Motion of dissipative optical fronts under the action of an oscillating pump

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The dynamics of domain walls in optical bistable systems with pump and loss is considered. It is shown that an oscillating component of the pump affects the average drift velocity of the domain walls. The cases of harmonic and biharmonic pumps are considered. It is demonstrated that in the case of biharmonic pulse the velocity of the domain wall can be controlled by the mutual phase of the harmonics. The analogy between this phenomenon and the ratchet effect is drawn. Synchronization of the moving domain walls by the oscillating pump in discrete systems is studied and discussed.

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I. INTRODUCTION

Stationary localized structures in nonlinear optical cavities have been actively studied in recent time because of their importance for various applications, for example, in optical information processing, and because of the fundamental interest motivated by rich dynamics of the considered systems [1].

A rich variety of nonlinear solitary structures are found in the systems of such a kind. Domain walls connecting two spatially uniform states are an important example of nonlinear solitary waves. They are found in the systems of different physical origin, in particular in optical systems; see Ref. [2]. The area of domain walls has been attracting much of the attention for a long time and is an actively developing area of research now; see Ref. [3] and references therein. The investigation of the domain walls is of interest because the dynamics of many optical systems is governed by the formation and evolution of the domain walls. For example, vector domain walls were experimentally discovered in lasers [4]. It was also found that active optical systems allow us to obtain the formation of dual-frequency domain walls (dark solitons) in fiber ring lasers [5]. This effect is of big academic interest and can possibly be used for the design of laser systems. Another interesting example of dissipative optical fronts is polarization domain walls [6]. The solitons of such a kind were investigated experimentally and theoretically in fiber lasers. It was shown that the dark-dark solitons form only if the dispersion of the fiber is normal, whereas bright-dark solitons were found for both normal and anomalous dispersions.

The motion of the domain walls can be seen as a mechanism of the switching between two spatially uniform states in bistable systems. The domain walls move extending the area of existence of an energetically favorable state. In its turn it depends on the pump which state is energetically favorable, and thus the direction of the domain wall motion can be controlled by the intensity of the pump. At the pump value called Maxwell point, the domain wall is at rest and so it is possible to say that at this pump the states connected by the domain wall are in a certain sense equivalent. The domain walls interact to each other and near Maxwell point they can form a rich plethora of bound states that are often referred to as bright or gray cavity solitons [7–16].

The formation and the dynamics of the solitary waves can be very different in discrete and periodic systems and this problem has been actively studied [17–25]. The difference can be dramatic if the size of the nonlinear structure is comparable with the intersite distance but even in the case when long-wave approximation is applicable the dynamics of the solitary waves can be different in the continuous and discrete systems.

For example, the translational invariance is broken in periodical systems and solitons can be pinned on the effective potential. Because of this, in discrete or periodic systems the domain walls can be at rest in a certain range of pumps around the Maxwell point. This results in the appearance of complex snaking patterns on the bifurcations diagrams of the solitons; see, for example, Ref. [23].

The goal of this paper is to study the dynamics of the domain walls under the action of the pump consisting of the temporally independent component and the component periodically oscillating in time. We restrict our consideration to the cases of harmonic and biharmonic oscillating component of the pump. A possible experimental setup where the domain walls can be studied experimentally is a one-dimensional nonlinear optical cavity (for example, a gas-filled cavity or a cavity made of a semiconductor) pumped from the top through a semitransparent mirror by a powerful laser beam; see Fig. 1. The oscillations of the pump have the frequency much lower compared to the optical frequency of the holding beam. Such low-frequency oscillations of the pump intensity can be created by an electro-optical or even by an optomechanical modulator. For instance, it is possible to use electro-optical or magneto-optical effects to control the transparency of a dielectric layer or the reflectivity of a mirror. This makes it possible to obtain an optical pump with the intensity varying in time. Let us mention here that for the observation of the effects discussed in the paper it is sufficient to provide a small modulation of the pump intensity (of order of 10^{-3}).

In this paper we focus on the effect of small oscillation of the holding beam intensity on the dynamics of the dissipative domain walls connecting two different spatially uniform states. It will be shown below that the oscillating pump can help to achieve precise control on the motion of the domain wall. This can be interesting from the point of view of the controllable formation of cavity solitons and for other applications.

