Non-Markovian barotropic-type and Hall-type fluctuation relations in crossed electric and magnetic fields

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In this paper we derive the non-Markovian barotropic-type and Hall-type fluctuation relations for noninteracting charged Brownian particles embedded in a memory heat bath and under the action of crossed electric and magnetic fields. We first obtain a more general non-Markovian fluctuation relation formulated within the context of a generalized Langevin equation with arbitrary friction memory kernel and under the action of a constant magnetic field and an arbitrary time-dependent electric field. It is shown that this fluctuation relation is related to the total amount of an effective work done on the charged particle as it is driven out of equilibrium by the applied time-dependent electric field. Both non-Markovian barotropic- and Hall-type fluctuation relations are then derived when the electric field is assumed to be also a constant vector pointing along just one axis. In the Markovian limit, we show explicitly that they reduce to the same results reported in the literature.

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I. INTRODUCTION

The fluctuation relations continue to be a topic of increasing interest in the understanding of the nonequilibrium dynamical behavior of physical, chemical, and biological systems, in which the stochastic fluctuations play a fundamental role. This indeed can be corroborated in the current literature where an important number of theoretical [1] and experimental studies, which elucidate different aspects of the fluctuation relations, can be found. In particular, a wide and interesting discussion was given recently in a paper by Bochkov and Kuzovlev [2] on the concepts of the generalized fluctuation-dissipation relations and fluctuation theorems for open and closed nonequilibrium thermodynamic systems. In that reference it is shown that the fluctuation theorems, Crooks relation and Jarzynski equality (JE), are alternative formulations or special cases of their theory reported in Refs. [2–5] and references therein. It is also shown explicitly that the Jarzynski [6] and Bochkov and Kuzovlev equalities supplement one another and coincide only when the free-energy difference $\Delta\mathcal{F} = 0$. It is worth noting that all of those interesting discussions are given in the spirit of nonequilibrium statistical mechanics of Markovian stochastic processes. However, a few years ago, in a paper by Ohkuma and Ohta, the fluctuation theorems obtained in a stochastic Markovian process could be generalized to a non-Markovian system governed by a generalized second-order Langevin equation with a Gaussian colored noise and arbitrary friction memory kernel [7]. Crooks fluctuation theorem [8], as well as the Jarzynski equality, were extended to this non-Markovian case. Similarly, in a paper by Mai and Dhar [9], it was proven that using also the generalized Langevin equation (GLE) with arbitrary memory kernel and under the action of arbitrary time-dependent driving force, the transient work fluctuation theorem (WFT), JE, and Crooks theorem are valid in an exact way. Very recently, the WFT has been extended to the case of a charged Brownian harmonic oscillator across a magnetic field and driven out of equilibrium by an arbitrary time-dependent applied electric field [10]. The theorem has shown to be valid using also a generalized Langevin equation (GLE) with arbitrary friction memory kernel. It is also true that a considerable amount of work related to all the above-mentioned stochastic concepts have been established and derived within the context of a Markovian dynamics [11–14]; only a few of them have been explored and derived within the context of a non-Markovian dynamics [7,9,10,15–21]. The major part of the experiments referred to in Refs. [22–34] deal with trapped nanoparticles surrounded by heat reservoirs. This calls our attention the very recent experiments related again to the concepts mentioned above. For instance, in a recent review [32], several concepts of statistical mechanics for systems that are out of equilibrium have been discussed from an experimental point of view. The validity of the fluctuation theorem for the relative entropy change occurring during relaxation from nonequilibrium steady state is explored in Ref. [33] and the measure of the work done by a trapped single particle which obeys the Crooks fluctuation theorem at an effective temperature is analyzed in Ref. [34]. In this latter case, the experiment is performed with an optically trapped single microparticle immersed in water in the presence of an external colored noise. On the other hand, the fluctuation theorems for the work, power, and total entropy change have also been confirmed theoretically for a charged particle trapped in harmonic potentials in the presence of a magnetic field, as can be corroborated in the works [35–40]. Also the fluctuation theorems have been proved to be valid for a classical system in the presence of a time-reversible symmetry-breaking field, such as an external time-dependent magnetic field, and nonconservative forces which cannot be derived from gradient of scalar potentials [41]. The merit of the work is to consider the system and a heat bath as a combined system

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and showing that the fluctuation theorems are valid even when the heat bath goes out of equilibrium during driving. Under these conditions the fluctuation theorems are demonstrated from both deterministic and stochastic, but Markovian, points of view. Although there is no Hamiltonian for the combined system due to the presence of nonconservative forces, it is possible to define the total energy for the system and show that the rate of change for the total energy is the same as the rate of change of the work done by all the external forces on the system. Additional comments of this reference will be given after our concluding remarks. It is worth noting that, as far as we know, the experimental confirmation of any of the aforementioned concepts in the presence of a constant or time-dependent magnetic field has not yet been performed, even in the Markovian case. So those given in the Markovian case and the present contribution suggest the execution of similar experiments as those executed in Refs. [22–34].

In 2008 other type of fluctuation relations named barotropic-type and Hall-type fluctuation relations were derived by Roy and Kumar [38] using the standard two-dimensional overdamped Langevin equation for noninteracting electrons in crossed electric and magnetic fields. The barotropic-type fluctuation relation was derived and established for the ratio $f(x,t)/f(-x,t)$ and the Hall-type fluctuation relation for the ratio $f(y,t)/f(-y,t)$, where $f(\zeta,t)$, with $\zeta = x,y$, is the probability of observing a given trajectory at (ζ,t) during the forward process and $f(-\zeta,t)$ the probability of observing its backward counterpart at $(-\zeta,t)$. It is our purpose in this paper to generalize both Markovian fluctuation relations to the case of a non-Markovian system characterized by a set of noninteracting charged particles governed by a generalized second-order Langevin equation with arbitrary friction memory kernel and under the action of crossed electric and magnetic fields. The applied magnetic field is assumed as a constant vector, usually pointing along the z axis, and the electric field as an arbitrary time-dependent vector responsible for the driving the system out of equilibrium. Due to this fact, the process on the plane x-y is independent of the process taking place along the z axis. We show that the non-Markovian barotropic- and Hall-type fluctuation relations are obtained from a more general fluctuation relation formulated within the two-dimensional configuration-space $\mathbf{x} = (x,y)$, in which the magnetic field plays a role. The general fluctuation relation is thus established for the ratio $f(\mathbf{x},t)/f(-\mathbf{x},t)$. It is also shown in a general way that this ratio is equal to the exponential of a time-dependent function which can be identified with the total amount of an effective dimensionless work done on the charged particle as it is driven out of equilibrium by the applied time-dependent electric field. The total amount of the effective dimensionless work is given as the sum of two contributions, namely the effective dimensionless translational and rotational mechanical work. As we will show, it also satisfies the Crooks relation in a similar way as those established in Refs. [7,9,10]. Due to the initial condition, two non-Markovian barotropic- and Hall-type fluctuation relations are derived when the applied electric field is a constant pointing just along the x axis. One barotropic- and one Hall-type fluctuation relation is derived if the canonical initial distribution is used, and the remaining two are derived when the initial condition is the Maxwell distribution. Finally, we study the Markovian limit in which the total amount of work has an explicit expression. In particular, the same barotropic- and Hall-type fluctuation relations derived by Roy and Kumar [38] are then recovered.

This work is organized as follows. In Sec. II, we introduce and solve the phase-space GLE with arbitrary friction memory kernel for a charged Brownian particle in crossed electric and magnetic fields. The solution is used to calculate the four-dimensional phase-space conditional probability density. In Sec. III we derive, according to the above-mentioned initial conditions, two non-Markovian fluctuation relations come from the joint probability density $f(\mathbf{x},t)$. The barotropic- and Hall-type fluctuation relations are obtained in Sec. IV and analyzed in the Markovian limit in order to compare them with the results derived in Ref. [38]. Some comments are given at the end of Sec. V. We conclude our work in Sec. VI and in two Appendixes we present some of the explicit calculations performed.

II. GLE FOR A CHARGED PARTICLE IN CROSSED ELECTRIC AND MAGNETIC FIELDS

Our theoretical model is related to an electrically charged Brownian particle of mass $m = 1$ and charge q embedded in a memory thermal bath of temperature T. Additionally, the particle is under the action of the Lorentz force $\mathbf{F}_L = (q/c)\mathbf{v} \times \mathbf{B} + q\mathbf{E}(t)$, where the applied magnetic field will be considered a constant vector pointing along the z axis, that is, $\mathbf{B} = (0,0,B)$, and $\mathbf{E}(t)$ an arbitrary time-dependent electric field. The electric field comes from two contributions: One is the time-dependent internal electric field $\mathbf{E}_{\text{in}}(t)$ responsible for the internal fluctuations and the other one is the time-dependent external electric field $\mathbf{E}_{\text{ex}}(t)$ responsible for driving the system out of equilibrium. The non-Markovian dynamics of a Brownian particle involving memory thermal interaction with its surroundings is in general characterized by a GLE containing an arbitrary friction memory kernel. For the present physical model it can be written as

$$\dot{x} = v_x, \tag{1}$$

$$\dot{y} = v_y, \tag{2}$$

$$\dot{z} = v_z, \tag{3}$$

$$\ddot{x} - \Omega\dot{y} + \int_0^t \gamma(t-t')\,\dot{x}(t')\,dt' - a_x(t) = f_x(t), \tag{4}$$

$$\ddot{y} + \Omega\dot{x} + \int_0^t \gamma(t-t')\,\dot{y}(t')\,dt' - a_y(t) = f_y(t), \tag{5}$$

$$\ddot{z} + \int_0^t \gamma(t-t')\,\dot{z}(t')\,dt' - a_z(t) = f_z(t), \tag{6}$$

where $\Omega = qB/c$ is the cyclotron frequency per unit mass, $\gamma(t)$ is the friction memory kernel per unit mass, $a_i(t)$ are the components of acceleration vector $\mathbf{a}(t) \equiv q\mathbf{E}_{\text{ex}}(t)$, and $f_i(t)$ are the components of the fluctuating force $\mathbf{f}(t) = q\mathbf{E}_{\text{in}}(t)$ per unit mass. This fluctuating force has zero mean value and satisfies the fluctuation-dissipation relation of the second kind [42] given by

$$\langle f_i(t) f_j(t')\rangle = k_B T\,\delta_{ij}\,\gamma(t-t'), \tag{7}$$

with k_B as the Boltzmann's constant. It should be mentioned that the validity of this relation guarantees that the stochastic processes (4) and (5) without the accelerations $a_i(t)$ must be stationary, as demanded by such a relation. The statement is briefly shown in Appendix A. The stationary character of the Eq. (6) without $a_z(t)$ was shown in Ref. [43]. The solution of Eqs. (4)–(6) for the positions (x,y,z) can be calculated using the Laplace transforms, such that

$$x(t) = \langle x(t)\rangle + \int_0^t \mathcal{H}_0(t-t')f_x(t')dt' - \Omega^2\int_0^t \mathcal{H}_2(t-t')f_x(t')dt' + \Omega\int_0^t \mathcal{H}_1(t-t')f_y(t')dt', \tag{8}$$

