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Soliton solution for the Landau-Lifshitz equation of a one-dimensional bicomponent magnonic crystal

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We investigate nonlinear localized magnetic excitations in a one-dimensional bicomponent magnonic crystal under a periodic magnetic field of spatially varying strength. The governing Landau-Lifshitz equation is transformed into a variable coefficient nonlinear Schrödinger (VCNLS) equation using stereographic projection. In general, the VCNLS equation is nonintegrable and by using Painlevé analysis, we obtain necessary conditions for the VCNLS equation to pass the Weiss-Tabor-Carnevale Painlevé test. A sufficient integrability condition is obtained by further exploring a transformation, which can map the VCNLS equation into the well-known standard nonlinear Schrödinger equation. The transformation builds a systematic connection between the solution of the standard nonlinear Schrödinger equation and VCNLS equation. The results show that the excitation of magnetization in the form of a soliton exists on the oscillatory background with a structure similar to the form of spin Bloch waves. Such a solution exists only when certain conditions on the coefficient of the VCNLS equation are satisfied. To corroborate the analytical results, we performed the numerical simulation by solving the governing VCNLS equation with integrability conditions using the split step Fourier method and the result agrees well with analytical results, and it suggests a way to control the dynamics of magnetization in the form of solitons by an appropriate spatial modulation of the nonlinearity coefficient in the governing VCNLS equation, which depends on the ferromagnetic materials which form the bicomponent magnonic crystal.

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I. INTRODUCTION

The study of nonlinear excitation of magnetization in ferromagnetic nanostructures in terms of solitary waves and solitons has attracted much interest in the past few decades [1–6]. The results reveal that the dynamics is governed by the Landau-Lifshitz equation which can be mapped to the nonlinear Schrödinger (NLS) family of equations [7]. Recently, the investigation on the formation and propagation of a soliton in nonlinear systems with spatial periodicity became a great topic of interest [8]. Bose-Einstein condensate (BEC) in optical lattices [9,10], soliton in photonic lattices [11], and so on are the typical models among them. Motivated by these considerations, in the present paper, we investigate the nature of excitation of magnetization in a one-dimensional bicomponent magnonic crystal. Magnonic crystal is a ferromagnetic medium in which magnetic properties are spatially varying in a periodic way along a definite direction [12]. In the linear studies, the observation and studies on frequency band-gap formation in magnonic crystals is well studied. The special feature of the composite ferromagnetic structure is the formation of an energy band gap in their spectrum of spin waves. The band gap represents a range of energy values in which the spin-wave excitations are forbidden from propagation [13]. The theoretical and experimental investigations on these periodic ferromagnetic structures are mostly devoted to linear phenomena [14]. The studies on the propagation of a soliton in a magnonic crystal are insufficient; there are only a few specific studies in the field that show the experimental and numerical simulation results based on a one-dimensional NLS equation. The bright and dark solitons were observed in yttrium iron garnet films with artificial periodicity [15–17]. Morozova *et al.* investigated the features of the formation of the soliton that are similar to Bragg solitons in the ferromagnetic one-dimensional periodic structure using coupled mode theory [18]. He *et al.* studied the modulation instability and gap solitons in ferromagnetic films under a periodic magnetic field using a multiscale expansion method [19]. The earlier studies are based on homogeneous ferromagnetic films and achieve periodicity by varying the thickness of the films or by applying a spatially varying periodic magnetic field [18,19]. In this present study, we consider an infinite one-dimensional magnonic crystal formed by a periodic array of distinct elements and investigate the nature of the excitation of magnetization in a one-dimensional bicomponent magnonic crystal and also study the impact of material parameters variation on the excitation of magnetization under a periodic magnetic field of spatially varying strength. The paper is organized as follows. In Sec. II, the one-dimensional magnonic crystal model under a periodic magnetic field of spatially varying strength is discussed and the dynamical equation is derived. The governing variable coefficient nonlinear Schrödinger (VCNLS) equation is analyzed through Painlevé analysis to obtain integrability conditions, and it is mapped into the standard NLS equation using a suitable transformation in Sec. III. In Sec. IV, the soliton solutions for a one-dimensional (1-D) magnonic crystal with a different ferromagnetic material combination is constructed, and the impact of material parameters on the localized excitation of magnetization is discussed. In Sec. V the analytical results are compared with the numerical simulation of the governing VCNLS equation using the split step Fourier

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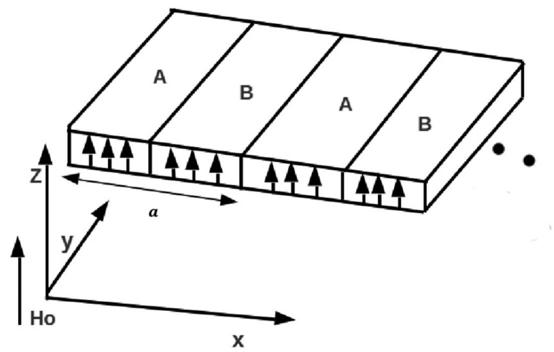

FIG. 1. Schematic drawing of a one-dimensional bicomponent magnonic crystal and its coordinate system. A and B are two ferromagnetic materials.

method (SSFM). Conclusions of the work are summarized in Sec. VI.

