Lévy flights between absorbing boundaries: Revisiting the survival probability and the shift from the exponential to the Sparre-Andersen limit behavior

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We revisit the problem of calculating the survival probability of Lévy flights in a finite interval with absorbing boundaries. Our approach is based on the master equation for discrete Lévy fliers, previously considered to treat the semi-infinite domain. We argue that, although the semi-infinite case can be treated exactly due to Wiener-Hopf factorization, the approximation involved in the problem with the finite interval is actually fairly good. We evidence the shift in the universal behavior of the long-term survival probability from the exponential decay in the presence of two absorbing barriers to the Sparre-Andersen power-law dependence in the single-barrier limit. In some cases, we also calculate the short- and intermediate-term behavior and present the explicit dependence of the survival probability on the Lévy flier's starting position. Our analytical results are confirmed by numerical simulations.

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I. INTRODUCTION

Lévy flights and walks have been largely employed to model anomalous diffusion, with applications pervading a broad variety of phenomena [1–18]. They are characterized by heavy-tailed distributions of step lengths, which allow for the existence of rare but extremely large jumps alternating with many short steps. Stochastic processes described by distributions with diverging second moment are governed by the generalized central limit theorem, with probability density function (pdf) after many jumps converging to the Lévy α-stable distribution and emergence of superdiffusive behavior [6,7]. In contrast, Brownian processes with finite second moment converge to the Gaussian pdf and display normal diffusion, as established by the central limit theorem. The degree of superdiffusion in Lévy processes is governed by the Lévy index α, defined in the range $0 < \alpha < 2$, with the Gaussian limit corresponding to the boundary value $\alpha = 2$.

Though similar in some aspects, Lévy flights and walks actually differ in a number of relevant features [6,7,11–13,18–20]. For instance, while Lévy flights are Markovian processes generally assigned to jumps of duration independent of the length, steps in Lévy walks are taken with constant speed, thus generating spatiotemporal correlation and non-Markovian character. This property leads the second moment of the pdf of Lévy walks in free space to diverge only in the infinite-time limit [18,19]. On the other hand, when constrained to a finite domain bounded by absorbing boundaries Lévy processes show superdiffusive behavior limited to an upper cutoff scale [15,16,21–23]. This bounded case is relevant as it is actually representative of a variety of realistic phenomena [11–13]. In this context, one important quantity is the survival probability, i.e., the probability of the walker or flier to remain unabsorbed after some time or after a given number of jumps [24]. This quantity is also intrinsically related to the mean first passage time to the boundary sites, which corresponds to the mean number of steps and mean length of the path of Lévy flights and walks, respectively [24–29].

The asymptotic behavior of the survival probability for large times or number of jumps surprisingly depends more on the nature of the boundary conditions than on the degree of diffusivity set by the Lévy index α. For example, in the case of semi-infinite domain in one dimension a theorem due to Sparre Andersen establishes [30–32] that the survival probability of symmetric Lévy fliers displays asymptotic decay for large number of steps n in the power-law form $1/\sqrt{n}$, irrespective of the value of α.

On the other hand, when the boundary conditions are set by absorbing boundaries in a finite one-dimensional domain, Lévy flights have been most generally studied in the continuous limit under the framework of fractional differential equations (FDEs), which display standard first-order time derivative and fractional α-dependent space derivative [11–13,33–41]. In this context, since the fractional derivative is defined as an integro-differential operator, its nonlocal character introduces a number of difficulties and subtleties in the presence of absorbing boundaries. For instance, the possibility of long jumps over the boundary sites [42] leads to a subtle distinction between the first time to arrive at the boundaries and the first passage time to the boundaries [43]. Further, this feature also implies the failure [43] of the method of images to solve the FDEs with absorbing boundaries for $0 < \alpha < 2$. The presence of the boundaries also hampers the use of the Wiener-Hopf technique [43,44] and the method of finite differences applied to the FDE [45,46]. Notwithstanding, one successful approach is to discretize the continuous fractional differential operator [27,28,38], with focus on the study of the set of eigenvalues and eigenfunctions of the FDE with absorbing boundaries. In this context, as the spectrum of eigenvalues is discrete and can be ordered, the survival probability for the Lévy flier in a finite domain displays long-term asymptotic exponential decay with the continuous time, in which the decay constant is identified with the inverse of the smallest eigenvalue (in absolute value) [38,39].

