Integrable dissipative exclusion process: Correlation functions and physical properties

N. Crampe,[1,*] E. Ragoucy,[2,†] V. Rittenberg,[3,‡] and M. Vanicat[2,§]

[1]*Laboratoire Charles Coulomb (L2C), UMR 5221 CNRS-Université de Montpellier, Montpellier, F-France*

[2]*Laboratoire de Physique Théorique LAPTh, CNRS and Université de Savoie, 9 chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux Cedex, France*

[3]*Physikalisches Institut der Universitaet Bonn, Nussallee 12 D-53115 Bonn, Federal Republic of Germany*

(Received 25 March 2016; published 1 September 2016)

We study a one-parameter generalization of the symmetric simple exclusion process on a one-dimensional lattice. In addition to the usual dynamics (where particles can hop with equal rates to the left or to the right with an exclusion constraint), annihilation and creation of pairs can occur. The system is driven out of equilibrium by two reservoirs at the boundaries. In this setting the model is still integrable: it is related to the open *XXZ* spin chain through a gauge transformation. This allows us to compute the full spectrum of the Markov matrix using Bethe equations. We also show that the stationary state can be expressed in a matrix product form permitting to compute the multipoints correlation functions as well as the mean value of the lattice and the creation-annihilation currents. Finally, the variance of the lattice current is computed for a finite-size system. In the thermodynamic limit, it matches the value obtained from the associated macroscopic fluctuation theory.

DOI: 10.1103/PhysRevE.94.032102

I. INTRODUCTION

In the context of biology (going from the microscopic size describing molecular dynamics in the cell to the macroscopic one for the evolution of different populations in competition), chemistry, physics, and mathematics, the reaction-diffusion models have been intensively studied. To understand the nature of the stationary state of such models, driven lattice gas models have been proposed [1–4] and then their generalization with dissipation [5–14]. For particular choices of the parameters, one-dimensional diffusive gas with dissipation can be mapped equivalently to a free fermions problem, which can be solved easily [15–27].

Recently [28], we have introduced an alternate one-dimensional integrable stochastic model. Most of the known integrable stochastic models are derived starting with representations of quotients of the Hecke [29] or Brauer [30] algebras. The new integrable stochastic model is of a different kind. It was pointed out to us by Pyatov [31] that in fact one deals with a special representation of a quotient of the Birman-Murakami-Wentzel algebra. This is new and it is still an open question how to generalize this model. It is the aim of this paper to understand what are the physical properties of the model having in mind possible generalizations. This task is simplified by the observation [28] that the probability distribution function in the stationary state can be written in terms of the matrix product ansatz. The dynamics of the model can be obtained using as usual, the Bethe ansatz or, as shown in the paper, the macroscopic fluctuation theory (MFT) [32,33].

The model is a one-parameter deformation of symmetric simple exclusion process (SSEP) model, allowing pairs of particles to be generated or get annihilated with equal rates. These rates are fixed by the parameter. We call the model DiSSEP, where Di stands for dissipative. Similar to SSEP, one can add sources and sinks at the end of the system keeping the integrability of the model. What can we expect to be the physics of the model? Since hopping takes place in a symmetric way, in the thermodynamic limit one can have only weakly correlation functions (the correlators see only the boundaries and not the distance between the particles). Finite-size effects, however, can be new since one can make the parameter dependent on the size of the system.

The stationary state is expressed using a matrix product ansatz, which allows us to compute the correlation functions and the mean value of the currents. A new feature is that we have two currents: one of them given by particles crossing the bonds and a second one given by particles leaving the system. A new feature appears also here in comparison to the SSEP with sinks and sources: the values of these currents are not homogeneous and depend on the site where we measure them. We describe in details the different behaviors of these physical quantities depending on the boundary parameters in the thermodynamic limit.

Using recursive relations between the correlation functions, we succeed in giving a closed analytical relation for the variance of the lattice current. This variance depends also on the site of the lattice, which makes it more involved in comparison to the SSEP. A byproduct of our research was the check of the applicability of the MFT to stochastic processes with creation and annihilation of particles. We have shown that in a particular case when the dynamics is of a diffusive kind the results obtained from lattice calculations and MFT coincide. This result is relevant since up to now the validity of MFT developed in Refs. [34–36] was confirmed only in purely diffusive models such as SSEP [37].

Finally, using Bethe ansatz approach, the spectrum of the associated Markow matrix is completely characterized by Bethe equations. Comparing with the exact diagonalization of the Markov matrix for small lattices, we can determine the Bethe state allowing us to compute the greatest nonvanishing eigenvalue. In this way, we can compute the spectral gap

*nicolas.crampe@univ-montp2.fr
†eric.ragoucy@lapth.cnrs.fr
‡vladimirrittenberg@yahoo.com
§matthieu.vanicat@lapth.cnrs.fr

 ©2016 American Physical Society

for large lattices (up to 150 sites) and we conjecture its thermodynamic limit by extrapolation.

The plan of the paper is as follows. In Sec. II, we describe precisely the DiSSEP, its symmetries, and its associated Markov matrix. Then, in Sec. III, we remark that for a particular choice of this parameter the eigenvectors and the eigenvalues become simpler and can be all computed explicitly. This allows us to infer the gap and the large deviation function for the current entering in the system from one of the reservoirs. Then, we start the study of the general case. In Sec. IV, we present the computation of the stationary state using the matrix product ansatz method that provides us analytical expressions for correlation functions. We compute also exactly the variance of the lattice current (which depends on the site where it is measured). We take the thermodynamic limit of the model in Sec. V: we show that the additional parameter must be rescaled with the length in order to have a competition between the hopping and the evaporation. We deduce from the previous microscopic computations the exact expressions for the densities, the currents, and the variance of the lattice current. In Sec. VI, we show that the latter is in agreement with the results obtained from macroscopic fluctuation theory. Finally, we relate in Sec. VII the Markov matrix of the DiSSEP to the Hamiltonian of the XXZ spin chain with triangular boundaries [38]. The eigenvalues of this Hamiltonian have been computed previously using Bethe ansatz methods. We present the associated Bethe equations and we solve them to get the spectral gap. In Sec. VIII we summarize our results and give an outlook for further research.

II. A ONE-PARAMETER DEFORMATION OF THE SSEP

A. Description of the model

We present here the stochastic process, DiSSEP. It describes particles evolving on a one-dimensional lattice composed of L sites and connected with two reservoirs at different densities on its extremities. There is a Fermi-like exclusion principle: there is at most one particle per site. Hence a configuration $\mathcal{C}$ of the system can be formally denoted by an L-tuple $(n_1,\ldots,n_L)$, where $n_i = 0$ if there is no particle at site i and $n_i = 1$ if the site is occupied. During each infinitesimal time dt, a particle in the bulk can jump to the left or to the right neighboring site with probability dt if it is unoccupied. A pair of neighbor particles can also be annihilated with probability $\lambda^2 \times dt$ and be created on unoccupied neighbor sites with probability $\lambda^2 \times dt$ (see Table I and Fig. 1). Let us mention that there is a slight change of notation in comparison to Ref. [28] in order to make the limit to the SSEP easier. At the two extremities of the lattice the dynamics is modified in order to take into account the interaction with the reservoirs: at the first site (connected with the left reservoir), during time dt, a particle is injected

TABLE I. Dynamical rules of DiSSEP.

Left	Bulk	Right
$0 \xrightarrow{\alpha} 1$	$01 \xleftrightarrow{1} 10$	$1 \xrightarrow{\beta} 0$
$1 \xrightarrow{\gamma} 0$	$00 \xleftrightarrow{\lambda^2} 11$	$0 \xrightarrow{\delta} 1$

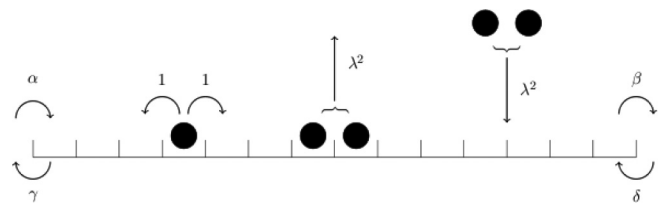

FIG. 1. Dynamical rules of the model.

with probability $\alpha \times dt$ if the site is empty and extracted with probability $\gamma \times dt$ if it is occupied. The dynamics is similar at last site (connected with the right reservoir) with injection rate δ and extraction rate β. The dynamical rules can be summarized in Table I, where 0 stands for vacancy and 1 stands for a particle. The transition rates between the configurations are written above the arrows.

We choose the coefficient of condensation and evaporation to be λ^2 and not λ for later convenience. Let us remark that the SSEP is recovered when the creation and annihilation rate λ^2 vanishes. The limit $\lambda^2 \to \infty$ provides a model with only condensation and evaporation.

The system is driven out of equilibrium by the boundaries. As shown in Sec. IV, there are particle currents in the stationary state for generic boundary rates α, β, γ, and δ. We will see in Eq. (65) that these choices of rates describe particle reservoirs with densities

$$\rho_a = \frac{\alpha}{\alpha+\gamma}, \quad \text{and} \quad \rho_b = \frac{\delta}{\beta+\delta}. \tag{1}$$

Remark: The system will reach a thermodynamic equilibrium if and only if both the densities of the reservoirs are equal to 1/2, that is $\alpha = \gamma$ and $\beta = \delta$. Indeed, the detailed balance is only satisfied in this case.

Symmetries of the model

We can make the following observations:

(1) Since we chose the evaporation/condensation rate to be λ^2, all the results shall be invariant under the transformation $\lambda \to -\lambda$.

(ii) The left-right symmetry of the chain is given by the transformations $\alpha \leftrightarrow \delta$, $\gamma \leftrightarrow \beta$ and a change of numbering of the sites $i \to L+1-i$.

(iii) The vacancy-particle symmetry translates into $\alpha \leftrightarrow \gamma$ and $\delta \leftrightarrow \beta$.

B. Markov matrix

We denote by $P_t(\mathcal{C})$ the probability for the system to be in configuration $\mathcal{C}$ at time t. $P_t(\mathcal{C})$ obeys the master equation

$$\begin{aligned}\frac{dP_t(\mathcal{C})}{dt} &= \sum_{\mathcal{C}'\neq\mathcal{C}} M(\mathcal{C},\mathcal{C}')P_t(\mathcal{C}') - \sum_{\mathcal{C}'\neq\mathcal{C}} M(\mathcal{C}',\mathcal{C})P_t(\mathcal{C}) \\ &= \sum_{\mathcal{C}'} M(\mathcal{C},\mathcal{C}')P_t(\mathcal{C}'),\end{aligned} \tag{2}$$

where $M(\mathcal{C},\mathcal{C}')$ is the transition rate from the configuration $\mathcal{C}'$ to the configuration $\mathcal{C}$. The second equality is obtained by setting $M(\mathcal{C},\mathcal{C}) = -\sum_{\mathcal{C}'\neq\mathcal{C}} M(\mathcal{C}',\mathcal{C})$. This equation can be

recast in a compact way: let us set

$$|P_t\rangle = \begin{pmatrix} P_t((0,\dots,0,0,0)) \\ P_t((0,\dots,0,0,1)) \\ P_t((0,\dots,0,1,0)) \\ \vdots \\ P_t((1,\dots,1,1,1)) \end{pmatrix} = \sum_{n_1,\dots,n_L\in\{0,1\}} P_t((n_1,\dots,n_L))|n_1\dots n_L\rangle, \quad (3)$$

with $|n_1\dots n_L\rangle = |n_1\rangle\otimes\cdots\otimes|n_L\rangle$, $|0\rangle = \binom{1}{0}$, and $|1\rangle = \binom{0}{1}$. In this formalism, each site of the lattice corresponds to one copy of $\mathbb{C}^2$ in the tensorial space $(\mathbb{C}^2)^{\otimes L}$. The master Eq. (2) is then rewritten as

$$\frac{d|P_t\rangle}{dt} = M\,|P_t\rangle. \quad (4)$$

The Markov matrix M corresponding to the dynamical rules (see Table I) can be expressed as

$$M = B_1 + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L, \quad (5)$$

where the local jump operators B, $\overline{B}$, and w are given by

$$B = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}, \quad (6)$$

and

$$w = \begin{pmatrix} -\lambda^2 & 0 & 0 & \lambda^2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \lambda^2 & 0 & 0 & -\lambda^2 \end{pmatrix}. \quad (7)$$

The subscripts in Eq. (5) indicate on which sites the local operators are acting.

III. STUDY OF THE PARTICULAR CASE $\lambda = 1$

Before studying the general model, we focus on the case $\lambda = 1$, where the calculations simplify drastically: it corresponds to the free fermion point of the model we introduced.