II. MODEL AND DYNAMICAL EQUATION

We consider an infinite one-dimensional bicomponent magnonic crystal represented by a system of alternating uniform ferromagnetic layers of two different materials A and B as shown in Fig. 1. The layers have different values of exchange length parameter J_{ex} and saturation magnetization M_{S} throughout the sample. Let $J_{\mathrm{ex,A}}$ and $J_{\mathrm{ex,B}}$ be the exchange lengths for the ferromagnetic materials A and B, respectively. Let $M_{\mathrm{S,A}}$ and $M_{\mathrm{S,B}}$ be the saturation magnetizations for the ferromagnetic materials A and B, respectively. The equation of motion of the magnetization in the 1-D bicomponent magnonic crystal is governed by the following Landau-Lifshitz (LL) equation [20]:

$$\frac{\partial \vec{M}(\vec{r},t)}{\partial t} = -\gamma \vec{M}(\vec{r},t) \times \vec{H}_{\mathrm{eff}}(\vec{r},t), \tag{1}$$

where γ is the gyromagnetic ratio, $\vec{M}(\vec{r},t)$ is the direction of magnetization, and $\vec{H}_{\mathrm{eff}}$ denotes the effective field. In general, the effective field is the sum of several components that includes the applied field, the anisotropy field, the demagnetization field, and the exchange field which are all dependent on space, and we assume the magnetization is nonuniform along the x direction:

$$\vec{H}_{\mathrm{eff}} = \vec{H}_0 + \vec{H}_{\mathrm{ani}} + \vec{H}_{\mathrm{d}} + \vec{H}_{\mathrm{ex}}. \tag{2}$$

The first component $\vec{H}_0$ is the applied magnetic field of spatially varying strength which is inhomogeneous in space along the x direction and applied along the z direction. $\vec{H}_{\mathrm{ani}}$ represents the anisotropy field, which is given by

$$\vec{H}_{\mathrm{ani}} = \beta(x) M_z \hat{z}, \tag{3}$$

where $\beta(x)$ represents the anisotropy constant.

The third component $\vec{H}_{\mathrm{d}}$ arises entirely from the demagnetizing field that corresponds to the shape anisotropy. In the case of a film in x-y plane, the demagnetization field is given by

$$\vec{H}_{\mathrm{d}} = \lambda M_z \hat{z}, \tag{4}$$

where $\lambda = -1$. The exchange field $\vec{H}_{\mathrm{ex}}$ given by [21]

$$\vec{H}_{\mathrm{ex}} = J_{\mathrm{ex}}(x) \vec{\nabla}^2 \vec{M}(\vec{r},t). \tag{5}$$

Here $J_{\mathrm{ex}}(x)$ is the exchange length. Thus, the total effective field $\vec{H}_{\mathrm{eff}}$ takes the form of

$$\vec{H}_{\mathrm{eff}} = H_0(x)\hat{z} + \beta(x) M_z \hat{z} + \lambda M_z \hat{z} + J_{\mathrm{ex}}(x) \vec{\nabla}^2 \vec{M}(\vec{r},t). \tag{6}$$

Upon using the above expression for the effective field, Eq. (6) in Eq. (1), we get

$$\frac{\partial \vec{M}(\vec{r},t)}{\partial t} = -\gamma \vec{M} \times [H_0(x)\hat{z} + \beta(x) M_z \hat{z} + \lambda M_z \hat{z} + J_{\mathrm{ex}}(x) \vec{\nabla}^2 \vec{M}(\vec{r},t)], \tag{7a}$$

$$M_x^2 + M_y^2 + M_z^2 = M_{\mathrm{S}}(x)^2, \tag{7b}$$

where $J_{\mathrm{ex}}(x)$, $M_{\mathrm{S}}(x)$, and $\beta(x)$ are material parameters which are periodic functions with a period equal to the magnonic crystal lattice constant a.

The exchange length $J_{\mathrm{ex}}(x)$ is represented as

$$J_{\mathrm{ex}}(x+a) = J_{\mathrm{ex}}(x) = \begin{cases} J_{\mathrm{ex,A}} & 0 \leqslant x < a/2, \\ J_{\mathrm{ex,B}} & a/2 \leqslant x < a. \end{cases}$$

The saturation magnetization $M_S(x)$ is represented as

$$M_{\mathrm{S}}(x+a) = M_{\mathrm{S}}(x) = \begin{cases} M_{\mathrm{S,A}} & 0 \leqslant x < a/2, \\ M_{\mathrm{S,B}} & a/2 \leqslant x < a. \end{cases}$$

Similarly, the anisotropy constant $\beta(x)$ is represented as

$$\beta(x+a) = \beta(x) = \begin{cases} \beta_{\mathrm{A}} & 0 \leqslant x < a/2, \\ \beta_{\mathrm{B}} & a/2 \leqslant x < a. \end{cases}$$

Since the Landau-Lifshitz equation is a continuous equation, the material parameters should be represented in continuous form. To make its continuous which are expanded in the form of the Fourier series

$$J_{\mathrm{ex}}(x) = J_{\mathrm{av}} + \sum_{n=1} \left(\frac{\Delta J}{n\pi}\right) [1 - (-1)^n] \sin\left(\frac{2n\pi x}{a}\right). \tag{8}$$

The saturation magnetization $M_{\mathrm{S}}(x)$ is represented as

$$M_{\mathrm{S}}(x) = M_{\mathrm{S,av}} + \sum_{n=1} \left(\frac{\Delta M_S}{n\pi}\right) [1 - (-1)^n] \sin\left(\frac{2n\pi x}{a}\right). \tag{9}$$

Similarly, the anisotropy constant $\beta(x)$ is represented as

$$\beta(x) = \beta_{\mathrm{av}} + \sum_{n=1} \left(\frac{\Delta \beta}{n\pi}\right) [1 - (-1)^n] \sin\left(\frac{2n\pi x}{a}\right), \tag{10}$$

where $\Delta J = J_{\mathrm{ex,A}} - J_{\mathrm{ex,B}}$, $\Delta M = M_{\mathrm{S,A}} - M_{S,B}$, and $\Delta \beta = \beta_{\mathrm{A}} - \beta_{\mathrm{B}}$. J_{av}, $M_{\mathrm{S,av}}$, and β_{av} represent the average exchange length, saturation magnetization, and anisotropy constant value of two ferromagnetic materials A and B, respectively. In this composite ferromagnetic structure the material parameters are varying gradually at the interface between the two ferromagnetic materials, and the average values of the material parameters represent a value of the material parameter of the periodic ferromagnetic system at the exact center of the interface between the two ferromagnetic materials.

