

Matrix-product ansatz for the totally asymmetric simple exclusion process with a generalized update on a ring

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We apply the matrix-product ansatz to study the totally asymmetric simple exclusion process on a ring with a generalized discrete-time dynamics depending on two hopping probabilities, p and \bar{p} . The model contains as special cases the TASEP with parallel update, when $\bar{p} = 0$, and with sequential backward-ordered update, when $\bar{p} = p$. We construct a quadratic algebra and its two-dimensional matrix-product representation to obtain exact finite-size expressions for the partition function, the current of particles, and the two-point correlation function. Our main new result is the derivation of the finite-size pair correlation function. Its behavior is analyzed in different regimes of effective attraction and repulsion between the particles, depending on whether $\bar{p} > p$ or $\bar{p} < p$. In particular, we explicitly obtain an analytic expression for the pair correlation function in the limit of irreversible aggregation $\bar{p} \rightarrow 1$, when the stationary configurations contain just one cluster.

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I. INTRODUCTION

The asymmetric simple exclusion process (ASEP) is one of the simplest exactly solved models of driven many-particle systems with particle conserving bulk stochastic dynamics [1,2]. In the extremely asymmetric case, when particles are allowed to move in one direction only, it reduces to the totally asymmetric simple exclusion process (TASEP). For its description in the context of interacting Markov processes we refer to Ref. [3]. In the course of time, ASEP and TASEP became paradigmatic models for understanding the broad variety of nonequilibrium phenomena. Devised to model kinetics of protein synthesis [4], TASEP and its numerous extensions have found many applications to vehicular traffic flow [5–8], biological transport [9–16], one-dimensional surface growth [17,18], forced motion of colloids in narrow channels [19,20], spintronics [21], transport of “data packets” on the internet [22], current through chains of quantum dots [23], limit order market [24], to mention some.

The stationary properties of the original TASEP in continuous time were exactly obtained by different methods. The first exact solution was based on a recurrence relation, obtained at special values of the parameters in Ref. [25], and generalized by Schütz and Domany [26]. Using this recursion, closed expressions for the average occupations of all sites were obtained. The stationary states of TASEP and its generalization to a reaction-diffusion process with two-site interactions were studied also by using the quantum Hamiltonian formalism [27]. A combinatorial approach and mapping on weighted lattice paths shed new light on the formulation and solution of particle hopping models in continuous and discrete time [28,29]. An effective way to exploit the recursive properties of the steady states of a variety of one-dimensional processes is the matrix-product ansatz (MPA). According to the MPA, the stationary configuration probabilities can be expressed as matrix elements of products of operators

representing particles and holes. Such a matrix-product representation of the steady-state probability distribution for TASEP was found by Derrida, Evans, Hakim, and Pasquier [30]. Their formalism involves two square matrices, D and E , which are infinite-dimensional in the general case and satisfy a quadratic algebra, known as the DEHP algebra. Krebs and Sandow [31] proved that the stationary state of any one-dimensional system with random-sequential dynamics involving nearest-neighbor hopping and single-site boundary terms can always be written in a matrix-product form. Fock representations of the general quadratic algebra were studied by Essler and Rittenberg [32], who found explicit representations in terms of infinite-dimensional tridiagonal matrices. Quadratic algebras involved in the MPA for continuous-time processes were studied also in Ref. [33].

The matrix-product ansatz marked a breakthrough in the solution of TASEP and ASEP in discrete time under periodic as well as open boundary conditions. For the definition of the different types of discrete-time updates we refer to Ref. [34]. First, by using the MPA, the case of sublattice-parallel update with deterministic bulk dynamics was solved [35]. The general case of ASEP with stochastic sublattice-parallel dynamics was studied in Ref. [36]. Next, the TASEP with ordered-sequential update was solved by mapping the corresponding algebra onto the DEHP algebra [37]; see also Ref. [38]. The case of parallel update (simultaneous updating of all sites) was solved by using two new versions of the matrix-product ansatz. One of these versions leads to a quartic algebra [39], in contrast to the previous cases, in which the algebra is quadratic. A different representation as a cubic algebra is obtained from a bond-oriented matrix-product ansatz [40]. In general, the MPA has become a powerful method for studying stationary states of different one-dimensional Markov processes out of equilibrium [41]. For example, it was used to solve TASEP with a defect particle [42], the multispecies TASEP with uniform [43] and inhomogeneous hopping rates [44], the discrete-time case with inhomogeneous rates in the bulk [45], and ASEP with internal degrees of freedom [46,47].

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A powerful technique for studying the dynamics of ASEP and some of its generalizations is the Bethe ansatz. According to it, the amplitudes of the eigenfunctions of the transfer matrix can be expressed as a nonlinear combination of properly defined plane waves. This method was used to exactly solve the open ASEP [48], a three-parametric family of hopping probabilities, which includes TASEP and ASEP [49], its discrete time versions with inhomogeneous and nonlocal transition rates [45], etc. A modified algebraic Bethe ansatz for the continuous-time TASEP with open boundaries, based on results known for integrable quantum spin chains, is given in Ref. [50].

It should be noted that the properties of the ASEP depend strongly on the choice of the boundary conditions, similarly to the case of systems with long-range interactions. This leads to inequivalence of the nonequilibrium statistical properties for open and closed systems. For instance, the MPA for continuous-time TASEP on a ring becomes trivial: the algebra has Abelian one-dimensional representation [34], while in the case of open boundaries the corresponding matrices are generically infinite-dimensional and noncommuting. The open system exhibits (in the thermodynamic limit) three stationary phases in the plane of particle input-output rates, with continuous or discontinuous transitions between them. Another example is the application the Bethe ansatz for obtaining the full current fluctuations in the periodic TASEP [51], and much later in the periodic ASEP [52]. We mention also that by using a new form of the Bethe ansatz the totally asymmetric exclusion process on a ring was solved for the nonstationary probabilities under arbitrary initial conditions and time intervals [53]. The full relaxation dynamics of the TASEP on a ring was solved also by the algebraic Bethe ansatz method [54]. For a review of some results obtained for different versions of ASEP on a ring by using the Bethe ansatz we refer to Ref. [55].

The steady state of TASEP with parallel update on the ring has a pair-factorized form and exhibits nearest-neighbor correlations [56]. A factorized form of the steady state of ASEP on a ring was found also in its generalizations including Langmuir kinetics for the attachment and detachment transitions of particles on the chain and in a reservoir, as well in the presence of memory reservoirs [57]. For open boundaries, the matrix-product representation was interpreted as a pair-factorized state on the ring modulated by a matrix-product state [58].

An important, exactly solvable generalization of the TASEP dynamics on a ring was found and studied by Wölki in 2005 [59]. That is the exactly solvable representative pw^{N-1} of the general class of so-called $p_1 p_2 \dots p_N$ models, under the discrete-time dynamics of which all clusters of particles are updated independently, $p_1 p_2 \dots p_N$ being the probability that a N -particle cluster will move as a whole one site ahead. Thus, each particle has its own hopping probability p_i , $1 \leq i \leq N$, depending on its position in the cluster. It was shown that the model with $p_1 = p$ and $p_2 = p_3 = \dots = p_{N-1} =: w$ satisfies the condition for models with discrete-mass transport to have a factorized steady state [60]. This model, to be denoted here, following Ref. [61], as gTASEP is the main object of study in the present paper (in our notation $w = \tilde{p}$). We note that the same model was thoroughly studied in the framework of the Bethe ansatz integrability in Ref. [62], as a particular

case of the general family formulated and investigated in Ref. [49].

Our main aim here is the construction and application of a quadratic algebra and its finite-dimensional matrix-product representation for the gTASEP on finite chains under periodic boundary conditions. The exact expressions for the main characteristics of the stationary state of the model are obtained for arbitrary fixed numbers of the lattice sites L and the particles N ($N \leq L$) on the ring. The correctness of our MPA representation is proved by a parallel combinatorial derivation. The finite-size two-point correlation function is calculated within the MPA.

The paper is organized as follows: in Sec. II we formulate the model, Sec. III presents the matrix-product algebra and its two-dimensional representation. Within a combinatorial approach, in Sec. IV we calculate the partition function and the average current of particles in the system. Section V contains derivation of the partition function, the average particle density, the current, and the pair correlation function by using the matrix-product formalism. A discussion of the approach, comparison with known results in several particular regimes of the model, as well as perspectives for further applications are given in Sec. VI. A more detailed derivation of the results is given in Ref. [63].

II. THE MODEL

We consider TASEP on a ring of L sites, labeled clockwise by the index $i = 1, 2, \dots, L$, where site 1 is the nearest-neighbor of site L in the clockwise direction. Each site of the lattice can be empty or occupied by just one particle.