In this work, we revisit the problem of calculating the survival probability of discrete Lévy flights in a finite domain with absorbing boundaries. Here, instead of working on the continuous FDE formalism, our approach is based on the master equation for discrete Lévy fliers, previously considered by Zumofen and Klafter [20] to treat the semi-infinite case.

Under this framework, it becomes clear how the long-term asymptotic exponential behavior of the survival probability in the presence of two absorbing barriers shifts to the Sparre-Andersen power-law decay in the single-barrier limit. In some cases, we also calculate the short- and intermediate-term behavior of the survival probability and present its explicit dependence on the Lévy flier's starting position. Our analytical results are confirmed by numerical simulations.

This work is organized as follows. In Sec. II we detail the general method and apply it explicitly to the Cauchy ($\alpha = 1$) and Gaussian ($\alpha = 2$) distributions of jump lengths. In the sequence, we present the calculation to the *general* case of a Lévy flier with Lévy index $0 < \alpha < 2$, identifying the change of universality in the long-term behavior of the survival probability from the single-barrier Sparre-Andersen power law to the double-barrier exponential decay. Finally, discussion and conclusions are presented in Sec. III.

II. SURVIVAL PROBABILITY OF LÉVY FLIERS WITH ABSORBING BARRIERS

We consider a Lévy flier that performs jumps of length ℓ distributed according to the Lévy α-stable pdf $p_\alpha(\ell)$ in a finite one-dimensional interval of length L (see Fig. 1). We take a symmetric pdf so that jumps of same length to the left and to the right are equiprobable. For convenience, we set the position of the absorbing barriers at $x = -x_0$ and $x = L - x_0$, with $0 < x_0 < L/2$, and the origin as the starting point. This means that x_0 represents the initial distance of the flier to the closest boundary. The Lévy flier is absorbed if a jump takes it to the regions $x \leqslant -x_0$ or $x \geqslant L - x_0$.

The calculations presented below follow the approach by Feller [44] and Zumofen and Klafter [20]. We denote by $Q_n(x)$ the pdf of the flier to remain unabsorbed in the interval $-x_0 < x < L - x_0$ after n discrete jumps. We thus set $Q_n(x) = 0$ for $x \leqslant 0$ and $x \geqslant L - x_0$. Conversely, $R_n^{(0)}(x)$ $[R_n^{(L)}(x)]$ represents the pdf to remain unabsorbed up to the $(n-1)$th jump, with absorption by the $x = -x_0$ $[x = L - x_0]$ boundary occurring precisely at the nth jump. Similarly, we write $R_n^{(0)}(x) = 0$ $[R_n^{(L)}(x) = 0]$ for $x > -x_0$ $[x < L - x_0]$. From these definitions, the master equation for the discrete Lévy flyer reads

$$R_{n+1}^{(0)}(x) + R_{n+1}^{(L)}(x) + Q_{n+1}(x) = \int_{-\infty}^{\infty} Q_n(y) p_\alpha(x-y)dy, \tag{1}$$

with initial conditions $Q_0(x) = \delta(x)$ and $R_0^{(0)}(x) = R_0^{(L)}(x) = 0$, and $\delta(x)$ denoting the Dirac delta function.

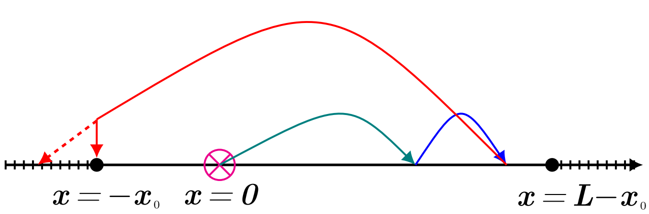

FIG. 1. Illustration of a Lévy flight in a finite domain of length L, with absorbing boundaries located at $x = -x_0$ and $x = L - x_0$. In this example, after starting from the origin the flier is absorbed by the left boundary as its third jump would lead it to the region $x \leqslant -x_0$.