The LL equation is a vector nonlinear partial differential equation and it is difficult to solve in its original form. By using stereographic projection, we transform the LL equation into a nonlinear equation of a complex function. Here assume that the magnetization is nonuniform along the x direction. In the component form Eq. (7) becomes

$$\frac{\partial M_x}{\partial t} = -\gamma\left\{H_0(x)M_y + \beta(x)M_zM_y + \lambda M_zM_y + J_{\rm ex}(x)\left(M_y\frac{\partial^2}{\partial x^2}M_z - M_z\frac{\partial^2}{\partial x^2}M_y\right)\right\}, \tag{11}$$

$$\frac{\partial M_y}{\partial t} = -\gamma\left\{-H_0(x)M_x - \beta(x)M_zM_x - \lambda M_zM_x - J_{\rm ex}(x)\left(M_x\frac{\partial^2}{\partial x^2}M_z - M_z\frac{\partial^2}{\partial x^2}M_x\right)\right\}, \tag{12}$$

$$\frac{\partial M_z}{\partial t} = -\gamma\left\{J_{\rm ex}(x)\left(M_x\frac{\partial^2}{\partial x^2}M_y - M_y\frac{\partial^2}{\partial x^2}M_x\right)\right\}. \tag{13}$$

We define

$$\psi(x,t) = \frac{M_x + iM_y}{M_{\rm S}(x)}, \tag{14}$$

$$\psi^*(x,t) = \frac{M_x - iM_y}{M_{\rm S}(x)}, \tag{15}$$

where ψ is a complex variable, and then we have

$$m_z(x,t) = (1-|\psi|^2)^{1/2}. \tag{16}$$

Consider a small deviations of magnetization from the equilibrium direction corresponding to $|\psi|^2 \ll 1$ and under the long wavelength approximation by keeping only the nonlinear terms of magnitude $|\psi|^2\psi$ [22]. Substituting Eqs. (14) to (16) into the component form of Eq. (7), we get

$$i\frac{\partial\psi}{\partial t} = \left(\frac{J_{\rm ex}(x)}{J_{\rm av}}\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right)\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2}[1-\beta(x)]\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right)|\psi|^2\psi - \left[\left(\frac{H_0(x)}{M_{\rm S,av}}\right) - [1-\beta(x)]\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right)\right]\psi. \tag{17}$$

Here, the temporal and spatial coordinates are rescaled by $t_o = 1/(\gamma M_{S,\rm av})$ and $l_o = \sqrt{(J_{\rm av})}$, respectively. Let

$$f(x) = \left(\frac{J_{\rm ex}(x)}{J_{\rm av}}\right)\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right),$$

$$g(x) = [1-\beta(x)]\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right),$$

$$h(x) = \left(\frac{H_0(x)}{M_{\rm S,av}}\right) - [1-\beta(x)]\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right).$$

Equation (17) becomes

$$i\frac{\partial\psi}{\partial t} - f(x)\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}g(x)|\psi|^2\psi + h(x)\psi = 0. \tag{18}$$

Equation (18) is the nonlinear Schrödinger equation with variable coefficients.

When $f(x) = g(x) = h(x) =$ constant, Eq. (18) reduces to the completely integrable nonlinear Schrödinger equation which admits N-soliton solutions [23]. In the absence of a cubic term, the above Eq. (18) is a linear periodic system which admits Bloch wave solutions and in the presence of a cubic term, it is a completely nonlinear problem. In general, the above VCNLS equation is nonintegrable and by using Painlevé analysis, we obtain necessary conditions for the VCNLS equation to be integrable and which is discussed in forthcoming section.

III. PAINLEVÉ ANALYSIS AND INTEGRABILITY CONDITIONS

Several tools such as Painlevé analysis [24], Lax pair [25], and similarity transformation techniques [26] are available to solve a VCNLS equation to obtain analytical solutions. Our analysis is based on the Painlevé test for partial differential equations, i.e., the Weiss-Tabor-Carnevale (WTC) test, which has been found to be a successful tool for investigating the integrability of partial differential equations. In this section, we use the WTC test to obtain an integrability condition for the VCNLS equation and then under this condition, we look for a transformation which converts the Eq. (18) to the standard NLS equation. In order to perform conveniently, we rewrite the Eq. (18) and its complex conjugate by replacing ψ by a and ψ^* by b we obtain

$$i\frac{\partial a}{\partial t} - f(x)\frac{\partial^2 a}{\partial x^2} + \frac{1}{2}g(x)a^2b + h(x)a = 0, \tag{19a}$$

$$-i\frac{\partial b}{\partial t} - f(x)\frac{\partial^2 b}{\partial x^2} + \frac{1}{2}g(x)b^2a + h(x)b = 0, \tag{19b}$$

where $f(x)$, $g(x)$, and $h(x)$ are real functions.

The next step is to seek the solution in the form of the Laurent series

$$a(x,t) = \sum_{j=0}^{\infty} a_j(x,t)\phi^{\alpha+j}(x,t), \tag{20a}$$

$$b(x,t) = \sum_{j=0}^{\infty} b_j(x,t)\phi^{\beta+j}(x,t), \tag{20b}$$

where a_j, b_j, and $\phi(x,t)$ are analytic functions. α and β are negative integers to be determined from the leading order analysis.

A. Leading order analysis

By the standard procedure, to determine the leading orders we substitute $a \sim a_0\phi^\alpha$ and $b \sim b_0\phi^\beta$ in Eq. (19) and compare the terms of smallest order

$$\alpha + \beta = -2,$$

which gives integer $\alpha = -1$ and $\beta = -1$, and in addition one can obtain the following relations:

$$2f(x)a_0\phi_x^2 - \frac{1}{2}g(x)a_0^2b_0 = 0, \tag{21a}$$

$$2f(x)b_0\phi_x^2 - \frac{1}{2}g(x)b_0^2a_0 = 0, \tag{21b}$$

and it gives

$$a_0b_0 = 4\left(\frac{f(x)}{g(x)}\right)\phi_x^2. \tag{22}$$

B. Resonance and arbitrary functions

Substituting Eq. (20) into Eq. (19) and for $j = 0$ equating the coefficients ϕ^{-3}, we obtain equations Eq. (21). For $j \geqslant 1$ equating the coefficients ϕ^{j-3} we get the following relations:

$$[f(x)(j^2-3j+2)\phi_x^2 - a_0 b_0 g(x)]a_r - \left(\frac{g(x)}{2}\right)a_0^2 b_r = 0, \tag{23a}$$

$$-\left(\frac{g(x)}{2}\right)b_0^2 a_r + [f(x)(j^2-3j+2)\phi_x^2 - a_0 b_0 g(x)]b_r = 0 \tag{23b}$$

where a_r and b_r are the analytic functions at resonance and obtain resonance level at $j = -1,0,3,4$. The resonance point at $j = -1$ corresponds to the arbitrariness of the singular manifold $\phi(x,t)$.