The dynamics of the model corresponds to the discrete-time backward-ordered update with probabilities p and \tilde{p} defined as follows. A particle can hop to a vacant nearest-neighbor site in the clockwise direction, or stay at its place. During each moment of time t , an update of the configuration of the whole system takes place in L consecutive steps, passing through successive updates of all the pairs of nearest-neighbor sites in the counterclockwise order $(L-1, L), \dots, (i, i+1), \dots, (1, 2), (L, 1)$. The probability of a hop along the bond $(i, i+1)$ depends on whether a particle has jumped from site $i+1$ to site $i+2$ in the previous step, when the bond $(i+1, i+2)$ was updated, or not.

(1) In the case when the site $i+1$ has not changed its occupation number, the probabilities are the standard ones: if site $i+1$ remains empty, then the jump of a particle from site i to site $i+1$ takes place with probability p , and the particle stays immobile with probability $1-p$; if site $i+1$ remains occupied, no jump takes place and the configuration of the bond $(i, i+1)$ is conserved.

(2) If in the previous step a particle has jumped from site $i+1$ to site $i+2$, thus leaving $i+1$ empty, then the jump of a particle from site i to site $i+1$ in the next step takes place with a different probability \tilde{p} , and the particle stays immobile with probability $1-\tilde{p}$.

Note that when $\tilde{p} = p$ one has the standard TASEP with backward-sequential update, and when $\tilde{p} = 0$ one has the TASEP with parallel update.

III. THE MATRIX-PRODUCT ALGEBRA

Under the above generalized dynamics, the left-hand side in each pair of nearest-neighbors $(i, i + 1)$, which is to be updated, can be either empty or occupied. With these two states we associate the matrices E and D , respectively, and introduce the column-vector

$$A = \begin{pmatrix} E \\ D \end{pmatrix}. \quad (1)$$

On the other hand, the right-hand side of such a pair $(i, i + 1)$ can be in three states: empty, being empty in the previous step of the update too, occupied, and empty but as a result of a particle hopping from site $i + 1$ to site $i + 2$ in the previous update step. With these three states we associate the matrices \hat{E} , \hat{D} , and F , respectively, and introduce the column-vector

$$\hat{A} = \begin{pmatrix} \hat{E} \\ \hat{D} \\ F \end{pmatrix}. \quad (2)$$

Thus, the state of the bond $(i, i + 1)$ to be updated is described by the direct matrix product,

$$A \otimes \hat{A}. \quad (3)$$

We conjecture the same update mechanism for each pair of nearest-neighbor sites as in the case of backward-sequential update, see Ref. [34],

$$\mathcal{T}[A \otimes \hat{A}] = \hat{A} \otimes A. \quad (4)$$

With the definition Eq. (2) of \hat{A} , this equation resembles also the stationarity mechanism suggested for the parallel update in Ref. [34].

The operator \mathcal{T} in Eq. (4) is a 6×6 matrix, defined by the probabilities of the possible elementary events,

$$\begin{aligned} \mathcal{P}(E\hat{E} \rightarrow \hat{E}E) &= 1, \\ \mathcal{P}(E\hat{D} \rightarrow \hat{E}D) &= 1, \\ \mathcal{P}(EF \rightarrow \hat{E}E) &= 1, \\ \mathcal{P}(D\hat{E} \rightarrow FD) &= p, \\ \mathcal{P}(D\hat{E} \rightarrow \hat{D}E) &= 1 - p, \\ \mathcal{P}(DF \rightarrow FD) &= \tilde{p}, \\ \mathcal{P}(DF \rightarrow \hat{D}E) &= 1 - \tilde{p}, \\ \mathcal{P}(D\hat{D} \rightarrow \hat{D}D) &= 1. \end{aligned} \quad (5)$$

Here $\mathcal{P}(\cdot)$ denotes the probability of the event in the brackets. Hence, we obtain the following quadratic algebra:

$$\hat{E}E = E\hat{E} + EF, \quad (6)$$

$$\hat{E}D = E\hat{D}, \quad (7)$$

$$FD = \tilde{p}DF + pD\hat{E}, \quad (8)$$

$$\hat{D}E = (1 - p)D\hat{E} + (1 - \tilde{p})DF, \quad (9)$$

$$\hat{D}D = D\hat{D}, \quad (10)$$

$$FE = 0. \quad (11)$$

The corresponding stochastic matrix \mathcal{T} has the explicit form

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - p & 0 & 1 - \tilde{p} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & \tilde{p} \end{pmatrix}. \quad (12)$$

To solve the quadratic algebra Eqs. (6)–(11), we make the ansatz

$$\hat{E} = E - F + c_1, \quad \hat{D} = D + c_2. \quad (13)$$

The constants c_i , $i = 1, 2$ will be determined later. The ansatz solves trivially Eq. (10), Eq. (6) is satisfied under the condition of Eq. (11), and Eqs. (7) and (8) become equivalent. Thus, the algebra reduces to the following three equations:

$$FD = c_1D - c_2E, \quad (14)$$

$$pDE - (p - \tilde{p})DF = c_1(1 - p)D + c_2E, \quad (15)$$

$$FE = 0. \quad (16)$$

We choose $c_2 = 0$, when

$$FD = c_1D, \quad (17)$$

$$pDE - (p - \tilde{p})DF = c_1(1 - p)D, \quad (18)$$

$$FE = 0. \quad (19)$$

A two-dimensional representation of the quadratic algebra Eqs. (17)–(19), depending on four free parameters d, e, f , and c_1 , is provided by the matrices

$$\begin{aligned} D = \hat{D} &= d \begin{pmatrix} 1 & 0 \\ \frac{pe}{f(1-p)} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} e & \frac{(p-\tilde{p})f}{p} \\ 0 & 0 \end{pmatrix}, \\ F &= \begin{pmatrix} 0 & f \\ 0 & \frac{pe}{1-p} \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} c_1 + e & \frac{(p-\tilde{p})f}{p} \\ 0 & c_1 - \frac{pe}{1-p} \end{pmatrix}. \end{aligned} \quad (20)$$

One of the most convenient forms of the above representation is obtained when $d = e = f = 1$, and $c_1 = \frac{p}{1-p}$. Then,

$$\begin{aligned} D = \hat{D} &= \begin{pmatrix} 1 & 0 \\ \frac{p}{1-p} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & \frac{p-\tilde{p}}{p} \\ 0 & 0 \end{pmatrix}, \\ F &= \begin{pmatrix} 0 & 1 \\ 0 & \frac{p}{1-p} \end{pmatrix}. \end{aligned} \quad (21)$$

Hence,

$$\hat{E} = \begin{pmatrix} \frac{1}{1-p} & -\frac{\tilde{p}}{p} \\ 0 & 0 \end{pmatrix}, \quad C = E + D = \begin{pmatrix} 2 & \frac{p-\tilde{p}}{p} \\ \frac{p}{1-p} & 0 \end{pmatrix}. \quad (22)$$

Note that, in view of the ansatz Eq. (13), we have $\hat{C} := \hat{E} + \hat{D} + F = C + c_1$, hence, $\hat{C}C = C\hat{C}$.

The most important properties of the representation Eq. (21) are

$$\begin{aligned} D^2 &= D, \quad E^2 = E, \quad \text{Tr}(DE) = \frac{1 - \tilde{p}}{1 - p} \equiv x, \\ \det(DE) &= 0. \end{aligned} \quad (23)$$

The eigenvalues of the degenerate matrix DE are $\lambda_1 = 0$ and $\lambda_2 = x$. It can be cast in diagonal form by means of a similarity

transformation $V^{-1}DEV$, where

$$V^{-1}DEV = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \quad V = x^{-1/2} \begin{pmatrix} \frac{p-\tilde{p}}{p} & 1 \\ -1 & \frac{p}{1-p} \end{pmatrix},$$

$$V^{-1} = x^{-1/2} \begin{pmatrix} \frac{p}{1-p} & -1 \\ 1 & \frac{p-\tilde{p}}{p} \end{pmatrix}. \quad (24)$$

Hence, $\text{Tr}(DE)^k = x^k$, $k = 1, 2, 3, \dots$, is the crucial feature used in the following consideration.

We note that another choice of the parameters, $d = 1$, $e = 1 - p$, $f = 1$, and $c_1 = p$, leads to a representation recently derived by P. Hrabák in Ref. [64] but it does not have all the nice properties of Eq. (23).

IV. COMBINATORIAL SOLUTION

Here we calculate the partition function and the average current within a purely combinatorial approach, involving a detailed analysis of all the possible configurations of a given number of particles on a finite periodic lattice, taking into account their statistical weights, and by application of the multinomial theorem.