By taking the Fourier transform of Eq. (1) to the k space, the generating function approach [44] can be applied, with, e.g., $Q(k,z) = \sum_{n=0}^{\infty} Q_n(k) z^n$, where z is defined in a range that assures the series convergence (certainly for $|z| < 1$, since the Q_n's are bounded). We thus obtain

$$\ln[1 - p_\alpha(k)z] = \ln\{1 - \sqrt{2\pi}[R^{(0)}(k,z) + R^{(L)}(k,z)]\} - \ln[\sqrt{2\pi}\, Q(k,z)]. \tag{2}$$

The contribution to the interval $-x_0 < x < L - x_0$ can be cast in the form

$$\ln[\sqrt{2\pi}\, Q(k,z)] = \sum_{m=1}^{\infty} \frac{z^m}{m} \int_{-x_0}^{L-x_0} P_m(x) e^{ikx} dx, \tag{3}$$

in which the lower and upper limits of the integral (i.e., the absorbing boundaries) must be excluded by definition and the pdf $P_m(x)$ to be at position x in free space after m jumps has Fourier transform $P_m(k) = [p_\alpha(k)]^m/\sqrt{2\pi}$. Equation (3) is exact in the $L \to \infty$ semi-infinite limit due to the Wiener-Hopf factorization of the contributions from the absorbing and nonabsorbing intervals [20,44]. In a finite interval, Eq. (3) represents a fairly good approximation in the large-L limit studied below. For example, the generating function $Q(k,z)$ of a random walker with symmetric exponential jump lengths, $p(\ell) = \exp(-|\ell|/d)/2d$, starting near the origin in a finite interval $0 \leqslant x \leqslant L$, can be obtained exactly using Eq. (2) [44]. The difference in $\ln[\sqrt{2\pi}\, Q(k,z)]$ between this exact result and the one calculated from Eq. (3) nullifies as $\sim e^{-\sqrt{1-z}L/d}$ in the $L \gg d$ limit.

By taking the Fourier transform of $P_m(x)$ and inserting the above relation for $P_m(k)$ into Eq. (3) we obtain

$$\ln[\sqrt{2\pi}\, Q(k,z)] = \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{z^m}{m} \int_{-x_0}^{L-x_0} dx\, e^{ikx} \times \int_{-\infty}^{\infty} dq\, e^{-iqx} [p_\alpha(q)]^m. \tag{4}$$

By solving the integral in x and applying the Cauchy principal value for improper integrals, we write

$$Q(k,z) = \frac{1}{\sqrt{2\pi}} \exp[i\phi(k,z)], \tag{5}$$

with the phase

$$\phi(k,z) = \frac{1}{2\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\ln[1 - p_\alpha(q)z]}{(q-k)} \times [e^{ix_0(q-k)} - e^{-i(L-x_0)(q-k)}]dq, \tag{6}$$

and P denoting the principal value. In Eq. (6) the Fourier transform of the symmetric Lévy α-stable pdf $p_\alpha(\ell)$ reads [6]

$$p_\alpha(q) = \frac{1}{\sqrt{2\pi}} \exp(-b|q|^\alpha), \tag{7}$$

with b a scale parameter and the Lévy index in the interval $0 < \alpha \leqslant 2$.

At this point, the time dependence of the generating function can be obtained from the expressions in terms of the discrete number of jumps by defining [20,24] $\psi_n(t)$ as the pdf for the nth jump to occur at time t. The following recursion relation holds:

$$\psi_n(t) = \int_0^t \psi(\tau)\psi_{n-1}(t-\tau)d\tau, \quad (8)$$

where $\psi(t)$ is the n-independent pdf of waiting times between consecutive jumps. The pdf $J(x,t)$ to arrive at an unabsorbed position x in time t thus reads

$$J(x,t) = \sum_{n=0}^{\infty} Q_n(x)\psi_n(t). \quad (9)$$

This expression allows one to write the pdf $Q(x,t)$ to be at x in time t as

$$Q(x,t) = \int_0^t J(x,\tau)\Psi(t-\tau)d\tau, \quad (10)$$

with $\Psi(t) = \int_t^\infty \psi(t')dt'$ representing the pdf of not having jumped until time t. We observe that the choice of the Poissonian form $\psi(t) = \exp(-t/\tau)/\tau$, with mean value τ, is compatible [11–13] with the associated FDE of Lévy flights, which includes a standard first-order continuous time derivative and fractional space derivative.