For $j = 0$ we have only one relation

$$a_0 b_0 = 4\left(\frac{f(x)}{g(x)}\right)\phi_x^2. \tag{24}$$

There are two unknown functions a_0 and b_0 but we have only one relation. So, one of the functions can be taken as arbitrary.

For $j = 1$, we obtain

$$a_1 = \left(\frac{1}{12 f\phi_x^2}\right)\{-8fa_{0x}\phi_x^3 + 6ia_0\phi_t\phi_x^2 - 2fa_0\phi_x^2\phi_{xx} + ga_0^2 b_{0x}\phi_x\}, \tag{25a}$$

$$b_1 = \left(\frac{1}{12 f\phi_x^2}\right)\{-8fb_{0x}\phi_x^3 - 6ib_0\phi_t\phi_x^2 - 2fb_0\phi_x^2\phi_{xx} + gb_0^2 a_{0x}\phi_x\}. \tag{25b}$$

For $j = 2$ we obtain

$$\begin{aligned}a_2 = \left(\frac{4}{3}\right)\left(\frac{1}{ga_0^2 b_0^2}\right)\bigg\{&-ia_0b_0a_{0t} - \left(\frac{g}{2}\right)a_0^2 b_0 a_1 b_1 \\ &- \left(\frac{g}{2}\right)a_0 b_0^2 a_1^2 + fa_0b_0a_{0xx} - \left(\frac{i}{2}\right)a_0^2 b_0 t - \left(\frac{1}{2}\right)a_0^2 f b_{0xx} \\ &- \left(\frac{1}{2}\right)ha_0^2 b_0 + \left(\frac{g}{4}\right)a_0^3 b_1^2\bigg\},\end{aligned} \tag{26a}$$

$$\begin{aligned}b_2 = \left(\frac{4}{3}\right)\left(\frac{1}{ga_0^2 b_0^2}\right)\bigg\{&ia_0b_0b_{0t} - \left(\frac{g}{2}\right)b_0^2 a_0 a_1 b_1 \\ &- \left(\frac{g}{2}\right)b_0 a_0^2 b_1^2 + fa_0b_0b_{0xx} + \left(\frac{i}{2}\right)b_0^2 a_0 t - \left(\frac{1}{2}\right)b_0^2 f a_{0xx} \\ &- \left(\frac{1}{2}\right)hb_0^2 a_0 + \left(\frac{g}{4}\right)b_0^3 a_1^2\bigg\}.\end{aligned} \tag{26b}$$

At $j = 3$, we get a linear relation between a_3 and b_3 and one of them is arbitrary:

$$\begin{aligned}b_0a_3 + a_0b_3 = \left(\frac{-2}{a_0 g}\right)\bigg\{&-f(2a_{2x}\phi_x + a_2\phi_{xx}) \\ &+ g\left[a_0a_1b_2 + b_0a_1a_2 + \frac{b_1}{2}(2a_0a_2 + a_1^2)\right] \\ &+ i(a_2\phi_t + a_{1t}) - fa_{1xx} + ha_1\bigg\},\end{aligned} \tag{27a}$$

$$\begin{aligned}b_0a_3 + a_0b_3 = \left(\frac{-2}{b_0 g}\right)\bigg\{&-f(2b_{2x}\phi_x + b_2\phi_{xx}) \\ &+ g\left[b_0b_1a_2 + a_0b_1b_2 + \frac{a_1}{2}(2b_0b_2 + b_1^2)\right] \\ &- i(b_2\phi_t + b_{1t}) - fb_{1xx} + hb_1\bigg\}.\end{aligned} \tag{27b}$$

Equating (27a) and (27b) and substituting Eqs. (24) to (26), we obtain a compatibility equation at $j = 3$:

$$\begin{aligned}&2f\phi_x(b_0a_{2x} - a_0b_{2x}) + f\phi_{xx}(b_0a_2 - a_0b_2) \\ &- g(b_0^2a_1a_2 - a_0^2b_1b_2) - \left(\frac{g}{2}\right)a_1b_1(b_0a_1 - a_0b1) \\ &- i\phi_t(b_0a_2 + a_0b_2) - i(b_0a_{1t} + a_0b_{1t}) \\ &+ f(b_0a_{1xx} - a_0b_{1xx}) - h(b_0a_1 - a_0b1) = 0.\end{aligned} \tag{28}$$

The above Eq. (28) is satisfied if and only if

$$\left(\frac{f_x}{f}\right)^2 + \left(\frac{f_xg_x}{fg}\right) + 4\left(\frac{g_x}{g}\right)^2 - \left(\frac{f_{xx}}{f}\right) - 2\left(\frac{g_{xx}}{g}\right) = 0. \tag{29}$$

From Eq. (29), we do not obtain clear explicit integrability conditions for the VCNLS equation in terms of its coefficients $f(x)$ and $g(x)$. So, we extend our analysis to the next resonance level.

