\int_0^\infty e^{-(\frac{v^4}{4D_v}-\frac{\alpha v^2}{2D_v})}v dv=2\pi\sqrt{D_v}e^{\frac{\alpha^2}{4D_v}}\int_{\frac{-\alpha}{2\sqrt{D_v}}}^\infty e^{-U^2}dU,\tag{B1}$$

where $U=\frac{v^2}{2\sqrt{D_v}}-\frac{\alpha}{2\sqrt{D_v}}$. Therefore,

$$\frac{1}{C}=2\pi\sqrt{D_v}e^{\frac{\alpha^2}{4D_v}}\left(\int_{\frac{-\alpha}{2\sqrt{D_v}}}^0 e^{-U^2}dU+\int_0^\infty e^{-U^2}dU\right)=\pi\sqrt{\pi D_v}\exp\left(\frac{\alpha^2}{4D_v}\right)\left[1+\mathrm{erf}\left(\frac{\alpha}{2\sqrt{D_v}}\right)\right],\tag{B2}$$

where we used the definition of the error function $\mathrm{erf}(x)=\int_0^x e^{-t^2}dt$ and the integral $\int_0^\infty e^{-t^2}dt=\sqrt{\pi}/2$. Also

$$\langle\mathbf{v}^2\rangle=2\pi Ce^{\frac{\alpha^2}{4D_v}}\int_0^\infty e^{-(\frac{v^2}{2\sqrt{D_v}}-\frac{\alpha}{2\sqrt{D_v}})^2}v^2 v dv=2\pi\sqrt{D_v}e^{\frac{\alpha^2}{4D_v}}\int_{\frac{-\alpha}{2\sqrt{D_v}}}^\infty e^{-U^2}2\sqrt{D_v}\left(U+\frac{\alpha}{2\sqrt{D_v}}\right)dU$$

$$=4\pi D_v e^{\frac{\alpha^2}{4D_v}}\left(\int_{\frac{-\alpha}{2\sqrt{D_v}}}^\infty\frac{\alpha}{2\sqrt{D_v}}e^{-U^2}dU+\int_{\frac{-\alpha}{2\sqrt{D_v}}}^\infty Ue^{-U^2}dU\right).\tag{B3}$$

The first integral is proportional to $1/C$ and the second integral can be calculated easily:

$$\int_{\frac{-\alpha}{2\sqrt{D_v}}}^\infty Ue^{-U^2}dU=\frac{1}{2}e^{\frac{-\alpha^2}{4D_v}}.\tag{B4}$$

Therefore,

$$\langle\mathbf{v}^2\rangle=\alpha+2\sqrt{\frac{D_v}{\pi}}\exp\left(-\frac{\alpha^2}{4D_v}\right)\left[1+\mathrm{erf}\left(\frac{\alpha}{2\sqrt{D_v}}\right)\right]^{-1}.\tag{B5}$$

This is different from the expression in Ref. [25] by a minus sign in the exponent of $\exp(-\frac{\alpha^2}{4D_v})$. We go ahead and use the same method as Refs. [25,44] to obtain $\langle\mathbf{v}^4\rangle$ and also $\langle\mathbf{v}^6\rangle$,

$$\langle\mathbf{v}^4\rangle=\frac{4D_v^2}{C^{-1}}\frac{\partial^2}{\partial\alpha^2}(C^{-1}),\tag{B6}$$

where C^{-1} follows Eq. (B2). And

$$\langle\mathbf{v}^6\rangle=\frac{8D_v^3}{C^{-1}}\frac{\partial^3}{\partial\alpha^3}(C^{-1}).\tag{B7}$$

So

$$\langle\mathbf{v}^4\rangle=2D_v+\alpha\langle\mathbf{v}^2\rangle,\tag{B8}$$

and

$$\langle\mathbf{v}^6\rangle=2\alpha D_v+(\alpha^2+4D_v)\langle\mathbf{v}^2\rangle.\tag{B9}$$

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