Now, by Laplace transforming Eq. (10) to space u, with $\psi(u) = 1/(1+\tau u)$ and $\psi_n(u) = [\psi(u)]^n$, we obtain $Q(x,u) = J(x,u)\Psi(u) = Q(x,z=\psi(u))\Psi(u)$, where $\Psi(u) = \tau/(1+\tau u)$. The Fourier transform of $Q(x,u)$ to space k implies

$$Q(k,u) = \Psi(u)Q(k,z=\psi(u)), \quad (11)$$

with $Q(k,z)$ given by Eqs. (5) and (6).

The survival probability $S(t)$ is finally obtained from $Q(k,u)$ by taking the inverse Laplace transform of

$$S(u) = \int_{-\infty}^{\infty} Q(x,u)dx = \sqrt{2\pi}\,Q(k=0,u), \quad (12)$$

where one has that $S(t=0) = 1$ and $S(t\to\infty) = 0$.

A. Cauchy flier: $\alpha = 1$

In the case of the Cauchy distribution of jump lengths, we consider $\alpha = 1$ in Eq. (7). Equation (6) thus yields

$$\phi(k=0,z) = -\frac{i}{\pi}\sum_{n=1}^{\infty}\frac{z^n}{n}\left\{\arctan\left(\frac{L}{nb - x_0(L-x_0)/nb}\right) + \frac{\pi}{2}\left[1+\mathrm{sgn}\left(\frac{x_0(L-x_0)}{nb}-1\right)\right]\right\}. \quad (13)$$

We notice above the presence of the required symmetry, $x_0 \leftrightarrow (L-x_0)$, concerning the starting distances to the boundaries. The sum converges for $|z|<1$ and any $u>0$, since $z = 1/(1+\tau u)$. By combining Eqs. (5) and (11)–(13), we find

$$S(u) = \frac{\tau}{(1+\tau u)}\exp\left\{\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(1+\tau u)^{-n}}{n} \times\left[\arctan\left(\frac{L}{nb - x_0(L-x_0)/nb}\right) + \frac{\pi}{2}\left[1+\mathrm{sgn}\left(\frac{x_0(L-x_0)}{nb}-1\right)\right]\right]\right\}. \quad (14)$$

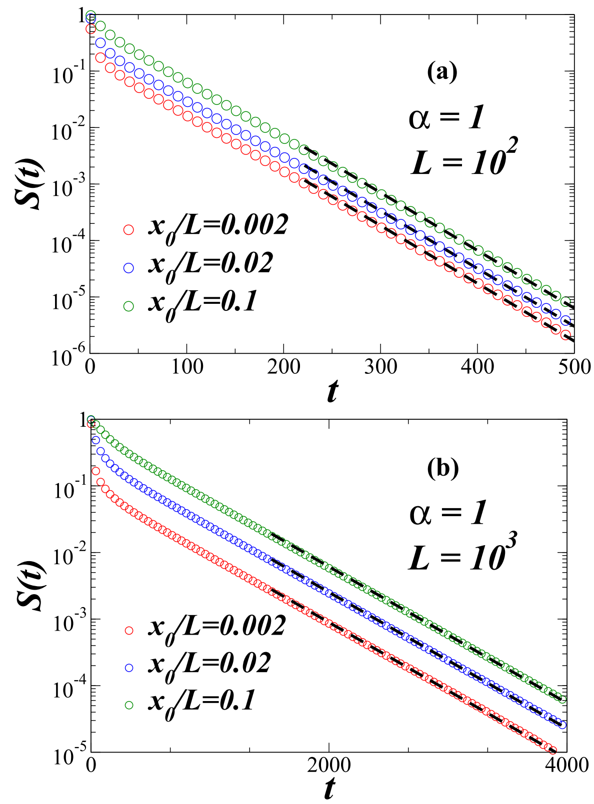

FIG. 2. Survival probability $S(t)$ as a function of time t for the $\alpha = 1$ Cauchy flier in a domain of length (a) $L = 10^2$ and (b) $L = 10^3$, with starting positions $x_0/L = 0.002, 0.02, 0.1$. The long-term exponential behavior, $S(t) \sim \exp(-t/\tau)$, is evidenced both from numerical simulation results (circles), with x_0-independent best-fit value $\tau^{-1} = 2.31/L$, and the analytical approach, with $\tau^{-1} = -\lambda_{k=1} \approx 3\pi/(4L) \approx 2.36/L$ [38] (dashed lines).