A. Eigenvectors and relaxation rate

For $\lambda = 1$, all the eigenvalues and the eigenvectors can be computed easily. Indeed, for $L \geqslant 3$, the 2^L eigenvectors are characterized by the set $\boldsymbol{\epsilon} = (\epsilon_1,\epsilon_2,\dots,\epsilon_L)$ with $\epsilon_i = \pm 1$ and are given by

$$\Omega(\boldsymbol{\epsilon}) = v(\epsilon_1,\epsilon_2,\alpha,\gamma)\otimes\begin{pmatrix}1\\ \epsilon_2\end{pmatrix}\otimes\begin{pmatrix}1\\ \epsilon_3\end{pmatrix}\otimes\cdots \cdots\otimes\begin{pmatrix}1\\ \epsilon_{L-1}\end{pmatrix}\otimes v(\epsilon_L,\epsilon_{L-1},\delta,\beta), \quad (8)$$

where $v(\epsilon,\epsilon',\mu,\nu) = \binom{\epsilon'+\nu}{1+\mu+f(\epsilon,\epsilon',\mu+\nu)}$ and $f(\epsilon,\epsilon',\tau) = \epsilon\epsilon' - 1 - \frac{\tau}{2}(1-\epsilon)$. The corresponding eigenvalues are

$$\Lambda(\boldsymbol{\epsilon}) = f(\epsilon_1,\epsilon_2,\alpha+\gamma) + \sum_{j=2}^{L-2}(\epsilon_j\epsilon_{j+1} - 1) + f(\epsilon_L,\epsilon_{L-1},\delta+\beta).$$

Let us remark that the ASEP on a ring with Langmuir kinetics has been treated similarly in Ref. [39]. From the previous results, we deduce that the stationary state is

$$\Omega(+,+,\dots,+) = \frac{1}{Z_L}\begin{pmatrix}1+\gamma\\ 1+\alpha\end{pmatrix}\otimes\begin{pmatrix}1\\ 1\end{pmatrix}^{\otimes L-2}\otimes\begin{pmatrix}1+\beta\\ 1+\delta\end{pmatrix},$$

where $Z_L = 2^{L-2}(2+\alpha+\gamma)(2+\beta+\delta)$ is the normalisation such that the entries be probabilities.

From this stationary state, we can compute the mean value of the injected current by the left reservoir (respectively, by the right reservoir)

$$j_{\text{left}} = \frac{\alpha-\gamma}{2+\alpha+\gamma}\left(\text{respectively, } j_{\text{right}} = \frac{\delta-\beta}{2+\delta+\beta}\right). \quad (9)$$

We see that the current has the sign of $\alpha-\gamma$ (respectively, $\delta-\beta$). As expected, it goes to the left when extraction is promoted, and to the right when injection is proeminent. It vanishes for $\alpha = \gamma$. The lattice current in the bulk vanishes.

We can also compute easily the first excited state whose eigenvalue provides the relaxation rate. Indeed, the greatest nonvanishing eigenvalue is

$$\begin{aligned} G &= -4 \quad \text{if } \alpha+\gamma > 2 \quad \text{and} \quad \beta+\delta > 2, \\ G &= -2-\beta-\delta \quad \text{if } \alpha+\gamma > 2 \quad \text{and} \quad \beta+\delta < 2, \\ G &= -2-\alpha-\gamma \quad \text{if } \alpha+\gamma < 2 \quad \text{and} \quad \beta+\delta > 2, \\ G &= -\alpha-\gamma-\beta-\delta \quad \text{if } \alpha+\gamma < 2 \quad \text{and} \quad \beta+\delta < 2. \end{aligned}$$

These results shall be generalized in Sec. VII B for any λ using the Bethe equations. The general result displayed on Fig. 9 matches the above values of the gap for $\lambda = 1$ ($\phi = 0$).

B. Current large deviation function

For this particular choice of λ, it is also possible to get the generating function of the cumulants of the current entering in the system from the left reservoir (the same result is also obtained by symmetry for the right reservoir). For general λ, one obtains the variance in Sec. IV C. It is well established that this generating function is the greatest eigenvalue of the following deformed Markov matrix [40],

$$M = B_1(s) + \sum_{k=1}^{L-1} w_{k,k+1} + \overline{B}_L, \quad (10)$$

where the local jump operator $B(s)$ is deformed as follows:

$$B = \begin{pmatrix} -\alpha & \gamma e^{-s} \\ \alpha e^{s} & -\gamma \end{pmatrix}. \quad (11)$$

One can show that the greatest eigenvalue is given by

$$E(s) = -\frac{2+\alpha+\gamma}{2} + \frac{1}{2}\sqrt{4+4\alpha e^{s}+4\gamma e^{-s}+(\alpha+\gamma)^2},$$

with the eigenvector

$$\Omega(s) = \begin{pmatrix}1+\gamma e^{-s}\\ 1+\alpha+E(s)\end{pmatrix}\otimes\begin{pmatrix}1\\ 1\end{pmatrix}^{\otimes L-2}\otimes\begin{pmatrix}1+\beta\\ 1+\delta\end{pmatrix}.$$

The rate-function $G(j)$ associated to the current is the Legendre transformation of this generating function of the

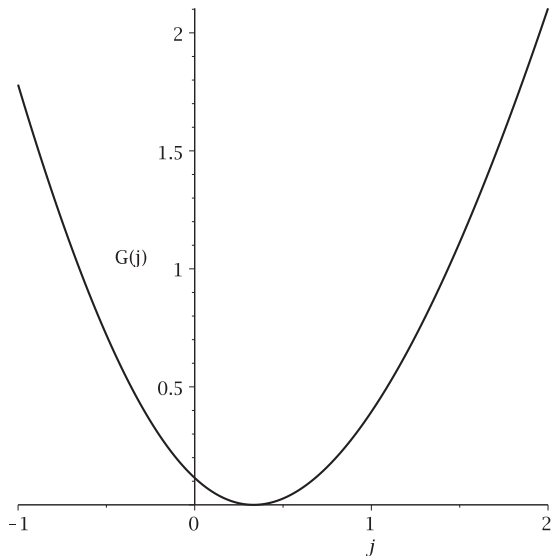

FIG. 2. Example of large deviation function $G(j)$ (on the plot $\alpha = 2$, $\gamma = 0.5$).

cumulants:

$$G(j) = s^* j - E(s^*), \quad \frac{d}{ds}E(s)\bigg|_{s=s^*} = j. \tag{12}$$

Then its explicit form is given by

$$G(j) = 1 + \frac{\alpha+\gamma}{2} - \sqrt{1 + \Delta(j) + \left(\frac{\alpha+\gamma}{2}\right)^2} + j \ln\left[\frac{\Delta(j)}{2\alpha} + \frac{j}{\alpha}\sqrt{1 + \Delta(j) + \left(\frac{\alpha+\gamma}{2}\right)^2}\right], \tag{13}$$

where

$$\Delta(j) = 2j^2 + \sqrt{4(\alpha\gamma + j^2 + j^4) + j^2(\alpha+\gamma)^2}. \tag{14}$$

Let us stress that Eq. (13) represents an exact result on the large deviation function of the current on the left boundary. The function $G(j)$ is convex and vanishes when j is equal to the mean value of the current on the left boundary given by Eq. (9) as expected; see Fig. 2. Note that it is not Gaussian.

IV. STATIONARY STATE AND PHYSICAL OBSERVABLES FOR ARBITRARY λ

We now turn to the general case (i.e., for generic λ). Although one cannot perform all the calculation presented in the $\lambda = 1$ case, one can still obtain interesting analytical results, in particular when focusing on the stationary state.

A. Matrix product ansatz

The integrability of the model reflects in the simple structure of its stationary state which can be built in a matrix product form. The power of this technique was revealed in a pioneering work [41], where the phase diagram of the totally asymmetric simple exclusion process (TASEP) was computed analytically, using a matrix product expression of the steady state wave function. It has led to numerous works and generalizations, among them can be mentioned the multispecies TASEP [42–44] and more complicated reaction-diffusion processes [26,45]. A review of these results can be found in Ref. [46]. In the framework of integrable Markov matrix, the stationary state can be expressed through a matrix product ansatz, which takes a simple form: the complete algebraic structure can be determined thanks to the Zamolodchikov-Faddeev and Ghoshal-Zamolodchikov relation as it was first seen in Ref. [47]. A systematic construction of the matrix product ansatz for integrable systems can be found in Ref. [28].

It was shown in Ref. [28] that the steady state $|\mathcal{S}\rangle$ of the present model can be built as follows:

$$|\mathcal{S}\rangle = \frac{1}{Z_L}\langle\langle W|\begin{pmatrix} E \\ D \end{pmatrix}^{\otimes L}|V\rangle\rangle, \tag{15}$$

where Z_L is the normalization factor,

$$Z_L = \langle\langle W|(E+D)^L|V\rangle\rangle. \tag{16}$$

The algebraic elements E and D belongs to an algebra composed of three generators E, D, and H. The commutation relations between these generators are given by

$$[D,E] = EH + HD, \quad [H,E] = [H,D] = \lambda^2(D^2 - E^2). \tag{17}$$

These relations are equivalent to the very useful telescopic relation

$$w\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} = \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} -H \\ H \end{pmatrix} - \begin{pmatrix} -H \\ H \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix}. \tag{18}$$

Notice here that, in contrast with the SSEP case (see Sec. IV D), where H is a scalar, the commutation relations between H and E, D are not trivial. The action of the generators E, D and H on the boundary vectors $\langle\langle W|$ and $|V\rangle\rangle$ is given by

$$\langle\langle W|(\alpha E - \gamma D) = \langle\langle W|H, \quad (\delta E - \beta D)|V\rangle\rangle = -H|V\rangle\rangle.$$

It is equivalent to

$$\langle\langle W|B\begin{pmatrix} E \\ D \end{pmatrix} = \langle\langle W|\begin{pmatrix} -H \\ H \end{pmatrix}, \tag{19}$$

and

$$\overline{B}\begin{pmatrix} E \\ D \end{pmatrix}|V\rangle\rangle = -\begin{pmatrix} -H \\ H \end{pmatrix}|V\rangle\rangle. \tag{20}$$

It is straightforward to see using Eqs. (18), (19), and (20) that $M|\mathcal{S}\rangle = 0$ with M defined by Eq. (5) and $|\mathcal{S}\rangle$ defined by Eq. (15). Indeed, we get a telescopic sum. We showed also in Ref. [28] that the algebra is consistent with the boundary equations by giving an explicit representation of the generators E, D, and H and of the boundary vectors $|V\rangle\rangle$ and $\langle\langle W|$. We have tried, without success, to get the probability distribution function for the stationary state in the case of the asymmetric hopping rates. This would have been very interesting. It is surprising since our matrix product ansatz is close to the one used to solve the symmetric simple exclusion process. Our failure is probably due to the nonintegrability of the system in this case.

Assuming that λ is nonvanishing, we make a change of basis from the generators E, D, and H to G_1, G_2, and G_3 as follows:

$$\begin{aligned} E &= G_1 + G_2 + G_3, \\ D &= G_2 - G_1 - G_3, \\ H &= 2\lambda(G_3 - G_1). \end{aligned} \tag{21}$$

This change of basis will simplify the calculation; moreover, it is natural in the general framework developed in Ref. [28]. The commutation relations between E, D, and H [Eq. (17)] are equivalent to

$$[G_1,G_3]=0, \quad G_2G_1=\phi\, G_1G_2, \quad \text{and} \quad G_3G_2=\phi\, G_2G_3, \tag{22}$$

with $\phi = \frac{1-\lambda}{1+\lambda}$.

The relations on the boundaries become

$$\begin{aligned} \langle\langle W|\big(G_1 - c\, G_2 - a\, G_3\big) &= 0, \\ (G_3 - b\, G_1 - d\, G_2)|V\rangle\rangle &= 0, \end{aligned} \tag{23}$$

with

$$\begin{aligned} a &= \frac{2\lambda-\alpha-\gamma}{2\lambda+\alpha+\gamma}, \quad c = \frac{\gamma-\alpha}{2\lambda+\alpha+\gamma}, \\ b &= \frac{2\lambda-\delta-\beta}{2\lambda+\delta+\beta}, \quad d = \frac{\beta-\delta}{2\lambda+\delta+\beta}. \end{aligned} \tag{24}$$

We can now give the main result of this paper. Indeed, in this new basis, it is possible to compute a closed expression for any word, for $p,q,r = 0,1,2,\ldots,$

$$\frac{\langle\langle W|G_1^pG_2^qG_3^r|V\rangle\rangle}{\langle\langle W|G_2^{p+q+r}|V\rangle\rangle} = \frac{\prod_{\ell=0}^{p-1}(c\,\phi^{p-1-\ell} + ad\,\phi^{q+r+\ell})\,\prod_{n=0}^{r-1}(d\,\phi^{r-1-n} + bc\,\phi^{q+p+n})}{\prod_{k=q}^{p+q+r-1}(1-ab\,\phi^{2k})}, \tag{25}$$

where by convention $\prod_{n=0}^{-1}(\cdot) = 1$. Let us mention that this formula is not valid if $ab\phi^k = 1$, which occurs for $\lambda = 0$ (SSEP case) or $\lambda \to \infty$. However, these limits can be performed for the physical quantities (see Sec. IV D).