A. The partition function

In the matrix-product representation each stationary configuration is represented as a string of the matrices E (for an empty site) and D (for an occupied site), e.g., $C = EEEDDEDEE \dots ED$, with length equal to the number of lattice sites L . Due to the projective properties of these matrices, the weight $W(C)$ of a configuration C is proportional to the number of clusters $k(C)$ in the configuration and does not depend on any other features of C . Thus,

$$W(C) \propto \text{Tr}(DE)^{k(C)} = x^{k(C)}. \quad (25)$$

A more detailed description of C is given by its cluster composition, which is represented by a partition $\mathbf{n}(C)$ of the fixed number of particles N :

$$\mathbf{n}(C) = (n_1(C), n_2(C), \dots, n_N(C)) : \sum_{j=1}^N j n_j(C) = N. \quad (26)$$

Here $\mathbf{n}(C)$ is a N -component vector with integer coordinates $n_j \geq 0$ denoting the number of clusters of size j , such that $\sum_{j=1}^N j n_j(C) = N$ (obviously, $n_N \in \{0, 1\}$). The total number of clusters in a given configuration C equals

$$k(C) = \sum_{j=1}^N n_j(C), \quad 1 \leq k(C) \leq \min\{N, L - N\}. \quad (27)$$

To calculate the partition function $Z(L, N)$, we have to determine the number of configurations of N particles with exactly k clusters on the ring of L labeled sites, so that $1 \leq k \leq \min\{N, L - N\}$. To solve the problem, we start by ordering the elements of the partition $\mathbf{n}_{N,k}$ into an ordered set $\mathbf{s}_{N,k}$, corresponding to the clockwise position on the ring of all the k clusters. Then, to each element of $\mathbf{s}_{N,k}$ we put into one-to-one correspondence an initial configuration, constructed as follows. First, we choose any of the k clusters as the first one, and place its first (leftmost) particle at site 1 of the ring. Next,

we realize that the number of compositions of the N particles into k clusters is

$$\binom{N-1}{k-1}, \quad (28)$$

and the composition of the $L - N$ empty sites into the same number k of clusters is

$$\binom{L-N-1}{k-1}. \quad (29)$$

Obviously, each of the empty clusters can separate any pair of particle clusters consecutively ordered on the ring after the first one, so the number of configurations with fixed position of the first cluster is given by the product of Eqs. (28) and (29). Finally, restoring the translational invariance of the configurations along the ring, we have to multiply the above number by L , because the origin can be taken at any site of the ring, and divide it by k , because the origin will k times occur at the leftmost site of one the k clusters. Thus, taking into account that the weight of a configuration is given by Eq. (25), we obtain the partition function of the model ($N \geq 1$, $L - N \geq 1$):

$$Z_{L,N}(p, \tilde{p}) = L \sum_{k=1}^{\min\{N, L-N\}} x^k \binom{L-N-1}{k-1} \frac{1}{k} \binom{N-1}{k-1}$$

$$= \frac{L}{L-N} \sum_{k=1}^{\min\{N, L-N\}} \binom{L-N}{k} \binom{N-1}{k-1} x^k, \quad (30)$$

where $x = (1 - \tilde{p})/(1 - p)$. In the particular cases when there are no particles or no holes on the ring one has to set $Z_{L,0} = Z_{L,L} = 1$.

Equation (30) represents the partition function $Z_{L,N}(p, \tilde{p})$ as a polynomial in $(1 - \tilde{p})/(1 - p) := x$. Sometimes it is convenient to express it as a polynomial in

$$v = \frac{\tilde{p} - p}{1 - p} = 1 - x.$$

To this end we expand $x^k = (1 - v)^k$ according to the binomial formula and make use of the identity

$$\sum_{k=1}^{\min\{N, L-N\}} \binom{L-N-1}{k-1} \binom{N-1}{k-1} \frac{1}{k} \binom{k}{m}$$

$$= \frac{1}{m} \binom{L-N-1}{m-1} \binom{L-1-m}{N-m} \quad (31)$$

and obtain the result

$$Z_{L,N}(p, \tilde{p}) = L \sum_{m=0}^{\min\{N, L-N\}} \frac{(-v)^m (L-m-1)!}{m!(N-m)!(L-N-m)!}$$

$$= \binom{L}{N} {}_2F_1(-N, -L+N; 1-L; v), \quad (32)$$

where ${}_2F_1(a, b; c; x)$ is the Gauss hypergeometric function.

Note that the above expression includes the special cases $Z_{L,0} = Z_{L,L} = 1$ and apparently exhibits the particle-hole symmetry $N \leftrightarrow L - N$ of the partition function. In addition,

since

$$Z_{L,N}(p, \tilde{p}) = \binom{L}{N} + L \sum_{m=1}^{\min\{N, L-N\}} \frac{(-v)^m (L-m-1)!}{m!(N-m)!(L-N-m)!}, \quad (33)$$

it is convenient to adopt the convention

$$Z_{0,N}(p, \tilde{p}) = \delta_{N,0}. \quad (34)$$

When $\tilde{p} = p$ ($v = 0$), the partition function becomes independent of the jump probability,

$$Z_{L,N}(p, p) = \binom{L}{N}. \quad (35)$$

Comparing Eq. (32) with the result for the corresponding zero-range process (ZRP) obtained in Ref. [61], we see that

$$Z_{L,N}(p, \tilde{p})^{\text{gTASEP}} = \frac{L}{L-N} Z_{L,N}(p, \tilde{p})^{\text{ZRP}}, \quad N \geq 1, \quad L-N \geq 1. \quad (36)$$

The factor $L/(L-N)$ is due to the different number of configurations in gTASEP and ZRP.

B. The current of particles

According to the update rules, a cluster of n particles yields the following average number of jumps per update (unit time):

$$\begin{aligned} j_n^{\text{cl}} &= p \left[(1-\tilde{p}) \sum_{k=1}^{n-1} k \tilde{p}^{k-1} + n \tilde{p}^{n-1} \right] \\ &= p \left[\sum_{k=1}^{n-1} k \tilde{p}^{k-1} - \sum_{k=1}^{n-1} k \tilde{p}^k + n \tilde{p}^{n-1} \right] \\ &= p \sum_{k=1}^n \tilde{p}^{k-1}. \end{aligned} \quad (37)$$

Having in mind the number of different configurations $\mathcal{N}_{\text{diff}}(\mathbf{n}_{N,k})$ calculated above for any given partition $\mathbf{n}_{N,k}$ of the number of particles N into k cluster, by summing up all the contributions we obtain for the current (the average total number of jumps per lattice site),

$$\begin{aligned} J_{L,N}(p, \tilde{p}) &= \frac{p}{Z_{L,N}(p, \tilde{p})} \sum_{k=1}^{\min\{N, L-N\}} x^k \binom{L-N-1}{k-1} \sum_{\mathbf{n}_{N,k}} \frac{(k-1)!}{\prod_{j=1}^N n_j!} \sum_{s=1}^N n_s \sum_{m=0}^{s-1} \tilde{p}^m \\ &= \frac{p}{Z_{L,N}(p, \tilde{p})} \sum_{k=1}^{\min\{N, L-N\}} x^k \frac{1}{k} \binom{L-N-1}{k-1} \sum_{m=0}^{N-1} \tilde{p}^m \sum_{\mathbf{n}_{N,k}} \frac{k! \sum_{s=m+1}^N n_s}{\prod_{j=1}^N n_j!}. \end{aligned} \quad (38)$$

To evaluate the sum

$$\sum_{\mathbf{n}_{N,k}} \frac{k!}{\prod_{j=1}^N n_j!} \sum_{s=m+1}^N n_s \quad (39)$$

we apply the operator

$$\sum_{s=m+1}^N z^s \frac{\partial}{\partial z^s}$$

to both sides of the multinomial identity,

$$\sum_{\mathbf{n}_k} \frac{k!}{\prod_{j=1}^N n_j!} z_1^{n_1} z_2^{n_2} \dots z_N^{n_N} = (z_1 + z_2 + \dots + z_N)^k, \quad (40)$$

where $\sum_{\mathbf{n}_k}$ denotes summation over the non-negative integers $n_j \geq 0$, $j = 1, 2, \dots, N$, under the constraint $\sum_j n_j = k$. The results reads

$$\sum_{\mathbf{n}_k} \frac{k!}{\prod_{j=1}^N n_j!} \left[\sum_{s=m+1}^N n_s \right] z_1^{n_1} z_2^{n_2} \dots z_N^{n_N} = k \sum_{s=m+1}^N z_s (z_1 + z_2 + \dots + z_N)^{k-1}. \quad (41)$$