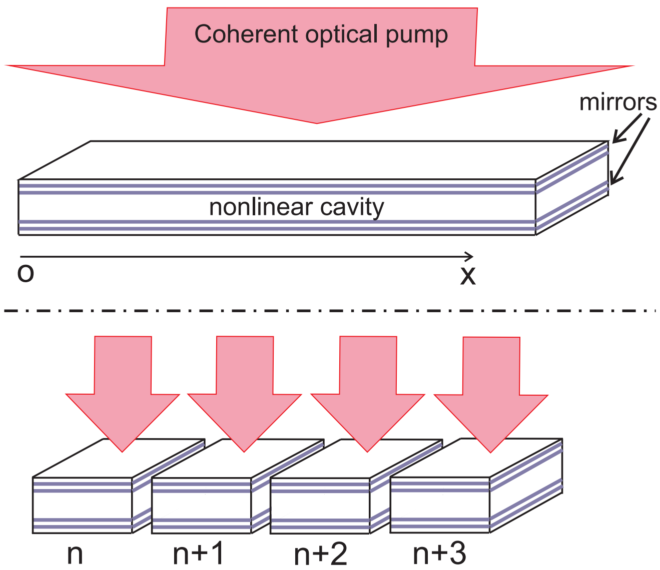

FIG. 1. The schematic view of a nonlinear cavity pumped by a laser beam is shown in the upper part of the figure. The bottom part of the figure shows an array of interacting nonlinear resonators pumped by coherent light.

An interesting effect discussed in the paper is the dependence of the domain wall velocity on the mutual phase of the temporal harmonics in the case of biharmonic pump. We develop an analogy between this effect and the effect of soliton ratchet that has been actively studied for topological [26–33] and nontopological [34–42] solitons in different physical systems ranging from Josephson junctions to optics and Bose-Einstein condensates. Soliton ratchet is also known in discrete systems where the phenomenon has some important peculiarities discussed in a number of papers [43–45].

The novelty of our research is that in the system considered in the present paper even a resting domain wall connects two different states and thus the symmetry typical for the systems demonstrating ratchet effect does not take place. However, it is possible to consider the dynamics of the domain wall in quasiparticle approximation and within the framework of this approach the symmetry is restored. So one can expect to see an effect similar but not completely analogous to the effect of soliton ratchet.

We also consider the effect of synchronization of domain walls by the oscillating component of the pump. The effect of the synchronization of the drift velocity was reported for the ratchet of particles [46] and for soliton ratchet [47]. In the present paper we study the phenomenon of synchronization for the domain walls connecting nonequivalent spatially uniform states. This effect can be useful for the precise control of the domain walls because the synchronization with the oscillating component of the pump makes it possible to count the number of hops of the domain wall over the sites and thus to know its exact position.

The paper is organized as follows. In Sec. II we consider a mathematical model and the spatially uniform states that can exist in the systems. The dynamics of the domain walls in the continuous systems under the action of the harmonic pump is considered in Sec. III. Section IV is devoted to the motion of the domain walls in the systems with biharmonic pump. Section V addresses the dynamics of the domain walls in the discrete systems. The main result of the paper is briefly summarized in Sec. VI.

II. MATHEMATICAL MODEL AND SPATIALLY UNIFORM STATES

Let us consider a system of coupled optical resonators pumped by an external laser beam described in the slow-varying amplitude approach by the model,

$$i\partial_t\psi_n + \delta\psi_n + \frac{\alpha\psi_n}{1+|\psi_n|^2} + c(\psi_{n+1}+\psi_{n-1}-2\psi_n) = P_n, \tag{1}$$

where each of the oscillators is descried by a complex slow-varying amplitude ψ_n, the real part of δ is linear detuning of the resonance frequency of the oscillators from the frequency of the pump, the imaginary part of δ is linear losses in the system, α is a nonlinear coefficient accounting for nonlinear change of the resonance frequencies, c is the coupling strength between the oscillators, and P_n is the amplitude of the external pump. Further, we assume that the pump is homogeneous in space $P_n = P$.

In the case of strong coupling the discreteness of the model is not important and one can describe the dynamics of the system by a continuous function $\psi[X]$ $\psi_n = \psi[\frac{n}{\sqrt{c}}]$. Here and through the whole paper we use square brackets to denote arguments of functions. The equation for ψ can then be written as

$$i\partial_t\psi + \delta\psi + \frac{\alpha\psi}{1+|\psi|^2} + \partial_X^2\psi = P. \tag{2}$$

Aiming to investigate the ratchet effect we consider the pumps periodic in time and for the sake of briefness and clarity we restrict our consideration to the cases of two-harmonic pumps $P = P_0 + a_1\sin[\omega t] + a_2\sin[2\omega t+\theta]$, where P_0 is a permanent pump, $a_{1,2}$ are the amplitudes of the fist and the second harmonics of the periodic component of the pump, and θ is the phase between the harmonics. In this paper we consider the case when P_0, a_1, and a_2 are real constants.

Bistability of the homogeneous states described by the Eq. (1) or (2) was studied in Ref. [22], where it was shown that in the case of saturable nonlinearity there is a region of parameters where both the upper and the lower states are stable. To investigate the dynamics of the fronts connecting nonequivalent states we choose the parameters belonging to this region $\delta = -0.3 + i$, $\alpha = -10$.