Proof of Eq. (25)*:* In order to compute $\langle\langle W|G_1^pG_2^qG_3^r|V\rangle\rangle$, we use a change of generators defined as follows

$$L_i = \frac{G_1}{\phi^i} - aG_3\phi^i, \quad \text{and} \quad R_k = \frac{G_3}{\phi^k} - bG_1\phi^k. \tag{26}$$

This is built so that L_i and R_k fulfill the following relations [derived straightforwardly from Eqs. (22) and (23)],

$$G_2^iL_i = L_0G_2^i, \quad R_kG_2^k = G_2^kR_0, \tag{27}$$

$$\begin{aligned} &[L_i,G_1] = [L_i,G_3] = [R_k,G_1] = [R_k,G_3] = 0, \\ &\text{and} \quad \langle\langle W|L_0 = c\langle\langle W|G_2, \quad R_0|V\rangle\rangle = dG_2|V\rangle\rangle. \end{aligned} \tag{28}$$

The change of generators Eq. (26) can be inverted to get

$$G_1 = \frac{\phi^{k+i}}{1-ab\phi^{2(k+i)}}\left(\frac{L_i}{\phi^k} + a\phi^iR_k\right), \tag{29}$$

and

$$G_3 = \frac{\phi^{k+i}}{1-ab\phi^{2(k+i)}}\left(b\phi^kL_i + \frac{R_k}{\phi^i}\right). \tag{30}$$

We can now begin the computation

$$\begin{aligned} &\langle\langle W|G_1^pG_2^qG_3^r|V\rangle\rangle \\ &= \frac{\phi^q}{1-ab\phi^{2q}}\langle\langle W|\left(\frac{L_0}{\phi^q} + aR_q\right)G_1^{p-1}G_2^qG_3^r|V\rangle\rangle \\ &= \frac{1}{1-ab\phi^{2q}}\langle\langle W|cG_2G_1^{p-1}G_2^qG_3^r|V\rangle\rangle \\ &\quad + \frac{1}{1-ab\phi^{2q}}\langle\langle W|G_1^{p-1}G_2^qG_3^rad\phi^qG_2|V\rangle\rangle \\ &= \frac{c\phi^{p-1} + ad\phi^{q+r}}{1-ab\phi^{2q}}\langle\langle W|G_1^{p-1}G_2^{q+1}G_3^r|V\rangle\rangle. \end{aligned}$$

The first equality is obtained using Eq. (29) with $i = 0$ and $k = q$ to transform the leftmost G_1. The second equality relies on the relation Eqs. (27) and (28). We get the last one using Eq. (22). This relation is a recursive relation between $\langle\langle W|G_1^pG_2^qG_3^r|V\rangle\rangle$ and $\langle\langle W|G_1^{p-1}G_2^{q+1}G_3^r|V\rangle\rangle$ that we can iterate to obtain

$$\begin{aligned} &\langle\langle W|G_1^pG_2^qG_3^r|V\rangle\rangle \\ &= \left(\prod_{l=0}^{p-1}\frac{c\phi^{p-1-l} + ad\phi^{r+q+l}}{1-ab\phi^{2(q+l)}}\right)\langle\langle W|G_2^{q+p}G_3^r|V\rangle\rangle. \end{aligned} \tag{31}$$

Performing similar computations with G_3 we obtain the following recursive relation:

$$\begin{aligned} &\langle\langle W|G_2^{q+p}G_3^r|V\rangle\rangle \\ &= \frac{\phi^{q+p}}{1-ab\phi^{2(q+p)}}\langle\langle W|G_2^{q+p}G_3^{r-1}\left(bL_{q+p} + \frac{R_0}{\phi^{q+p}}\right)|V\rangle\rangle \\ &= \frac{d\phi^{r-1} + bc\phi^{q+p}}{1-ab\phi^{2(q+p)}}\langle\langle W|G_2^{q+p+1}G_3^{r-1}|V\rangle\rangle, \end{aligned}$$

to get

$$\begin{aligned} &\langle\langle W|G_2^{q+p}G_3^r|V\rangle\rangle \\ &= \left(\prod_{n=0}^{r-1}\frac{d\phi^{r-1-n} + bc\phi^{q+p+n}}{1-ab\phi^{2(q+p+n)}}\right)\langle\langle W|G_2^{p+q+r}|V\rangle\rangle. \end{aligned} \tag{32}$$

Recombining Eqs. (31) and (32) together, the desired result, Eq. (25), is proved.

Let us stress that since G_1, G_2, G_3 form a basis, the knowledge of all words built on them allows us to reconstruct all words built on E and D using the two first relations of Eq. (21). In particular, we are able to compute exactly physical observables, as it is illustrated below.

B. Calculation of physical observables: Correlation functions, currents

As usual the matrix product ansatz permits us to compute the physical observables (see Ref. [41]). For example, the one-point correlation function (or density) is expressed as follows in terms of the matrix product ansatz,

$$\langle n_i\rangle = \frac{\langle\langle W|C^{i-1}DC^{L-i}|V\rangle\rangle}{\langle\langle W|C^{L}|V\rangle\rangle}, \tag{33}$$

where we have used the usual notation $C = E + D$. Let us remark that, in our basis, $C = 2G_2$. It is well-known that similar expressions exist also for all the correlations or the currents (see below).

Thanks to Eq. (25) of any words, it is now possible to compute physical observables for DiSSEP. In particular, we can compute the one-point correlation function,

$$\begin{aligned}\langle n_i\rangle &= \frac{1}{2}\frac{\langle\langle W|G_2^{i-1}(-G_1+G_2-G_3)G_2^{L-i}|V\rangle\rangle}{\langle\langle W|G_2^{L}|V\rangle\rangle}\\ &= \frac{1}{2} - \frac{c\phi^{i-1}+ad\phi^{L+i-2}+d\phi^{L-i}+bc\phi^{2L-i-1}}{2(1-ab\phi^{2L-2})},\end{aligned} \tag{34}$$

the connected two-point correlation function, for $1 \leqslant i < j \leqslant L$,

$$\langle n_i n_j\rangle_c = \langle n_i n_j\rangle - \langle n_i\rangle\langle n_j\rangle \tag{35}$$

$$= \frac{\langle\langle W|C^{i-1}DC^{j-i-1}DC^{L-j}|V\rangle\rangle}{\langle\langle W|C^{L}|V\rangle\rangle} - \frac{\langle\langle W|C^{i-1}DC^{L-i}|V\rangle\rangle}{\langle\langle W|C^{L}|V\rangle\rangle}\frac{\langle\langle W|C^{j-1}DC^{L-j}|V\rangle\rangle}{\langle\langle W|C^{L}|V\rangle\rangle} \tag{36}$$

$$= \frac{\phi^{L+j-i-3}(1-\phi^2)(d+bc\phi^{L-1})(c+ad\phi^{L-1})}{4(1-ab\phi^{2(L-1)})^2(1-ab\phi^{2(L-2)})} \times(1+b\phi^{2(L-j)})(1+a\phi^{2(i-1)}), \tag{37}$$

and the connected three-point correlation function, for $1 \leqslant i < j < k \leqslant L$,

$$\begin{aligned}\langle n_i n_j n_k\rangle_c &= \langle n_i n_j n_k\rangle - \langle n_i\rangle\langle n_j n_k\rangle - \langle n_j\rangle\langle n_i n_k\rangle - \langle n_k\rangle\langle n_i n_j\rangle\\ &\quad + 2\langle n_i\rangle\langle n_j\rangle\langle n_k\rangle\\ &= -\frac{\phi^{L+k-i-5}(1-\phi^2)^2(d+bc\phi^{L-1})(c+ad\phi^{L-1})}{8(1-ab\phi^{2(L-1)})^3(1-ab\phi^{2(L-2)})(1-ab\phi^{2(L-3)})}\\ &\quad\times(1+b\phi^{2(L-k)})(1+a\phi^{2(i-1)})\\ &\quad\times[\phi^{L-j}(d+bc\phi^{L-3})(1+2a\phi^{2(j-1)}+ab\phi^{2(L-1)})\\ &\quad+\phi^{j-1}(c+ad\phi^{L-3})(1+2b\phi^{2(L-j)}+ab\phi^{2(L-1)})].\end{aligned} \tag{38}$$

Note that for generic i, j, k, the two- and three-point correlation functions satisfy both a set of closed linear relations:

$$\begin{aligned}&(1-\lambda^2)(\langle n_{i-1}n_j\rangle_c + \langle n_{i+1}n_j\rangle_c + \langle n_i n_{j-1}\rangle_c + \langle n_i n_{j+1}\rangle_c)\\ &= 4(1+\lambda^2)\langle n_i n_j\rangle_c,\end{aligned} \tag{39}$$

and

$$\begin{aligned}&(1-\lambda^2)(\langle n_{i-1}n_j n_k\rangle_c + \langle n_{i+1}n_j n_k\rangle_c + \langle n_i n_{j-1}n_k\rangle_c\\ &\quad+\langle n_i n_{j+1}n_k\rangle_c + \langle n_i n_j n_{k-1}\rangle_c + \langle n_i n_j n_{k+1}\rangle_c)\\ &= 6(1+\lambda^2)\langle n_i n_j n_k\rangle_c.\end{aligned} \tag{40}$$

We can also compute the particle currents. There are two different currents: the lattice current, which stands for the number of particles going through the bond between site i and $i+1$ per unit of time, and the evaporation-condensation current, which stands for the number of particles evaporating or condensing at sites i and $i+1$ per unit of time. The lattice current is given by

$$\begin{aligned}&\left\langle J^{\text{lat}}_{i\to i+1}\right\rangle\\ &= \frac{\langle\langle W|C^{i-1}(DE-ED)C^{L-i-1}|V\rangle\rangle}{\langle\langle W|C^{L}|V\rangle\rangle}\\ &= \frac{1-\phi}{2}\frac{bc\phi^{2L-i-2}+d\phi^{L-i-1}-ad\phi^{L+i-2}-c\phi^{i-1}}{1-ab\phi^{2L-2}}.\end{aligned} \tag{41}$$

Counting positively the pairs of particles that condensate on the lattice and negatively the pairs that evaporate, we get for the evaporation-condensation current,

$$\begin{aligned}&\left\langle J^{\text{cond}}_{i,i+1}\right\rangle\\ &= \frac{\langle\langle W|C^{i-1}\lambda^2(E^2-D^2)C^{L-i-1}|V\rangle\rangle}{\langle\langle W|C^{L}|V\rangle\rangle}\\ &= \frac{(1-\phi)^2}{2(1+\phi)}\frac{bc\phi^{2L-i-2}+d\phi^{L-i-1}+ad\phi^{L+i-2}+c\phi^{i-1}}{1-ab\phi^{2L-2}}.\end{aligned} \tag{42}$$

Note that the above expressions behave as expected under the three symmetries:

(1) The symmetry $\lambda \to -\lambda$, which translates into $\phi \to 1/\phi$, $a \to 1/a$, $c \to -c/a$, $b \to 1/b$, and $d \to -d/b$, leaves them invariant.

(2) The left-right symmetry, which becomes $a \leftrightarrow b$, $c \leftrightarrow d$, and $i \to L+1-i$, changes the sign of the lattice current, keeps the condensation current and the density invariant.

(3) The particle-hole symmetry, which reads $a \to a$, $b \to b$, $c \to -c$, and $d \to -d$, changes the sign of both currents and transforms $\langle n_i\rangle$ into $1-\langle n_i\rangle$.

The physical quantities computed above are not all independent. The particle conservation law at site i reads

$$\left\langle J^{\text{lat}}_{i-1\to i}\right\rangle - \left\langle J^{\text{lat}}_{i\to i+1}\right\rangle + \left\langle J^{\text{cond}}_{i-1,i}\right\rangle + \left\langle J^{\text{cond}}_{i,i+1}\right\rangle = 0, \tag{43}$$

which can be seen on the matrix product ansatz using relations Eq. (17). From the identity $[D,E] = [D,C]$, one then deduces

$$\left\langle J^{\text{lat}}_{i\to i+1}\right\rangle = \langle n_i\rangle - \langle n_{i+1}\rangle, \tag{44}$$

and using $E^2 - D^2 = C^2 - CD - DC$, one gets

$$\left\langle J^{\text{cond}}_{i,i+1}\right\rangle = \lambda^2(1-\langle n_i\rangle - \langle n_{i+1}\rangle). \tag{45}$$

From these three relations, one obtains

$$\langle J^{\rm cond}_{i-1,i}\rangle - \langle J^{\rm cond}_{i,i+1}\rangle + \lambda^2\big(\langle J^{\rm lat}_{i-1\to i}\rangle + \langle J^{\rm lat}_{i\to i+1}\rangle\big) = 0 \tag{46}$$

and

$$\langle n_{i-1}\rangle + \langle n_{i+1}\rangle - 2\langle n_i\rangle + \lambda^2(2 - \langle n_{i-1}\rangle - \langle n_{i+1}\rangle - 2\langle n_i\rangle) = 0. \tag{47}$$

C. Fluctuations of the currents

As mentioned previously, there are closed linear relations between the two- and three-point correlation functions, which allow one to compute the cumulant of the currents. In this section, we present the computations of the second cumulant of the lattice current between sites i and $i+1$. Let us note that it depends on the site, because of the evaporation-condensation process. As usual for such a purpose [40], we use the deformed Markovian matrix defined as follows:

$$M^s = B_1 + \sum_{k=1}^{i-1} w_{k,k+1} + w^s_{i,i+1} + \sum_{k=i+1}^{L-1} w_{k,k+1} + \overline{B}_L,$$

with

$$w^s = \begin{pmatrix} -\lambda^2 & 0 & 0 & \lambda^2 \\ 0 & -1 & e^s & 0 \\ 0 & e^{-s} & -1 & 0 \\ \lambda^2 & 0 & 0 & -\lambda^2 \end{pmatrix}.$$