The small-u behavior of $S(u)$ thus yields the long-time ($t/\tau \gg 1$) asymptotic dependence of the survival probability,

$$S(t) \sim A(\xi)\exp(-t/\tau), \quad (15)$$

in which A is a constant prefactor dependent on $\xi = x_0(L-x_0)$. By comparing this result with the FDE formalism, the decay constant is identified with the inverse of the smallest eigenvalue (in absolute value) of the Cauchy FDE with absorbing boundaries, $\tau = -1/\lambda_{k=1}$ [38].

These findings are confirmed by numerical simulation results, shown in Fig. 2. In the simulations, a Cauchy flier with $\alpha = 1$ performed jumps in the domain $-x_0 < x < L - x_0$ and was absorbed (i.e., the simulation ended) if it stepped either at the region $x \leqslant -x_0$ or $x \geqslant L - x_0$. In Fig. 2 we display results for $L = 10^2$ [Fig. 2(a)] and $L = 10^3$ [Fig. 2(b)], with $x_0/L = 0.002, 0.02, 0.1$, $b = 1$, so that $L \gg b$, and averages taken over 10^8 runs in each case. As indicated by Eq. (15) and also shown in Fig. 2, in the long-term asymptotic regime the effect of the flier's starting distance x_0 is present in the prefactor but not in the x_0-independent decay rate. Indeed, the starting point influences mostly the initial transient regime, implying a collapse of the curves $S(t)/A(\xi)$ in the long-time limit. We also notice that the best-fit numerical value $\tau^{-1} = 2.31/L$ shows good agreement with the approximate result $\tau^{-1} = -\lambda_{k=1} \approx$

$3\pi/(4L) \approx 2.36/L$ obtained within the FDE framework [38]. Our findings corroborate and extend on previous numerical results on Lévy fliers in a finite domain [47].

B. Gaussian limit: $\alpha = 2$

The Gaussian limit corresponds to take $\alpha = 2$ in Eq. (7). As mentioned, in contrast with the diverging second moment of Lévy α-stable distributions with $0 < \alpha < 2$, the standard deviation of the Gaussian pdf of jump lengths is finite, given by $\sigma = \sqrt{2b}$ [6]. This fact adds one more relevant scale to the problem, σ, besides the natural ones, x_0 and L.

As a consequence, the analysis of the long-term survival probability in the Gaussian case must be split into two limit situations, representing the cases in which the flier might access or not the boundary that is initially faraway. In particular, the latter situation corresponds to the single-boundary limit.

For convenience, we set in the following $x_0 \ll L$. Therefore, the single-boundary limit should be recovered by considering a rather small standard deviation $\sigma \ll L$, so that the initially very distant boundary located at the position $x = L - x_0$ cannot be reached asymptotically. On the other hand, by setting a sufficiently large value of σ one allows the barrier which was initially faraway to be effectively reached.

We start by calculating the phase from Eq. (6):

$$\phi(k=0,z) = -\frac{i}{2}\sum_{n=1}^{\infty}\frac{z^n}{n}\left[\mathrm{erf}\left(\frac{L-x_0}{\sigma\sqrt{2n}}\right) + \mathrm{erf}\left(\frac{x_0}{\sigma\sqrt{2n}}\right)\right], \tag{16}$$

where $\mathrm{erf}(x)$ denotes the error function. As in the Cauchy case, the series convergence is assured for $|z| < 1$.

We first consider values $x_0 \ll L$ and $\sigma < L$, so that both barriers can be effectively accessed by the Gaussian flier. In this case $S(u)$ reads

$$S(u) \approx \frac{\tau}{(1+\tau u)}\exp\left[\frac{L}{\sqrt{2\pi}\sigma}\sum_{n=1}^{\infty}\frac{(1+\tau u)^{-n}}{n^{3/2}}\right]. \tag{17}$$

The long-time behavior of the survival probability with two barriers is therefore

$$S(t) \sim B(\zeta)\exp(-t/\tau), \tag{18}$$

where now the prefactor B is a function of $\zeta = L/\sigma$.