The bifurcation diagram showing the dependence of the absolute value of the spatially uniform field $|\psi|$ on the amplitude of the pump P_0 is shown in Fig. 2. Note that here the pump does now contain time-dependent component $p_1 = p_2 = 0$. Both the upper and the lower states are dynamically stable, the intermediate state is unstable, more details on the stability of the states can be found in Ref. [22]. In this work we carefully checked the stability of the domain walls by solving numerically the eigenvalue problem governing the spectral stability of the nonlinear structures. We also performed long-time numerical simulations to be sure that for the chosen set of parameters the solitons are stable.

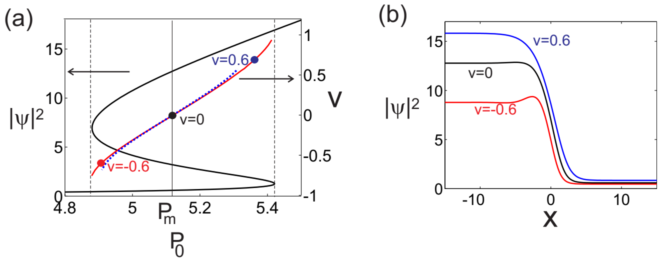

FIG. 2. (a) The thick black line shows the bifurcation diagram for the homogeneous states (left vertical axis). The red line (right vertical axis) is the dependency of the velocity of the front-connecting stable homogeneous state on the pump. Field distributions corresponding the points marked by the circles are shown in panel (b). Thin blue dotted line shows the dependence of the drift velocity on the permanent component of the pump $\langle v\rangle[P_0]$ for the case when the total pump has a component oscillating in time with frequency $\omega = 0.25$, the amplitude of the oscillating component is $a_1 = 0.12$. The dependency $\langle v\rangle[P_0]$ and the domain wall field distributions are shown for the continuous model. Vertical thin dashed lines mark the area of bistability and a solid thin line marks the so-called Maxwell point. (b) The field distributions of the fronts connecting stable homogeneous states and moving with the velocities $v = -0.6$ (red line, $P = 4.913$), $v = 0$ (black line, $P = P_m = 5.122$), and $v = 0.6$ (blue line, $P = 5.337$). The other parameters are $\delta = -0.3 + i$, $\alpha = -10$.

It is known that in the systems described by Eq. (2) the dynamical solutions in the form of switching waves between the lower and the upper states can exist. The formation of the domain walls is provided by the balance of the dispersion, nonlinear Kerr effect, the driving force (the holding beam pumping the systems), and the losses. The advantage of these solitary waves is that they are attractors and this facilitates their experimental observation. Let us mention an important difference of the considered domain walls from the solitary structures in the systems with gain. It is known that in the lasing systems there are domain walls connecting physically equivalent spatially uniform states and thus the phase of the field changes by π at these dissipative fronts; see Ref. [48]. In this paper we consider the systems pumped directly by the holding beam of coherent light [accounted by the right-hand side of Eqs. (1) and (2)]. The pump of such a kind breaks the phase symmetry fixing the phases of the spatially uniform solutions on the right and on the left from the domain walls.

At a given pump one of the states prevails and this state spreads filling the whole system. However, there may be a special value of pump, so-called Maxwell point P_m when the front connecting the lower and the upper states is at rest. If the pump becomes lower than the Maxwell point $P < P_m$ then the lower state prevails and the front sets to motion expanding the area of the existence of the lower state. If the pump gets higher than the Maxwell point $P > P_m$ then the upper state wins and propagation of the switching wave increases the domain of the existence of the upper state.

Numerically calculated dependence of the velocity of the switching wave on the pump is shown in Fig. 2(a), the Maxwell point is indicated as P_m. The profiles of the stationary kinks are shown in Fig. 2(b) for moving kinks and for the resting one.

III. MOTION OF THE FRONTS UNDER THE ACTION OF A HARMONIC PUMP

Let us now consider how weak temporal oscillations of the pump affect the dynamics of the domain walls. We focus our attention on the case when the permanent component of the pump is in the vicinity of the Maxwell point.

In the case of the ratchet of the particles and for the fronts connecting equivalent states the dependency of the velocity v on a control parameter η has the symmetry $v[\eta] = -v[-\eta]$. From this it follows that if the control parameter has only one temporal harmonics $\eta \sim \sin[\omega t]$ then the average velocity is equal to zero.

Working with the dissipative fronts we can consider the varying part of the pump as a control parameter. However, in this case the homogeneous states connected by the front are not equivalent and the aforementioned symmetry is absent. However, let us show that this symmetry appears for weak and adiabatically slow variations of the pump around Maxwell point.

Indeed, it is possible to calculate numerically the dependence of the front velocity on the pump; see the previous section. At Maxwell point the velocity is equal to zero and thus the leading term in the Taylor expansion of the dependency of the velocity on the pump is $v \approx \frac{\partial v}{\partial P}[P - P_m]$. This formula is valid for the constant pump P; however, it should also work for slow-varying pumps $P = P_0 + p_1 \sin[\omega t]$. Taking the permanent component of the pump to be equal to Maxwell point $P_0 = P_m$, we obtain that the velocity of the front is $v = \frac{\partial v}{\partial P} p_1 \sin[\omega t]$ and the average velocity is equal to zero $\langle v\rangle = 0$.