Let $|v^s\rangle$ be the eigenstate of M^s with highest eigenvalue

$$M^s\,|v^s\rangle = \mu(s)\,|v^s\rangle\,. \tag{48}$$

$\mu(s)$ is the generating function for the cumulants of the lattice current between sites i and $i+1$. We introduce the following notation for vectors

$$\langle\{j\}| = (\langle 0| + \langle 1|)^{\otimes(j-1)} \otimes \langle 1| \otimes (\langle 0| + \langle 1|)^{\otimes(L-j)},$$
$$\langle\{j,k\}| = (\langle 0| + \langle 1|)^{\otimes(j-1)} \otimes \langle 1| \otimes (\langle 0| + \langle 1|)^{\otimes(k-j-1)} \otimes \langle 1| \otimes (\langle 0| + \langle 1|)^{\otimes(L-k)},$$

and similarly for $\langle\{j_1,j_2,\ldots,j_M\}|$. In words, these vectors represent configurations with one particle at site j_1, j_2, $\ldots$, j_M, and anything else on the other sites. Remark that this definition applies whatever the order on $j_1,\ldots,j_M$, and thus extends the one given in the above equations. By extension, we note $\langle\emptyset| = (\langle 0| + \langle 1|)^{\otimes L}$. Then, we define the components:

$$T_j(s) \equiv T_j = \frac{\langle\{j\}|v^s\rangle}{\langle\emptyset|v^s\rangle};$$
$$U_{jk}(s) \equiv U_{jk} = \frac{\langle\{j,k\}|v^s\rangle}{\langle\emptyset|v^s\rangle} \tag{49}$$
$$\text{and}\quad V_{jkl}(s) \equiv V_{jkl} = \frac{\langle\{j,k,l\}|v^s\rangle}{\langle\emptyset|v^s\rangle}.$$

Note that by construction, U and V are symmetric, e.g., $U_{jk} = U_{kj}$. Now, projecting Eq. (48) on $\langle\emptyset|$, we get

$$\mu(s) = (e^{-s}-1)(T_{i+1} - U_{i,i+1}) + (e^s - 1)(T_i - U_{i,i+1}). \tag{50}$$

We also project Eq. (48) on $\langle\{j\}|$ for $j=1$,

$$\mu(s)\,T_1 = \alpha(1-T_1) - \gamma T_1 + \lambda^2(1 - T_1 - T_2) + T_2 - T_1 + (e^{-s}-1)(U_{1,i+1} - V_{1,i,i+1}) + (e^s - 1)(U_{1,i} - V_{1,i,i+1}), \tag{51}$$

for $1 < j < i$ and $i+1 < j < L$,

$$\mu(s)\,T_j = \lambda^2(2 - 2T_j - T_{j+1} - T_{j-1}) + T_{j+1} - 2T_j + T_{j-1} + (e^{-s}-1)(U_{j,i+1} - V_{j,i,i+1}) + (e^s - 1)(U_{j,i} - V_{j,i,i+1}), \tag{52}$$

for $j = i$,

$$\mu(s)\,T_i = \lambda^2(2 - 2T_i - T_{i+1} - T_{i-1}) + T_{i+1} - 2T_i + T_{i-1} + (e^{-s}-1)(T_{i+1} - U_{i,i+1}), \tag{53}$$

for $j = i+1$,

$$\mu(s)\,T_{i+1} = \lambda^2(2 - 2T_{i+1} - T_{i+2} - T_i) + T_{i+2} - 2T_{i+1} + T_i + (e^s - 1)(T_i - U_{i,i+1}), \tag{54}$$

and for $j = L$,

$$\mu(s)\,T_L = \delta(1-T_L) - \beta T_L + \lambda^2(1 - T_L - T_{L-1}) + T_{L-1} - T_L + (e^{-s}-1)(U_{i+1,L} - V_{i,i+1,L}) + (e^s - 1)(U_{i,L} - V_{i,i+1,L}). \tag{55}$$

These equations are solved iteratively, expanding all quantities as series in s. We set

$$\mu(s) = \mu^{(0)} + s\,\mu^{(1)} + \frac{s^2}{2}\mu^{(2)} + o(s^2),$$
$$T_j(s) = T_j^{(0)} + s\,T_j^{(1)} + o(s),$$
$$U_{j,k}(s) = U_{j,k}^{(0)} + s\,U_{j,k}^{(1)} + o(s).$$

In the above expansions, $\mu^{(0)} = 0$ is the greatest eigenvalue of the undeformed Markov matrix and $\mu^{(1)} = \langle J^{\rm lat}_{i\to i+1}\rangle$ is the mean value of the lattice current measured between the site i and $i+1$, where the deformation occurs. We recall that $\langle J^{\rm lat}_{i\to i+1}\rangle$ has been computed in Eq. (41). The value of $T_j^{(0)} = \langle n_j\rangle$ has also been already calculated; see Eq. (34). Similarly, $U_{j,k}^{(0)}$ is linked to the two-points correlation function; see Eq. (37).

We wish to compute $\mu(s)$ up to order 2, which corresponds to the variance of the lattice current. We get it through the expansion of Eq. (50) up to order 2:

$$\mu^{(1)} = T_i^{(0)} - T_{i+1}^{(0)}, \tag{56}$$

$$\mu^{(2)} = 2\big(T_i^{(1)} - T_{i+1}^{(1)}\big) + T_i^{(0)} + T_{i+1}^{(0)} - 2\,U_{i,i+1}^{(0)}. \tag{57}$$

Equation (56) just reproduces Eq. (44) between the mean values of the lattice current and of the density.

To get $T_j^{(1)}$, one considers Eqs. (51)–(55) at first order in s. They only involve $T_j^{(1)}$, $T_j^{(0)}$, and $U_{jk}^{(0)}$, and can be solved

recursively in $T_j^{(1)}$. We get

$$\begin{pmatrix} T_{i+1}^{(1)} \\ T_i^{(1)} \end{pmatrix} = \frac{\phi^L}{1-ab\phi^{2L-2}} \begin{pmatrix} \frac{b\phi^{L-i-1}+\phi^{i+1-L}}{\phi^2-1} & \frac{a\phi^i+\phi^{-i}}{\phi^2-1} \\ \frac{b\phi^{L-i}+\phi^{i-L}}{\phi^2-1} & \frac{a\phi^{i-1}+\phi^{1-i}}{\phi^2-1} \end{pmatrix} \times \begin{pmatrix} \sum_{l=0}^{i-1}(a\phi^l+\phi^{-l})I_{l+1} \\ \sum_{l=0}^{L-i-1}(b\phi^l+\phi^{-l})I_{L-l} \end{pmatrix},$$

with

$$I_j = \frac{(\phi+1)^2}{4\phi} \times \begin{cases} \mu^{(1)}\, T_i^{(0)} + T_{i+1}^{(0)} - U_{i,i+1}^{(0)}, & \text{for } j=i, \\ \mu^{(1)}\, T_{i+1}^{(0)} - T_i^{(0)} + U_{i,i+1}^{(0)}, & \text{for } j=i+1, \\ \mu^{(1)}\, T_j^{(0)} + U_{j,i+1}^{(0)} - U_{j,i}^{(0)}, & \text{otherwise.} \end{cases}$$

Plugging these values into Eq. (57), we get the analytical expression of the variance of the lattice current:

$$\mu^{(2)} = T_i^{(0)} + T_{i+1}^{(0)} - 2\,U_{i,i+1}^{(0)} + \frac{2}{1+\phi} \times \left\{ \frac{\phi^i(b\phi^{2L-2i-1}-1)}{1-ab\phi^{2L-2}} \sum_{\ell=0}^{i-1}(a\phi^\ell+\phi^{-\ell})I_{\ell+1} - \frac{\phi^{L-i}(a\phi^{2i-1}-1)}{1-ab\phi^{2L-2}} \sum_{\ell=0}^{L-i-1}(b\phi^\ell+\phi^{-\ell})I_{L-\ell} \right\}. \quad (58)$$

Using the explicit form of I_j, one can compute the sums in Eq. (58) to perform the thermodynamic limit for $\mu^{(2)}$; see Sec. V. Let us conclude this subsection by mentioning that the higher cumulants may be computed in principle by similar methods. However, the computations become much harder and are beyond the scope of this paper.

D. Comparison with SSEP

As mentioned previously the DiSSEP is a deformation of the SSEP that can be easily recovered when taking $\lambda = 0$. This limit already reveals at the level of the matrix product ansatz algebra: the commutation relations between E, D, and H Eq. (17) become simpler when $\lambda = 0$. We have indeed $[H,E] = [H,D] = 0$. Hence, H can be chosen equal to the identity. In this case we recover the well-known relation $[D,E] = D + E$, relevant in the construction of the steady state of the SSEP. Remark that the generators G_1, G_2, and G_3 in contrast "diverge" when taking the limit $\lambda \to 0$ [this can be seen by inverting the change of basis (21)].

We can also take the limit $\lambda \to 0$ at the level of the physical observables. For the one- and two-points correlation function we get

$$\lim_{\lambda\to 0} \langle n_i \rangle = \frac{\rho_a(L+B-i) + \rho_b(i-1+A)}{L+A+B-1},$$

with

$$A = \frac{1}{\alpha+\gamma}, \quad B = \frac{1}{\beta+\delta},$$

and

$$\lim_{\lambda\to 0} \langle n_i n_j \rangle_c = -\frac{(i-1+A)(B+L-j)(\rho_a-\rho_b)^2}{(L+A+B-1)^2(L+A+B-2)},$$

which are in agreement with the known expressions for the SSEP [48,49]. For the lattice and evaporation-condensation currents we get

$$\lim_{\lambda\to 0} \left\langle J_{i\to i+1}^{\text{lat}} \right\rangle = \frac{\rho_a-\rho_b}{L+A+B-1} \quad \text{and} \quad \lim_{\lambda\to 0} \left\langle J_{i,i+1}^{\text{cond}} \right\rangle = 0,$$

also in agreement with the SSEP results.

We can also take carefully the limit of Eq. (58) to recover the variance of the lattice current for the SSEP model [48]:

$$\mu_{2,\text{SSEP}} = \frac{\rho_a+\rho_b}{L+A+B-1} - \frac{2\left(\rho_a^2+\rho_a\rho_b+\rho_b^2\right)}{3(L+A+B-2)} + \frac{\rho_a^2+\rho_b^2}{(L+A+B-1)(L+A+B-2)} + \frac{(A-3A^2+2A^3+B-3B^2+2B^3)(\rho_a-\rho_b)^2}{3(L+A+B-1)^3(L+A+B-2)} - \frac{(\rho_a-\rho_b)^2}{3(L+A+B-1)^2(L+A+B-2)}.$$

V. THERMODYNAMIC LIMIT

In this section, we study the thermodynamic limit of the previous model when there exists a competition between the diffusion of particles and the evaporation and condensation of pairs.

A. Scaling of the parameters

In order to maintain the competition in the continuous limit, we have to scale properly the parameters of the model. In other words, the mean time for a particle to go through the lattice by diffusion must be comparable to the time for it to be evaporated.

Let us write the time evolution of the one point correlation function for $1 < i < L$:

$$\frac{d\langle n_i\rangle}{dt} = \langle n_{i-1}(1-n_i)\rangle + \langle (1-n_i)n_{i+1}\rangle - \langle (1-n_{i-1})n_i\rangle - \langle n_i(1-n_{i+1})\rangle + \lambda^2(\langle (1-n_{i-1})(1-n_i)\rangle + \langle (1-n_i)(1-n_{i+1})\rangle - \langle n_{i-1}n_i\rangle - \langle n_i n_{i+1}\rangle) \quad (59)$$

$$= \langle n_{i-1}\rangle + \langle n_{i+1}\rangle - 2\langle n_i\rangle + \lambda^2(2 - \langle n_{i-1}\rangle - \langle n_{i+1}\rangle - 2\langle n_i\rangle). \quad (60)$$

Note that although the two-point correlation functions cancel when going from Eqs. (59) to (60), the mean-field approximation is not exact in the sense that the connected two-point function does not vanish; see Eq. (37).

Remark: The one-point correlation function verifies a closed set of equation as for the SSEP, in contrast with the ASEP case where the equations couple the one-point function and the two-points function. This property remains valid for the higher-order correlation functions Eq. (39), which allows in principle to compute them. However, for the multipoints correlation functions, solving this set of equation can be very

hard. This points out the usefulness of the matrix product ansatz, which makes the calculations much easier.

We want to take the large L limit in Eq. (60). We set $i = Lx$ with $x \in [0\,,1]$ and $\langle n_i \rangle = \rho(x)$. We get

$$\frac{d\rho}{dt}(x) = \frac{1}{L^2}[\rho''(x) + o(1)] + 2\lambda^2[1 - 2\rho(x) + o(1)]. \quad (61)$$

We see on the previous equation that we have to take $\lambda = \lambda_0/L$ in order to have a balance between diffusion and creation-annihilation. After a rescaling of the time $t \to L^2 t$, we obtain

$$\frac{\partial\rho}{\partial t} = \frac{\partial^2\rho}{\partial x^2} + 2\lambda_0^2(1 - 2\rho), \quad (62)$$

with the boundary conditions $\rho(0) = \rho_a$ and $\rho(1) = \rho_b$. This equation shows that the correlation length for this scaling is finite. Indeed, the stationary density obtained in Eq. (65) decays as $\overline{\rho}(x) - 1/2 \sim \exp(-2\lambda_0 x)$ for x far from the boundaries. The correlation length can thus be defined as $1/(2\lambda_0)$.