Next, by setting here $z_j = z^j$ ($j = 1, 2, \dots, N$), we obtain

$$\sum_{\mathbf{n}_k} \frac{k!}{\prod_{j=1}^N n_j!} \left[\sum_{s=m+1}^N n_s \right] z^{\sum_j j n_j} = k \sum_{s=m+1}^N z^s (z + z^2 + \dots + z^N)^{k-1}. \quad (42)$$

Therefore, when the number of particles is fixed, $\sum_j j n_j = N$, the sum on the left-hand side of Eq. (42) at $z = 1$ must equal the coefficient of z^N in the expansion of the right-hand side,

$$k [z^N] \sum_{s=m+1}^N z^s (z + z^2 + \dots + z^N)^{k-1} = k \sum_{s=m+1}^N \binom{N-s-1}{k-2} = k \binom{N-m-1}{k-1}. \quad (43)$$

Thus, the expression for the current Eq. (38) simplifies to

$$\begin{aligned}
 J_{L,N}(p, \tilde{p}) &= \frac{p}{Z_{L,N}(p, \tilde{p})} \sum_{k=1}^M x^k \binom{L-N-1}{k-1} \sum_{m=0}^{N-k} \tilde{p}^m \binom{N-m-1}{k-1} \\
 &= \frac{x p}{Z_{L,N}(p, \tilde{p})} \sum_{m=0}^{N-1} \tilde{p}^m \sum_{k=0}^{M-1} x^k \binom{L-N-1}{k} \binom{N-m-1}{k},
 \end{aligned} \tag{44}$$

where $M \equiv \min\{N, L - N\} \geq 1$.

It is instructive to rewrite the above expression as a polynomial in $v = 1 - x$. To this end we write $x^k = x(1 - v)^{k-1}$, and expand the last expression according to the Newtonian binomial. By using the identity

$$\sum_{k=0}^{M-1} \binom{k}{n} \binom{L-N-1}{k} \binom{N-m-1}{k} = \binom{L-N-1}{n} \binom{L-m-n-2}{N-m-n-1}, \tag{45}$$

we obtain

$$\begin{aligned}
 J_{L,N}(p, \tilde{p}) &= \frac{x p}{Z_{L,N}(p, \tilde{p})} \sum_{m=0}^{N-1} \tilde{p}^m \sum_{n=0}^{M-1} \frac{(-v)^n}{n!} \frac{(L-m-n-2)!}{(N-m-n-1)!(L-N-n-1)!} \\
 &= \frac{x p}{Z_{L,N}(p, \tilde{p})} \sum_{m=0}^{N-1} \tilde{p}^m \binom{L-m-2}{N-m-1} {}_2F_1(1+m-N, N+1-L; 2+m-L; v) \\
 &= \frac{x p}{Z_{L,N}(p, \tilde{p})} \binom{L-2}{N-1} F_1(1-N, N+1-L, 1, 2-L; v, \tilde{p}) \\
 &= p \left(\frac{1-\tilde{p}}{1-p} \right) \frac{N(L-N)}{L(L-1)} \frac{F_1(1-N, N+1-L, 1, 2-L; v, \tilde{p})}{{}_2F_1(-N, -L+N; 1-L; v)}.
 \end{aligned} \tag{46}$$

This expression coincides with the gTASEP current derived in Ref. [61] under a mapping of the ZRP on the gTASEP.

V. MATRIX-PRODUCT DERIVATION

A. The partition function

According to the matrix-product ansatz, the grand canonical partition function is given by

$$Z_L(p, \tilde{p}) = \text{Tr}(C^L), \tag{47}$$

with $C = E + D$. To obtain the partition function in the case of fixed number of particles N , we introduce a chemical potential μ of the particles, and define

$$C(\mu) = E + \mu D = \begin{pmatrix} 1 + \mu & \frac{p-\tilde{p}}{p} \\ \frac{\mu p}{1-p} & 0 \end{pmatrix}. \tag{48}$$

Now the partition function for the generalized TASEP, on a ring of L sites with fixed number of particles N , we can write in the form

$$Z_{L,N}(p, \tilde{p}) = [\mu^N] \text{Tr}\{C^L(\mu)\}, \tag{49}$$

where the symbol $[\mu^N]$ denotes the coefficient of the μ^N term in the polynomial in μ . Since the eigenvalues of the 2×2 matrix $C(\mu)$ are

$$\lambda_{1,2}(\mu; p, \tilde{p}) = \frac{1}{2} [1 + \mu \pm \sqrt{(1 + \mu)^2 - 4\mu v}], \tag{50}$$

where

$$x = \frac{1-\tilde{p}}{1-p}, \quad v := 1-x = \frac{\tilde{p}-p}{1-p}, \tag{51}$$

after some algebra we obtain

$$\begin{aligned}
 Z_{L,N}(p, \tilde{p}) &= [\mu^N] \{ \lambda_1^L(\mu; p, \tilde{p}) + \lambda_2^L(\mu; p, \tilde{p}) \} \\
 &= [\mu^N] 2^{-(L-1)} \sum_{m=0}^{\lfloor L/2 \rfloor} \binom{L}{2m} (1 + \mu)^{L-2m} \\
 &\quad \times [(1 + \mu)^2 - 4\mu v]^m \\
 &= 2^{-(L-1)} \sum_{m=0}^{\lfloor L/2 \rfloor} \binom{L}{2m} \sum_{n=0}^m \binom{m}{n} 4^n (-v)^n \\
 &\quad \times \binom{L-2n}{N-n}.
 \end{aligned} \tag{52}$$

Next, we change the order of summation over m and n and make use of the identity

$$\sum_{m=n}^{\lfloor L/2 \rfloor} \binom{L}{2m} \binom{m}{n} = 2^{L-2n-1} \frac{L}{L-2n} \binom{L-n-1}{n}, \tag{53}$$

to obtain the result

$$\begin{aligned}
 Z_{L,N}(\tilde{p}, p) &= \sum_{n=0}^{\min\{N, L-N\}} \binom{L-n-1}{n} \binom{L-2n}{N-n} \frac{L}{L-2n} (x-1)^n \\
 &= L \sum_{n=0}^{\min\{N, L-N\}} (-1)^n \frac{(L-n-1)!}{(N-n)!(L-N-n)! n!} v^n,
 \end{aligned} \tag{54}$$

which is identical to Eq. (32).

B. The local density

We show here how our matrix-product expression for the particle density $\rho_{L,N} = N/L$ produces several identities involving sums over sets of reduced partition functions. We start from the definition of the average particle density of the model:

$$\rho_{L,N} = Z_{L,N}^{-1} [\mu^{N-1}] \text{Tr}[DC^{L-1}(\mu)], \quad (55)$$

which is, obviously, constant over the ring, equal to N/L . The above trace is readily calculated by using the diagonal form of the matrix $C(\mu)$,

$$U^{-1}(\mu)C(\mu)U(\mu) = \begin{pmatrix} \lambda_1(\mu) & 0 \\ 0 & \lambda_2(\mu) \end{pmatrix}, \quad (56)$$

where (for the sake of brevity we omit the argument μ)

$$U(\mu) = \left[\frac{\tilde{p} - p}{p(\lambda_1 - \lambda_2)} \right]^{1/2} \begin{pmatrix} 1 & 1 \\ \frac{p\lambda_2}{\tilde{p}-p} & \frac{p\lambda_1}{\tilde{p}-p} \end{pmatrix},$$

$$U^{-1}(\mu) = \left[\frac{\tilde{p} - p}{p(\lambda_1 - \lambda_2)} \right]^{1/2} \begin{pmatrix} \frac{p\lambda_1}{\tilde{p}-p} & -1 \\ -\frac{p\lambda_2}{\tilde{p}-p} & 1 \end{pmatrix}. \quad (57)$$

Now, the similarity transformation of D with the matrix $U(\mu)$ yields

$$U^{-1}DU = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 - v & \lambda_1 - v \\ v - \lambda_2 & v - \lambda_2 \end{pmatrix}. \quad (58)$$

Thus,

$$\text{Tr}[U^{-1}DUU^{-1}C^{L-1}(\mu)U] = \frac{1}{\lambda_1 - \lambda_2} [(\lambda_1 - v)\lambda_1^{L-1} - (\lambda_2 - v)\lambda_2^{L-1}] = \left(\sum_{m=0}^{L-1} \lambda_1^m \lambda_2^{L-1-m} - v \sum_{m=0}^{L-2} \lambda_1^m \lambda_2^{L-2-m} \right). \quad (59)$$

Next, we use the equalities,

$$\sum_{k=0}^n \lambda_1^k \lambda_2^{n-k} = \begin{cases} \sum_{m=0}^{\lfloor n/2 \rfloor} (\mu v)^m [\lambda_1^{n-2m} + \lambda_2^{n-2m}], & n = \text{odd}, \\ \sum_{m=0}^{n/2-1} (\mu v)^m [\lambda_1^{n-2m} + \lambda_2^{n-2m}] + (\mu v)^{n/2}, & n = \text{even}, \end{cases} \quad (60)$$

and take into account that in the remainder we will need the expression for

$$[\mu^q] \sum_{k=0}^n \lambda_1^k \lambda_2^{n-k} = \sum_{m=0}^{\lfloor n/2 \rfloor} v^m Z_{n-2m, q-m}, \quad (61)$$

which is independent of the parity of n . In deriving Eq. (61) we have taken into account that for n even one has $[\mu^q](\mu v)^{n/2} = \mu^{n/2} \delta_{q, n/2}$, which equals the summand $v^m Z_{n-2m, q-m}$ at the upper limit $m = n/2$; see Eq. (34).