Generalizing these results let us derive the formula for the average velocity of the motion of the front under the action of the periodically varying pump $P = P_0 + p[t]$:

$$\begin{aligned}\langle v\rangle &= \frac{1}{T}\int_0^T v[P_0 + p[t]]dt \\ &= \frac{1}{T}\int_0^T v[P_0] + \frac{\partial v}{\partial P} p[t] + \frac{1}{2}\frac{\partial^2 v}{\partial P^2} p[t]^2 + ...dt \\ &= v[P_0] + \frac{\partial v}{\partial P}\langle p\rangle + \frac{1}{2}\frac{\partial^2 v}{\partial P^2}\langle p^2\rangle + ... \qquad (3)\end{aligned}$$

Here the derivatives $\frac{\partial v}{\partial P}$ are calculated for $P = P_0$, and the sign $\langle \dots \rangle$ denotes averaging over time.

For the harmonic pump $P = P_0 + a_1 \sin[\omega t]$ formula, Eq. (3) gives in leading approximation the following expression for $\langle v\rangle$:

$$\langle v\rangle = v[P_0] + \frac{1}{4}\frac{\partial^2 v}{\partial P^2} a_1^2. \qquad (4)$$

This expression tells that the oscillating pump results in the nonzero drift velocity of the domain wall. For low amplitudes of the oscillating component of the pump the direction of the motions depends on the sign of the derivative $\frac{\partial^2 v}{\partial P^2}$ and is proportional to the square of the amplitude of the oscillating pump.

To check the dependence Eq. (3) and to investigate nonadiabatic dynamics we performed direct numerical simulations

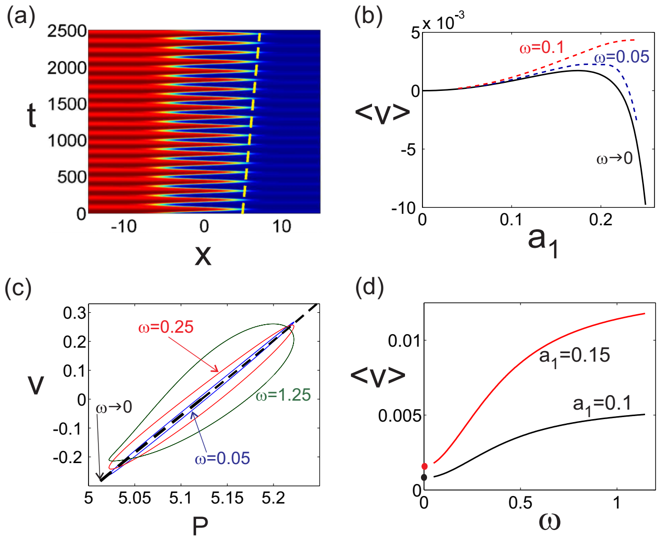

FIG. 3. (a) The intensity distribution $|\psi(x)|^2$ as a function of t for the pump $P = P_0 + a_1 \sin[\omega t]$, $a_1 = 0.1$, $\omega = 0.05$, $P_0 = P_m = 5.122$. Dashed yellow line is the guide for the eye showing the drift of the averaged position of the front. (b) The dependence of the drift velocity $\langle v \rangle$ on the amplitude of the oscillating component of the pump a_1. Black solid line corresponds to Eq. (3) valid for the adiabatic motion when the frequency $\omega \to 0$, the blue dashed line is obtained from direct numerical simulations for the pump oscillation frequency $\omega = 0.05$, the red dashed line represents results of numerical simulations for $\omega = 0.1$. (c) The curves representing the instant velocity $v(t)$ of the front against the instant value of the pump $P = P_0 + a_1 \sin[\omega t]$. The thick dashed black line shows the adiabatic dependence for $\omega \to 0$, thin blue, red, and green lines show the dependencies obtained from direct numerical simulations for $\omega = 0.05$, $\omega = 0.25$, and $\omega = 1.25$. (d) The dependence of the drift velocity $\langle v \rangle$ on the frequency of the oscillations of the pump ω for the amplitudes of the oscillating part of the pump $a_1 = 0.1$ and $a_1 = 0.15$. The black and red dots show the drift velocities obtained from Eq. (3) for the adiabatic cases. The other parameters are the same as described in the caption for Fig. 2.

of the dynamics of the front. We took the permanent part of the pump to be very close to Maxwell point and added a small sinusoidal signal. The dynamics of the front is shown in Fig. 3(a). It is seen that the boundary between the lower and the upper state oscillates and there is a slow drift toward the lower state.