At $j = 4$, we get a linear relation between a_4 and b_4 and one of them is arbitrary:

$$\begin{aligned}-b_0a_4 + a_0b_4 = \left(\frac{-2}{a_0 g}\right)\bigg\{&ia_{2t} + 2ia_3\phi_t - f[a_{2xx} + 4a_{3x}\phi_x \\ &+ 2\phi_{xx}a_3] + g\bigg[a_0a_1b_3 + \left(\frac{b_2}{2}\right)(a_1^2 + 2a_0a_2) \\ &+ \left(\frac{b_1}{2}\right)(2a_0a_3 + 2a_1a_2) \\ &+ \left(\frac{1}{2}\right)b_0(2a_1a_3 + a_2^2)\bigg] + ha_2\bigg\},\end{aligned} \tag{30a}$$

$$\begin{aligned}-b_0a_4 + a_0b_4 = \left(\frac{2}{b_0 g}\right)\bigg\{&-ib_{2t} - 2ib_3\phi_t - f[b_{2xx} + 4b_{3x}\phi_x \\ &+ 2\phi_{xx}b_3] + g\bigg[b_0b_1a_3 + \left(\frac{a_2}{2}\right)(b_1^2 + 2b_0b_2) \\ &+ \left(\frac{a_1}{2}\right)(2b_0b_3 + 2b_1b_2) \\ &+ \left(\frac{1}{2}\right)a_0(2b_1b_3 + b_2^2)\bigg] + hb_2\bigg\}.\end{aligned} \tag{30b}$$

Equating (30a) and (30b) and substituting Eqs. (24) to (26) and also a_3 and b_3 of the resonance at $j = 3$. we get a compatibility equation at $j = 4$:

$$\begin{aligned}&4\phi_x f(b_0a_{3x} + a_0b_{3x}) + 2f\phi_{xx}(b_0a_3 + a_0b_3) \\ &+ f(b_0a_{2xx} + a_0b_{2xx}) - g[2a_0b_0a_1b_3 + 2a_0b_0a_2b_2 \\ &+ 2a_0b_0b_1a_3 + b_0a_1b_1a_2 + a_0a_1b_1b_2 + b_0^2a_1a_3 + a_0^2b_1b_3 \\ &+ \tfrac{1}{2}b_0b_2a_1^2 + \tfrac{1}{2}a_0a_2b_1^2 + \tfrac{1}{2}b_0^2a_2^2 + \tfrac{1}{2}a_0^2b_2^2] - i(b_0a_{2t} - a_0b_{2t}) \\ &- 2i\phi_t(b_0a_3 - a_0b_3) - h(b_0a_2 + a_0b_2) = 0.\end{aligned} \tag{31}$$

The above equation is satisfied if and only if the following conditions are satisfied:

$$\left(\frac{f_x}{f}+2\frac{g_x}{g}\right)=0, \tag{32}$$

which integrates to give

$$f(x)=\frac{k}{g(x)^2}, \tag{33}$$

$$h(x)=\left(\frac{k}{2}\right)\left[\frac{g_{xx}}{g^3}-\frac{3}{2}\frac{g_x^2}{g^4}\right], \tag{34}$$

where k is an integration constant. By employing the Painlevé method for the governing VCNLS Eq. (18), the integrability conditions are obtained. Here,

$$f(x)=\left(\frac{J_{\rm ex}(x)}{J_{\rm av}}\right)\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right),$$

$$g(x)=[1-\beta(x)]\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right),$$

$$h(x)=\left(\frac{H_0(x)}{M_{\rm S,av}}\right)-[1-\beta(x)]\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right).$$

The difference in anisotropy constant values β between the constituent materials tends to be low and the effect of this inhomogeneity is minimum. Hence assume this component to be negligible. Then,

$$f(x)=\left(\frac{J_{\rm ex}(x)}{J_{\rm av}}\right)\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right),$$

$$g(x)=\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right),$$

$$h(x)=\left(\frac{H_0(x)}{M_{\rm S,av}}\right)-\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right).$$

Substituting $f(x)$, $g(x)$, and $h(x)$ in Eqs. (33) and (34) we get the integrability conditions as

$$\left(\frac{J_{\rm ex}(x)}{J_{\rm av}}\right)=\left(k\frac{M_{\rm S,av}^3}{M_{\rm S}^3(x)}\right), \tag{35}$$

and

$$\left(\frac{H_0(x)}{M_{\rm S,av}}\right)-\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right) = M_{\rm S,av}^2\left(\frac{-3M_{\rm S}^{'2}(x)+2M_{\rm S}(x)M_{\rm S}^{''}(x)}{4M_{\rm S}^4(x)}\right), \tag{36}$$

i.e.,

$$\frac{H_0(x)}{M_{\rm S,av}}=\left(\frac{M_{\rm S}(x)}{M_{\rm S,av}}\right)+M_{\rm S,av}^2\left(\frac{-3M_{\rm S}^{'2}(x)+2M_{\rm S}(x)M_{S}^{''}(x)}{4M_{\rm S}^4(x)}\right). \tag{37}$$

The above integrability conditions are consistent with [9] and [24]. The material parameter exchange constant $J_{\rm ex}(x)$, which is inhomogeneous in space related to the saturation magnetization $M_{\rm S}(x)$ is given by Eq. (35). The material combination of two different ferromagnetic materials that we have chosen to form a magnonic crystal is such that the material combination satisfies the above integrability condition, Eq. (35). From Eq. (37) it is noted that the form of the periodic applied magnetic field is determined by the ferromagnetic materials which form the magnonic crystal. The applied magnetic field is periodic in space and its periodicity forms a periodic potential for the spin waves. The form of the potential is related to saturation magnetization $M_{\rm S}(x)$ which is given by Eq. (37).

Further, we have to construct the solution of the VCNLS equation by using a transformation which converts Eq. (18) into a standard NLS equation. We look for the transformation of the form [9],

$$\psi(x,t)=r(x)q(X(x),T(t)), \tag{38}$$

where $X=X(x)$ and $T=T(t)$ and $r(x)$ are the real functions to be determined.