Thus, the result for the particle density $\rho_{L,N} = N/L$ can be cast in the form of the identity

$$\frac{N}{L} Z_{L,N}(v) = Z_{L,N}(v) + \sum_{m=0}^{\lfloor L/2 \rfloor} v^m [Z_{L-1-2m, N-m-1}(v) - Z_{L-2m, N-m}(v)]. \quad (62)$$

Next, we derive a different expression, which follows from the definition of the average density of empty sites:

$$1 - \rho_{L,N}(v) = Z_{L,N}^{-1} [\mu^N] \text{Tr}[EC^{L-1}(\mu)], \quad (63)$$

which is, obviously, constant over the ring, equal to $1 - N/L$.

The above trace is readily calculated by using the similarity transform of the matrix E with the matrix $U(\mu)$, see Eq. (57),

$$U^{-1}EU = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1(1 - \lambda_2) & \lambda_1(1 - \lambda_1) \\ -\lambda_2(1 - \lambda_2) & -\lambda_2(1 - \lambda_1) \end{bmatrix}. \quad (64)$$

Then

$$[\mu^N] \text{Tr}(\mu U^{-1}EC^{L-1}(\mu)U) = [\mu^N] \frac{1}{\lambda_1 - \lambda_2} [(\lambda_1^L - \lambda_2^L) - \mu v (\lambda_1^{L-1} - \lambda_2^{L-1})]$$

$$= [\mu^N] \left[\sum_{m=0}^{L-1} \lambda_1^m \lambda_2^{L-1-m} - \mu v \sum_{m=0}^{L-2} \lambda_1^m \lambda_2^{L-2-m} \right]. \quad (65)$$

Due to the equality Eq. (61), the expression for the hole density $1 - N/L$ does not depend on the parity of L :

$$(1 - N/L) Z_{L,N}(v) = Z_{L,N}(v) + \sum_{m=0}^{\lfloor L/2 \rfloor} v^m [Z_{L-1-2m, N-m}(v) - Z_{L-2m, N-m}(v)]. \quad (66)$$

Finally, by taking the difference of Eqs. (62) and (66), we find

$$(2N/L - 1) Z_{L,N}(v) = \sum_{m=0}^{\lfloor L/2 \rfloor} v^m [Z_{L-1-2m, N-1-m}(v) - Z_{L-1-2m, N-m}(v)]. \quad (67)$$

Thus, here we have established three identities, Eqs. (62), (66), and (67), involving the particle density N/L for all $L \geq 1$ and $1 \leq N \leq L$.

C. The pair correlation function

The matrix-product form of the two-point particle-particle correlation function of the model is given by the expression

$$G_{L,N}(\tau_i = 1, \tau_j = 1) = [\mu^N] \frac{1}{Z_{L,N}} \text{Tr}[C^{i-1}(\mu) \mu D C^{j-i-1} \mu D C^{L-j}(\mu)] = [\mu^N] \frac{1}{Z_{L,N}} \text{Tr}[\mu D C^{j-i-1} \mu D C^{L-j+i-1}(\mu)]. \quad (68)$$

Consider first the nearest-neighbor particle-particle correlations, when $j = i + 1$. In this case Eq. (68) reduces to

$$G_{L,N}(\tau_i = 1, \tau_{i+1} = 1) = [\mu^N] \frac{1}{Z_{L,N}} \text{Tr}[\mu^2 D^2 C^{L-2}(\mu)] = \frac{1}{Z_{L,N}} [\mu^{N-2}] \text{Tr}[D C^{L-2}(\mu)] = \frac{N-1}{L-1} \frac{Z_{L-1,N-1}}{Z_{L,N}}. \quad (69)$$

Here we have taken into account the relationship $D^2 = D$ and used the definition of the particle density for a ring of length $L - 1$ sites having $N - 1$ particles; see Eq. (55).

Hence, one readily obtains the nearest-neighbor particle-hole correlation function

$$\begin{aligned} G_{L,N}(\tau_i = 1, \tau_{i+1} = 0) &= [\mu^N] \frac{1}{Z_{L,N}} \text{Tr}[\mu D E C^{L-2}(\mu)] = \frac{1}{Z_{L,N}} [\mu^N] \text{Tr}[\mu D C^{L-1}(\mu) - \mu D \mu D C^{L-2}(\mu)] \\ &= \frac{N}{L} - \frac{N-1}{L-1} \frac{Z_{L-1,N-1}}{Z_{L,N}}. \end{aligned} \quad (70)$$

It is interesting to note that quite a different in form representation for that function follows from the direct evaluation of the trace in the first line of Eq. (70). To this end we make use of the matrices $U(\mu)$ and $U^{-1}(\mu)$, which diagonalize $C(\mu)$, see Eqs. (56) and (57), to obtain

$$\text{Tr}\{D E C^n\} = \text{Tr}\{U^{-1} D E U \text{diag}\{\lambda_1^n, \lambda_2^n\}\}, \quad (71)$$

where $\text{diag}\{\lambda_1^n, \lambda_2^n\}$ denotes a diagonal 2×2 matrix with eigenvalues λ_1^n and λ_2^n . Taking into account that

$$U^{-1} D E U = \frac{\nu}{\lambda_1 - \lambda_2} \begin{pmatrix} (1 - \lambda_2)(\lambda_1/\nu - 1) & (1 - \lambda_1)(\lambda_1/\nu - 1) \\ (1 - \lambda_2)(1 - \lambda_2/\nu) & (1 - \lambda_1)(1 - \lambda_2/\nu) \end{pmatrix}, \quad (72)$$

we obtain

$$\begin{aligned} \text{Tr}\{D E C^n\} &= \frac{\nu}{\lambda_1 - \lambda_2} \left[- (1 + \lambda_1 \lambda_2 / \nu) (\lambda_1^n - \lambda_2^n) + \lambda_1 \lambda_2 (\lambda_1^{n-1} - \lambda_2^{n-1}) + (\lambda_1^{n+1} - \lambda_2^{n+1}) / \nu \right] \\ &= x \sum_{m=0}^n \lambda_1^m \lambda_2^{n-m} = \begin{cases} x \sum_{m=0}^{\lfloor n/2 \rfloor} (\mu \nu)^m [\lambda_1^{n-2m} + \lambda_2^{n-2m}], & n = \text{odd} \\ x \sum_{m=0}^{n/2-1} (\mu \nu)^m [\lambda_1^{n-2m} + \lambda_2^{n-2m}] + x (\mu \nu)^{n/2}, & n = \text{even}. \end{cases} \end{aligned} \quad (73)$$

In the above derivation we have used the equalities $x = 1 - \nu$, $\lambda_1 \lambda_2 = \mu \nu$, and $(1 + \mu) \lambda_2 = \mu \nu + \lambda_2^2$. Thus, taking into account equality Eq. (61), we find

$$G_{L,N}(\tau_i = 1, \tau_{i+1} = 0) = [\mu^{N-1}] \frac{1}{Z_{L,N}} \text{Tr}[D E C^{L-2}(\mu)] = \frac{1-\nu}{Z_{L,N}} \sum_{m=0}^{\lfloor (L-2)/2 \rfloor} \nu^m Z_{L-2-2m, N-1-m} = \frac{N}{L} - \frac{N-1}{L-1} \frac{Z_{L-1, N-1}}{Z_{L,N}}. \quad (74)$$