Without rescaling λ and the time with respect to L (or for $\lambda = \lambda_0/L^\mu$ with $\mu < 1$ and rescaling the time $t \to L^{2\mu}t$), the diffusive term drops out and the density satisfies

$$\frac{\partial\rho}{\partial t} = 2\lambda^2(1 - 2\rho). \quad (63)$$

In the case where $\lambda = \lambda_0/L^\mu$ for $\mu > 1$ with the rescaling of the time $t \to L^2 t$, the system becomes a pure diffusive model for large L and one gets for the density

$$\frac{\partial\rho}{\partial t} = \frac{\partial^2\rho}{\partial x^2}. \quad (64)$$

B. Thermodynamic limit of the observables

From Eq. (34), we can compute the expression of the one-point correlation function in the continuous limit ($i = Lx$ and $\lambda = \lambda_0/L$),

$$\overline{\rho}(x) := \lim_{L\to\infty} \langle n_{Lx} \rangle = \frac{1}{2} + \frac{1}{2\sinh 2\lambda_0}(q_1 e^{-2\lambda_0(x-1/2)} + q_2 e^{2\lambda_0(x-1/2)}), \quad (65)$$

where

$$q_1 = (\rho_a + \rho_b - 1)\sinh(\lambda_0) - (\rho_b - \rho_a)\cosh(\lambda_0), \quad (66)$$

and

$$q_2 = (\rho_a + \rho_b - 1)\sinh(\lambda_0) + (\rho_b - \rho_a)\cosh(\lambda_0). \quad (67)$$

It is easy to check that it satisfies the stationary version of Eq. (62).

We can also compute the two-point correlation function in this limit. One can see that it scales as $\frac{1}{L}$, i.e., it has weak correlations. We get

$$c_2(x,y) := \lim_{L\to\infty} L \times \langle n_{Lx} n_{Ly} \rangle_c = \frac{2\lambda_0\, q_1 q_2}{(\sinh 2\lambda_0)^3} \sinh 2\lambda_0(1 - y)\sinh 2\lambda_0 x. \quad (68)$$

For $\lambda_0 \ll 1$, this two-point correlation function behaves algebraically with respect to x and y, whereas it behaves exponentially and is short range for $\lambda_0 \gg 1$.

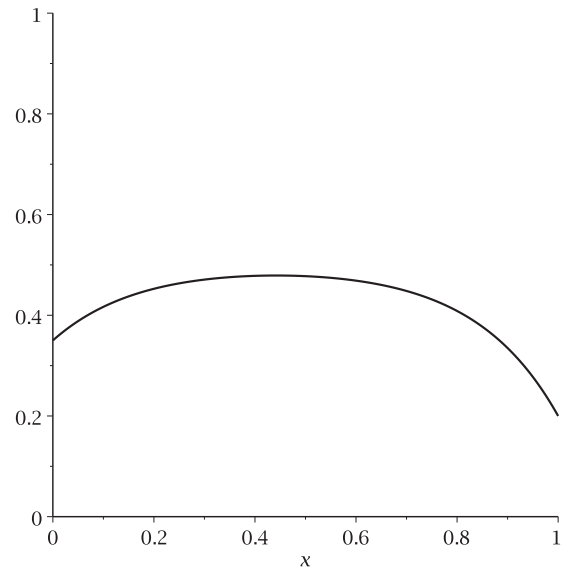

FIG. 3. Density for $\rho_a = 0.35$, $\rho_b = 0.2$, and $\lambda_0 = 3$.

The limit of the particle currents are given by

$$j^{\text{lat}}(x) := \lim_{L\to\infty} L \times \langle J^{\text{lat}}_{Lx\to Lx+1} \rangle = \frac{\lambda_0}{\sinh 2\lambda_0}(q_1 e^{-2\lambda_0(x-1/2)} - q_2 e^{2\lambda_0(x-1/2)}), \quad (69)$$

and

$$j^{\text{cond}}(x) := \lim_{L\to\infty} L^2 \times \langle J^{\text{cond}}_{Lx,Lx+1} \rangle = \frac{-\lambda_0^2}{\sinh 2\lambda_0}(q_1 e^{-2\lambda_0(x-1/2)} + q_2 e^{2\lambda_0(x-1/2)}). \quad (70)$$

Remark that these expressions are consistent with the fact that when the system reaches a thermodynamic equilibrium, that is for $\rho_a = \rho_b = 1/2$ (or equivalently $q_1 = q_2 = 0$), both currents vanish.

The particle conservation law Eq. (43) becomes in the thermodynamic limit

$$-\frac{dj^{\text{lat}}}{dx}(x) + 2j^{\text{cond}}(x) = 0,$$

which is satisfied by the expressions above. In the same way, relation Eqs. (44) and (45) become in the thermodynamic limit

$$\frac{d\overline{\rho}}{dx}(x) + j^{\text{lat}}(x) = 0\,, \quad j^{\text{cond}}(x) = \lambda_0^2[1 - 2\overline{\rho}(x)]. \quad (71)$$

1. *Behavior of the density and the currents*

Depending on the values of q_1 and q_2 defined in Eqs. (66) and (67), the behavior of the density may change:

(i) the density is not monotonic when $e^{-2\lambda_0} < \frac{q_1}{q_2} < e^{2\lambda_0}$, which implies that q_1 and q_2 have the same sign. In that case, it possesses an extremum at $\overline{x}$ satisfying $e^{4\lambda_0(\overline{x}-1/2)} = \frac{q_1}{q_2}$. The lattice current vanishes at this point.

(a) the density presents a maximum,

$$\overline{\rho}(\overline{x}) = \frac{1}{2} - \frac{\sqrt{q_1 q_2}}{\sinh(2\lambda_0)}, \quad (72)$$

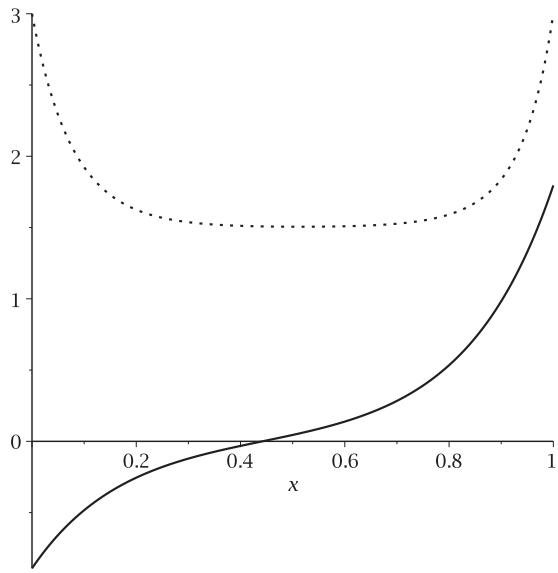

FIG. 4. Mean value (—) and variance (· · ·) of the lattice current for $\rho_a = 0.35$, $\rho_b = 0.2$, and $\lambda_0 = 3$.

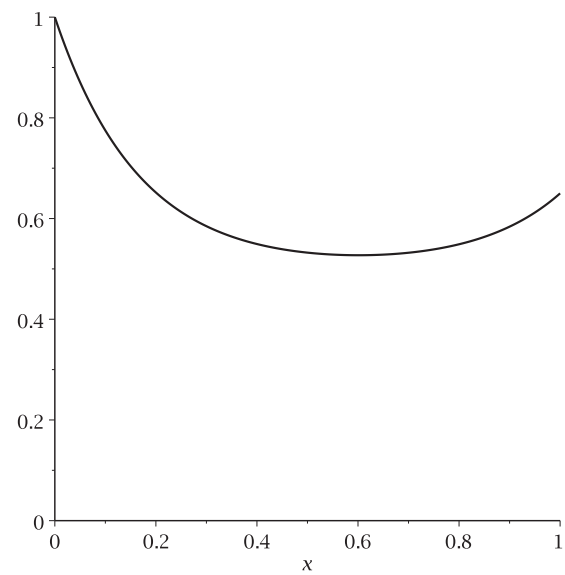

FIG. 5. Density for $\rho_a = 1$, $\rho_b = 0.65$, and $\lambda_0 = 3$.

when $q_1,q_2 < 0$. Let us remark that in this case, the density is everywhere smaller than $1/2$. Example of such behavior can be seen on Fig. 3.

The lattice current changes direction at the point $\overline{x}$, as expected since the lattice current goes from high density to low density. At this point, the condensation current is minimal but positive, since the density is smaller than $1/2$, so that condensation is promoted.

(b) It presents a minimum,

$$\overline{\rho}(\overline{x}) = \frac{1}{2} + \frac{\sqrt{q_1 q_2}}{\sinh(2\lambda_0)}, \tag{73}$$

when $q_1,q_2 > 0$. In this case, the density is everywhere greater than $1/2$.

The condensation current is negative but maximal, so that the evaporation is minimal. As previously, the lattice current changes sign at $\overline{x}$, still going from high density to low density. Example of such behavior can be seen on Fig. 5.

(ii) The density is monotonic from ρ_a to ρ_b when $\frac{q_1}{q_2} < e^{-2\lambda_0}$ or $\frac{q_1}{q_2} > e^{2\lambda_0}$. In this case, the lattice current never vanishes. Example of such behavior can be seen on Fig. 7.

The condensation current follows the same pattern, due to Eq. (71). The lattice current behaves as follows:

(i) it is not monotonic when $e^{-2\lambda_0} < -\frac{q_1}{q_2} < e^{2\lambda_0}$, which implies that q_1 and q_2 have opposite sign. There is an extremum at $\overline{x}$ satisfying $e^{4\lambda_0(\overline{x}-1/2)} = -\frac{q_1}{q_2}$. The condensation current vanishes at this point.

(a) When $q_1 < 0$, the lattice current presents a maximum,

$$j^{\text{lat}}(\overline{x}) = -\frac{2\lambda_0}{\sinh(2\lambda_0)}\sqrt{-q_1 q_2}. \tag{74}$$

(b) When $q_1 > 0$, it presents a minimum (see Fig. 8),

$$j^{\text{lat}}(\overline{x}) = \frac{2\lambda_0}{\sinh(2\lambda_0)}\sqrt{-q_1 q_2}. \tag{75}$$

(ii) The lattice current is monotonic when $-\frac{q_1}{q_2} < e^{-2\lambda_0}$ or $-\frac{q_1}{q_2} > e^{2\lambda_0}$; see Figs. 4 and 6.

2. *Variance of the lattice current*

The thermodynamic limit of the variance of the lattice current, computed exactly in Eq. (58) for any size, takes the form

$$\begin{aligned}\mu_2(x) = {} & 2q_1\, q_2\, \lambda_0^2 \left\{ (2x-1)\, \frac{\sinh[2\,\lambda_0\,(2\,x-1)]}{[\sinh(2\,\lambda_0)]^3} - \frac{\cosh(2\,\lambda_0)\,\cosh[2\,\lambda_0\,(2x-1)]+1}{[\sinh(2\,\lambda_0)]^4} \right\} \\ & - q_2^2 \lambda_0\, \frac{e^{4\,\lambda_0\,x} + e^{-4\,\lambda_0\,(1-x)} - e^{4\,\lambda_0\,(2\,x-1)} + 3}{4[\sinh(2\,\lambda_0)]^3} - q_1^2 \lambda_0\, \frac{e^{4\,\lambda_0\,(1-x)} + e^{-4\,\lambda_0\,x} - e^{4\,\lambda_0\,(1-2\,x)} + 3}{4[\sinh(2\,\lambda_0)]^3} \\ & + \frac{\lambda_0\,\cosh(2\,\lambda_0\,x)\cosh[2\,\lambda_0\,(1-x)]}{\sinh(2\,\lambda_0)}.\end{aligned} \tag{76}$$

As all physical quantities of the model, the variance is invariant under the transformation $q_1 \leftrightarrow q_2$ and $x \to 1-x$, which is the left-right symmetry. The particle-hole symmetry amounts to change $q_1 \to -q_1$ and $q_2 \to -q_2$: it leaves μ_2 invariant,

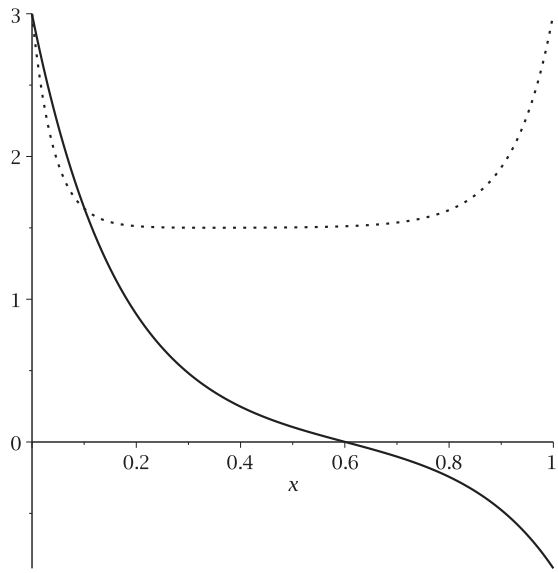

FIG. 6. Mean value (—) and variance (· · ·) of the lattice current for $\rho_a = 1$, $\rho_b = 0.65$, and $\lambda_0 = 3$.

transforms $\overline{\rho}(x)$ into $1 - \overline{\rho}(x)$, and changes the sign of the currents. The symmetry $\lambda \to -\lambda$ reads $\lambda_0 \to -\lambda_0$ and $q_1 \leftrightarrow (-q_2)$ and leaves all quantities invariant.