$$y(t) = \langle y(t)\rangle + \int_0^t \mathcal{H}_0(t-t')f_y(t')dt' - \Omega^2\int_0^t \mathcal{H}_2(t-t')f_y(t')dt' - \Omega\int_0^t \mathcal{H}_1(t-t')f_x(t')dt', \tag{9}$$

$$z(t) = \langle z(t)\rangle + \int_0^t \mathcal{H}_0(t-t')f_z(t')dt', \tag{10}$$

where for nonrandom initial conditions

$$\langle x(t)\rangle = x_0 + [\mathcal{H}_0(t) - \Omega^2\mathcal{H}_2(t)]v_{x0} + \Omega\mathcal{H}_1(t)v_{y0} + \int_0^t \mathcal{H}_0(t-t')a_x(t')dt' - \Omega^2\int_0^t \mathcal{H}_2(t-t')a_x(t')dt' + \Omega\int_0^t \mathcal{H}_1(t-t')a_y(t')dt', \tag{11}$$

$$\langle y(t)\rangle = y_0 - \Omega\mathcal{H}_1(t)v_{x0} + [\mathcal{H}_0(t) - \Omega^2\mathcal{H}_2(t)]v_{y0} + \int_0^t \mathcal{H}_0(t-t')a_y(t')dt' - \Omega^2\int_0^t \mathcal{H}_2(t-t')a_y(t')dt' - \Omega\int_0^t \mathcal{H}_1(t-t')a_x(t')dt'. \tag{12}$$

$$\langle z(t)\rangle = z_0 + \mathcal{H}_0(t)v_{z0} + \int_0^t \mathcal{H}_0(t-t')a_z(t')dt', \tag{13}$$

with $x_0 = x(0)$, $y_0 = y(0)$, $v_{x0} = v_x(0) = \dot{x}_0$, $v_{y0} = v_y(0) = \dot{y}_0$. The functions $\mathcal{H}_0(t)$, $\mathcal{H}_1(t)$, and $\mathcal{H}_2(t)$ are, respectively, the inverse Laplace transform of $\hat{\mathcal{H}}_0(s)$, $\hat{\mathcal{H}}_1(s)$, and $\hat{\mathcal{H}}_2(s)$, with

$$\hat{\mathcal{H}}_0(s) = \frac{1}{s[s+\hat{\gamma}(s)]}, \tag{14}$$

$$\hat{\mathcal{H}}_1(s) = \frac{1}{s[(s+\hat{\gamma}(s))^2+\Omega^2]}, \tag{15}$$

$$\hat{\mathcal{H}}_2(s) = \frac{1}{s(s+\hat{\gamma}(s))\,[(s+\hat{\gamma}(s))^2+\Omega^2]}, \tag{16}$$

and $\hat{\gamma}(s)$ is the Laplace transform of the friction memory kernel $\gamma(t)$. The solution for the velocities (v_x,v_y,v_z) are calculated with the help of the functions $\mathcal{H}_0(t)$, $\mathcal{H}_1(t)$, and $\mathcal{H}_2(t)$ and their corresponding derivatives at time $t=0$. This can be done in the following way: We evaluate Eqs. (11)–(13), as well as their time derivatives at $t=0$, in the absence of the external force. This easily leads to the following set of equations:

$$v_{x0}[\mathcal{H}_0(0) - \Omega^2\mathcal{H}_2(0)] = -v_{y0}\Omega\mathcal{H}_1(0), \tag{17}$$

$$v_{y0}[\mathcal{H}_0(0) - \Omega^2\mathcal{H}_2(0)] = v_{x0}\Omega\mathcal{H}_1(0), \tag{18}$$

$$v_{z0}\mathcal{H}_0(0) = 0, \tag{19}$$

and

$$v_{x0} = v_{x0}[\dot{\mathcal{H}}_0(0) - \Omega^2\dot{\mathcal{H}}_2(0)] + v_{y0}\Omega\dot{\mathcal{H}}_1(0), \tag{20}$$

$$v_{y0} = v_{y0}[\dot{\mathcal{H}}_0(0) - \Omega^2\dot{\mathcal{H}}_2(0)] - v_{x0}\Omega\dot{\mathcal{H}}_1(0), \tag{21}$$

$$v_{z0} = v_{z0}\dot{\mathcal{H}}_0(0). \tag{22}$$

It is easy to see from Eqs. (19) and (22) that $\mathcal{H}_0(0) = 0$ and $\dot{\mathcal{H}}_0(0) = 1$. Equations (17), (18), (20), and (21) show that $[\mathcal{H}_0(0) - \Omega^2\mathcal{H}_2(0)] = 0$, $\mathcal{H}_1(0) = 0$, $[\dot{\mathcal{H}}_0(0) - \Omega^2\dot{\mathcal{H}}_2(0)] = 1$, and $\dot{\mathcal{H}}_1(0) = 0$; thus $\mathcal{H}_2(0) = 0$ and $\dot{\mathcal{H}}_2(0) = 0$. It is now clear from Eqs. (8)–(10) that

$$v_x(t) = \langle v_x(t)\rangle + \int_0^t \dot{\mathcal{H}}_0(t-t')f_x(t')dt' - \Omega^2\int_0^t \dot{\mathcal{H}}_2(t-t')f_x(t')dt' + \Omega\int_0^t \dot{\mathcal{H}}_1(t-t')f_y(t')dt', \tag{23}$$

$$v_y(t) = \langle v_y(t)\rangle + \int_0^t \dot{\mathcal{H}}_0(t-t')f_y(t')dt' - \Omega^2\int_0^t \dot{\mathcal{H}}_2(t-t')f_y(t')dt' - \Omega\int_0^t \dot{\mathcal{H}}_1(t-t')f_x(t')dt', \tag{24}$$

$$v_z(t) = \langle v_z(t)\rangle + \int_0^t \dot{\mathcal{H}}_0(t-t')f_z(t')dt'\,x, \tag{25}$$

where

$$\langle v_x(t)\rangle = [\dot{\mathcal{H}}_0(t) - \Omega^2\dot{\mathcal{H}}_2(t)]v_{x0} + \Omega\dot{\mathcal{H}}_1(t)v_{y0} + \int_0^t \dot{\mathcal{H}}_0(t-t')a_x(t')dt' - \Omega^2\int_0^t \dot{\mathcal{H}}_2(t-t')a_x(t')dt' + \Omega\int_0^t \dot{\mathcal{H}}_1(t-t')a_y(t')dt', \tag{26}$$

$$\langle v_y(t)\rangle = -\Omega\dot{\mathcal{H}}_1(t)v_{x0} + [\dot{\mathcal{H}}_0(t) - \Omega^2\dot{\mathcal{H}}_2(t)]v_{y0} + \int_0^t \dot{\mathcal{H}}_0(t-t')a_y(t')dt' - \Omega^2\int_0^t \dot{\mathcal{H}}_2(t-t')a_y(t')dt' - \Omega\int_0^t \dot{\mathcal{H}}_1(t-t')a_x(t')dt'. \tag{27}$$

$$\langle v_z(t)\rangle = \dot{\mathcal{H}}_0(t)v_{z0} + \int_0^t \dot{\mathcal{H}}_0(t-t')a_z(t')dt'. \tag{28}$$

The process given by Eq. (6) is independent of those given by Eqs. (4) and (5), which represent a coupled system of two equations in which the magnetic field plays a role. The fluctuation relations which we are interested in arise from an analytical study of these two latter processes. For this reason, from now on we will pay attention only to these. Once we have calculated the solutions of Eqs. (4)–(6), we can obtain the conditional probability density (CPD) in the phase space $(\mathbf{x},\mathbf{u})$, that is, $P(\mathbf{x},\mathbf{u},t|\mathbf{x}_0,\mathbf{u}_0)$, where $\mathbf{x}=(x,y)$, $\mathbf{u}=(v_x,v_y)$, $\mathbf{x}_0=(x_0,y_0)$, and $\mathbf{u}_0=(v_{x0},v_{y0})$. The corresponding algebra is given in Appendix B and we conclude that this phase-space CPD becomes

$$P(\mathbf{R},\mathbf{S}) = \frac{1}{4\pi^2(FG-H^2-I^2)} \times \exp\left[-\frac{(F|\mathbf{S}|^2-2H\mathbf{R}\cdot\mathbf{S}-2I(\mathbf{R}\times\mathbf{S})_z+G|\mathbf{R}|^2)}{2(FG-H^2-I^2)}\right], \quad (29)$$

where each term of the exponential is explicitly given in the same Appendix B.

III. NON-MARKOVIAN FLUCTUATION RELATION

The non-Markovian fluctuation relation for the joint probability density $f(\mathbf{x},t)$ can be obtained first from the general integration over initial conditions

$$f(\mathbf{x},t) = \int P(\mathbf{x},t|\mathbf{x}_0,\mathbf{u}_0) f(\mathbf{x}_0,\mathbf{u}_0,0)\, d\mathbf{x}_0\, d\mathbf{u}_0, \quad (30)$$

where $f(\mathbf{x}_0,\mathbf{u}_0,0)$ is an arbitrary initial distribution function and $P(\mathbf{x},t|\mathbf{x}_0,\mathbf{u}_0)\equiv P(\mathbf{R})$ is the configuration-space CPD, which in turns is calculated from the marginal integration as

$$P(\mathbf{R}) = \int P(\mathbf{R},\mathbf{S})\, d\mathbf{S}. \quad (31)$$

It is easy to show that

$$P(\mathbf{R}) = \frac{1}{2\pi\, k_B T\hat{F}} \exp\left(-\frac{|\mathbf{R}|^2}{2k_B T\hat{F}}\right), \quad (32)$$

where $\hat{F}$ is obtained from Eq. (29) such that $F = k_B T\,\hat{F}$, with

$$\hat{F} = -[\mathcal{H}_0(t)-\Omega^2\mathcal{H}_2(t)]^2 - \Omega^2\mathcal{H}_1^2(t) + 2\int_0^t \mathcal{H}_0(t')dt' - 2\Omega^2\int_0^t \mathcal{H}_2(t')dt'. \quad (33)$$

According to Eqs. (B14) and (B15), the $\mathbf{R}$ vector can also be written as

$$\mathbf{R} = \mathbf{x}-\langle\mathbf{x}\rangle = \mathbf{x}-\mathbf{x}_0-\mathbb{H}(t)\mathbf{u}_0-\int_0^t \mathbb{H}(t-t')\mathbf{a}(t')dt', \quad (34)$$

where

$$\mathbb{H}(t) = \begin{pmatrix} [\mathcal{H}_0(t)-\Omega^2\mathcal{H}_2(t)] & \Omega\mathcal{H}_1(t) \\ -\Omega\mathcal{H}_1(t) & [\mathcal{H}_0(t)-\Omega^2\mathcal{H}_2(t)] \end{pmatrix}. \quad (35)$$