Now we turn to the pair correlations in the case of general separation between the sites, $j - i - 1 = n \geq 1$, and for brevity of notation denote $L - j + i - 1 = L - n - 2 = m$. With the use the similarity transformation Eq. (58) for D , and the diagonal form of the matrix C , we obtain for the trace in Eq. (68),

$$[\mu^N] \text{Tr}[\mu D C^n(\mu) \mu D C^m(\mu)] = \frac{[\mu^{N-2}]}{(\lambda_1 - \lambda_2)^2} [\lambda_1^{n+1} - \lambda_2^{n+1} - \nu(\lambda_1^n - \lambda_2^n)] [\lambda_1^{m+1} - \lambda_2^{m+1} - \nu(\lambda_1^m - \lambda_2^m)]. \quad (75)$$

Next, having in mind that

$$\lambda_1^n - \lambda_2^n = (\lambda_1 - \lambda_2) \sum_{k=0}^{n-1} \lambda_1^k \lambda_2^{n-1-k}, \quad (76)$$

and applying equality Eq. (61), we find

$$\begin{aligned}
 [\mu^{N-2}] \text{Tr}[DC^n(\mu)DC^m(\mu)] &= \sum_{q=0}^{N-2} [\mu^q] \left(\sum_{k=0}^n \lambda_1^k \lambda_2^{n-k} - \nu \sum_{k=0}^{n-1} \lambda_1^k \lambda_2^{n-1-k} \right) [\mu^{N-2-q}] \left(\sum_{l=0}^m \lambda_1^l \lambda_2^{m-l} - \nu \sum_{l=0}^{m-1} \lambda_1^l \lambda_2^{m-1-l} \right) \\
 &= \sum_{q=0}^{N-2} [Z_{n+1,q+1} + \sum_{k=0}^{[(n+1)/2]} \nu^k [Z_{n-2k,q-k} - Z_{n+1-2k,q+1-k}]] \\
 &\quad \times \left[Z_{m+1,N-1-q} + \sum_{l=0}^{[(m+1)/2]} \nu^l [Z_{m-2l,N-2-q-l} - Z_{m+1-2l,N-1-q-l}] \right]. \tag{77}
 \end{aligned}$$

Here we have taken into account that for n even $[n/2] = [(n+1)/2] = n/2$, and for n odd $Z_{n-2k,q-k}$ vanishes at the upper limit $k = (n+1)/2$; similarly, for m odd $Z_{m-2l,N-2-q-l}$ vanishes at the corresponding the upper limit $l = (m+1)/2$.

The above expression essentially simplifies by noting that the substitution $L = n+1$, $N = q+1$ in Eq. (62) yields for the summand in the first square brackets in the right-hand side of Eq. (77),

$$Z_{n+1,q+1} + \sum_{k=0}^{[(n+1)/2]} \nu^k [Z_{n-2k,q-k} - Z_{n+1-2k,q+1-k}] = \frac{q+1}{n+1} Z_{n+1,q+1}. \tag{78}$$

The expression for the summand in the second square brackets in the right-hand side of Eq. (77) follows from Eq. (78) under the replacement $n \rightarrow m = L-2-n$, and $q \rightarrow N-2-q$:

$$Z_{m+1,N-1-q} + \sum_{l=0}^{[(m+1)/2]} \nu^l [Z_{m-2l,N-2-q-l} - Z_{m+1-2l,N-1-q-l}] = \frac{N-1-q}{L-1-n} Z_{L-1-n,N-1-q} \quad (m = L-2-n). \tag{79}$$

Thus, we obtain for the pair correlation between particles at sites i and $j = i+1+r$:

$$\begin{aligned}
 G_{L,N}(\tau_i = 1, \tau_{i+1+r} = 1; \tilde{p}, p) &:= F_{L,N}^{1,1}(r; \nu) = \frac{1}{Z_{L,N}(\tilde{p}, p)} \sum_{q=\max\{0, r-L+N\}}^{\min\{r, N-2\}} \frac{q+1}{r+1} Z_{r+1, q+1}(\tilde{p}, p) \frac{N-1-q}{L-1-r} Z_{L-1-r, N-1-q}(\tilde{p}, p) \\
 &= \frac{1}{Z_{L,N}(\nu)} \sum_{q=\max\{0, r-L+N\}}^{\min\{r, N-2\}} \binom{r}{q} \binom{L-2-r}{N-2-q} {}_2F_1(-q-1, q-r; -r; \nu) \\
 &\quad \times {}_2F_1(q+1-N, -L+N-q+r; -L+2+r; \nu). \tag{80}
 \end{aligned}$$

This expression is valid for any $r \geq 0$. Indeed, at $r = 0$ it reduces to

$$F_{L,N}(0; \nu) = \frac{1}{Z_{L,N}(\tilde{p}, p)} Z_{1,1}(\tilde{p}, p) \frac{N-1}{L-1} Z_{L-1, N-1}(\tilde{p}, p), \tag{81}$$

which, in view of $Z_{1,1}(\tilde{p}, p) = 1$, coincides with the pair correlation function Eq. (69).

Obviously, the pair correlation function Eq. (80) is invariant with respect to exchanging the place of the two distances between the particles on the ring, $r \leftrightarrow L-2-r$.

Remarkably, in the case of the backward sequential update, when $\tilde{p} = p$ ($\nu = 0$), the pair correlation function $F_{L,N}^{1,1}(r; 0)$ becomes constant, independent of both the distance r and the jump probability p . Indeed, in this case, taking into account Eq. (35), we obtain

$$F_{L,N}^{1,1}(r; p, p) = \frac{1}{Z_{L,N}(p, p)} \sum_{q=\max\{0, r-L+N\}}^{\min\{r, N-2\}} \frac{q+1}{r+1} \binom{r+1}{q+1} \frac{N-1-q}{L-1-r} \binom{L-1-r}{N-1-q} = \binom{L}{N}^{-1} \binom{L-2}{N-2} = \frac{N(N-1)}{L(L-1)}. \tag{82}$$

Another exact analytic expression follows in the limit $\tilde{p} \rightarrow 1$, i.e., $x = (1-\tilde{p})/(1-p) \rightarrow 0$. This case models a deterministic (irreversible) aggregation of one-dimensional driven lattice gas. Now, it is convenient to use representation Eq. (30) for the partition function, since

$$\begin{aligned}
 Z_{L,N}(\tilde{p} \rightarrow p) &= Lx + O(x^2), \quad \text{when } N \neq 0, L, \quad \text{and} \\
 Z_{L,L}(\tilde{p}, p) &= 1, \quad L \geq 1. \tag{83}
 \end{aligned}$$

Hence, assuming $N < L$, a nonzero contribution in the sum over q in Eq. (80) for the pair correlation function will come from the following terms:

(1) The first partition function in the numerator becomes $Z_{r+1, r+1} = 1$, and the other multipliers are nonzero. This takes place for all $0 \leq r \leq N-2$, when $q = r$;

(2) The second partition function $Z_{L-1-r, L-1-r} = 1$ and all the prefactors are nonvanishing. That occurs for all $L-N \leq r \leq L-2$, when $q = r-L+N$.

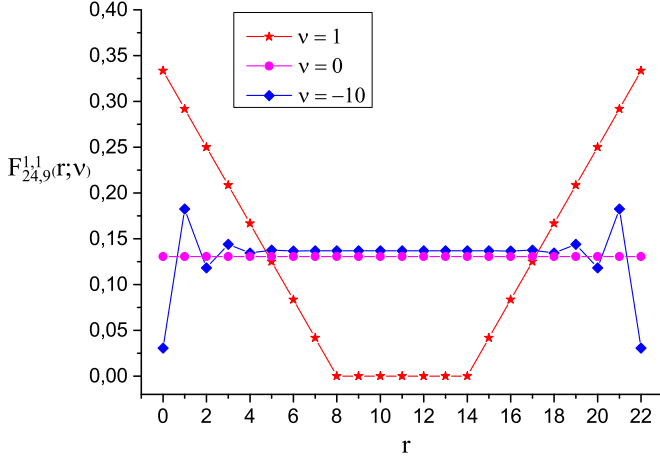


FIG. 1. The dependence on the distance r of the particle-particle correlation function $F_{24,9}^{1,1}(r; \nu)$ on a ring of $L = 24$ sites with $N = 9$ particles: (a) in the limit $\nu \rightarrow 1$, when only configurations containing a single cluster survive (red stars); (b) at $\nu = 0$, when the correlations are constant (magenta disks), and (c) at $\nu = -10$, when marked anticorrelations appear at the wings (blue rotated squares).