We measured the velocity of the drift as a function of the amplitude of the oscillating component of the pump. The results are presented in Fig. 3(b). It is seen that for low frequencies the numerically calculated dependency follows closely to the dependency calculated by Eq. (3). However, for higher frequencies the discrepancy increases.

It is worth noting here that the average velocity goes to zero quadratically $\sim p_1^2$ for low pumps as it follows from the Taylor expansion in Eq. (3). For higher amplitudes of the pump more terms in the Taylor expansion must be kept, for very high pumps the formula has to used in the form $\langle v \rangle = \frac{1}{T}\int_0^T v[P_0 + p[t]]dt$.

From numerical simulations we can obtain the dependence of the instantaneous velocity of front n time $v(t)$. The dependencies $v[t]$ and $P[t]$ defines a closed curve in $v{-}P$ plane. In Fig. 3(c) these curves are plotted in the same graph as the dependency of the stationary velocity of the front on the amplitude of the permanent pump. One can see that, indeed, for low frequencies of the pump the instantaneous velocity of the front is close to the value of the stationary velocity of the front calculated for the permanent pump. For higher frequencies the curves deviate significantly from the dependency of the stationary velocity on the pump. This means that for higher frequencies the instantaneous velocity cannot be approximated by the stationary velocity for the same pump.

We also measured the dependencies of the average velocity of the frequency of the pump; see Fig. 3(d). The numerical simulations show that the average velocity grows with the increase of the frequency of the pump.

IV. MOTION OF THE FRONTS UNDER THE ACTION OF THE BIHARMONIC SIGNAL

In this section we consider the influence of the biharmonic signal $P = P_0 + a_1 \sin[\omega t] + a_2 \sin[2\omega t + \theta]$ on the dynamics of the domain walls and as before we start with the adiabatic case. In leading approximation, Eq. (3) gives for the drift velocity the dependence

$$\langle v \rangle = v[P_0] + \frac{1}{4}\frac{\partial^2 v}{\partial P^2}\left(a_1^2 + a_2^2\right) + \frac{1}{12}\frac{\partial^3 v}{\partial P^3} a_2 a_1^2 \cos[\theta]. \tag{5}$$

It is important that in this case the drift velocity depends also on the mutual phase θ of the harmonics. It resembles so-called ratchet effect when a particle is set into a directed motion under the action of biharmonic force. The ratchet effect is associated with the break of the asymmetry of the force acting on the particle. In the case of the biharmonic force the asymmetry depends on the mutual phase of the harmonic and thus the particle motion can be controlled by the phase.

This effect is also well known for solitons and kinks. However, to our best knowledge this effect was described only for the case when the kinks connect two equivalent states and so the kinks are at rest in the absence of the oscillating pump. The case considered here is different: the domain walls connect two nonequivalent states. Nevertheless, the domain walls are sensitive to the asymmetry of the pump. This make it possible to mimic the ratchet effect acting by the asymmetric pump on the dissipative domain walls connecting nonequivalent states.

The effect is especially clear at the Maxwell point when the upper and the low states become equivalent in the sense that they can be in equilibrium and the domain wall between these states is at rest. However, as it is seen from Eq. (5), the symmetry $\langle v \rangle[\theta] = -\langle v \rangle[\theta + \pi(2n+1)]$ taking place for the conventional ratchet effect is broken in the case of domain walls.

This can be understood taking into account that the oscillating pump modifies the spatially uniform states and it can be said that the oscillating pump shifts the Maxwell point. It is seen from Eq. (5) that the phase-independent contribution from the oscillating components can be compensated by the change of the time-independent component of the pump. The

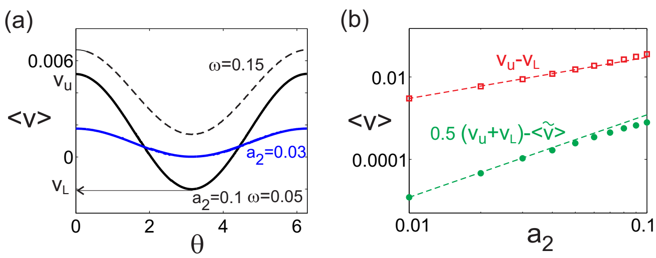

FIG. 4. (a) The dependencies of the drift velocity on the mutual phase of the harmonics of the pump. The solid black line corresponds to the fundamental frequency of the pump $\omega = 0.05$, the amplitude of the second harmonic of the pump is $a_2 = 0.1$. The solid blue curve corresponds to the frequency $\omega = 0.05$ but smaller amplitude of the second harmonic of the pump $a_2 = 0.03$. The nonadiabatic case is illustrated by the dashed curve calculated for $\omega = 0.15$ and $a_2 = 0.1$. The amplitude of the first harmonic of the pump is $a_1 = 0.1$ for all dependencies. The maximum and the minimum velocities for the case $\omega = 0.05$, $a_2 = 0.1$ are marked as v_U and v_L, correspondingly. (b) Red dashed line shows the analytical dependency of $v_U - v_L$ on the amplitude of the second harmonic of pump a_2. The red circles show the differences $v_U - v_L$ obtained from direct numerical simulations. The green line and green circles are the analytical and numerical dependencies of $0.5(v_U - v_L) - \langle\tilde{v}\rangle$ on the amplitude second harmonic of the pump.

shift of the Maxwell point is another mechanism leading to the change of the drift velocity.