A. Canonical initial distribution

The selection of the initial distribution is a matter of choice and, in principle, we can propose the simplest expression by means of choosing $\delta(\mathbf{x}_0)\delta(\mathbf{u}_0)$. However, if we recall that the system is embedded in a thermal bath at temperature T under the action of crossed electric and magnetic fields, then it is convenient to suggest a distribution which can take into account also the influence of the magnetic field. Hence, we will assume that it can be written as a product of functions in the configuration and velocity spaces. Here we will assume the position dependence as a Gaussian function with a width determined by the temperature T and the magnetic field and a δ function in the velocity $\mathbf{u}_0$ in such a way that

$$f(\mathbf{x}_0,\mathbf{u}_0,t) = \frac{\gamma_0^2(1+C^2)}{2\pi k_B T}\exp\left[-\frac{\gamma_0^2(1+C^2)|\mathbf{x}_0|^2}{2k_B T}\right]\delta(\mathbf{u}_0), \quad (36)$$

where $C=\Omega/\gamma_0$ is a dimensionless factor, with γ_0 as the friction constant. On substitution of this initial distribution function into Eq. (30) and defining the dimensionless function $F_0=\gamma_0^2(1+C^2)\hat{F}$, we can show after some straightforward algebra that the joint probability density becomes

$$f(\mathbf{x},t) = \frac{1}{2\pi\bar{\sigma}(t)}\exp\left(-\frac{|\mathbf{x}-\langle\mathbf{x}\rangle|^2}{2\bar{\sigma}(t)}\right), \quad (37)$$

where the variance $\bar{\sigma}(t) = k_B T(1+F_0)/\gamma_0^2(1+C^2)$ and

$$\langle\mathbf{x}\rangle = \int_0^t \mathbb{H}(t-t')\mathbf{a}(t')\,dt'. \quad (38)$$

It must be noticed that $\bar{\sigma}(t)=\bar{\sigma}_{xx}(t)=\bar{\sigma}_{yy}(t)$. Hence, the non-Markovian fluctuation relation for this probability density is established as

$$\frac{f(\mathbf{x},t)}{f(-\mathbf{x},t)} = \exp\left[\frac{2\,\mathbf{x}\cdot\langle\mathbf{x}\rangle}{\bar{\sigma}(t)}\right]. \quad (39)$$

To understand the physical meaning contained in Eq. (39), we show in the following that the argument of the exponential is related to the total amount of dimensionless energy $\mathcal{E}(t)$ that the particle can exchange with the memory heat bath in the form of effective dimensionless work performed along a single stochastic trajectory $\mathbf{x}(t)$ [11]. Here we understand as an *effective dimensionless work* $W_e(t)$ the work done by the external electric field scaled by an arbitrary time-dependent function. Let us thus define the dimensionless energy $\mathcal{E}(t)$ as

$$\mathcal{E}(t) = \frac{2\,\mathbf{x}\cdot\langle\mathbf{x}\rangle}{\bar{\sigma}(t)} = \frac{2}{\bar{\sigma}(t)}\mathbf{x}\cdot\int_0^t \mathbb{H}(t-t')q\mathbf{E}_{ex}(t')\,dt', \quad (40)$$

where Eq. (38) has been used. Due to the algebraic structure of expression (40) it appears to be closely related to the mechanical work done by the external electric force, except for the time-dependent functions $\bar{\sigma}(t)$ and $\mathbb{H}(t)$. This can be seen in the following way: Let us suppose that the time-dependent electric field is given by $\mathbf{E}_{\rm ex}(t)=\varphi_e(t)\mathbf{E}_{\rm ex}$, where $\varphi_e(t)$ is an arbitrary dimensionless function of time and $\mathbf{E}_{ex}$ a constant electric field given by $\mathbf{E}_{\rm ex}=(E_x^{\rm ex},E_y^{\rm ex})$. On substitution of this expression into Eq. (40) we obtain

$$W_e(t) = \frac{2}{\bar{\sigma}_{xx}(t)}\left[\left(qE_x^{\rm ex}x+qE_y^{\rm ex}y\right)\int_0^t \varphi_e(t')\mathbb{H}_{11}(t')dt' + \left(qE_y^{\rm ex}x-qE_x^{\rm ex}y\right)\int_0^t \varphi_e(t')\mathbb{H}_{12}(t')dt'\right], \quad (41)$$

where $\mathbb{H}_{11}(t)=\mathbb{H}_{22}(t)$ and $\mathbb{H}_{21}(t)=-\mathbb{H}_{12}(t)$ are the elements of matrix $\mathbb{H}(t)$ given by Eq. (35). It is clear that

$qE_y^{ex}x - qE_x^{ex}y = |\mathbf{x} \times q\mathbf{E}_{ex}|$, and thus we can write

$$W_e(t) = \frac{\Psi_1(t)}{k_B T} q\mathbf{E}_{ex} \cdot \mathbf{x} + \frac{\Psi_2(t)}{k_B T} |\mathbf{x} \times q\mathbf{E}_{ex}|, \quad (42)$$

where

$$\Psi_1(t) = \frac{2\gamma_0^2(1+C^2)}{1+F_0} \int_0^t \varphi_e(t')[\mathcal{H}_0(t-t') - \Omega^2 \mathcal{H}_2(t-t')]\, dt', \quad (43)$$

$$\Psi_2(t) = \frac{2\gamma_0^2(1+C^2)}{1+F_0} \int_0^t \Omega \varphi_e(t') \mathcal{H}_1(t-t')\, dt', \quad (44)$$

are dimensionless functions of time. As can be seen, these functions show a dependence on both the memory heat bath and magnetic field through the functions $\mathcal{H}_0(t)$, $\mathcal{H}_1(t)$, and $\mathcal{H}_2(t)$, which in turn represent the inverse Laplace transform of the respective functions $\hat{\mathcal{H}}_0(s)$, $\hat{\mathcal{H}}_1(s)$, and $\hat{\mathcal{H}}_2(s)$, as shown in Eqs. (14)–(16). It is now clear from Eq. (42) that $W_t(t) \equiv q\mathbf{E}_{ex} \cdot \mathbf{x}$ represents the usual mechanical translational work done on the charged particle by the constant electric force, and therefore $W_e^t(t) \equiv [\Psi_1(t)/k_B T] W_t(t)$ represents what we can call an *effective dimensionless translational work*. In a similar way, the second term of Eq. (42) must be related to the usual definition of the mechanical rotational work in the following way: We first notice that the modulus $\tau \equiv |\mathbf{x} \times q\mathbf{E}_{ex}|$ is the magnitude of the torque $\boldsymbol{\tau} = \mathbf{x} \times q\mathbf{E}_{ex}$ done by the electric force. Next we can check that the term $\Psi_2(t)|\mathbf{x} \times q\mathbf{E}_{ex}| = \tau\theta(t)$ must be related to the rotational work as follows: Let us suppose in a very simple case that $\Psi_2(t) \sim \Omega t$ so, if we identify $\Omega t = \theta(t)$, then $W_e^r(t) \equiv \Psi_2(t)|\mathbf{x} \times q\mathbf{E}_{ex}| = \tau\theta(t)$ represents the usual mechanical rotational work done by the electric torque. So, the term $W_e^r(t) \equiv [\Psi_2(t)/k_B T]|\mathbf{x} \times q\mathbf{E}_{ex}|$ can be identified with what we can call the *effective dimensionless rotational work*. In conclusion, the effective dimensionless work, as given by Eq. (42), can be written as $W_e(t) = W_e^t(t) + W_e^r(t)$, where its effective character is due to the non-Markovian dimensionless functions $\Psi_1(t)$ and $\Psi_2(t)$. On the other hand, it can also be checked that the effective dimensionless mechanical rotational work can be expressed as

$$\frac{\Psi_2(t)}{k_B T} |\mathbf{x} \times q\mathbf{E}_{ex}| = \frac{\hat{\Psi}_2(t)}{k_B T} C\, |\mathbf{x} \times q\mathbf{E}_{ex}|, \quad (45)$$

where $\hat{\Psi}_2(t) = \Psi_2(t)/C$. Now $C\,|\mathbf{x} \times q\mathbf{E}_{ex}| = (q^2/c\gamma_0)\,\mathbf{x} \cdot (\mathbf{E}_{ex} \times \mathbf{B})$, which shows in an explicit way how the magnetic and electric fields are coupled to give a contribution. Alternatively $qC|\mathbf{x} \times \mathbf{E}_{ex}| = (q^2/c\gamma_0)\,\mathbf{E}_{ex} \cdot (\mathbf{x} \times \mathbf{B})$, which is an expression that shows the torque caused by the magnetic field (this torque is on the x-y plane). In a general way, we now can identify the argument of the exponential (39) as $W_e(t) = 2\,\mathbf{x} \cdot \langle \mathbf{x} \rangle / \bar{\sigma}(t)$.

Additionally, we can show that the non-Markovian effective dimensionless work $W_e(t)$ also satisfies the transient WFT, and, similarly, it satisfies the total amount of non-Markovian work as shown in Refs. [7,9,10] in the non-Markovian case. To see this, we notice that the variance of the effective dimensionless work, defined as $\sigma_{W_e}^2(t) = \langle W_e^2(t) \rangle - \langle W_e(t) \rangle^2$, also satisfies the relation $\sigma_{W_e}^2(t) = 2\langle W_e(t) \rangle$, as can easily be corroborated. In principle, we can also formulate a fluctuation relation for $W_e(t)$, taking into account that it is a Gaussian random variable (GRV), since $\mathbf{x}$ is also a GRV. Therefore, the probability density for the effective dimensionless work $W_e(t)$ satisfies

$$P(W_e(t)) = \frac{1}{\sqrt{2\pi\sigma_{W_e}^2}} \exp\left(-\frac{[W_e(t) - \langle W_e(t) \rangle]^2}{2\sigma_{W_e}^2}\right). \quad (46)$$

After substituting $\sigma_{W_e}^2(t) = 2\langle W_e(t) \rangle$, we show that the non-Markovian fluctuation relation for the dimensionless effective work also satisfies $P[W_e(t)]/P[-W_e(t)] = e^{W_e(t)}$, which is very similar to the Crooks fluctuation relation except by the term $\Delta\mathcal{F}$, corresponding to the free-energy difference between two equilibrium states. In our case, $\Delta\mathcal{F} = 0$, consistent with the Bohr-van Leeuwen theorem on the absence of orbital diamagnetism in a classical system of charged particles in thermodynamic equilibrium [10,35,37]. Quite similar expressions have been derived for the non Markovian transient WFT relation, among other such relations, in Ref. [7], in Ref. [9] for a linear and nonlinear Brownian harmonic oscillator, and very recently for a linear harmonic oscillator in crossed electric and magnetic fields [10]. For the reasons given above, we can suggest writing the non-Markovian fluctuation relation (39) in the following way:

$$\frac{f(\mathbf{x},t)}{f(-\mathbf{x},t)} = e^{W_e(t)}, \quad (47)$$

which quantifies the probability of observing a given trajectory at $(\mathbf{x},t)$ during the forward process and the probability of observing its backward counterpart at $(-\mathbf{x},t)$. $W_e(t)$ is the total amount of an effective dimensionless work done on the charged particle as it is driven out of equilibrium by the time-dependent electric field. It has been obtained in a general way for a charged Brownian particle under the action of crossed electric and magnetic fields, characterized by a planar GLE with arbitrary friction memory kernel.