Thus, we obtain

$$F_{L,N}^{1,1}(r; 1) = \begin{cases} (N-1-r)/L, & 0 \leq r \leq N-2 \\ (r-L+N+1)/L, & L-N \leq r \leq L-2, \end{cases} \quad (84)$$

and $F_{L,N}^{1,1}(r; 1) \equiv 0$, if $N-2 < r < L-N$. This behavior is illustrated in Fig. 1 for the case of $L = 24$ and $N = 9$. The shape of the correlation function is readily explainable by the fact that in the limit $\tilde{p} \rightarrow 1$ the configurations of the lowest order in $x \rightarrow 0$ are those in which all the N particles constitute a single cluster.

In the considered limit $\tilde{p} \rightarrow 1$, the value of the nearest-neighbor correlation function is $F_{L,N}^{1,1}(0; 1) = (N-1)/L$, and the maximum distance at which nonvanishing particle-particle correlations occur is $r = N-2$, when $F_{L,N}^{1,1}(N-2; 1) = 1/L$. The above facts are in exact conformity with the single-cluster stationary state of the model in the deterministic aggregation regime.

Another interesting observation concerns the case $2N \geq L+2$, when the summation in Eq. (80) allows for simultaneous contribution from both partition functions in the numerator. That takes place for distances $L-N \leq r \leq N-2$, and the sum of the two results in the right-hand side of Eq. (84) gives the constant value: Thus, we obtain

$$F_{L,N}^{1,1}(r; 1) = (2N-L)/L, \quad \text{when} \\ L-N \leq r \leq N-2. \quad (85)$$

Actually, this nonzero flat bottom of the pair correlation function extends in the somewhat larger interval $L-N-1 \leq r \leq N-1$, since each of the endpoints of that interval contributes the same value,

$$F_{L,N}^{1,1}(r = L-N-1; 1) = F_{L,N}^{1,1}(r = N-1; 1) \\ = (2N-L)/L, \quad (86)$$

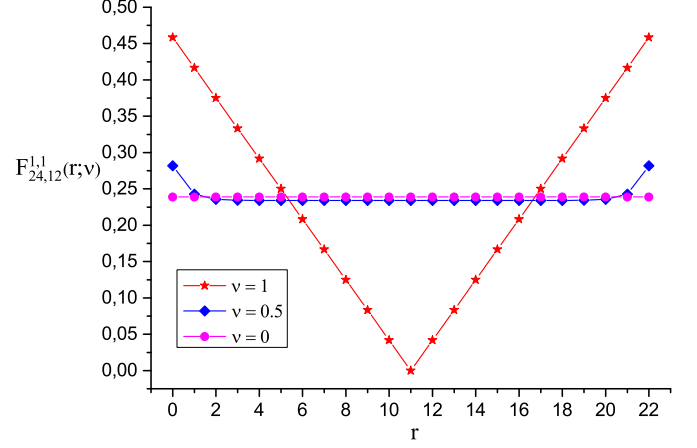


FIG. 2. The dependence on the distance r of the particle-particle correlation function $F_{24,12}^{1,1}(r; \nu)$ on a ring of $L = 24$ sites with $N = 12$ particles: (a) in the limit $\nu \rightarrow 1$, when only single-cluster configurations contribute (red stars), (b) at $\nu = 0.5$, when the graph is considerably more smooth and flat (blue rotated squares), and (c) at $\nu = 0$, when the correlations are constant (magenta disks).

coming from the second partition function the numerator of Eq. (80), when $q = r = L - N - 1$, and from the first partition function when $r = N - 1$ and $q = 2N - L - 1$. Thus, the flat bottom at value $(2N-L)/L > 0$ occurs whenever $2N \geq L+1$; when $2N = L$, the function $F_{L,L/2}^{1,1}(r; 1)$ is V-shaped, and vanishes at the single-site bottom at $r = L/2 - 1$; see Fig. 2.

Obviously, positive values of $\nu < 1$ lead to more smooth and flat graph of the correlation function which, as $\nu \rightarrow 0$, approaches the constant value Eq. (82) describing a completely uniform distribution of particles in the stationary state; see Fig. 2.

On the other hand, negative values of ν mark the tendency of splitting clusters of particles into smaller ones by the stochastic dynamics. An extreme case is provided by the parallel dynamics, when $\tilde{p} = 0$, hence $\nu = -p/(1-p)$. Nearest-neighbor particle-hole effective attraction, or particle-particle anticorrelations like those observed on Fig. 1 at $\nu = -10$, were analytically obtained in the thermodynamic limit for TASEP with parallel update and ring geometry in Ref. [65]; see also Ref. [56].

D. The current

To prove the matrix-product representation for the current Eq. (46),

$$J_{L,N}(p, \tilde{p}) = \frac{P}{Z_{L,N}(p, \tilde{p})} \sum_{k=0}^{N-1} \tilde{p}^k [\mu^{N-k}] \text{Tr}\{\mu DEC^{L-2-k}\}, \quad (87)$$

we make use of Eq. (73) for $\text{Tr}\{DEC^n(\mu)\}$ and equality Eq. (61) for $[\mu^q] \text{Tr}\{DEC^n(\mu)\}$. For brevity of notation, we

set $a = L - 2 - k$, $b = N - 1 - k$, and write

$$\begin{aligned} J_{L,N}(p, \tilde{p}) &= \frac{P}{Z_{L,N}(p, \tilde{p})} \sum_{k=0}^{N-1} \tilde{p}^k [\mu^{N-k-1}] \text{Tr}\{DEC^{L-2-k}\} = \frac{xP}{Z_{L,N}(p, \tilde{p})} \sum_{k=0}^{N-1} \tilde{p}^k \sum_{m=0}^{\lfloor a/2 \rfloor} v^m [\mu^{b-m}] [\lambda_1^{a-2m} + \lambda_2^{a-2m}] \\ &= \frac{xP}{Z_{L,N}(p, \tilde{p})} \sum_{k=0}^{N-1} \tilde{p}^k \sum_{m=0}^{\lfloor a/2 \rfloor} v^m Z_{a-2m, b-m}(p, \tilde{p}). \end{aligned} \quad (88)$$

To bring the above expression to the form of Eq. (46), we consider

$$\begin{aligned} \sum_{m=0}^{\lfloor a/2 \rfloor} v^m Z_{a-2m, b-m}(p, \tilde{p}) &= \sum_{m=0}^{\lfloor a/2 \rfloor} v^m (a-2m) \sum_{n \geq 0} \frac{(-v)^n (a-2m-n-1)!}{n!(b-m-n)!(a-b-m-n)!} \\ &= \sum_{j=0}^{M(a,b)} \frac{(-v)^j}{(b-j)!(a-b-j)!} \sum_{m=0}^j \frac{(-1)^m (a-2m)(a-m-j-1)!}{(j-m)!}. \end{aligned} \quad (89)$$

Here, in exchanging the order of summation over m and j , we have taken into account that $\lfloor a/2 \rfloor \geq M(a,b) \equiv \min\{a-b, b\}$. Now we prove the equality

$$\sum_{m=0}^j (-1)^m \frac{(a-2m)(a-1-j-m)!}{(j-m)!} - (-1)^j \delta_{a,2j} = \frac{(a-j)!}{j!} \quad (a \geq 2j), \quad (90)$$

valid for $a \geq 2j$. The calculation of the sum here is straightforward when we consider the cases of $a \geq 2j+1$ and $a = 2j$ separately.

(1) First we assume $a \geq 2j+1$ and set $n = a - 2j - 1 \geq 0$. Then the left-hand side of Eq. (90) becomes

$$\sum_{m=0}^j (-1)^m \frac{[1+n+2(j-m)](n+j-m)!}{(j-m)!} = (-1)^j \sum_{m=0}^j (-1)^m \frac{(1+n+2m)(n+m)!}{m!}. \quad (91)$$

Now we perform the summation as follows:

$$\begin{aligned} (-1)^j \sum_{m=0}^j (-1)^m \frac{(1+n+2m)(n+m)!}{m!} &= (-1)^j (n+1)! \sum_{m=0}^j (-1)^m \left[\binom{n+m+1}{n+1} + \binom{n+m}{n+1} \right] \\ &= (-1)^j (n+1)! \left[-\sum_{m=1}^{j+1} (-1)^m \binom{n+m}{n+1} + \sum_{m=1}^j (-1)^m \binom{n+m}{n+1} \right] \\ &= \frac{(n+1+j)!}{j!} = \frac{(a-j)!}{j!}, \end{aligned} \quad (92)$$

which is exactly the right-hand side of Eq. (90).

(2) Finally, let $a = 2j$. Then the left-hand side of Eq. (90) becomes

$$\sum_{m=0}^j (-1)^m \frac{2(j-m)(j-1-m)!}{(j-m)!} - (-1)^j = 2 \sum_{m=0}^j (-1)^m - (-1)^j = 1, \quad (93)$$

which equals the right-hand side of Eq. (90) at $a = 2j$. This completes the proof of the equality under the given condition.