On the other hand, if the boundary at $x = L - x_0$ is initially very far from the flier, then, by considering $x_0 \ll \sigma \ll L$ in Eq. (16) we find in the single-barrier limit,

$$S(u) \approx \frac{\sqrt{\tau/u}}{\sqrt{1+\tau u}}\left[1 + \frac{x_0}{\sqrt{2\pi}\sigma}\sum_{n=1}^{\infty}\frac{(1+\tau u)^{-n}}{n^{3/2}}\right], \tag{19}$$

which gives rise to

$$S(t) \approx I_0(t/2\tau)\exp(-t/2\tau) + \frac{x_0}{\sqrt{2\pi}\sigma}\sum_{n=1}^{\infty}\left(\frac{t}{\tau}\right)^n\frac{{}_1F_1(n+1/2;n+1;-t/\tau)}{n^{3/2}\Gamma(n+1)}. \tag{20}$$

Above, $I_0(x)$, $\Gamma(x)$, and ${}_1F_1(a;b;x)$ denote the modified Bessel function of zeroth order, gamma function, and hypergeometric function, respectively.

We remark that Eq. (20) is also approximately valid in the short- and intermediate-t regimes (not only in the long-term limit). Interestingly, as the time t becomes progressively much larger than the time scale τ, the first term above dominates over the second, so that, by applying the asymptotic form $I_0(x) \approx \exp(x)/\sqrt{2\pi x}, x \gg 1$, we obtain the long-term $t/\tau \gg 1$ regime of the single-barrier limit for the Gaussian flier [20]

$$S(t) \sim \left(\frac{\tau}{\pi t}\right)^{1/2}. \tag{21}$$

The contrast between Eqs. (18) and (21) evidences a remarkable shift of the long-term behavior of the survival probability of a Gaussian flier from the exponential to the power-law decay, depending respectively on whether the initially distant barrier can be effectively accessed or not along the flier's path. In other words, as the ratio of the probabilities of being absorbed by the initially faraway and close barriers tends to zero as $L/\sigma \to \infty$, the functional form of $S(t)$ at long times progressively shifts from $S \sim e^{-t}$ to $S \sim 1/\sqrt{t}$. We show below that this shift is not exclusive of the Gaussian flier, but also applies to general Lévy fliers between absorbing boundaries.

These distinct long-term regimes of the survival probability of a Gaussian flier are clearly seen in Fig. 3. In Fig. 3(a) the long-term exponential decay of $S(t)$ is evidenced for $L = 10^2$, $x_0/L = 0.01, 0.1, 0.2$, and $\sigma = 1$. In this regime, both boundaries can be reached by the Gaussian flier, as described by the exponential behavior of Eq. (18). Numerical results represent averages over 10^8 runs. The best-fit numerical value $\tau^{-1} = 9.55/L^2$ is in close agreement with the result $\tau^{-1} = -\lambda_{k=1} = \pi^2/L^2 \approx 9.87/L^2$ from the FDE framework [38].

On the other hand, by making the initially faraway boundary essentially inaccessible to the Gaussian flier through setting $L = 10^3 \gg \sigma \gg x_0$, the long-time power-law decay establishes, with exponent $-1/2$, as shown in Fig. 3(b) for $x_0 = 0.05, 0.5$. For the smallest value of x_0, we also observe in Fig. 3(b) a good agreement between Eq. (20) and the numerical data already for short times, $t \gtrsim 10$. In this case, we took into account 10^3 terms in the sum of Eq. (20) and $\tau = 1$ (results kept essentially unaltered when considering 5×10^3 terms). Consistently with the above findings, we have also numerically checked that the number of fliers that ended up being absorbed by the faraway boundary at $x = L - x_0$ was null in this regime.

C. General Lévy flier: $0 < \alpha < 2$

We now generalize the previous analyses to the case of a Lévy flier with Lévy index in the interval $0 < \alpha < 2$. For Lévy flights between two absorbing boundaries, the same procedure above leads to

$$S(u) = \frac{\tau}{(1+\tau u)}\exp\left\{\sum_{n=1}^{\infty}\frac{(1+\tau u)^{-n}}{2\pi n} \times [I_{n,\alpha}(x_0) + I_{n,\alpha}(L-x_0)]\right\}, \tag{22}$$