3. SSEP limit

Finally, by taking the limit $\lambda_0 \to 0$ in the previous quantities, we recover the well-known SSEP expressions [48]:

$$\begin{aligned}
&\overline{\rho}_{\mathrm{SSEP}}(x) = \rho_a(1-x) + \rho_b x, \quad j^{\mathrm{lat}}_{\mathrm{SSEP}}(x) = \rho_a - \rho_b,\\
&c_{2,\mathrm{SSEP}}(x,y) = -x(1-y)(\rho_a - \rho_b)^2,\\
&\mu_{2,\mathrm{SSEP}}(x) = \rho_a + \rho_b - \tfrac{2}{3}\left(\rho_a^2 + \rho_a\rho_b + \rho_b^2\right).
\end{aligned} \tag{77}$$

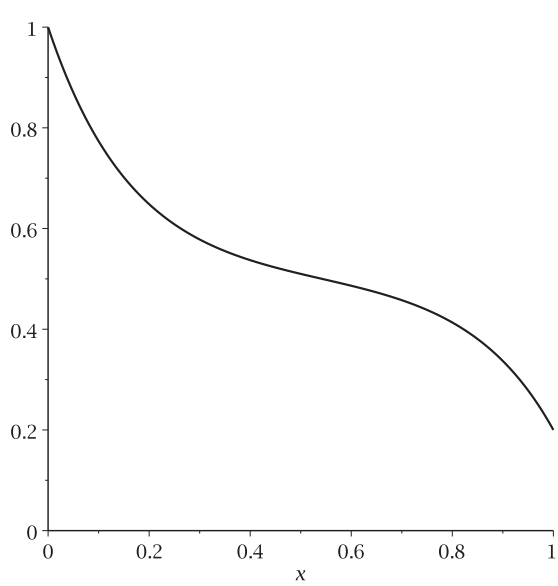

FIG. 7. Density for $\rho_a = 1$, $\rho_b = 0.2$, and $\lambda_0 = 3$.

VI. COMPARISON WITH MACROSCOPIC FLUCTUATION THEORY

A. Presentation of the macroscopic fluctuation theory

The model presented in this paper belongs to a larger class of models describing lattice gas with diffusive dynamics and evaporation-condensation in the bulk, which are driven out of equilibrium by two reservoirs at different densities. In the thermodynamic limit, the dynamics of these models can be understood (under proper assumptions) through the so-called macroscopic fluctuation theory (MFT). More precisely the large deviation functional for the density profile and the particle currents has been computed in Refs. [32,33] based on the pioneering works for diffusive models, Refs. [34–36]. The aim of this section is to extract from this general framework the local variance of the current on the lattice for the DiSSEP model and check its exact agreement with the value of $\mu_2(x)$ computed previously; see Eq. (76) from the microscopic point of view.

Let us start by briefly presenting the key ingredients of the MFT related to our model. A detailed presentation can be found in Refs. [32,33]. It has been shown that the microscopic behavior of the system can be averaged in the thermodynamic limit and can be described at the macroscopic level by a small number of relevant parameters: $D(\rho)$, $\sigma(\rho)$, $A(\rho)$, and $C(\rho)$. These parameters depend on the microscopic dynamics of the model and have to be computed for each different model. The two first are related to the diffusive dynamics on the lattice: $D(\rho)$ is the diffusion coefficient and $\sigma(\rho)$ is the conductivity. For the DiSSEP, the diffusive dynamics is the same as for the SSEP and hence these coefficients take the values $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$. The two other parameters $A(\rho)$ and $C(\rho)$ are related to the creation-annihilation dynamics. $A(\rho)$ can be understood intuitively as the mean number of particles annihilated per site and per unit of time when the density profile is identically flat and equal to ρ in the system, whereas $C(\rho)$ stands for the mean number of particles created. A rigorous definition of these parameters can be found in Ref. [32]. For the DiSSEP, we have $A(\rho) = 2\lambda_0^2\rho^2$ and $C(\rho) = 2\lambda_0^2(1-\rho)^2$.

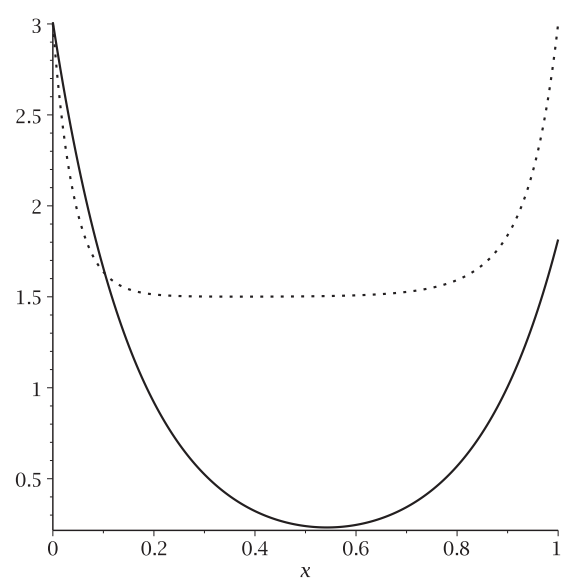

FIG. 8. Mean value (—) and variance (· · ·) of the lattice current for $\rho_a = 1$, $\rho_b = 0.2$, and $\lambda_0 = 3$.

When the number of sites L goes to infinity, the probability of observing a given history of the density profile ρ, of the lattice current $\dot{Q}$, and of the creation-annihilation current $\dot{K}$ during the time interval $[0,T]$,[1] can by written as

$$\mathbb{P}_{[0,T]}(\{\rho,\dot{Q},\dot{K}\}) \sim \exp[-L\mathcal{I}_{[0,T]}(\rho,\dot{Q},\dot{K})], \tag{78}$$

with the large deviation functional

$$\mathcal{I}_{[0,T]}(\rho,\dot{Q},\dot{K}) = \int_0^T dt \int_0^1 dx \left\{ \frac{(\dot{Q}(x,t)+D(\rho(x,t))\partial_x\rho(x,t))^2}{2\sigma(\rho(x,t))} + \Phi\big(\rho(x,t),\dot{K}(x,t)\big)\right\},$$

where

$$\Phi(\rho,\dot{K}) = \frac{1}{2}\left[A(\rho)+C(\rho)-\sqrt{\dot{K}^2+4A(\rho)C(\rho)} + \dot{K}\ln\left(\frac{\sqrt{\dot{K}^2+4A(\rho)C(\rho)}+\dot{K}}{2C(\rho)}\right)\right].$$

The factor $1/2$ in the definition of Φ is a slight modification in comparison to Ref. [32] due to the fact that we consider here creation-annihilation of pairs of particles instead of creation-annihilation of single particles.

The quantities ρ, $\dot{Q}$, and $\dot{K}$ are related through the conservation equation

$$\partial_t\rho(x,t) = -\partial_x\dot{Q}(x,t)+\dot{K}(x,t), \tag{79}$$

and the value of ρ is fixed on the boundaries $\rho(0,t)=\rho_a$, $\rho(1,t)=\rho_b$. The minimum of the large deviation functional $\mathcal{I}_{[0,T]}$ is achieved when the particle currents take their typical values, that is $\dot{Q}(x,t) = -D(\rho(x,t))\partial_x\rho(x,t)$ and $\dot{K}(x,t) = C(\rho(x,t))-A(\rho(x,t))$. The typical evolution of the density profile is hence given by

$$\partial_t\rho(x,t) = \partial_x(D(\rho(x,t))\partial_x\rho(x,t)) + C(\rho(x,t))-A(\rho(x,t)),$$

which matches Eq. (62) for the DiSSEP.

B. Computation of the variance of the lattice current

Using the previous formalism and following Ref. [32], it is possible to compute the local variance of the lattice current $\dot{Q}$ in the stationary regime. Due to the fact that the dynamics of the model does not conserve the number of particle, this current and its fluctuations depend on the position in the system. Hence, given a function $\tau(x)$, we want to compute the generating function of the cumulants of the integrated current $\int_0^T dt\int_0^1 dx\ \tau(x)\ \dot{Q}(x,t)$ for T going to infinity:

$$\mathcal{F}(\{\tau\}) = \lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{LT} \times \ln\mathbb{E}_{[0,T]}\left[\exp\left(\int_0^T dt\int_0^1 dx\tau(x)\dot{Q}(x,t)\right)\right].$$

[1]We keep here all the notations used in Ref. [32]. The link with the quantities previously computed is given by the fact that in the stationary state the mean value of $\dot{Q}(x)$ is $j^{\rm lat}(x)$ and the mean value of $\dot{K}(x)$ is $2j^{\rm cond}(x)$.

The previous expression can be simplified using Eq. (78) and a saddle point method. It reduces to maximize a functional over the time dependent fields ρ, $\dot{Q}$, and $\dot{K}$. Assuming that the extrema of this functional is achieved for time independent profiles, we end up with the following expression (the reader is invited to refer to Ref. [32] for the details):

$$\mathcal{F}(\{\tau\}) = \sup_{\rho,\dot{Q}}\left(\int_0^1 dx\tau(x)\dot{Q}(x) - \hat{\mathcal{I}}(\rho,\dot{Q})\right), \tag{80}$$

with

$$\hat{\mathcal{I}}(\rho,\dot{Q}) = \int_0^1 dx\left\{\frac{(\dot{Q}(x)+D[\rho(x)]\partial_x[\rho(x)]^2}{2\sigma[\rho(x)]} + \Phi\big(\rho(x),\partial_x\dot{Q}(x)\big)\right\}. \tag{81}$$

To compute the local variance of the lattice current $\dot{Q}$ at the point y, it is enough to take $\tau(x)=\delta(x-y)$ and expand $\mathcal{F}(\{\epsilon\tau\})$ up to order ϵ^2. For a small perturbation ϵ, the fields are expected to be close to their typical value,

$$\rho(x) = \overline{\rho}(x)+\epsilon\frac{f(x)}{D[\overline{\rho}(x)]},$$
$$\dot{Q}(x) = -D[\overline{\rho}(x)]\partial_x\overline{\rho}(x)+\epsilon q(x),$$

with the constraint $f(0)=f(1)=0$ due to the boundaries. We then obtain

$$\mathcal{F}(\{\epsilon\tau\}) = -\epsilon D[\overline{\rho}(y)]\partial_x\overline{\rho}(y)+\frac{\epsilon^2}{2}\mu_2(y), \tag{82}$$

with the variance of the lattice current at the point y,

$$\mu_2(y) = 2\sup_{q,f}\left[q(y)-\int_0^1 dx\left(\frac{[q(x)+f'(x)]^2}{2\sigma[\overline{\rho}(x)]} + \frac{[q'(x)+U(x)f(x)]^2}{4\{A[\overline{\rho}(x)]+C[\overline{\rho}(x)]\}}\right)\right] \tag{83}$$

and $U(x) = \frac{A'[\overline{\rho}(x)]-C'[\overline{\rho}(x)]}{D[\overline{\rho}(x)]} = 4\lambda_0^2$. We make the following change of variables to solve this optimization problem:

$$\varphi(x) = \frac{q(x)+f'(x)}{\sigma[\overline{\rho}(x)]},$$
$$\psi(x) = \frac{q'(x)+U(x)f(x)}{2\{A[\overline{\rho}(x)]+C[\overline{\rho}(x)]\}}, \tag{84}$$

so that the Euler-Lagrange equations become for the DiSSEP,

$$\psi'(x) = \varphi(x)-\delta(x-y),$$
$$\varphi'(x) = 4\lambda_0^2\psi(x). \tag{85}$$

Note that there are slight modifications in Eqs. (82) and (84) with respect to Ref. [32], in accordance with the modification of Φ [see discussion after Eq. (79)].

These equations can be solved analytically, and we get

$$\psi(x) = \theta(x\leqslant y)\frac{\sinh(2\lambda_0x)\cosh[2\lambda_0(1-y)]}{\sinh 2\lambda_0} + \theta(x>y)\frac{\sinh[2\lambda_0(x-1)]\cosh(2\lambda_0y)}{\sinh 2\lambda_0},$$

$$\varphi(x) = \theta(x \leqslant y)\,\frac{2\lambda_0\cosh(2\lambda_0 x)\cosh[2\lambda_0(1-y)]}{\sinh 2\lambda_0} + \theta(x > y)\frac{2\lambda_0\cosh[2\lambda_0(x-1)]\cosh(2\lambda_0 y)}{\sinh 2\lambda_0}.$$

The function $q(x)$ can be also computed analytically by solving

$$q''(x) - 4\lambda_0^2 q(x) = \partial_x(2\{A[\overline{\rho}(x)] + C[\overline{\rho}(x)]\}\psi(x)) - 4\lambda_0^2\sigma[\overline{\rho}(x)]\varphi(x).$$

Note that it depends on y; see, for instance, the above expressions of $\psi(x)$ and $\phi(x)$. It allows us to deduce the expression of $q(x)$ at the special point y [as needed in Eq. (82)],

$$q(y) = \int_0^1 dx(\sigma[\overline{\rho}(x)]\varphi(x)^2 + 2\{A[\overline{\rho}(x)] + C[\overline{\rho}(x)]\}\psi(x)^2),$$

with φ and ψ are given above. Hence, for the DiSSEP, the variance of the current lattice computed from MFT is

$$\begin{aligned}\mu_2(y) &= \int_0^1 dx(\sigma[\overline{\rho}(x)]\varphi(x)^2 + 2\{A[\overline{\rho}(x)] + C[\overline{\rho}(x)]\}\psi(x)^2)\\ &= \frac{4\lambda_0^2\cosh^2 2\lambda_0(1-y)}{\sinh^2 2\lambda_0}\int_0^1 dx\{\sigma[\overline{\rho}(x)] + \sinh^2 2\lambda_0 x\}\\ &\quad + \frac{4\lambda_0^2\cosh^2 2\lambda_0 y}{\sinh^2 2\lambda_0}\int_0^1 dx\{\sigma[\overline{\rho}(x)] + \sinh^2 2\lambda_0(1-x)\}.\end{aligned}$$

Using the explicit form for σ, we show that this result obtained from MFT matches perfectly the previous result, Eq. (76), computed exactly from a microscopic description of the model.