Substituting Eq. (38) into Eq. (18), we obtain set of following equations:

$$f(x)r_{xx}+h(x)r(x)=0, \tag{39}$$

$$2r_xX_x+r(x)X_{xx}=0, \tag{40}$$

$$T_t=f(x)X_x^2=g(x)r(x)^2, \tag{41}$$

and

$$iq_T-q_{XX}+\tfrac{1}{2}|q|^2q=0. \tag{42}$$

By using the integrability conditions, the above constraint equations are solved and give

$$r(x)=\sqrt{\frac{1}{g(x)}}, \tag{43}$$

$$X(x)=\int g(x)dx, \quad \text{and} \quad T(t)=t. \tag{44}$$

Then, we obtain the solution of Eq. (18) by the known solution of the standard NLS equation, $q(X,T)$,

$$\psi(x,t)=\sqrt{\frac{1}{g(x)}}q(X(x),T(t)). \tag{45}$$

Here, $q(X,T)$ is the solution of the standard NLS equation, Eq. (42), and there exists several methods to solve the standard NLS equation, such as the classical inverse scattering transform (IST), the Darboux transformation (DBT), the Hirota bilinear method, and so on. In this article, the Hirota bilinear method is used to construct the dark one soliton solution:

$$q(X(x),T(t))=\frac{1}{\sqrt{2}}\{c-2id\tanh[d(X-cT)]\} \times\exp\left[\frac{-i}{2}(c^2+4d^2)T\right]. \tag{46}$$

IV. RESULTS AND DISCUSSION

From Eq. (45), we obtain

$$\psi(x,t)=\frac{1}{\sqrt{2g(x)}}\left\{c-2id\tanh\left[d\left(\int g(x)dx-ct\right)\right]\right\} \times\exp\left[\frac{-i}{2}(c^2+4d^2)t\right], \tag{47}$$

where the parameters c and d correspond to the velocity and depth of the dark soliton, respectively, and

$$g(x) = 1 + \sum_{n=1} \left(\frac{M_{\mathrm{S,A}} - M_{\mathrm{S,B}}}{M_{\mathrm{S,av}} n\pi} \right) [1 - (-1)^n] \sin\left(\frac{2n\pi x}{a} \right), \tag{48}$$

which is written as

$$g(x) = 1 + \sum_{n=1} l_n \sin\left(\frac{2n\pi x}{a} \right), \tag{49}$$

$$l_n = \left(\frac{M_{\mathrm{S,A}} - M_{\mathrm{S,B}}}{M_{\mathrm{S,av}} n\pi} \right) [1 - (-1)^n], \tag{50}$$

where l_n is the control parameter which determines the nature of the magnonic crystal. $M_{S,A}$ and $M_{S,B}$ are the saturation magnetizations for the ferromagnetic materials A and B, respectively. $M_{S,av}$ is the average saturation magnetization value of two ferromagnetic materials A and B. The material parameter $M_S(x)$ varies smoothly at the interface between the two ferromagnetic materials, and the average value represents the saturation magnetization value of the periodic ferromagnetic system at the exact center interface point between the two ferromagnetic materials.

As we mentioned earlier, the Landau-Lifshitz equation is a continuous equation which describes the equation of motion of magnetization in a ferromagnetic medium. Here we consider a periodic ferromagnetic system in which the material parameter varies periodically. In order to make it continuous and incorporate it into the LL equation, we use Fourier series to represent the periodic material variation into continuous form. In the above equation, $g(x)$ represents continuous form for the variation of saturation magnetization value at all points in the periodic ferromagnetic system having equal widths. By using two different ferromagnetic materials to form a periodic ferromagnetic structure, the form of $g(x)$ changes accordingly. The results show that the amplitude of the soliton solution depends on the nonlinearity coefficient $g(x)$, which means that the soliton can be spatially modulated and admits several interesting phenomena.

TABLE I. Experimentally observed material parameters for analytical and numerical calculations [20].

Material	Saturation Magnetization M_{S} (10^6 A/m)	Exchange Length J_{ex} (nm)
Fe	1.752	3.30
Co	1.445	4.78
Py	0.860	7.64

From ψ, we obtain the components of magnetization as

$$m_x(x,t) = \left(\frac{\psi + \psi*}{2} \right), \tag{51}$$

$$m_y(x,t) = \left(\frac{\psi - \psi*}{2i} \right), \tag{52}$$

$$m_z(x,t) = (1 - |\Psi|^2)^{\frac{1}{2}}. \tag{53}$$

A magnonic crystal system formed by a periodic array of distinct ferromagnetic elements and its structure is formed with lattice constant $a = 500$ nm, by choosing the width of the each layer as 250 nm. The magnon density profile $|\Psi|^2 = \frac{1}{2g(x)}[c^2 + 4d^2 \tanh^2 \{d[\int g(x)dx - ct]\}]$, for the two materials (a) 250Fe-250Co and (b) 250Co-250Py is shown in Fig. 2. The material parameters used for calculations are taken from Ref. [20] shown in Table I.

The excitation of magnetization in the composite magnonic crystal structure with the above mentioned integrability conditions is in the form of a soliton that exists in the oscillatory background with a structure similar to the form of spin Bloch waves [27]. The periodic applied magnetic field of spatially varying strength of the form given in Eq. (37) which is the condition for integrability of the governing VCNLS equation is shown in Fig. 3 for the combination of (a) 250Fe-250Co and (b) 250Co-250Py. The parameter $g(x)$ determines the nature of the magnonic crystal and periodic applied magnetic field which act as a periodic potential for spin waves. The periodic potential has different depths for the different combinations of the magnonic crystal structure and form of the potential given in Eq. (37) and shown in Fig. 3. Experimentally it is achieved by

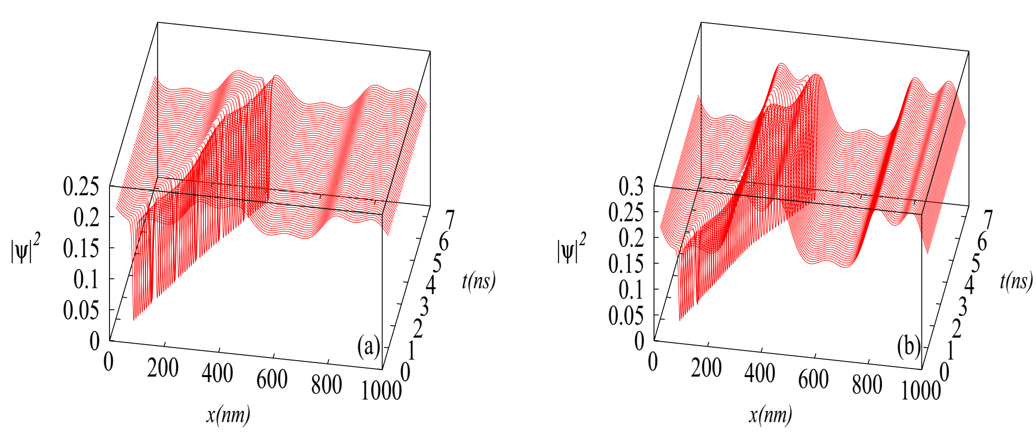

FIG. 2. The profile of magnon density $|\Psi|^2 = m_x^2 + m_y^2$, in the form of a soliton with a background of spin Bloch waves in three dimensions with parameters $c = 0.03$ and $d = 0.3$ for the two materials (a) 250Fe-250Co and (b) 250Co-250Py.