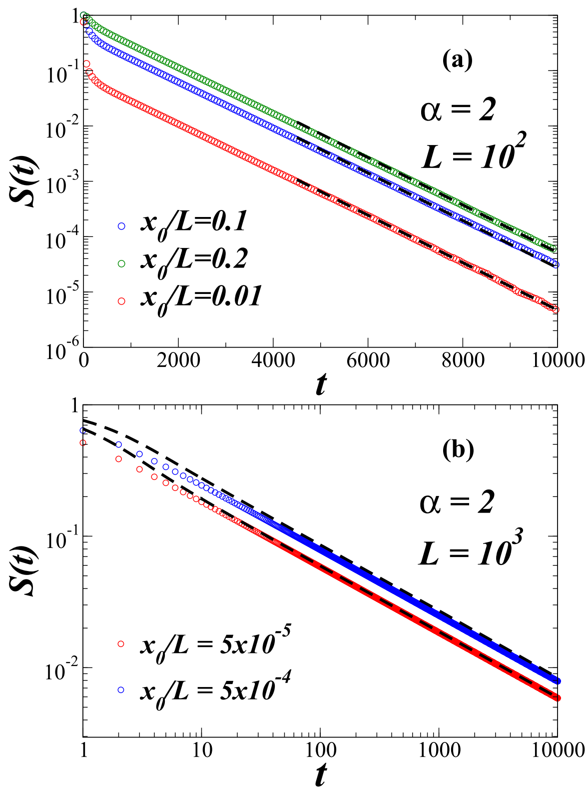

FIG. 3. Survival probability $S(t)$ as a function of time t for the $\alpha = 2$ Gaussian flier. (a) In a domain of length $L = 10^2$, with starting positions $x_0/L = 0.01, 0.1, 0.2$ and $\sigma = 1$, the initially faraway boundary located at $x = L - x_0$ can be effectively reached. The long-term exponential behavior, $S(t) \sim \exp(-t/\tau)$, is evidenced from numerical simulation results (circles), with best-fit value $\tau^{-1} = 9.55/L^2$, as well as from the analytical approach, with $\tau^{-1} = -\lambda_{k=1} \approx \pi^2/L^2 \approx 9.87/L^2$ [38] (dashed lines). (b) In a $10\times$ larger domain, $L = 10^3$, with $x_0/L = 5 \times 10^{-5}, 5 \times 10^{-4}$, and $\sigma = 1$, the faraway boundary is not effectively reached. The analytical expression for $S(t)$, Eq. (20), is depicted in dashed lines, and the long-term behavior shifts to the Sparre-Andersen power-law form $S(t) \sim 1/t^{\nu}$, with x_0-independent exponent $\nu = 1/2$ (theory) and $\nu = 0.49$ (best-fit numerical value).

where the α-dependent integrals above are generally defined as

$$I_{n,\alpha}(x) = \mathrm{P}\int_{-\infty}^{\infty} \frac{\sin(y)}{y} e^{-nb|y|^{\alpha}/|x|^{\alpha}} dy. \tag{23}$$

We observe that as long as the argument of the functions $I_{n,\alpha}(x)$ remains positive and finite, they display monotonic decrease with n, as shown in Fig. 4 for several α. In this case, an analysis similar to those of the previous subsections leads to the asymptotic exponential behavior of $S(t)$, but with the prefactor as in Eq. (15) dependent on α as well.

The situation changes drastically in the single-boundary limit in which $L \to \infty$. Now, since the following limit applies, $I_{n,\alpha}(x \to \infty) = \pi$ (see inset of Fig. 4), Eq. (22) becomes

$$S(u) = \frac{\sqrt{\tau/u}}{\sqrt{1+\tau u}} \exp\left\{\sum_{n=1}^{\infty} \frac{(1+\tau u)^{-n}}{2\pi n} I_{n,\alpha}(x_0)\right\}, \tag{24}$$

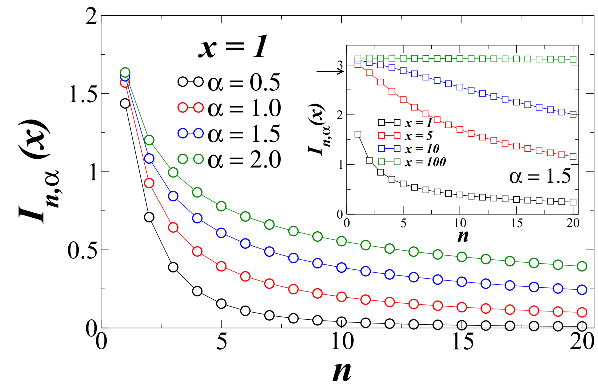

FIG. 4. Integrals $I_{n,\alpha}(x)$, defined in Eq. (23), as a function of n, associated with the Laplace transform of the survival probability, Eq. (22), of a Lévy flier with index $0 < \alpha \leqslant 2$. When the argument x is positive and finite, $I_{n,\alpha}(x)$ presents monotonic decrease with n, as seen for $\alpha = 0.5, 1.0, 1.5, 2.0$ and $x = 1$. Inset: as the argument $x \to \infty$, convergence to the limit $I_{n,\alpha}(x \to \infty) = \pi$ is achieved.