B. Maxwell initial distribution

We now consider the case for which the initial condition is the Maxwell distribution. In this case, we have for a mass $m = 1$ that

$$f(\mathbf{x}_0,\mathbf{u}_0,t) = \frac{1}{2\pi\, k_B T} \exp\left(-\frac{|\mathbf{u}_0|^2}{2k_B T}\right) \delta(\mathbf{x}_0). \quad (48)$$

On substitution of Eqs. (32) and (48) into (30) we now get

$$f(\mathbf{x},t) = \frac{1}{2\pi\, \hat{\sigma}(t)} \exp\left(-\frac{|\mathbf{x} - \langle \mathbf{x} \rangle|^2}{2\hat{\sigma}(t)}\right), \quad (49)$$

where now the variance is given by

$$\hat{\sigma}(t) = 2k_B T \int_0^t \mathbb{H}_{11}(t')\, dt' = 2k_B T \left(\int_0^t \left[\mathcal{H}_0(t') - \Omega^2 \int_0^{t'} \mathcal{H}_2(t') \right] dt' \right), \quad (50)$$

but the mean vale $\langle \mathbf{x} \rangle$ is the same as Eq. (38). Again $\hat{\sigma}(t) = \hat{\sigma}_{xx}(t) = \hat{\sigma}_{yy}(t)$. In this case, the fluctuation relation becomes

$$\frac{f(\mathbf{x},t)}{f(-\mathbf{x},t)} = \exp\left[\frac{2\,\mathbf{x}\cdot\langle \mathbf{x}\rangle}{\hat{\sigma}(t)}\right]. \tag{51}$$

In a manner similar to that done before, for an arbitrary time-dependent electric field $\mathbf{E}_{\text{ex}}(t) = \varphi_e(t)\mathbf{E}_{\text{ex}}$, with $\mathbf{E}_{\text{ex}}$ as constant, we can show that the effective dimensionless work $\hat{W}_e(t) \equiv \frac{2\,\mathbf{x}\cdot\langle\mathbf{x}\rangle}{\hat{\sigma}(t)}$ is given by

$$\hat{W}_e(t) = \frac{\Pi_1(t)}{k_B T} q\mathbf{E}_{\text{ex}}\cdot\mathbf{x} + \frac{\Pi_2(t)}{k_B T}|\mathbf{x}\times q\mathbf{E}_{\text{ex}}|, \tag{52}$$

where

$$\Pi_1(t) = \frac{\int_0^t \varphi_e(t')\mathbb{H}_{11}(t-t')\,dt'}{\int_0^t \mathbb{H}_{11}(t')\,dt'}, \tag{53}$$

$$\Pi_2(t) = \frac{\int_0^t \varphi_e(t')\mathbb{H}_{12}(t-t')\,dt'}{\int_0^t \mathbb{H}_{11}(t')\,dt'}. \tag{54}$$

Again, we can define $\hat{W}_e^t(t) \equiv (\Pi_1(t)/k_B T)q\mathbf{E}_{\text{ex}}\cdot\mathbf{x}$ as an *effective dimensionless translational work* and $\hat{W}_e^r(t) \equiv \Pi_2(t)|\mathbf{x}\times q\mathbf{E}_{\text{ex}}|$ an *effective dimensionless rotational work*. The non-Markovian fluctuation relation then reads

$$\frac{f(\mathbf{x},t)}{f(-\mathbf{x},t)} = e^{\hat{W}_e(t)}. \tag{55}$$

Also, we easily show that the effective dimensionless work $\hat{W}_e(t)$ satisfies the Crooks relation $P[\hat{W}_e(t)]/P[-\hat{W}_e(t)] = e^{\hat{W}_e(t)}$. According to the above-obtained theoretical results, we can conclude the following: Even when the effective dimensionless works $W_e(t)$ and $\hat{W}_e(t)$, given, respectively, by Eqs. (42) and (52), depend on the details of the heat bath, each one curiously satisfies the universal Crooks relation, because the latter has been established in this case without the use of the Jarzynski's definition of the work.

IV. NON-MARKOVIAN BAROTROPIC AND HALL-TYPE FLUCTUATION RELATIONS

It is indeed interesting to analyze an expression for $W_e(t)$ and $\hat{W}_e(t)$ when the electric field is just a constant for which the dimensionless function $\varphi_e(t) = 1$. Under these conditions, the effective work $W_e(t)$ now becomes

$$W_e(t) = \frac{\Phi_1(t)}{k_B T} q\mathbf{E}_{\text{ex}}\cdot\mathbf{x} + \frac{\Phi_2(t)}{k_B T}|\mathbf{x}\times q\mathbf{E}_{\text{ex}}|\,, \tag{56}$$

with $\Phi_1(t)$ and $\Phi_2(t)$ being the same as $\Psi_1(t)$ and $\Psi_2(t)$, respectively, except that $\varphi_1(t) = 1$, that is,

$$\Phi_1(t) = \frac{2\gamma_0^2(1+C^2)}{1+F_0}\int_0^t [\mathcal{H}_0(t-t') - \Omega^2\mathcal{H}_2(t-t')]\,dt', \tag{57}$$

$$\Phi_2(t) = \frac{2\gamma_0^2(1+C^2)}{1+F_0}\int_0^t \Omega\mathcal{H}_1(t-t')\,dt'\,. \tag{58}$$

We now consider the fluctuation relations associated to the marginal probability densities $f(x,t)$ and $f(y,t)$, which are obtained through the marginal integration of Eq. (37) over y and x coordinates, respectively. In this case, we obtain the quotients

$$\frac{f(x,t)}{f(-x,t)} = e^{W_x^e(t)}, \tag{59}$$

$$\frac{f(y,t)}{f(-y,t)} = e^{W_y^e(t)}, \tag{60}$$

where now

$$W_x^e(t) = \frac{\Phi_1(t)}{k_B T} qxE_x^{\text{ex}} + \frac{\hat{\Phi}_2(t)}{k_B T}\frac{q^2}{c\gamma_0} xE_y^{\text{ex}}B, \tag{61}$$

$$W_y^e(t) = \frac{\Phi_1(t)}{k_B T} qyE_y^{\text{ex}} - \frac{\hat{\Phi}_2(t)}{k_B T}\frac{q^2}{c\gamma_0} yE_x^{\text{ex}}B, \tag{62}$$

with $\hat{\Phi}_2(t) = \Phi_2(t)/C$. Both expressions assume an interesting form when we consider the particle as an electron $q=-e$, the electric field pointing along the x axis, $E_x^{\text{ex}} = E$, and $E_y^{\text{ex}} = 0$. In this case, Eqs. (59) and (60) reduce to

$$\frac{f(x,t)}{f(-x,t)} = \exp\left[-\Phi_1(t)\frac{eEx}{k_B T}\right], \tag{63}$$

$$\frac{f(y,t)}{f(-y,t)} = \exp\left[-\hat{\Phi}_2(t)\frac{e^2EBy}{c\gamma_0 k_B T}\right]. \tag{64}$$

These last non-Markovian fluctuation relations are the two main results of our present contribution because relation (63) is the one we can call the *non-Markovian longitudinal or barotropic-type fluctuation relation* and relation (64) is the *non-Markovian Hall-type fluctuation relation*. As we will show below, they are quite similar to those obtained in the Markovian case [38], except for the dimensionless non-Markovian time-dependent functions $\Phi_1(t)$ and $\hat{\Phi}_2(t)$, which clearly contain the influence of both the memory effects of the heat bath and the magnetic field, as mentioned above. As can be seen, relation (64) comes from the effect of both fields, the electric field along the x axis and the magnetic field pointing along the z axis. The crossed effect produces fluctuations along the y axis, which is a signature of the usual Hall effect. It is also seen that relation (63) comes from the effective dimensionless translational work, whereas relation (64) comes from the effective dimensionless rotational work. For the effective dimensionless work $\hat{W}_e(t)$, we get the following interesting result:

$$\hat{W}_e(t) = \frac{1}{k_B T} q\mathbf{E}_{\text{ex}}\cdot\mathbf{x} + \frac{\Theta_2(t)}{k_B T}|\mathbf{x}\times q\mathbf{E}_{\text{ex}}|\,, \tag{65}$$

where $\Theta_2(t)$ is the same as $\Pi_2(t)$ but with $\varphi_e(t) = 1$, i.e.,

$$\Theta_2(t) = \frac{\int_0^t \Omega\mathcal{H}_1(t-t')\,dt'}{\int_0^t [\mathcal{H}_0(t') - \Omega^2\mathcal{H}_2(t')]\,dt'}. \tag{66}$$

In a similar way, can can write

$$\hat{W}_x^e(t) = \frac{1}{k_B T} qxE_x^{\text{ex}} + \frac{\hat{\Theta}_2(t)}{k_B T}\frac{q^2}{c\gamma_0} xE_y^{\text{ex}}B, \tag{67}$$

$$\hat{W}_y^e(t) = \frac{1}{k_B T} qyE_y^{\text{ex}} - \frac{\hat{\Theta}_2(t)}{k_B T}\frac{q^2}{c\gamma_0} yE_x^{\text{ex}}B, \tag{68}$$

with $\hat{\Theta}(t) = \Theta_2(t)/C$. Again, if we consider the particular case for which the particle is an electron $q = -e$, the components of the electric field as $E_x^{\rm ex} = E$, and $E_y^{\rm ex} = 0$, then we obtain the non-Markovian barotropic- and Hall-type fluctuation relations in the form

$$\frac{f(x,t)}{f(-x,t)} = \exp\left(-\frac{eEx}{k_B T}\right), \tag{69}$$

$$\frac{f(y,t)}{f(-y,t)} = \exp\left[-\hat{\Theta}_2(t)\,\frac{e^2 EBy}{c\gamma_0\,k_B T}\right]. \tag{70}$$

As can be seen in these expressions, the barotropic-type fluctuation relation is the same as that obtained by Roy and Kumar in the Markovian case, but the Hall-type fluctuation relation still remains dependent on the details of the heat bath details. The algebraic structure of this dependence is obviously more complicated than that given in the Markovian case, because $\hat{\Theta}_2(t)$ contains the influence of the magnetic and the arbitrary forms of the friction memory kernel.