We performed direct numerical simulations of Eq. (1) and measured the dependencies of the domain wall drift velocity on the mutual phase of the harmonics of the pump. These dependencies are shown in Fig. 4(a) for different values of the amplitudes and the fundamental frequency of the pump. These dependencies are very much similar to the dependencies typical for conventional ratchet effect.

For low frequencies the measured velocities are in good agreement with the approximation given by Eq. (5). To demonstrate that the adiabatic motion of the domain wall is well described by the developed theory we did the following measurements. From numerical modeling we can measure the maximum v_u and the minimum v_L drift velocities. From Eq. (5) it follows that the difference of the velocities is $v_U - v_l = \frac{1}{6}\frac{\partial^3 v}{\partial P^3} a_2 a_1^2$, so it is proportional to a_2 and to square of a_1. Figure 4(b) shows the dependency calculated by the formula by the red dashed line. The red circles on this panel show the velocities difference obtained from numerical simulations. It is seen the agreement is very good for low values of a_1. The dependencies are plotted in double logarithmic scales to make all power dependencies to be straight lines.

We also checked that the phase-independent contribution of the second harmonic of the pump to the drift velocity is proportional to a_2^2. Denoting the drift velocity in the absence of the second harmonic of the pump as $\langle\tilde{v}\rangle$ we can derive that $0.5(v_u + v_L) - \langle\tilde{v}\rangle = \frac{1}{4}\frac{\partial^2 v}{\partial P^2} a_2^2$. This dependency is shown in Fig. 4(b) by green dashed line. The corresponding values obtained from the direct numerical simulations are shown by the green circles. It is seen that the agreement between the analytical and numerical results is good.

V. MOTION OF THE KINKS IN THE DISCRETE SYSTEMS

In this section we consider how discreteness affects the propagation of the fronts under the action of the oscillation of the pump. The discreteness breaks the translational symmetry, and the domain wall connecting the upper and the lower spatially uniform states can be at rest in a range of the pumps. This can be considered as pinning of the domain walls in the discrete lattice.

We performed numerical simulations of Eq. (1) for the pump consisting of a temporally independent and an oscillating part. The dependency of the drift velocity on the amplitude of temporally independent part P_0 of the pump is shown in Fig. 5. The velocity is defined as the number of sites that the domain wall passes in a unit of time. An important feature seen in the figure is the steps on the dependency $\langle v\rangle[P_0]$.

The steps can be explained as synchronizations of the oscillations appearing when a domain wall is moving in discrete system with the oscillations of the pump. The oscillation of the field ψ that occur because of the motion of the domain wall in discrete systems we will refer below as eigen oscillations of the domain wall. The time interval between the moments when the center of the domain wall coincide with one of the sites is equal to $T = \frac{1}{\langle v\rangle}$. This means that the eigen oscillations have the fundamental frequency $\Omega_0 = \frac{2\pi}{T} = 2\pi\langle v\rangle$ and the harmonics $\Omega_n = 2\pi\langle v\rangle n$, n is integer.

The dependency of the domain wall velocity on the pump is nonlinear and thus the pump oscillating with frequency ω produces the harmonics $\omega_m = m\omega$, m is integer. The eigen oscillations can be synchronized with the oscillating pump when the frequency of the harmonic of the pump coincide with the frequency of a harmonic of the eigen oscillations. From this it can be derived that in the synchronization regime the velocity of the domain wall is defined by the frequency of the pump. Thus, the positions of the synchronization steps on dependency $\langle v\rangle(P_0)$ the are given by the ratio

$$\langle v\rangle_{nm} = \frac{\omega m}{2\pi n}, \tag{6}$$

where $\langle v\rangle_{nm}$ is the positions of the step corresponding to the synchronization of the mth harmonic of the pump with nth harmonic of the eigenoscillations of the domain wall.

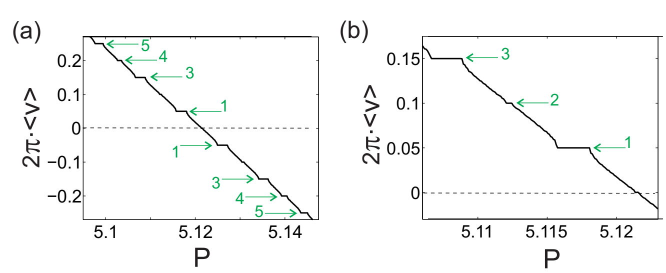

FIG. 5. The dependency of the domain drift velocity on the time-independent pump in the discrete system is shown in panel (a). The pump contains an oscillating component with the amplitude $a_1 = 0.1$ and the frequency $\omega_0 = 0.005$. The synchronization steps are marked by the index of the eigenoscillation harmonics synchronized with the oscillations of the external pump. Panel (b) shows a zoom of the panel (a). The coupling coefficient is $c = 0.3$.