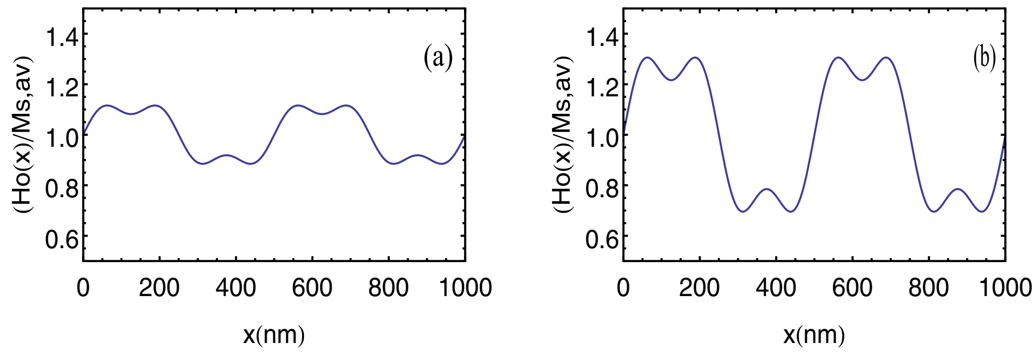

FIG. 3. Periodic applied magnetic field of spatially varying strength which is the condition for integrability from Eq. (37) for the combination of (a) 250Fe-250Co and (b) 250Co-250Py.

periodically screening the uniform field [28]. Figure 4 shows the propagation of the magnon density profile in the form of a soliton on the background of spin Bloch waves at different times: Figs. 4(a) and 4(b) for the combination of 250Fe-250Co at times $t = 1$ and $t = 7$ ns, respectively; and Figs. 4(c) and 4(d) for 250Co-250Py at times $t = 1$ and $t = 7$ ns, respectively. For time $t = 1$ ns, the soliton position is located at 62 and 78 nm for 250Fe-250Co and 250Co-250Py, respectively. At time $t = 7$ ns, the position of the localized soliton profile varies and is located at 420 and 438 nm, respectively, for 250Fe-250Co and 250Co-250Py. Thus, the Fig. 4 clearly shows that the velocity of the soliton propagating in the composite magnonic crystal structure varies with respect to different combinations. Also one can utilize the composite structure effect to control and change the velocity of the soliton accordingly.

V. NUMERICAL RESULTS

In order to corroborate our analytical results, the numerical computation of the soliton profile is carried out by solving the governing VCNLS equation, Eq. (18), with integrability conditions using the SSFM [29]. The split step Fourier method is a pseudospectral numerical method, to solve nonlinear partial differential equations and it relies on computing the solution in small steps. In the SSFM, the corresponding nonlinear partial differential equation is split into linear and nonlinear parts. The nonlinear part is solved in the time domain and the linear part solved in the frequency domain by means of a Fourier transform. In this section, we have solved the Eq. (18) with integrability conditions and the corresponding soliton profiles are plotted in Fig. 5. The material parameters used for calculations are taken from Ref. [20] shown in Table I. Figure 5

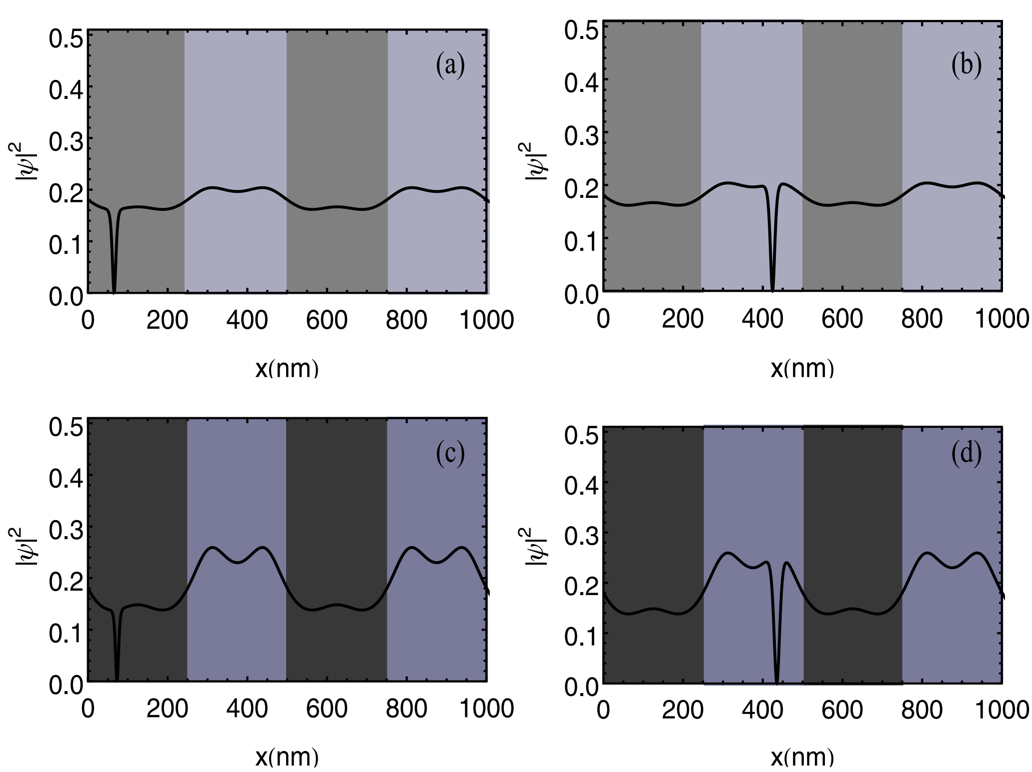

FIG. 4. The spatial profile of magnon density $|\Psi|^2 = m_x^2 + m_y^2$, in the form of a soliton on the background of spin Bloch waves in two dimensions (2D) with parameters $c = 0.03$ and $d = 0.3$, at different times (a) $t = 1$ and (b) $t = 7$ for 250Fe-250Co, and (c) $t = 1$ and (d) $t = 7$ ns for 250Co-250Py. The localized soliton position varies with respect to time for the combinations of 250Fe-250Co and 250Co-250Py.