Markovian fluctuation relations

The above general results can be considerably simplified when the friction kernel is memoryless, i.e., $\gamma(t) = \gamma_0\,\delta(t)$, with γ_0 being the friction constant. In this case, we readily show after some algebra that

$$\mathcal{H}_0(t) = \frac{1}{\gamma_0}(1 - e^{-\gamma_0 t}), \tag{71}$$

$$\mathcal{H}_1(t) = \frac{1}{b^2} - \frac{1}{b^2}e^{-\gamma_0 t}\cos\Omega t - \frac{\gamma_0}{\Omega b^2}e^{-\gamma_0 t}\sin\Omega t, \tag{72}$$

$$\mathcal{H}_2(t) = \frac{1}{\gamma_0 b^2} - \frac{1}{\gamma_0 b^2}e^{-\gamma_0 t} - \frac{\gamma_0}{\Omega^2 b^2}e^{-\gamma_0 t} - \frac{1}{\Omega b^2}e^{-\gamma_0 t}\sin\Omega t + \frac{\gamma_0}{\Omega^2 b^2}e^{-\gamma_0 t}\cos\Omega t, \tag{73}$$

where $b^2 = \Omega^2 + \gamma_0^2 = \gamma_0^2(1 + C^2)$. The factor $F_0 = \gamma_0^2(1 + C^2)\hat{F}$ is obtained from Eq. (33), which renders

$$F_0 = (2\gamma_0 t - 3 + 4e^{-\gamma_0 t} - e^{-2\gamma_0 t}) - 4e^{-\gamma_0 t}(1 - \cos\Omega t) + \frac{4C^2}{1 + C^2}(1 - e^{-\gamma_0 t}\cos\Omega t) - \frac{4C}{1 + C^2}e^{-\gamma_0 t}\sin\Omega t. \tag{74}$$

The above time-dependent dimensionless functions become

$$\Phi_1(t) = \frac{2}{1 + F_0}\left[\gamma_0 t + \frac{1 - C^2}{1 + C^2}(e^{-\gamma_0 t}\cos\Omega t - 1) - \frac{2C}{1 + C^2}e^{-\gamma_0 t}\sin\Omega t\right], \tag{75}$$

$$\hat{\Phi}_2(t) = \frac{2}{1 + F_0}\left[\gamma_0 t + \frac{2}{1 + C^2}(e^{-\gamma_0 t}\cos\Omega t - 1) + \frac{(1 - C^2)}{C(1 + C^2)}e^{-\gamma_0 t}\sin\Omega t\right], \tag{76}$$

and

$$\hat{\Theta}_2(t) = \frac{\hat{\Phi}_2(t)}{\Phi_1(t)}. \tag{77}$$

As can be seen, the first term of the right-hand side of Eq. (42), when expressions (75) and (76) are therein substituted, contains terms of the form $q\mathbf{E}_{\rm ex}\cdot\mathbf{x}\,\gamma_0 t$, $(e^{-\gamma_0 t}\cos\Omega t)q\mathbf{E}_{\rm ex}\cdot\mathbf{x}$, and $(e^{-\gamma_0 t}\sin\Omega t)q\mathbf{E}_{\rm ex}\cdot\mathbf{x}$. The first one represents the time increment of the work, whereas the last two correspond to the oscillatory-modulated dissipation of the work since $e^{-\gamma_0 t}q\mathbf{E}_{\rm ex}\cdot\mathbf{x}$ is its dissipation. The second term of the right-hand side of Eq. (42), which is the same as Eq. (45), contains terms of the forms $\tau\,C\gamma_0 t$, $(e^{-\gamma_0 t}\cos\Omega t)\,\tau$, and $(e^{-\gamma_0 t}\sin\Omega t)\,\tau$. It is clear that $\tau\,C\gamma_0 t = \tau\theta(t)$, which represents what we have called the rotational work done on the charged particle by the electric torque, whereas the last two can be identified as the oscillatory-modulated dissipation of the rotational work. The results given in this section allow us to analyze two limiting cases (short- and large-time regimes) which help to disentangle the terms in the general expression.

In the *short-time regime*, characterized by $\gamma_0 t \ll 1$ and $\Omega t \ll 1$, it can be shown that $F_0 \to 0$, $\Phi_1(t) \to 0$, and $\hat{\Phi}_2(t) \to 0$ and thus $W(\gamma_0 t \ll 1, \Omega t \ll 1) \to 0$. If we recall that this regime corresponds to the ballistic regime in the Brownian movement of the particle, then we have that $f(\mathbf{x},t)/f(-\mathbf{x},t) \to 1$ and hence $f(\mathbf{x},t) = f(-\mathbf{x},t)$. In this case, the probability of observing trajectories during the forward processes is the same as their backward counterparts. This can be understood in the following way: At the very beginning (short times) of the process, the Brownian particle does not yet feel the presence of the surrounding medium, so it moves as a free particle in the absence of friction and the process becomes reversible. The short-time limit is then equivalent to consider the zero-friction case $\gamma_0 \to 0$. In this limit it is also clear that $f(x,t) = f(-x,t)$ and $f(x,t) = f(-y,t)$

On the other hand, in the *large-time regime* $\gamma_0 t \gg 1$, we can see that $1 + F_0 \to 2\gamma_0 t$, $\Phi_1(t) \to 1$, and $\hat{\Phi}_2(t) \to 1$ or $\Phi_2(t) \to C$. Therefore,

$$W(\gamma_0 t \gg 1) = \frac{1}{k_B T}(q\mathbf{E}_{\rm ex}\cdot\mathbf{x} + C|\mathbf{x}\times q\mathbf{E}_{\rm ex}|), \tag{78}$$

but also

$$W_x(\gamma_0 t \gg 1) = \frac{1}{k_B T}qE_x^{\rm ex}x + \frac{q^2}{k_B T\,c\gamma_0}E_y^{\rm ex}Bx,$$

$$W_y(\gamma_0 t \gg 1) = \frac{1}{k_B T}qE_y^{\rm ex}y - \frac{q^2}{k_B T\,c\gamma_0}E_x^{\rm ex}By. \tag{79}$$

If we now make $q = -e$, $E_x^{\rm ex} = E$, and $E_y^{\rm ex} = 0$, then it is easy to see that Eqs. (63) and (64) reduce, respectively, to

$$\frac{f(x,t)}{f(-x,t)} = \exp\left(-\frac{eEx}{k_B T}\right), \tag{80}$$

which was named the *longitudinal or barotropic-type fluctuation relation*, and

$$\frac{f(y,t)}{f(-y,t)} = \exp\left(-\frac{e^2 EBy}{c\gamma_0\,k_B T}\right), \tag{81}$$

the corresponding transversal or *Hall-type fluctuation relation* [38], as expected. These are the reasons why the fluctuation relations (63) and (64) have been named in a similar way. It is only in the Markovian and high-friction case that both the effective dimensionless translational and rotational work are identified with the corresponding mechanical definitions of the translational and rotational work, as see in Eq. (78).

In the case of the fluctuation relations given by Eqs. (69) and (70), only the Hall-type fluctuation relation must be compared with that obtained by Roy and Kumar in the high-friction-limiting case. In the *short-time regime*, characterized by $\gamma_0 t \ll 1$ and $\Omega t \ll 1$, there is a indetermination in $\hat{\Theta}_2(t)$ since $\Phi_1(t) \to 0$ and $\hat{\Phi}_2(t) \to 0$. This ballistic regime can be avoided because it corresponds to the reversible regimen in which the particle still does not feel the presence of the surroundings. In the *large-time regime* or irreversible regime characterized by $\gamma_0 t \gg 1$, $\Phi_1(t) \to \gamma_0 t$ and $\hat{\Phi}_2(t) \to \gamma_0 t$, but $\hat{\Theta}_2(t) \to 1$. In this case, the non-Markovian Hall-type fluctuation relation (70) reduces to the same result obtained by Roy and Kumar, as expected.

As commented in the Introduction, Pradhan [41] showed the validity of the fluctuation relations for classical systems in the presence of a time-reversal symmetry-breaking field, such as an external time-dependent magnetic one, and nonconservative forces which cannot be derived from gradient of scalar potentials. In that paper it is shown explicitly that the concept of the work is the same as $W = \int_0^\tau (dE/dt)dt$, where E is the total energy for the combined system (CS) and W the work done by all the external forces on the system. What we can see in the study of Ref. [41] is the explicit absence of the friction force $-\gamma_0 \vec{v}_i$ which is not a function of the position $\vec{r}_i$. However, this effect is taken into account when the system is coupled to the heat bath, thus making both a combined system. Now, we are interested in checking what the expression of the work W is when the external magnetic field is a constant without specifying its orientation. In this simple case there is no interaction among particles but only the presence of an external potential with control parameter of the form $V = -\mathbf{r} \cdot \boldsymbol{\lambda}(t)$, with the control parameter being $\boldsymbol{\lambda}(t) = q\mathbf{E}_{ex}(t)$. Then the phase-space Newton's second law for the CS would be

$$\dot{\mathbf{r}} = \mathbf{v}, \tag{82}$$

$$m\dot{\mathbf{v}} = \frac{q}{c}\mathbf{v} \times \mathbf{B} + q\mathbf{E}_{\rm ex}(t), \tag{83}$$

where bold letters stand for vector quantities. It is clear in this case that the external potential with the control parameter contributes to the rate of change of the total energy as $dE/dt = -(\partial V/\partial \boldsymbol{\lambda}) \cdot \dot{\boldsymbol{\lambda}} = \mathbf{r} \cdot q\dot{\mathbf{E}}_{\rm ex}(t)$. The work in this case becomes $W = \int_0^\tau \mathbf{r} \cdot q\dot{\mathbf{E}}_{\rm ex}(t)\,dt$, which is consistent with the definition of the work introduced by Jarzynski [6] for Hamiltonian systems, that is, $W = \int_0^\tau [\partial H(\lambda(t),t)/\partial t]dt = \int_0^\tau [\partial H(\lambda(t),t)/\partial \lambda]\dot{\lambda}(t)dt$, where $H(\lambda(t),t)$ is the Hamiltonian of the system. For the system (82) and (83), the Hamiltonian reads $H(\boldsymbol{\lambda}(t),t) = V = \mathbf{r} \cdot \boldsymbol{\lambda}(t)$, and again $W = \int_0^\tau \mathbf{r} \cdot q\dot{\mathbf{E}}_{\rm ex}(t)\,dt$. Even more, if the independent Brownian particles are in a harmonic potential with arbitrary dragging of its minimum of the form $V = (k/2)|\mathbf{r} - \boldsymbol{\lambda}(t)|^2$, the time-dependent protocol would be $\boldsymbol{\lambda}(t) = (q/k)\mathbf{E}_{\rm ex}(t)$. This potential has one conservative part, $(k/2)|\mathbf{r}|^2$, and the other one has the contribution of the external potential with a control parameter of the form $(k/2)[-2\mathbf{r} \cdot \boldsymbol{\lambda}(t) + |\boldsymbol{\lambda}(t)|^2]$. However, its contribution to the rate of change of the total energy becomes $dE/dt = -(\partial V/\partial \boldsymbol{\lambda}) \cdot \dot{\boldsymbol{\lambda}} = -k[\mathbf{r} - \boldsymbol{\lambda}(t)] \cdot \dot{\boldsymbol{\lambda}}(t)$, and the work now reads $W = -k\int_0^\tau [\mathbf{r} - \boldsymbol{\lambda}(t)] \cdot \dot{\boldsymbol{\lambda}}(t)$, which is again consistent with the definition of the work given by Jarzynski and used in several papers [10,35–37,39]. Another example studied in Ref. [41] to verify the Crooks relation and Jarzynski equality is that of a Brownian motion of a particle bounded on a ring but in the presence of a time-dependent magnetic field. Both relations are proven to be valid using the concept of the work given in the same paper [41].