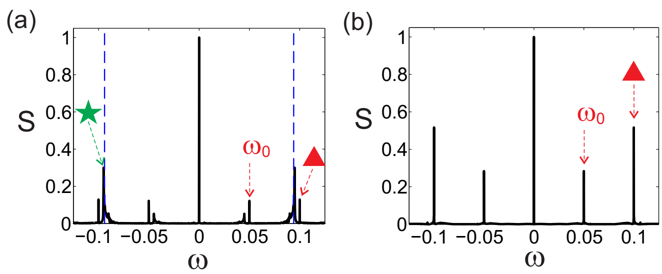

FIG. 6. (a) The spectra of the velocity of the nonsynchronized domain wall. The pump amplitudes are $P_0 = 5.0895$, $a_1 = 0.0015$, the frequency of the pump is $\omega_0 = 0.05$. (b) The spectrum of the velocity of the synchronized domain wall, the parameters are the same as for (a) but the amplitude of the oscillating pump is $a_1 = 0.005$. The coupling coefficient $c = 0.3$.

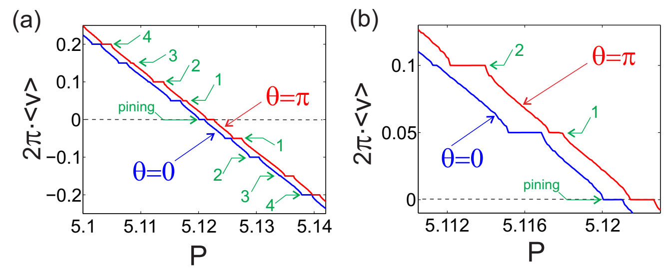

FIG. 7. (a) The dependencies of the domain drift velocity on the time-independent pump in the discrete system are shown for the case of biharmonic pump $P = P_0 + a_1 \sin[\omega t] + a_2 \sin[2\omega t + \theta]$. The blue line corresponds to $\theta = 0$, the red line corresponds to $\theta = \pi$. (b) Zoom of the dependencies shown in panel (a). The parameters are $c = 0.5$, $a_1 = 0.1$, $a_2 = 0.05$, and $\omega = 0.005$.

The width of the steps depends on the efficiency of the locking between the pump and the eigen oscillations of the domain wall. It is worth mentioning that the oscillations of the pump can suppress the pinning of the domain wall on the lattice.

Synchronization between high harmonics is weak and the corresponding steps are very small and not visible in the plotted dependency $\langle v\rangle[P_0]$. The step with $m = 0$ can be interpreted as the pining of the domain wall on the lattice and it is seen in Fig. 5 that this step can be very small.

The synchronization can be diagnosed by the temporal spectrum of the velocity of the domain wall. The spectrum is calculated as follows. By numerical simulations we find the temporal dependency of the instantaneous velocity for very long time and then we calculate the Fourier transform of this dependency.

A typical spectrum for the pump outside the synchronization steps is shown in Fig. 6(a). The spectral line corresponding to the frequency of the pump is clearly seen, it is marked in the figure as ω_0, the second harmonic of the pump is marked by the red triangle. The most intense spectral line associated with the eigenoscillations of the domain wall is marked as green star. This line is broadened, its maximum is marked by a vertical dashed blue line.

Let us note that there are also spectral lines generated by the mixing of the harmonics of the pump with the eigenocillations of the domain wall field. The frequencies of the pump and the frequency of the eigenoscillations of the domain wall are not commensurable and the motion is quasiperiodic.

The spectral picture changes in the case of the stronger oscillating pump when synchronization occurs; see Fig. 6(b). It is seen that the spectral lines of the eigenoscillations merge with the spectral lines generated by the pump. So the spectrum becomes equidistant and this means that in the synchronous regime the variations of the domain wall velocity is periodic.

In the case of the biharmonic pump the velocity of the domain wall depends on the mutual phase of the harmonics; see Fig. 7, showing the dependencies of the drift velocity on the permanent component of the pump P_0. It is seen that the width of the same synchronization steps depends on the phase θ. For example, for $\theta = 0$ the step marked by 2 is suppressed, whereas the same step is pronounced for $\theta = \pi$; see Fig. 7(b).

The dependency of the velocity on the mutual phase opens a way to control the motion of the domain walls by the phase of the oscillating components of the pump. Probably the most important case is when in the absence of the oscillating pump the domain wall is pinned. The discreteness produces an effective periodic potential for the domain wall and the oscillating pump makes the domain wall hopping from one minimum of the potential to another. This allows us to move the domain wall very slowly and since the hopping is synchronized with the oscillations of the pump it is possible to count the number of hops. This gives a precise control over the domain wall.