Now, we insert equality Eq. (90) for $a > 2j$ into Eq. (89) to obtain

$$\sum_{m=0}^{\lfloor a/2 \rfloor} v^m Z_{a-2m, b-m}(p, \tilde{p}) = \sum_{j=0}^{M(a,b)} \frac{(-v)^j (a-j)!}{j!(b-j)!(a-b-j)!}. \quad (94)$$

Finally, after restoring the original values of a and b , and inserting the above result into Eq. (88), we recover representation Eq. (46) for the current.

Let us show now how Eq. (88) reproduces in the thermodynamic limit the well-known result for the backward-ordered sequential update. When $\tilde{p} = p$, hence $x = 1$ and $v = 0$, Eq. (88) reduces to

$$J_{L,N}(p, p) = \frac{P}{Z_{L,N}(p, p)} \sum_{k=0}^{N-1} p^k Z_{L-2-k, N-1-k}(p, p). \quad (95)$$

Note that Eq. (32) at $\nu = 0$ yields

$$Z_{L,N}(p,p) = \frac{L!}{N!(L-N)!} = \binom{L}{N}. \quad (96)$$

Therefore, Eq. (95) can be written as

$$J_{L,N}(p,p) = p \binom{L-2}{N-1} \binom{L}{N}^{-1} \sum_{k=1}^{N-1} p^k \binom{L-2-k}{N-1-k} \binom{L-2}{N-1}^{-1} = p \frac{N(L-N)}{L(L-1)} \left[1 + \sum_{k=1}^{N-1} c_{L,N}(k) p^k \right], \quad (97)$$

where

$$c_{L,N}(k) = \prod_{n=1}^k \frac{N-n}{L-1-n}. \quad (98)$$

Taking the limit $L \rightarrow \infty$ with $N = \rho L$ at fixed ρ and k , we obtain

$$\lim_{L \rightarrow \infty} c_{L,\rho L}(k) = \rho^k, \quad (99)$$

and

$$\lim_{L \rightarrow \infty} J_{L,\rho L}(p,p) = p\rho(1-\rho) \sum_{k=0}^{\infty} (\rho p)^k = p \frac{\rho(1-\rho)}{1-\rho p} \quad (100)$$

is the well-known result for the thermodynamic current in the TASEP with backward-ordered update [34].

Finally, from the general expression Eq. (88) we derive the well-known result for the thermodynamic limit of the current in TASEP with parallel update. In this case $\tilde{p} = 0$ and Eq. (88) reduces to

$$J_{L,N}(p,0) = \frac{p}{Z_{L,N}(p,0)} \sum_{k=1}^{M(L,N)} x^k \binom{L-N-1}{k-1} \binom{N-1}{k-1}, \quad (101)$$

where $x = (1-p)^{-1}$, hence $p = 1-x^{-1}$. On the other hand, from the expression for the partition function Eq. (30) at $\tilde{p} = 0$ we have

$$Z_{L,N}(p,0) = L \sum_{k=1}^{M(L,N)} \frac{x^k}{k} \binom{L-N-1}{k-1} \binom{N-1}{k-1}. \quad (102)$$

From Eqs. (101) and (102) it follows that

$$J_{L,N}(p,0) = \frac{p x}{L} \frac{\partial}{\partial x} \ln Z_{L,N}(1-x^{-1},0) \Big|_{x=(1-p)^{-1}}. \quad (103)$$

This remarkable expression allows us to evaluate $\lim_{L \rightarrow \infty} J_{L,\rho L}(p,0)$ by using the leading order approximation for $Z_{L,\rho L}(p,0)$ as $L \rightarrow \infty$. To this end, we set in Eq. (102) $N = \rho L$ and $k = yL$, and evaluate the sum by the corresponding Laplace integral:

$$Z_{L,\rho L}(p,0) \propto \int_{y=1/L}^{\min\{1-\rho,\rho\}} dy \exp[L S(x,y)], \quad (104)$$

where

$$\begin{aligned} S(x,y) &= (1-\rho) \ln(1-\rho) + \rho \ln \rho \\ &+ y \ln x - (1-\rho-y) \ln(1-\rho-y) \\ &- (\rho-y) \ln(\rho-y) - 2y \ln y. \end{aligned} \quad (105)$$

Hence, one readily finds the equation for the stationary point of $S(x,y)$ as a function of y :

$$(1-x^{-1})y^2 - y + \rho(1-\rho) = 0, \quad (106)$$

and the solution

$$y = \bar{y}(x,\rho) \equiv \frac{1 - \sqrt{1 - 4(1-x^{-1})\rho(1-\rho)}}{2(1-x^{-1})}, \quad (107)$$

at which $S(x,y)$ attains its maximum with respect to y in the interval $1/L \leq y \leq \min\{1-\rho,\rho\}$. Thus, we obtain

$$\lim_{L \rightarrow \infty} \frac{1}{L} \ln Z_{L,\rho L}(p,0) = S(x,\bar{y}(x,\rho)), \quad (108)$$

which implies

$$\begin{aligned} \lim_{L \rightarrow \infty} J_{L,\rho L}(p,0) &= \bar{y}(x = (1-p)^{-1}, \rho) \\ &= \frac{1 - \sqrt{1 - 4p\rho(1-\rho)}}{2}. \end{aligned} \quad (109)$$

This is exactly the thermodynamic limit for the current of particles in the TASEP with parallel update; see, e.g., Ref. [34].

VI. DISCUSSION

We have studied a version of the TASEP with a generalized discrete-time dynamics described by two hopping probabilities, p and \tilde{p} , within the MPA approach to stationary stochastic states. The model is considered on a ring of finite number of sites, labeled in counterclockwise order from 1 to L . The configurations with a fixed number of particles N are updated in a cluster-oriented clockwise order, starting with a particle which has a vacant nearest-neighbor site in the hopping direction. If the particle is isolated, or if it is the first particle to be updated in a cluster of particles, then it can jump to its nearest-neighbor site in the counterclockwise direction with probability p , and stay immobile with probability $1-p$. On the other hand, if the particle is not the rightmost one in a cluster of particles (before the update), and if the site in front of it has been emptied in the same update, then the particle can jump ahead with probability \tilde{p} , or stay immobile with probability $1-\tilde{p}$. Therefore, $k \geq 1$ particles will chip off a cluster of length $n > k$ with probability $(1-\tilde{p})\tilde{p}^{k-1}$. Following Ref. [61], we denote the studied model by gTASEP. The gTASEP contains as special cases the TASEP with parallel update, when $\tilde{p} = 0$, and with sequential backward-ordered update, when $\tilde{p} = p$. It belongs to the most general class of discrete-mass transport models, and was recognized as its exactly solvable representative $p w^{N-1}$ in the unpublished Ref. [64].

In Ref. [64] the gTASEP was solved in the thermodynamic limit by using a mean-field theory, and for finite L and N in the partially deterministic case $p = 1$. Unfortunately, our matrix representation of the underlying algebra is singular at $p = 1$ and we cannot cover that case.

In Ref. [61] the stationary properties of the model were obtained by using a mapping onto the zero-range process. The pair correlation function was derived by using the transfer-matrix method in the grand canonical ensemble with a subsequent choice of the fugacity yielding the prescribed average density of particles in the thermodynamic limit.

In contrast, we have worked entirely in the ensemble with fixed number of particles on a finite ring, which has lead to more involved calculations. The main aim of the paper was to construct a two-dimensional matrix-product representation for the gTASEP and to use it for the derivation of exact finite-size expressions for the partition function, the current of particles and the two-point correlation function. Our results for the partition function and the current were checked against expressions independently derived by combinatorial methods, as well as by comparison with the results of Ref. [61] and the well-known ones for the particular cases of parallel and

backward-ordered updates. The obtained expression for the gTASEP partition function is related to the corresponding ZRP by the constant factor $L/(L - N)$, just as the densities and the currents of the two processes are related. This constant factor is due to the different number of configurations in the TASEP and ZRP. Our main new result is the derivation of the finite-size pair correlation function by the MPA method. Its behavior is analyzed in different regimes of effective attraction and repulsion between the particles, depending on whether $\tilde{p} > p$ or $\tilde{p} < p$. In particular, we have explicitly obtained an analytic expression for the pair correlation function in the limit of irreversible aggregation $\tilde{p} \rightarrow 1$, which has confirmed the expectation that in that limit the stationary configurations contain just one cluster.

As a future continuation of the present study we intend to attempt the construction of (infinite-dimensional) matrix-product representations of the gTASEP on open chains.

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