In conclusion, the expression of the work W defined in Ref. [41] and used to prove the validity of the fluctuation theorems is quite similar to that defined by Jarzynski [6]. In our present contribution, the expression of the effective dimensionless works $W_e(t)$ and $\hat{W}_e(t)$ are not derived from the Jarzynski's definition [6], which is equivalent to that derived by Pradhan [41]; they arise in a natural way from our theoretical formulation and also satisfy the Crooks relation, which is a curious result because the proof herein given does not stem from Jarzynski's work definition, which is equivalent to that given by Pradhan. The proof also stems from the fact that $\sigma^2_{W_e}(t) = 2\langle W_e(t)\rangle$, $\sigma^2_{\hat{W}_e}(t) = 2\langle \hat{W}_e(t)\rangle$, in a similar way as done in Refs. [10,35,37,39]. Finally, we would want to note that the generalization of the theoretical formulation of Ref. [41] to a non-Markovian system does not seem to be an easy task to accomplish since the problem would be then how to incorporate the memory effects in the CS without the explicit presence of the friction force. This problem is beyond the purposes of our present contribution. The problem of boundary conditions and that of a particle on a ring are interesting problems which could in principle be studied in the context of non-Markovian processes. However, we are not sure how much more difficult those problems can be when an arbitrary friction memory kernel and a time-dependent magnetic field are taken into account.

V. CONCLUDING REMARKS

The barotropic- and Hall-type fluctuation relations derived for a Markovian two-dimensional system of noninteracting electrons under the action of crossed electric and magnetic fields in a high-friction limit [38] have been generalized to the case of a non-Markovian system of noninteracting charged Brownian particles, also under the action of crossed electric and magnetic fields, as given by Eqs. (63), (64), (69), and (70). These relations have been derived from a more general fluctuation relation (47) and (55) established within the context of a phase-space second-order GLE with arbitrary friction memory kernel as given by Eqs. (1), (2), (4), and (5). The fluctuation relation (47) is derived when the initial distribution is assumed to be a canonical one with the influence of the magnetic field, whereas the fluctuation relation (55) is obtained when the initial condition is the Maxwell distribution. In both cases we have explicitly shown that the arguments of the exponentials are related with the total amount of dimensionless energy that the particle can exchange with the memory heat bath in the form of effective dimensionless work [$W_e(t)$ and $\hat{W}_e(t)$] done on the charged particle as it is driven out of equilibrium by a time-dependent electric field. The effective character of the works $W_e(t)$ and $\hat{W}_e(t)$ is due to the presence of the dimensionless, time-dependent functions $\Psi_1(t)$, $\Psi_2(t)$, $\Pi_1(t)$, and $\Pi_2(t)$, which are dependent of the memory heat bath and the magnetic field, as shown in Eqs. (43), (44), (53), and (54). Both effective dimensionless works $W_e(t)$ and $\hat{W}_e(t)$ have two contributions, namely an effective dimensionless mechanical translational work $W_e^t(t)$ or $\hat{W}_e^t(t)$ plus an effective dimensionless mechanical rotational work $W_e^r(t)$ or $\hat{W}_e^r(t)$. We have shown that even when the effective works $W_e(t)$ and $\hat{W}_e(t)$ contain the details of the heat bath, each satisfies the universal Crooks relation, although this latter one does not come from Jarzynski's work definition, which is equivalent to that given by Pradhan: It arises from our theoretical formulation in a quite natural way, which is indeed a surprising result. Even more, an interesting result arises in the general expression given by the effective work $\hat{W}_e(t)$ when the electric field is a constant. This is because, in this case, its translational contribution simply becomes $\hat{W}_e^t(t) = q\mathbf{E}\cdot\mathbf{x}/k_BT$ and thus it is independent of the details of the heat bath, which is not the case of its rotational contribution as shown in Eq. (65). In the Markovian limit there is no memory dependence but only constant friction and a magnetic field dependence as shown in Eqs. (75), (76), and (77). Even in this case we can talk about the concept of the total amount of an effective dimensionless work. In this Markovian case, but also in the high-friction limit, the dimensionless functions have the limit $\Phi_1(t), \hat{\Phi}_2(t), \hat{\Theta}_2(t) \to 1$, and then the barotropic-type (63) and Hall-type [(64) and (70)] fluctuation relations reduce respectively to Eqs. (80) and (81), as expected. For an initial canonical distribution, it is in this latter limiting case that both the effective translational and rotational works become just the usual mechanical work. However, if the initial distribution is a Maxwellian one, only the rotational work becomes the usual one, because the usual translational mechanical work arises in natural way in our theoretical formulation. Maybe the non-Markovian fluctuation relations established in this work, as well as the Hall-type fluctuation relation established by Roy and Kumar, could give some information to experimentally calculate the transport coefficients, such as the mobility, in a fluid with or without memory. With the present contribution we are extending the range of applicability of the fluctuation relations to non-Markovian systems in which the magnetic field plays a relevant role.

Last, we have additionally complemented our work with the fact that the processes given by Eqs. (4) and (5) without the applied time-dependent electric field become stationary in the large time limit if the fluctuation-dissipation relation (FDR) of the second kind is valid. The statement is explicitly proven in Appendix A. To the best of the authors knowledge, the second kind of FDR for Markovian processes has been derived generally by explicit use of microscopic dynamics [4]. However, a classical derivation of the FDR for macroscopic non-Markovian dynamics in the presence of an external time-dependent force field and under the action of a constant magnetic field has not been reported elsewhere. For a classical derivation in the presence of a time-dependent force field, but without the presence of a magnetic field, we can refer to the paper by Grabert *et al.* [44].

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APPENDIX A: THE GLE WITHOUT THE APPLIED TIME-DEPENDENT ELECTRIC FIELD IS STATIONARY

We shown in this Appendix that the GLEs (4) and (5) without the external electric field are stationary if the fluctuation-dissipation relation of the second kind is valid. To do this, we write Eqs (4) and (5) in the equivalent form,

$$\dot{v}_x - \Omega v_y + \int_0^t \gamma(t-t')\, v_x(t')\, dt' = f_x(t), \tag{A1}$$

$$\dot{v}_y + \Omega v_x + \int_0^t \gamma(t-t')\, v_y(t')\, dt' = f_y(t). \tag{A2}$$

The solution of these equations are again obtained via the Laplace transform technique, yielding

$$\begin{aligned} v_x(t) &= [\chi_0(t) - \Omega^2\chi_2(t)]v_{x0} + \Omega\chi_1(t)v_{y0} \\ &\quad + \int_0^t \chi_0(t-t')f_x(t')dt' + \Omega\int_0^t \chi_1(t-t')f_y(t')dt' \\ &\quad - \Omega^2\int_0^t \chi_2(t-t')f_x(t')dt', \end{aligned} \tag{A3}$$

$$\begin{aligned} v_y(t) &= [\chi_0(t) - \Omega^2\chi_2(t)]v_{y0} - \Omega\chi_1(t)v_{x0} \\ &\quad + \int_0^t \chi_0(t-t')f_y(t')dt' - \Omega\int_0^t \chi_1(t-t')f_x(t')dt' \\ &\quad - \Omega^2\int_0^t \chi_2(t-t')f_y(t')dt', \end{aligned} \tag{A4}$$

where v_{x0}, v_{y0} are the initial conditions and $\chi_0(t)$, $\chi_1(t)$, and $\chi_2(t)$ the inverse Laplace transform of $\hat{\chi}_0(s)$, $\hat{\chi}_1(s)$, and $\hat{\chi}_2(s)$, respectively, with

$$\hat{\chi}_0(s) = \frac{1}{s + \hat{\gamma}(s)}, \tag{A5}$$

$$\hat{\chi}_1(s) = \frac{1}{[s + \hat{\gamma}(s)]^2 + \Omega^2}, \tag{A6}$$

$$\hat{\chi}_2(s) = \frac{1}{[s + \hat{\gamma}(s)]\{[s + \hat{\gamma}(s)]^2 + \Omega^2\}}, \tag{A7}$$

with $\hat{\gamma}(s)$ standing again as the Laplace transform of $\gamma(t)$. Following the similar algebraic steps given in Sec. II, it can also be shown that $\chi_0(0) = 1$, $\chi_1(0) = 0$, $\chi_2(0) = 0$, $\dot{\chi}_0(0) = 0$, $\dot{\chi}_1(0) = 1$, and $\dot{\chi}_2(0) = 0$. To prove the stationary character of Eqs. (A1) and (A2) in the large time limit we use the strategy proposed in Ref. [43]. It assumes that the initial condition for the velocity is a Maxwellian distribution function and independent for each initial condition v_{x0}, v_{y0}, and v_{z0}, that is, $P(v_{i0}) = (1/\sqrt{2\pi\, k_BT})\exp(-v_{i0}^2/2k_BT)$, Thus $\langle v_{i0}\rangle = 0$ and $\langle v_{i0}^2\rangle = k_BT$. Herein, for simplicity in the notation, we also understand the average $\langle\cdots\rangle$ when it is applied to the corresponding initial condition. From the solutions (23) and (24), it can be easily verified that the correlation function $\langle v_x(t_1)v_x(t_2)\rangle$ is the same as $\langle v_y(t_1)v_y(t_2)\rangle$ and $\langle v_x(t_1)v_y(t_2)\rangle = 0$, where

$$\begin{aligned} \langle v_x(t_1)v_x(t_2)\rangle &= k_BT\chi_0(t_1)\chi_0(t_2) - k_BT\Omega^2\chi_0(t_1)\chi_2(t_2) \\ &\quad - k_BT\Omega^2\chi_0(t_2)\chi_2(t_1) + k_BT\Omega^4\chi_2(t_1)\chi_2(t_2) \\ &\quad + k_BT\Omega^2\chi_1(t_1)\chi_1(t_2) + I_0 + I_1 + I_2 - I_3 - I_4, \end{aligned} \tag{A8}$$

and

$$I_0 = \int_0^{t_2}\int_0^{t_1} \chi_0(t_2-s_2)\chi_0(t_1-s_1) \times \langle f_x(s_1)f_x(s_2)\rangle ds_1 ds_2, \tag{A9}$$

$$I_1 = \Omega^2\int_0^{t_2}\int_0^{t_1} \chi_1(t_2-s_2)\chi_1(t_1-s_1) \times \langle f_y(s_1)f_y(s_2)\rangle ds_1 ds_2, \tag{A10}$$

$$I_2 = \Omega^4\int_0^{t_2}\int_0^{t_1} \chi_2(t_2-s_2)\chi_2(t_1-s_1) \times \langle f_x(s_1)f_x(s_2)\rangle ds_1 ds_2, \tag{A11}$$

$$I_3 = \Omega^2\int_0^{t_2}\int_0^{t_1} \chi_0(t_2-s_2)\chi_2(t_1-s_1) \times \langle f_x(s_1)f_x(s_2)\rangle ds_1 ds_2, \tag{A12}$$

$$I_4 = \Omega^2\int_0^{t_2}\int_0^{t_1} \chi_0(t_1-s_1)\chi_2(t_2-s_2) \times \langle f_x(s_1)f_x(s_2)\rangle ds_1 ds_2. \tag{A13}$$