Let us now consider the ratchet effect that can take place in the discussed discrete systems. We take a permanent pump close to the Maxwell point so that in the absence of the biharmonic pump the domain wall is at rest. Then we switch on the oscillating pump and measure how the velocity of the domain wall depends on the mutual phase between the first and the second harmonic of the pump.

The dependencies of the drift velocities on the mutual phase is shown in Fig. 8 for different values of the permanent component of the pump P_0. It is seen that the dependencies are not symmetric, which does not look very surprising considering that the homogeneous states connected by the domain wall are not equivalent. Let us, however, notice that for the fixed amplitude of the oscillating component of the pump the dependency can be made more symmetric by precise tuning of the permanent component of the pump.

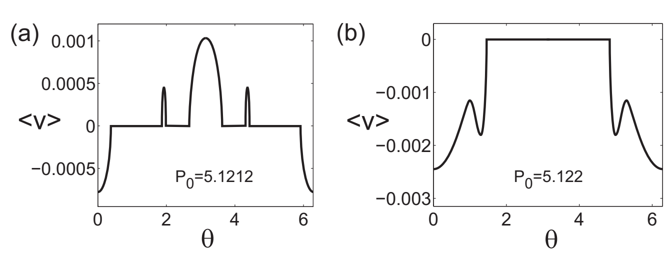

FIG. 8. The dependencies of the drift velocity on the mutual phase θ of the harmonics of the oscillating part of the pump. Panel (a) is for the permanent pump $P_0 = 5.1212$, panel (b) is for the permanent pump $P_0 = 5.122$. The other parameters are $c = 0.5$, $a_1 = 0.1$, $a_2 = 0.05$, and $\omega = 0.05$.

Despite the fact that the ratchet effect is partially disguised by co-effects it is seen that the drift velocity of the domain wall can be controlled by the phase and not only the value but the sign of the domain wall velocity can be changed. In general, the observed dependencies can be treated as superposition of the ratchet effect, the effect of the modification of the spatially uniform states by the oscillating pump and the effect of the pinning of the domain wall on the effective potential created by the discreteness of the system.

VI. CONCLUSION

Now let us briefly summarize the main results of the paper. The motion of the domain walls connecting two different stable spatially uniform states is studied in the dissipative system. The dependency of the velocity on the pump is found numerically.

The influence of the harmonic oscillation of the pump on the motion of the domain wall is studied. It is shown in the adiabatic limit that the change of the drift velocity can be found perturbatively and that the leading term in the expansion is proportional to the square of the amplitude of the oscillations of the pump. The dynamics of the walls under the action of the pump with higher frequencies is also investigated.

It is shown that in the more complex case of the biharmonic oscillating part of the pump the dynamics of the domain wall depends not only on the amplitudes of the harmonics but on their mutual phase. It is checked that the developed perturbative approach describes the velocity of the domain wall well if the frequency of the pump is low and the adiabatic approximation is applicable. The analogy between the observed effect and the effect of the soliton ratchet is discussed.

The dynamics becomes richer in the case of a discrete model because the motion of the domain wall excites oscillations of the field. It is demonstrated that these oscillations can be locked by the oscillating pump and thus the velocity of the domain wall becomes defined by the frequency of the pump.

The motion of the domain walls in discrete systems in the presence of the biharmonic oscillating part of the pump is also studied. It is shown that in this case the motion of the domain walls depends on the mutual phase of the harmonics and an effect similar to soliton ratchet can be observed. It is shown that the velocity of the domain wall can be controlled by the phase and the sign of the velocity can be changed by the appropriate choice of the phase. However, in the discrete case the dependency of the velocity on the phase is usually asymmetric and very fine tuning of the time-independent part of the pump is needed to obtain relatively symmetric dependence of the domain wall velocity on the phase of the harmonics of the pump.

We would like to note that the collision of the domain walls, in particular the collisions controlled by the ratchet effect, is an open and interesting topic of research. For example, the dissipative solitons is often treated as the bound states of the domain walls and thus the problem of the controllable motion of the domain walls is directly related to the formation of dissipative solitons. It is anticipated that the presence of the oscillating pump should affect the formation of the dissipative solitons significantly. However, this problem is out of the scope of the present paper and will be considered elsewhere.

Apart from their fundamental interest, the effects discussed in the paper can possibly find practical applications. In particular, the controllable motion of the optical domain walls opens a way to create bound states of the domain walls (dissipative solitons). This can be used for example for the design of optical logic. Another possible application of the effect is, for instance, optical manipulation. It is known that dielectric particles are attracted to the area of more intense optical field. Moving the domain wall it is possible to change the distribution of the optical field and by this to control the positions of the dielectric particles on the surface of the cavity.

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