Let us remark that a similar comparison between MFT for diffusive model [34–36] and microscopic exact computations has been performed in Ref. [37] for the SSEP. The result obtained here is a confirmation of the MFT developed in Refs. [32,33] for a system with diffusion and dissipation.

VII. SPECTRUM OF THE MARKOV MATRIX

A. Link with the *XXZ* spin chain, integrability, and Bethe equations

The model introduced above (DiSSEP) possesses the distinctive feature of being integrable, i.e., the Markov matrix M governing the process belongs to a set of commuting operators. Let us recall briefly the main objects to get this set. The detailed construction for this particular model can be found in Ref. [28]. This set is constructed through a generating operator depending on a spectral parameter, the so-called transfer matrix $t(x)$. The building blocks of this transfer matrix are the R matrix, obeying the Yang-Baxter equation, and the boundary matrices K and $\widetilde{K}$ satisfying, respectively, the reflection equation and the dual reflection equation. These equations ensure the commutation of the transfer matrix for different values of the spectral parameter as it was shown in Ref. [50]: $[t(x),t(y)] = 0$. The Markov matrix is then obtained as the first moment of the transfer matrix: $M \propto t'(1)$.

The integrability of this model is also revealed through its unexpected connexion with the XXZ model. To be more precise, let us introduce the following Hamiltonian H:

$$\begin{aligned}H &= (\alpha-\gamma)\sigma_1^+ - \frac{\alpha+\gamma}{2}(\sigma_1^z+1) - \frac{\lambda^2-1}{2}\\ &\quad\times\sum_{k=1}^{L-1}\left[\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y - \frac{\lambda^2+1}{\lambda^2-1}(\sigma_k^z\sigma_{k+1}^z - 1)\right]\\ &\quad + (\delta-\beta)\sigma_L^+ - \frac{\delta+\beta}{2}(\sigma_L^z+1),\end{aligned} \tag{86}$$

where $\sigma^{x,y,z,+,-}$ are the Pauli matrices. It corresponds to the open XXZ spin chain with upper triangular boundaries. This Hamiltonian H is conjugated to the Markov matrix M defined in Eq. (5). Namely, one has

$$H = Q_1 Q_2 \ldots Q_L M Q_1^{-1} Q_2^{-1} \ldots Q_L^{-1}, \tag{87}$$

where $Q = \left(\begin{smallmatrix}-1 & 1\\ 1 & 1\end{smallmatrix}\right)$. Let us also mention that the XXZ Model for particular choices of boundaries is conjugated to the Markov matrix of the open ASEP. However, for the boundaries present in Eq. (86), the conjugation provides non-Markovian boundaries.

We deduce from Eq. (87) that the spectrum of M is identical to the one of H. Moreover, the eigenvalues (but not the eigenvectors) of XXZ spin chain with upper triangular boundaries are the same that the ones for diagonal boundaries and one can use the results of Refs. [50–52]. Let us mention that equality between the spectrums of two different models has been used previously to study models with only evaporation [24]. Note also that for $\lambda^2 = 1$, the bulk Hamiltonian becomes diagonal, and the full Hamiltonian triangular, allowing to get its spectrum easily without Bethe ansatz, in accordance with the results of Sec. III.

The eigenvalues of H with diagonal boundaries can be parametrized in two different ways depending on the choice of the pseudovacuum:

(i) For the pseudovacuum with all the spins up and in the notations of the present paper, the eigenvalues of H are given by

$$E = -\alpha-\beta-\gamma-\delta + 4(\phi-1)^2\sum_{i=1}^{N}\frac{u_i}{(u_i-\phi^2)(u_i-1)}, \tag{88}$$

where $N = 0,1,\ldots,L$ and u_i are the Bethe roots. The Bethe roots must satisfy the following Bethe equations:

$$\frac{u_i+a\phi^2}{\phi(au_i+1)}\,\frac{u_i+b\phi^2}{\phi(bu_i+1)}\left[\frac{\phi(u_i-1)}{u_i-\phi^2}\right]^{2L} = \prod_{\substack{j=1\\ j\neq i}}^{N}\frac{\phi^2(\phi^2u_i-u_j)(u_iu_j-1)}{(u_i-\phi^2u_j)(u_iu_j-\phi^4)}, \tag{89}$$

where $i = 1,2,\ldots,N$ and a and b are defined in Eq. (24).

(ii) For the pseudovacuum with all the spins down, the eigenvalues of H are given by

$$E = 4(\phi-1)^2\sum_{i=1}^{N}\frac{v_i}{(v_i-\phi^2)(v_i-1)}, \tag{90}$$

where v_i satisfy the following Bethe equations:

$$\frac{av_i+\phi^2}{\phi(v_i+a)}\,\frac{bv_i+\phi^2}{\phi(v_i+b)}\left[\frac{\phi(v_i-1)}{v_i-\phi^2}\right]^{2L} = \prod_{\substack{j=1\\ j\neq i}}^{N}\frac{\phi^2(\phi^2v_i-v_j)(v_iv_j-1)}{(v_i-\phi^2v_j)(v_iv_j-\phi^4)}. \tag{91}$$

Let us stress again that although the spectrum of the XXZ spin chain is the same for diagonal or upper boundaries, the eigenvectors are different. For the XXZ spin chains with upper triangular boundaries, the eigenvectors associated to the parametrization Eqs. (88) and (89) of the eigenvalues were computed only recently by algebraic Bethe ansatz in Refs. [38,53] based on the previous results for the XXX spin chain [54–56]. The computation of the eigenvectors associated to the parametrization Eqs. (90) and (91) is still an open problem.

B. Computation of the spectral gap

In this section, we want to study the dynamical properties of the model: using the previous Bethe equations, we study the approach to the stationary state at large times for a large system. We must compute the eigenvalue, denoted by G, for the first excited state (i.e., the one with the greatest nonvanishing eigenvalue).

We start by presenting the main results for the gap then we give the sketch of the numerical evidences for them.

(i) In the case when evaporation rate λ is independent of the size of the system L, there is a nonvanishing gap. The values of this gap depends on the boundaries parameters and on λ.

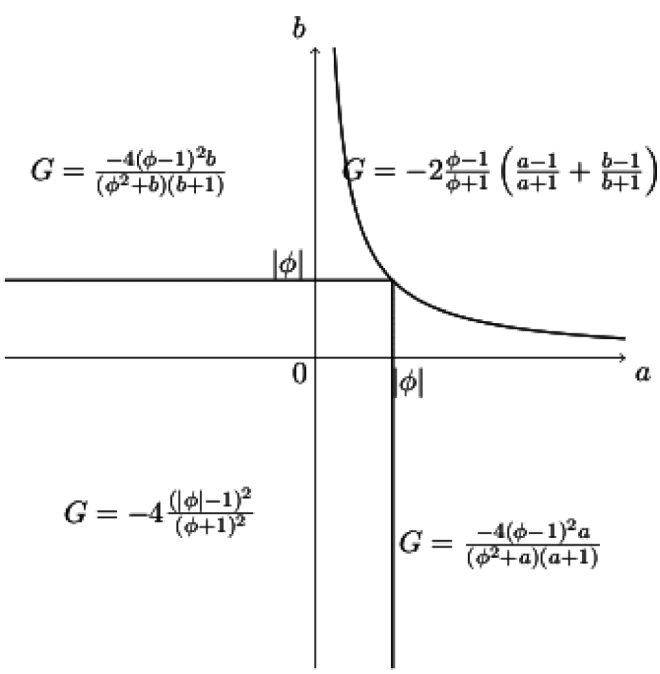

FIG. 9. Value of the gap G depending on the parameters a, b, and ϕ. The equation of the curve is $b=\phi^2/a$. [This particular figure is drawn for $\phi=1/4$ ($\lambda=0.6$), even though similar one is valid for any ϕ.]

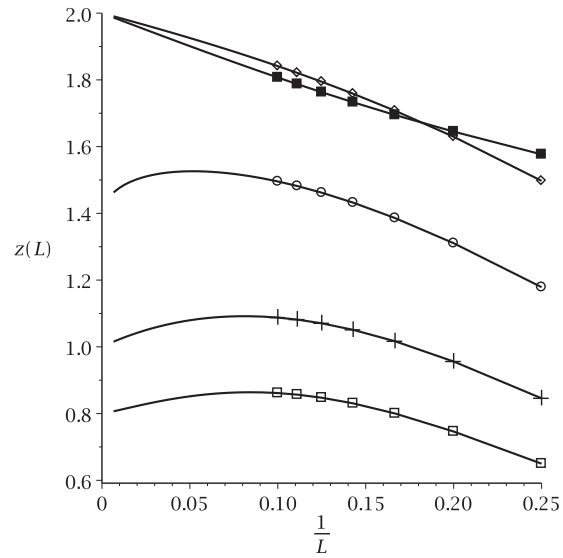

FIG. 10. Behavior of the gap in the thermodynamic limit when λ behaves as $\frac{1}{L^\mu}$: plot of $z(L)=\frac{\ln(G_L/G_{L-1})}{\ln((L-1)/L)}$ as a function of $\frac{1}{L}$. The lines represent the values obtained from Bethe ansatz for $4\leqslant L\leqslant 150$, while the dots correspond to direct diagonalization of H for $4\leqslant L\leqslant 10$ (■, $\mu=2$; ◇, $\mu=1$; ○, $\mu=0.7$; +, $\mu=0.5$; □, $\mu=0.4$).

We present these different values of the gaps on Fig. 9. They are consistent with the analytical result obtained for $\lambda=1$ (see Sec. III).

(ii) If the rate λ behaves as $\frac{1}{L^\mu}$ for large system, the model is gapless and we get

$$G\sim\frac{1}{L^{2\mu}}\quad\text{for}\quad 0<\mu<1, \tag{92}$$

and

$$G\sim\frac{1}{L^2}\quad\text{for}\quad \mu\geqslant 1. \tag{93}$$

We show in Fig. 10 numerical evidence for such a behavior. We plot $z(L)=\frac{\ln(G_L)-\ln(G_{L-1})}{\ln(L-1)-\ln(L)}$ as a function of $\frac{1}{L}$: as $\frac{1}{L}$ tends to 0, it tends to 2μ (respectively, 2) for $\mu<1$ (respectively, $\mu\geqslant 1$). The $\frac{1}{L^2}$ behavior of the gap for $\mu>1$ is expected since the system becomes in this case a diffusive model in the thermodynamic limit as discussed in Sec. V A.

To prove these results, we must study in detail the Bethe Eqs. (89) and (91). The comparison of the eigenvalues obtained by the exact diagonalization of M or by the numerical resolutions of the Bethe equations for small system (up to 10 sites), show that the gap is obtained for $N=1$ in Eqs. (90) and (91) or is equal to $G=-\alpha-\beta-\gamma-\delta=-2\frac{\phi-1}{\phi+1}(\frac{a-1}{a+1}+\frac{b-1}{b+1})$ [which corresponds to $N=0$ in Eqs. (88) and (89)]. We assume that this behavior holds for any L then we must solve only Eq. (89) for $N=1$. This Bethe equation can be written as the vanishing of a polynomial of degree $2L+2$ with respect to v_1. This polynomial has two obvious roots ϕ and $-\phi$, which are not physical since they corresponds to a vanishing "eigenvector." The remaining factor is a polynomial of degree $2L$ with respect to v_1, which can be transformed, thanks to Eq. (90) (and up to a normalization), to a polynomial of degree L with respect to E. Then, the Bethe Eq. (91) for

$N = 1$ becomes

$$\sum_{p=0}^{L} \frac{(1+\phi)^{2p}\, E^p}{4^p} \sum_{q=0}^{L-p} \phi^{2q} \Bigg[ab \binom{p+q}{q} \binom{L-q-2}{p-2} + (a+b) \binom{p+q-1}{q} \binom{L-q-1}{p-1} + \binom{p+q-2}{q} \binom{L-q}{p} \Bigg] = 0. \tag{94}$$

The left-hand side of the previous equation is a factor of the characteristic polynomial of the Hamiltonian H Eq. (86) or of the Markov matrix M Eq. (5). It is possible now to find numerically the roots of the polynomial (94) for large system (up to 150 sites) and pick up the largest ones. Performing this computation for different values of λ and of the boundary parameters, we obtain the results for the gap summarized previously; see Fig. 9.