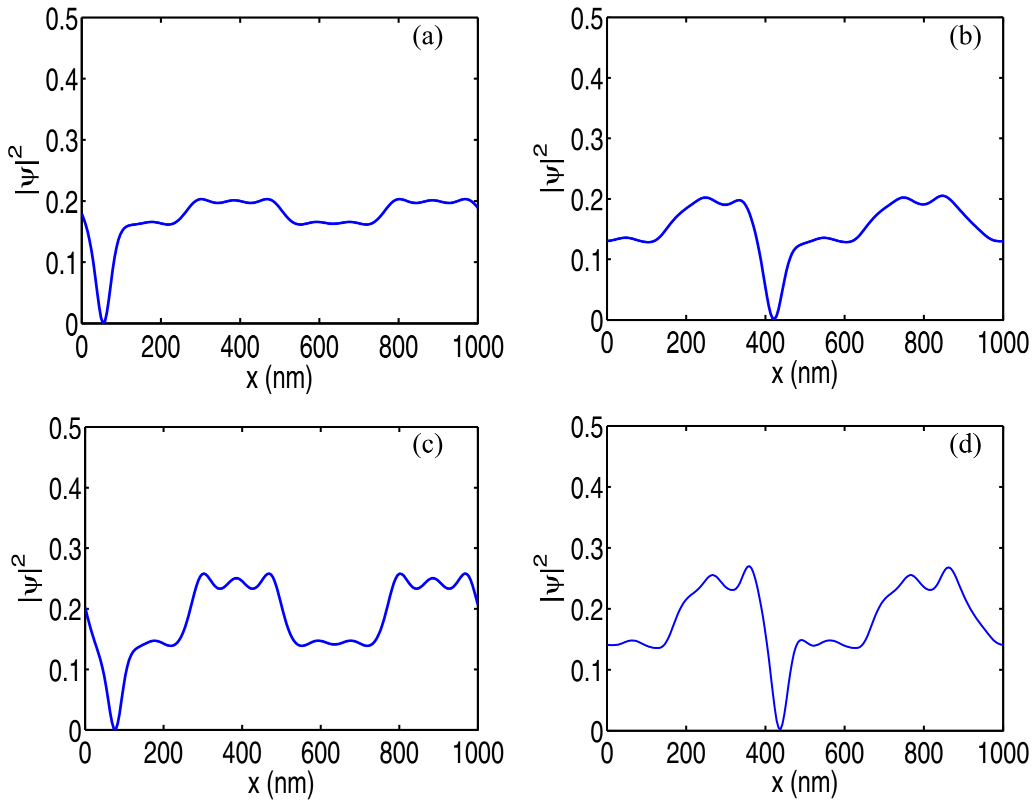

FIG. 5. Profile of magnon density $|\Psi|^2 = m_x^2 + m_y^2$, in the form of a soliton on the background of spin Bloch waves in 2D with parameters $c = 0.03$ and $d = 0.3$ based on a numerical study, for the material combination 250Fe-250Co at different times (a) $t = 1$ and (b) $t = 7$ ns, and for the material combination 250Co-250Py at times (c) $t = 1$ and (d) $t = 7$ ns.

shows numerical computation of the propagation of magnon density profile $|\Psi|^2$ in the form of a soliton on the background of spin Bloch waves at different times: Figs. 5(a) and 5(b) for the combination of 250Fe-250Co at times $t = 1$ and $t = 7$ ns, respectively; and Figs. 5(c) and 5(d) for 250Co-250Py at times $t = 1$ and $t = 7$ ns, respectively. For $t = 1$ ns, the soliton profile located at 59 and 74 nm and for $t = 7$ ns, it moves and located at 418 and 436 nm for materials with combinations 250Fe-250Co and 250Co-250Py, respectively. The numerical solution of the Eq. (18) given in Fig. 5 agrees well with the analytical solution of the soliton profile which is presented in Fig. 4. The stability nature of the soliton solution depends on parameters which are material dependent and it's completely under our control. By choosing the ferromagnetic materials which forms the magnonic crystal of our interest, it is possible to tune the spatially modulated amplitude soliton. These spatially modulated amplitude soliton solutions with oscillatory background describe the nonlinear localized excitation of magnetization in a one-dimensional magnonic crystal.

VI. CONCLUSIONS

In this paper, we have investigated the dynamics of magnetization in a one-dimensional magnonic crystal by transforming the governing Landau-Lifshitz equation into the VCNLS equation, and with the aid of Painlevé analysis we constructed the soliton solution that exists on the oscillatory background with a structure similar to the form of spin Bloch waves. Such solutions exists in certain constraint conditions on the coefficients of the VCNLS equation. The results show that the amplitude of the soliton solution has a spatial period on the background of spin Bloch waves. For the different combinations of the magnonic crystal structure, the position of the localized soliton profile varies with respect to time. It clearly shows that the velocity of the soliton propagates in the composite magnonic crystal structure varies with respect to different material combinations. Thus it gives an additional degree of freedom to control the propagation of the soliton in this composite ferromagnetic structure provided it must satisfy the integrability conditions. We have verified the analytical results with a numerical simulation by solving the governing VCNLS equation with integrability conditions using the split step Fourier method. The result of the numerical simulation agrees well with the analytical results. The spatial distribution of the soliton profile is determined by the free parameter $g(x)$, which is dependent on the saturation magnetization M_S values of the two ferromagnetic materials which are used to form the magnonic crystal. For different combination of magnonic crystal, the form of nonlinear coefficient $g(x)$ varies and also the form of the periodic applied magnetic field can be changed to satisfy the integrability conditions. Hence the desirable amplitude modulation of the soliton can be achieved.

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