On substitution of the fluctuation-dissipation relation given by (7) and following the same algebraic procedure as done in Ref. [43], we obtain the double Laplace transform of each I_i, giving as result

$$\mathcal{I}_0 = k_B T\left[\frac{\hat\gamma(s)\hat\chi_0(s)\hat\chi_0(s')}{s+s'} + \frac{\hat\gamma(s')\hat\chi_0(s')\hat\chi_0(s)}{s+s'}\right], \tag{A14}$$

$$\mathcal{I}_1 = \Omega^2 k_B T\left[\frac{\hat\gamma(s)\hat\chi_1(s)\hat\chi_1(s')}{s+s'} + \frac{\hat\gamma(s')\hat\chi_1(s')\hat\chi_1(s)}{s+s'}\right], \tag{A15}$$

$$\mathcal{I}_2 = \Omega^4 k_B T\left[\frac{\hat\gamma(s)\hat\chi_2(s)\hat\chi_2(s')}{s+s'} + \frac{\hat\gamma(s')\hat\chi_2(s')\hat\chi_2(s)}{s+s'}\right], \tag{A16}$$

$$\mathcal{I}_3 = \Omega^2 k_B T\left[\frac{\hat\gamma(s)\hat\chi_0(s)\hat\chi_2(s')}{s+s'} + \frac{\hat\gamma(s')\hat\chi_2(s')\hat\chi_0(s)}{s+s'}\right], \tag{A17}$$

$$\mathcal{I}_4 = \Omega^2 k_B T\left[\frac{\hat\gamma(s)\hat\chi_2(s)\hat\chi_0(s')}{s+s'} + \frac{\hat\gamma(s')\hat\chi_0(s')\hat\chi_2(s)}{s+s'}\right]. \tag{A18}$$

Using now the identities (A5)–(A7) and after some algebra, we finally show that

$$\langle v_x(t_1)v_x(t_2)\rangle = \langle v_y(t_1)v_y(t_2)\rangle = k_B T[\chi_0(t_2-t_1) - \Omega^2\chi_2(t_2-t_1)]. \tag{A19}$$

In reality, the arguments of $\chi_0(t)$ and $\chi_2(t)$ must be $|t_2-t_1|$; however, we just consider the case $t_2 > t_1$. Equation (A19) shows clearly that the stochastic dynamics (A1) and (A2) are stationary at large times. For times $t_1 = t_2 = t$, Eq. (A19) becomes $\langle v_x^2(t)\rangle = k_B T$, which also means that the initial Maxwellian distribution persists, as expected.

APPENDIX B: PHASE-SPACE CPD $P(\mathbf{x},\mathbf{u},t|\mathbf{x}_0,\mathbf{u}_0)$

Due to the fact that the processes $(\mathbf{x},\mathbf{u})$ are Gaussian, the phase-space CPD satisfies the Gaussian distribution function

$$P(\mathbf{x},\mathbf{u},t|\mathbf{x}_0,\mathbf{u}_0) = \frac{1}{(2\pi)^2\sqrt{\det\boldsymbol{\sigma}(t)}}\exp\left(-\frac{1}{2}\boldsymbol{\alpha}^T\boldsymbol{\sigma}^{-1}(t)\boldsymbol{\alpha}\right), \tag{B1}$$

where $\boldsymbol{\alpha}$ is a vector such that $\alpha_i = \xi_i - \langle\xi_i\rangle$, and $\boldsymbol{\sigma}(t) \equiv \sigma_{ij}(t)$ is the variance and covariance matrix such that $\sigma_{ij}(t) = \langle\alpha_i\alpha_j\rangle \equiv \langle[\xi_i - \langle\xi_i\rangle][\xi_j - \langle\xi_j\rangle]\rangle$, with $\xi_{i,j} = x,y,v_x,v_y$. From the solutions (8) and (9) and then (23) and (24), it is easy to see that $\sigma_{xx}(t) = \sigma_{yy}(t)$, $\sigma_{v_xv_x}(t) = \sigma_{v_yv_y}(t)$, $\sigma_{xv_x}(t) = \sigma_{yv_y}(t)$, $\sigma_{xv_y}(t) = -\sigma_{yv_x}(t)$, $\sigma_{xy}(t) = \sigma_{yx}(t) = 0$, and $\sigma_{v_xv_y}(t) = \sigma_{v_yv_x}(t) = 0$. If we define

$$\begin{aligned} F &\equiv \sigma_{xx}(t) = \langle[x-\langle x\rangle]^2\rangle,\\ G &\equiv \sigma_{v_xv_x}(t) = \langle[v_x-\langle v_x\rangle]^2\rangle,\\ H &\equiv \sigma_{xv_x}(t)(t) = \langle[x-\langle x\rangle][v_x-\langle v_x\rangle]\rangle,\\ I &\equiv \sigma_{xv_y}(t)(t) = \langle[x-\langle x\rangle][v_y-\langle v_y\rangle]\rangle, \end{aligned} \tag{B2}$$

ththen the elements of matrix σ_{ij} given in Eq. (B2) are explicitly given by

$$\begin{aligned} F &= \langle[x-\langle x\rangle]^2\rangle\\ &= \frac{2}{\beta}\bigg[\int_0^t \mathcal{H}_0(t')dt'\int_0^{t'}\mathcal{H}_0(t'')\gamma(t'-t'')dt''\\ &\quad + \Omega^2\int_0^t \mathcal{H}_1(t')dt'\int_0^{t'}\mathcal{H}_1(t'')\gamma(t'-t'')dt''\\ &\quad + \Omega^4\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt''\\ &\quad - \Omega^2\int_0^t \mathcal{H}_0(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt''\\ &\quad - \Omega^2\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'}\mathcal{H}_0(t'')\gamma(t'-t'')dt''\bigg], \end{aligned} \tag{B3}$$

$$\begin{aligned} G &= \langle[v_x-\langle v_x\rangle]^2\rangle\\ &= \frac{2}{\beta}\bigg[\int_0^t \dot{\mathcal{H}}_0(t')dt'\int_0^{t'}\dot{\mathcal{H}}_0(t'')\gamma(t'-t'')dt''\\ &\quad + \Omega^2\int_0^t \dot{\mathcal{H}}_1(t')dt'\int_0^{t'}\dot{\mathcal{H}}_1(t'')\gamma(t'-t'')dt''\\ &\quad + \Omega^4\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt''\\ &\quad - \Omega^2\int_0^t \dot{\mathcal{H}}_0(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt''\\ &\quad - \Omega^2\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}\dot{\mathcal{H}}_0(t'')\gamma(t'-t'')dt''\bigg], \end{aligned} \tag{B4}$$

$$\begin{aligned} H &= \langle[x-\langle x\rangle][v_x-\langle v_x\rangle]\rangle\\ &= \frac{1}{\beta}\bigg[\int_0^t \mathcal{H}_0(t')dt'\int_0^{t'}\dot{\mathcal{H}}_0(t'')\gamma(t'-t'')dt''\\ &\quad + \int_0^t \dot{\mathcal{H}}_0(t')dt'\int_0^{t'}\mathcal{H}_0(t'')\gamma(t'-t'')dt'' \end{aligned}$$

$$
\begin{aligned}
&-\Omega^2\int_0^t \mathcal{H}_0(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt''\\
&-\Omega^2\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}\mathcal{H}_0(t'')\gamma(t'-t'')dt''\\
&-\Omega^2\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'}\dot{\mathcal{H}}_0(t'')\gamma(t'-t'')dt''\\
&-\Omega^2\int_0^t \dot{\mathcal{H}}_0(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt''\\
&+\Omega^2\int_0^t \mathcal{H}_1(t')dt'\int_0^{t'}\dot{\mathcal{H}}_1(t'')\gamma(t'-t'')dt''\\
&+\Omega^2\int_0^t \dot{\mathcal{H}}_1(t')dt'\int_0^{t'}\mathcal{H}_1(t'')\gamma(t'-t'')dt''\\
&+\Omega^4\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt''\\
&+\Omega^4\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt''\Bigg], \qquad \text{(B5)}
\end{aligned}
$$

$$
\begin{aligned}
I &= \langle[x-\langle x\rangle][v_y-\langle v_y\rangle]\rangle\\
&=\frac{1}{\beta}\Bigg[-\Omega\int_0^t \mathcal{H}_0(t')dt'\int_0^{t'}\dot{\mathcal{H}}_1(t'')\gamma(t'-t'')dt''\\
&-\Omega\int_0^t \dot{\mathcal{H}}_1(t')dt'\int_0^{t'}\mathcal{H}_0(t'')\gamma(t'-t'')dt''\\
&+\Omega^3\int_0^t \mathcal{H}_2(t')dt'\int_0^{t'}\dot{\mathcal{H}}_1(t'')\gamma(t'-t'')dt''\\
&+\Omega^3\int_0^t \dot{\mathcal{H}}_1(t')dt'\int_0^{t'}\mathcal{H}_2(t'')\gamma(t'-t'')dt''\\
&-\Omega^3\int_0^t \mathcal{H}_1(t')dt'\int_0^{t'}\dot{\mathcal{H}}_2(t'')\gamma(t'-t'')dt''\\
&-\Omega^3\int_0^t \dot{\mathcal{H}}_2(t')dt'\int_0^{t'}\mathcal{H}_1(t'')\gamma(t'-t'')dt''\\
&+\Omega\int_0^t \mathcal{H}_1(t')dt'\int_0^{t'}\dot{\mathcal{H}}_0(t'')\gamma(t'-t'')dt''\\
&+\Omega\int_0^t \dot{\mathcal{H}}_0(t')dt'\int_0^{t'}\mathcal{H}_1(t'')\gamma(t'-t'')dt''\Bigg]. \qquad \text{(B6)}
\end{aligned}
$$

To explicitly calculate the matrix elements $\sigma_{ij}(t)$, we take the time derivative of each one of them and use the identities given in Eqs. (14)–(16). After very long algebra, it can be shown that

$$
\begin{aligned}
\beta F &= -[\mathcal{H}_0(t)-\Omega^2\mathcal{H}_2(t)]^2-\Omega^2\mathcal{H}_1^2(t)\\
&\quad+2\int_0^t\mathcal{H}_0(t')dt'-2\Omega^2\int_0^t\mathcal{H}_2(t')dt', \qquad \text{(B7)}
\end{aligned}
$$

$$
\beta G = \left[1-\Omega^2\dot{\mathcal{H}}_1^2(t)\right]-[\dot{\mathcal{H}}_0(t)-\Omega^2\dot{\mathcal{H}}_2(t)]^2, \qquad \text{(B8)}
$$

$$
\begin{aligned}
\beta H &= \mathcal{H}_0(t)[1-\dot{\mathcal{H}}_0(t)]-\Omega^2\mathcal{H}_2(t)\\
&\quad+\Omega^2[\mathcal{H}_0(t)\dot{\mathcal{H}}_2(t)+\dot{\mathcal{H}}_0(t)\mathcal{H}_2(t)]\\
&\quad-\Omega^2\mathcal{H}_1(t)\dot{\mathcal{H}}_1(t)-\Omega^4\mathcal{H}_2(t)\dot{\mathcal{H}}_2(t), \qquad \text{(B9)}
\end{aligned}
$$

$$
\begin{aligned}
\beta I &= -\Omega\mathcal{H}_1(t)+\Omega[\mathcal{H}_0(t)\dot{\mathcal{H}}_1(t)-\dot{\mathcal{H}}_0(t)\mathcal{H}_1(t)]\\
&\quad+\Omega^3[\mathcal{H}_1(t)\dot{\mathcal{H}}_2(t)-\dot{\mathcal{H}}_1(t)\mathcal{H}_2(t)]. \qquad \text{(B10)}
\end{aligned}
$$