VIII. CONCLUSION

The DiSSEP model (see Fig. 1) was recently introduced in Ref. [28]. It was shown that the model is integrable and that the probability distribution function describing the stationary state can be written using the matrix product Ansatz. In the present paper we discuss in detail the properties of the model. If one chooses the symmetric hopping rate equal to 1, the physics is dependent on the parameter λ whose square is the common rate for annihilation and creation of pair of particles.

It can be shown that the Hamiltonian (Markov matrix) has the same spectrum as a XXZ spin 1/2 quantum chain with nondiagonal boundary terms [see Eq. (86)]. One observes that if λ vanishes, one gets the ferromagnetic XXX model, which corresponds to the well-known SSEP model. A natural idea is to study the system in the weak dissipation limit $\lambda \sim 1/L^{\mu}$, where L is the size of the system. We have studied the effect of this Ansatz on the physics of the model. We remind the reader that the weak ASEP model [57] is defined in a similar way. The size dependence enters in the forward-backward hopping asymmetry

As a warmup exercise we have studied in detail the $\lambda = 1$ case. In the bulk, the Hamiltonian is diagonal in this case and the spectrum can easily be computed. The current large deviation function on the first bond has been derived. The function is convex and behaves like $j \ln(j)$ for large values of the current j.

The case where λ is arbitrary was considered next. Using the matrix product Ansatz, we have obtained Eq. (25) for the average value of an arbitrary monomial of the generators of the quadratic algebra Eq. (22) in a specific representation Eq. (23). This expression allows us to compute any correlator. We give the expressions for the average values of the local density Eq. (34), of the two-point and three-point densities correlators Eq. (37), respectively, Eq. (38). We also show [see Eq. (47)] that to derive the average density profile, the mean-field result is exact. There are two kinds of currents in our problem. The first one is given by particles crossing a bond between two sites, the second one is given by the pair of particles that leave or enter the system by the creation and annihilation processes. Both expressions are given [see Eqs. (41) and (42)]. The variance of the first current was also computed; the result can be found in Eq. (58).

In order to study the properties of the model in the large L limit, as mentioned before, we take λ vanishing like $L^{-\mu}$. Using Eqs. (34) and (37) one sees that the correlation length is proportional to $1/\lambda$, which suggests that the system is gapless for $\mu > 0$. Using the Bethe ansatz, we have shown that this is indeed the case. The energy gap behavior is given in Eqs. (92) and (93). We could be tempted to look closely at the value $\mu = 1/2$ when the gap vanishes like $1/L$, suggesting conformal invariance. This is not the case as one can see from the behavior of the average density and the two-point correlation function, which do not have the expected behavior [58]. We have decided to consider in detail the case $\mu = 1$, which corresponds to a critical dynamic exponent $z = 2$ corresponding to diffusive processes. The results are given in Sec. V.

We have also compared our results with those that can be obtained using the macroscopic fluctuation theory [32,33]. The variance of the current computed using this method coincides with the lattice calculation described earlier in the text.

Finally, we would like to point out two generalizations that look interesting to us. The first one is to consider asymmetric hopping rates. The system will probably be not integrable but mean-field, and Monte Carlo simulations will reveal new physics. The second generalization deals with the multispecies problem keeping integrability. This implies a generalization of the results obtained in Ref. [28].

ACKNOWLEDGMENTS

It is a pleasure to warmly thank L. Ciandrini and C. Finn for fruitful discussions and suggestions.

[1] S. Katz, J. L. Lebowitz, and H. Spohn, Nonequilibrium steady states of stochastic lattice gas models of fast ionic conductors, J. Stat. Phys. **34**, 497 (1984).

[2] B. Schmittmann and R. K. P. Zia, Statistical mechanics of driven diffusive systems, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, San Diego, 1995), Vol. 17.

[3] G. M. Schütz, Exactly solvable models for many-body systems far from equilibrium, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 2000), Vol. 19.

[4] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge, 2010).

[5] Y. Elskens and H. L. Frisch, Annihilation kinetics in the one-dimensional ideal gas, Phys. Rev. A **31**, 3812 (1985).

[6] Z. Rácz, Diffusion-Controlled Annihilation in the Presence of Particle Sources: Exact Results in one Dimension, Phys. Rev. Lett. **55**, 1707 (1985).

[7] H. Hinrichsen, V. Rittenberg, and H. Simon, Universality properties of the stationary states in the one-dimensional coagulation-diffusion model with external particle input, J. Stat. Phys. **86**, 1203 (1997).

[8] A. Parmeggiani, T. Franosch, and E. Frey, The totally asymmetric simple exclusion process with langmuir kinetics, Phys. Rev. E **70**, 046101 (2004).

[9] A. De Masi, P. Ferrari, and J. Lebowitz, Rigorous Derivation of Reaction-Diffusion Equations with Fluctuations, Phys. Rev. Let. **55**, 1947 (1985); Reaction-diffusion equations for interacting particle systems, J. Stat. Phys. **44**, 589 (1986).

[10] E. Bertin, An exactly solvable dissipative transport model, J. Phys. A **39**, 1539 (2006).

[11] D. Levanony and D. Levine, Correlation and response in a driven dissipative model, Phys. Rev. E **73**, 055102(R) (2006).

[12] Y. Shokef and D. Levine, Energy distribution and effective temperatures in a driven dissipative model, Phys. Rev. E **74**, 051111 (2006).

[13] J. Farago, Energy profile fluctuations in dissipative nonequilibrium stationary states, J. Stat. Phys. **118**, 373 (2005).

[14] J. Farago and E. Pitard, Injected power fluctuations in 1D dissipative systems, J. Stat. Phys. **128**, 1365 (2007).

[15] M. Barma, M. D. Grynberg, and R. B. Stinchcombe, Jamming and Kinetics of Deposition-Evaporation Systems and Associated Quantum Spin Models, Phys. Rev. Lett. **70**, 1033 (1993).

[16] C. R. Doering and D. ben-Avraham, Interparticle distribution functions and rate equations for diffusion-limited reactions, Phys. Rev. A **38**, 3035 (1988).

[17] M. D. Grynberg, T. J. Newman, and R. B. Stinchcombe, Exact solutions for stochastic adsorption-desorption models and catalytic surface processes, Phys. Rev. E **50**, 957 (1994).

[18] M. D. Grynberg and R. B. Stinchcombe, Dynamic Correlation Functions of Adsorption Stochastic Systems with Diffusional Relaxation, Phys. Rev. Lett. **74**, 1242 (1995).

[19] G. M. Schütz, Diffusion-annihilation in the presence of a driving field, J. Phys. A **28**, 3405 (1995).

[20] M. J. de Oliveira, Exact density profile of a stochastic reaction-diffusion process, Phys. Rev. E **60**, 2563 (1999).

[21] K. Sasaki and T. Nakagawa, Exact results for a diffusion-limited pair annihilation process on a one-dimensional lattice, J. Phys. Soc. Jpn. **69**, 1341 (2000).

[22] M. Mobilia and P.-A. Bares, Exact solution of a class of one-dimensional nonequilibrium stochastic models, Phys. Rev. E **63**, 056112 (2001).

[23] A. Ayyer and K. Mallick, Exact results for an asymmetric annihilation process with open boundaries, J. Phys. A **43**, 045003 (2010).

[24] F. C. Alcaraz, M. Droz, M. Henkel, and V. Rittenberg, Reaction-diffusion processes, critical dynamics and quantum chains, Ann. Phys. **230**, 250 (1994).

[25] H. Hinrichsen, K. Krebs, and I. Peschel, Solution of a one-dimensional diffusion-reaction model with spatial asymmetry, Z. Physik B **100**, 105 (1996).

[26] H. Hinrichsen, S. Sandow, and I. Peschel, On matrix product ground states for reaction-diffusion models, J. Phys. A **29**, 2643 (1996).

[27] J. E. Santos, G. M. Schütz, and R. B. Stinchcombe, Diffusion-annihilation dynamics in one spatial dimension, J. Chem. Phys. **105**, 2399 (1996).

[28] N. Crampe, E. Ragoucy, and M. Vanicat, Integrable approach to simple exclusion processes with boundaries. Review and progress, J. Stat. Mech. (2014), P11032.

[29] F. C. Alcaraz and V. Rittenberg, Reaction-diffusion processes as physical realizations of hecke algebras, Phys. Lett. B **314**, 377 (1993).

[30] M. J. Martins, B. Nienhuis, and R. Rietman, Intersecting Loop Model as a Solvable Super Spin Chain, Phys. Rev. Lett. **81**, 504 (1998).

[31] P. Pyatov (private communication).

[32] T. Bodineau and M. Lagouge, Current large deviations in a driven dissipative model, J. Stat. Phys. **139**, 201 (2010).

[33] T. Bodineau and M. Lagouge, Large deviations of the empirical currents for a boundary driven reaction diffusion model, Ann. Appl. Probab. **22**, 2282 (2012).

[34] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Fluctuations in Stationary Non Equilibrium States of Irreversible Processes, Phys. Rev. Lett. **87**, 040601 (2001).

[35] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory for stationary non-equilibrium states, J. Stat. Phys. **107**, 635 (2002).

[36] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory, Rev. Mod. Phys. **87**, 593 (2015).

[37] B. Derrida, Microscopic versus macroscopic approaches to non-equilibrium systems, J. Stat. Mech. (2011) P01030.

[38] S. Belliard, Modified algebraic Bethe ansatz for XXZ chain on the segment – I: Triangular cases, Nucl. Phys. B **892**, 1 (2015).

[39] J. Sato and K. Nishinari, Relaxation dynamics of the asymmetric simple exclusion process with Langmuir kinetics on a ring, Phys. Rev. E **93**, 042113 (2016).

[40] M. D. Donsker and S. R. S. Varadhan, Asymptotic evaluation of certain markov process expectations for large time I, Commun. Pure Appl. Math. **28**, 1 (1975); Asymptotic evaluation of certain markov process expectations for large time II, *ibid.* **28**, 279 (1975); Asymptotic evaluation of certain Markov process expectations for large time III, *ibid.* **29**, 389 (1976); Asymptotic evaluation of certain markov process expectations for large time IV, *ibid.* **36**, 183 (1983).

[41] B. Derrida, M. R. Evans, V. Hakim, and V. Pasquier, Exact solution of a 1d asymmetric exclusion model using a matrix formulation, J. Phys. A **26**, 1493 (1993).

[42] S. Prolhac, M. R. Evans, and K. Mallick, Matrix product solution of the multispecies partially asymmetric exclusion process, J. Phys. A **42**, 165004 (2009).

[43] M. R. Evans, P. A. Ferrari, and K. Mallick, Matrix Representation of the stationary measure for the multispecies TASEP, J. Stat. Phys. **135**, 217 (2009).

[44] N. Crampe, K. Mallick, E. Ragoucy, and M. Vanicat, Open two-species exclusion processes with integrable boundaries, J. Phys. A **48**, 175002 (2015).

[45] A. P. Isaev, P. N. Pyatov, and V. Rittenberg, Diffusion algebras, J. Phys. A **34**, 5815 (2001).

[46] R. A. Blythe and M. R. Evans, Nonequilibrium steady states of matrix-product form: A solver's guide, J. Phys. A **40**, R333 (2007).

[47] T. Sasamoto and M. Wadati, Stationary states of integrable systems in matrix product form, J. Phys. Soc. Jpn. **66**, 2618 (1997).

[48] B. Derrida, B. Doucot, and P.-E. Roche, Current fluctuations in the one dimensional Symmetric Exclusion Process with open boundaries, J. Stat. Phys. **115**, 717 (2004).
[49] B. Derrida, Non-equilibrium steady states: Fluctuations and large deviations of the density and of the current, J. Stat. Mech. (2007) P07023.
[50] E. K. Sklyanin, Boundary conditions for integrable quantum systems, J. Phys. A **21**, 2375 (1988).
[51] M. Gaudin, *La fonction d'onde de Bethe* (Masson, Paris, 1983).
[52] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter, and G. R. W. Quispel, Surface exponents of the quantum XXZ, Ashkin-Teller and Potts models, J. Phys. A **20**, 6397 (1987).
[53] R. A. Pimenta and A. Lima-Santos, Algebraic Bethe ansatz for the six vertex model with upper triangular K-matrices, J. Phys. A **46**, 455002 (2013).
[54] N. Crampe and E. Ragoucy, Generalized coordinate Bethe ansatz for nondiagonal boundaries, Nucl. Phys. B **858**, 502 (2012).
[55] S. Belliard, N. Crampe, and E. Ragoucy, Algebraic Bethe ansatz for open XXX model with triangular boundary matrices, Lett. Math. Phys. **103**, 493 (2013).
[56] S. Belliard and N. Crampe, Heisenberg XXX Model with general boundaries: eigenvectors from algebraic bethe ansatz, SIGMA **9**, 072 (2013).
[57] B. Derrida, C. Enaud, C. Landim, and S. Olla, Fluctuations in the weakly asymmetric exclusion process with open boundary conditions, J. Stat. Phys. **118**, 795 (2005).
[58] F. C. Alcaraz and V. Rittenberg, Correlation functions in conformal invariant stochastic processes, J. Stat. Mech. (2015) P11012.