which leads to $S(t)$ given by the first term of Eq. (20). As a consequence, the long-time behavior of $S(t)$ displays the $1/\sqrt{t}$ dependence as in of Eq. (21), consistent with the Sparre-Andersen theorem.

III. DISCUSSION AND CONCLUSIONS

We start the discussion by comparing the above findings for the double-barrier case with those reported by Zumofen and Klafter for a Lévy flier in the semi-infinite domain [20]. In order to provide a proper comparison, we first generalize the results of Ref. [20], which are valid for $x_0 \to 0$, to also include the explicit dependence on the flier's starting distance x_0. Further, we also calculate the dependence of $S(t)$ in the short- and intermediate-t regimes, thus extending the long-time asymptotic results reported in [20].

In the semi-infinite space, with the single absorbing boundary placed at $x = -x_0$, we obtain for the $\alpha = 1$ Cauchy flier starting at the origin,

$$S(t) \approx I_0(t/2\tau)\exp(-t/2\tau) + \frac{x_0}{b}\sum_{n=1}^{\infty}\left(\frac{t}{\tau}\right)^n \frac{{}_1F_1(n+1/2; n+1; -t/\tau)}{\pi n^2 \Gamma(n+1)}, \tag{25}$$

for any t and $x_0 \ll b$. The long-term $t \gg \tau$ limit of Eq. (25) thus reads

$$S(t) \sim \left(\frac{\tau}{\pi t}\right)^{1/2}\left(1 + \frac{\pi x_0}{6b}\right), \tag{26}$$

in agreement with the result $S(t) \sim (\tau/\pi t)^{1/2}$ reported in [20] for $x_0 \to 0$.

On the other hand, for the $\alpha = 2$ Gaussian case in the semi-infinite domain the generalization of the result of Ref. [20] to include the small-x_0 dependence in any t leads precisely to Eq. (20), as expected. In this case, the same long-term behavior $S(t) \sim (\tau/\pi t)^{1/2}$, $x_0 \to 0$, is obtained, in agreement with [20] as well.

Remarkably, this universal (i.e., α-independent) $1/\sqrt{t}$-asymptotic behavior of the survival probability of Lévy fliers in the semi-infinite domain (including the Gaussian case as the $\alpha = 2$ limit) became a signature of the influence of this specific boundary condition on the flight dynamics, as rigorously derived by Sparre Andersen more than 60 years ago [30,31]. This result is by no means trivial, and helped to evidence the failure [43] of some powerful analytical techniques to solve FDEs, such as the method of images, which predicted [11–13] a nonuniversal asymptotic behavior of $S(t)$ as $1/t^{1/\alpha}$ for Lévy fliers in the semi-infinite domain.

On the other hand, regarding the case of Lévy fliers in a finite interval with absorbing boundaries investigated in this work, the failure of the method of images and other techniques also compels one to rely on other analytical approaches such as the master equation for discrete Lévy flights or the discretization of the continuous fractional differential operator in the FDE formalism [38]. In this context, an expression of universality emerges in the presence of double barriers, which involves a much faster exponential decay of the asymptotic survival probability, regardless of the particular details of the flight dynamics, such as the specific form of the integrals $I_{n,\alpha}(x)$ in Eq. (22).

In conclusion, by comparing the results on the finite and semi-infinite domains, our calculations based on the master equation approach for discrete Lévy flights evidenced the shift in the universality of the asymptotic survival probability, from the power-law Sparre-Andersen $1/\sqrt{t}$ dependence to the exponential e^{-t} decay. Our results were supported by explicit calculations and numerical simulations on the Cauchy ($\alpha = 1$) and Gaussian ($\alpha = 2$) dynamics, which provided in some cases the dependence on the flyer's starting distance for any time, and also by the general long-term analysis of Lévy fliers with index $0 < \alpha < 2$.

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