It can easily be corroborated that $\frac{1}{2}\dot{F}=H$. According to above expressions, we can obtain the matrix $\boldsymbol{\sigma}(t)=\sigma_{ij}(t)$, which we write as

$$
\boldsymbol{\sigma}(t)=\begin{pmatrix} F & 0 & H & I\\ 0 & F & -I & H\\ H & -I & G & 0\\ I & H & 0 & G\end{pmatrix}, \qquad \text{(B11)}
$$

and its inverse,

$$
\boldsymbol{\sigma}^{-1}(t)=\frac{1}{FG-H^2-I^2}\begin{pmatrix} G & 0 & -H & -I\\ 0 & G & I & -H\\ -H & I & F & 0\\ -I & -H & 0 & F\end{pmatrix}. \qquad \text{(B12)}
$$

Under these conditions it is now clear that the phase-space CPD $P(\mathbf{x},\mathbf{u},t|\mathbf{x}_0,\mathbf{u}_0)$ established in Eq. (B1) can be written as

$$
\begin{aligned}
P(\mathbf{R},\mathbf{S}) &= \frac{1}{4\pi^2(FG-H^2-I^2)}\\
&\quad\times\exp\left[-\frac{(F|\mathbf{S}|^2-2H\mathbf{R}\cdot\mathbf{S}-2I(\mathbf{R}\times\mathbf{S})_z+G|\mathbf{R}|^2)}{2(FG-H^2-I^2)}\right], \qquad \text{(B13)}
\end{aligned}
$$

where $\mathbf{R}\cdot\mathbf{S}$ is the scalar product of the vectors $\mathbf{R}=(R_1,R_2)$, $\mathbf{S}=(S_1,S_2)$, and $(\mathbf{R}\times\mathbf{S})_z$ the z component of the cross product $\mathbf{R}\times\mathbf{S}$, such that

$$
\begin{aligned}
R_1 &= x-\langle x\rangle = x-x_0-[\mathcal{H}_0(t)-\Omega^2\mathcal{H}_2(t)]v_{x0}\\
&\quad-\Omega\mathcal{H}_1(t)v_{y0}-\int_0^t\mathcal{H}_0(t-t')a_x(t')dt'\\
&\quad+\Omega^2\int_0^t\mathcal{H}_2(t-t')a_x(t')dt'-\Omega\int_0^t\mathcal{H}_1(t-t')a_y(t')dt', \qquad \text{(B14)}
\end{aligned}
$$

$$
\begin{aligned}
R_2 &= y-\langle y\rangle = y-y_0-[\mathcal{H}_0(t)-\Omega^2\mathcal{H}_2(t)]v_{y0}\\
&\quad+\Omega\mathcal{H}_1(t)v_{x0}-\int_0^t\mathcal{H}_0(t-t')a_y(t')dt'\\
&\quad+\Omega^2\int_0^t\mathcal{H}_2(t-t')a_y(t')dt'+\Omega\int_0^t\mathcal{H}_1(t-t')a_x(t')dt', \qquad \text{(B15)}
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= v_x-\langle v_x\rangle = v_x-[\dot{\mathcal{H}}_0(t)-\Omega^2\dot{\mathcal{H}}_2(t)]v_{x0}\\
&\quad-\Omega\dot{\mathcal{H}}_1(t)v_{y0}-\int_0^t\dot{\mathcal{H}}_0(t-t')a_x(t')dt'\\
&\quad+\Omega^2\int_0^t\dot{\mathcal{H}}_2(t-t')a_x(t')dt'-\Omega\int_0^t\dot{\mathcal{H}}_1(t-t')a_y(t')dt', \qquad \text{(B16)}
\end{aligned}
$$

$$
\begin{aligned}
S_2 &= v_y-\langle v_y\rangle = v_y-[\dot{\mathcal{H}}_0(t)-\Omega^2\dot{\mathcal{H}}_2(t)]v_{y0}\\
&\quad+\Omega\dot{\mathcal{H}}_1(t)v_{x0}-\int_0^t\dot{\mathcal{H}}_0(t-t')a_y(t')dt'\\
&\quad+\Omega^2\int_0^t\dot{\mathcal{H}}_2(t-t')a_y(t')dt'+\Omega\int_0^t\dot{\mathcal{H}}_1(t-t')a_x(t')dt'. \qquad \text{(B17)}
\end{aligned}
$$

[1] R. Klages, W. Just, and C. Jarzynski, *Reviews of Nonlinear Dynamics and Complexity* (Wiley, Weinheim, Germany, 2013).

[2] G. N. Bochkov and Yu. E. Kuzovlev, Phys. Usp. **56**, 590 (2013).

[3] G. N. Bochkov and Yu. E. Kuzovlev, Sov. Phys. JETP **45**, 125 (1977); Zh. Eksp. Teor. Fiz. **72**, 238 (1977); Radiophys. Quant. Electron. **27**, 811 (1984).

[4] G. N. Bochkov and Yu. E. Kuzovlev, Physica A **106**, 443 (1981); **106**, 480 (1981).

[5] Yu. E. Kuzovlev, arXiv:1106.0589v1[cond-mat.stat-mech].

[6] C. Jarzynski, Phys. Rev. Lett. **78**, 2690 (1997); Phys. Rev. E **56**, 5018 (1997); J. Stat. Phys. **98**, 77 (2000).

[7] T. Ohkuma and T. Ohta, J. Stat. Mech: Theor. Exp. (2007) P10010.

[8] G. E. Crooks, J. Stat. Phys. **90**, 1481 (1998); Phys. Rev. E **60**, 2721 (1999); **61**, 2361 (2000).

[9] T. Mai and A. Dhar, Phys. Rev. E **75**, 061101 (2007).

[10] J. I. Jiménez-Aquino, Phys. Rev. E **92**, 022149 (2015).

[11] U. Seifert, Phys. Rev. Lett. **95**, 040602 (2005); Eur. Phys. J. B **64**, 423 (2008); Rep. Prog. Phys. **75**, 126001 (2012).

[12] Marco Baiesi, Tim Jacobs, Christian Maes, and Nikos S. Skantzos, Phys. Rev. E **74**, 021111 (2006).

[13] P. Pradhan and U. Seifert, Eur. Phys. Lett. **89**, 37001 (2010).

[14] Chulan Kwon, Jae Dong Noh, and Hyunggyu Park, Phys. Rev. E **88**, 062102 (2013).

[15] T. Speck and U. Seifert, J. Stat. Mech. (2007) L09002.

[16] F. Zamponi, F. Bonetto, L. F. Cugliandolo, and J. Kurchan, J. Stat. Mech. (2005) P09013.

[17] A. Puglisi and D. Villamaina, Europhys. Lett. **88**, 30004 (2009).

[18] T. Ohkuma, J. Stat. Mech: Theor. Exp. (2009) P10015.

[19] J. I. Jiménez-Aquino and R. M. Velasco, Phys. Rev. E **87**, 022112 (2013); Entropy **16**, 1917 (2014).

[20] M. K. Sen, A. Baura, and B. C. Bag, Eur. Phys. J. B **83**, 381 (2011).

[21] Arnab Pal and Sanjib Sabhapandit, Phys. Rev. E **90**, 052116 (2014).

[22] G. M. Wang, E. M. Sevick, E. Mittag, D. J. Searles, and D. J. Evans, Phys. Rev. Lett. **89**, 050601 (2002).

[23] J. Liphardt, S. Dumont, S. B. Smith, I. Tinoco, Jr., and C. Bustamente, Science **296**, 1832 (2002).

[24] D. Collin, F. Ritort, C. Jarzynski, S. B. Smith, I. Tinoco, Jr., and C. Bustamante, Nature (London) **437**, 231 (2005).

[25] F. Douarche, S. Ciliberto, A. Petrosyan, and I. Rabbiosi, Europhys. Lett. **70**, 593 (2005).

[26] F. Douarche, S. Joubaud, N. B. Garnier, A. Petrosyan, and S. Ciliberto, Phys. Rev. Lett. **97**, 140603 (2006).

[27] V. Blickle, T. Speck, L. Helden, U. Seifert, and C. Bechinger, Phys. Rev. Lett. **96**, 070603 (2006).

[28] F. Ritort, J. Phys. Condens. Matter **18**, R531 (2006); Adv. Chem. Phys. **137**, 31 (2008).

[29] S. Ciliberto, S. Joubaud, and A. Petrosyan, J. Stat. Mech: Theor. Exp. (2010) P12003.

[30] J. R. Gomez-Solano, L. Bellon, A. Petrosyan, and S. Ciliberto, Eur. Phys. Lett. **189**, 60003 (2010).

[31] J. R. Gomez-Solano, A. Petrosyan, and S. Ciliberto, Phys. Rev. Lett. **106**, 200602 (2011).

[32] S. Ciliberto, R. Gomez-Solano, and A. Petrosyan, Condens. Matter Phys. **4**, 235 (2013).

[33] J. Gieseler, R. Quidant, C. Dellago, and L. Novotny, Nat. Nanotech. **9**, 358 (2014).

[34] Pau Mestres, Ignacio A. Martinez, Antonio Ortiz-Ambriz, Raul A. Rica, and Edgar Roldan, Phys. Rev. E **90**, 032116 (2014).

[35] A. M. Jayannavar and M. Sahoo, Phys. Rev. E **75**, 032102 (2007).

[36] A. Saha and A. M. Jayannavar, Phys. Rev. E **77**, 022105 (2008).

[37] J. I. Jiménez-Aquino, R. M. Velasco, and F. J. Uribe, Phys. Rev. E **78**, 032102 (2008).

[38] D. Roy and N. Kumar, Phys. Rev. E **78**, 052102 (2008).

[39] J. I. Jiménez-Aquino, F. J. Uribe, and R. M. Velasco, J. Phys. A **43**, 255001 (2010).

[40] J. I. Jiménez-Aquino, Phys. Rev. E **82**, 051118 (2010).

[41] Punyabrata Pradhan, Phys. Rev. E **81**, 021122 (2010).

[42] R. Kubo, Rep. Prog. Phys. **29**, 255 (1966).

[43] Ronald Forrest Fox, J. Math. Phys. **18**, 2331 (1977).

[44] H. Grabert, P. Hänggi, and P. Talkner, Phys. Lett. A **66**, 255 (1978).