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Repulsive particles under a general external potential: Thermodynamics by neglecting thermal noise

Mauricio S. Ribeiro[1,*] and Fernando D. Nobre[1,2,†]
[1]*Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil*
[2]*National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil*

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A recent proposal of an effective temperature θ, conjugated to a generalized entropy s_q, typical of nonextensive statistical mechanics, has led to a consistent thermodynamic framework in the case $q = 2$. The proposal was explored for repulsively interacting vortices, currently used for modeling type-II superconductors. In these systems, the variable θ presents values much higher than those of typical room temperatures T, so that the thermal noise can be neglected ($T/\theta \simeq 0$). The whole procedure was developed for an equilibrium state obtained after a sufficiently long-time evolution, associated with a nonlinear Fokker-Planck equation and approached due to a confining external harmonic potential, $\phi(x) = \alpha x^2/2$ ($\alpha > 0$). Herein, the thermodynamic framework is extended to a quite general confining potential, namely $\phi(x) = \alpha|x|^z/z$ ($z > 1$). It is shown that the main results of the previous analyses hold for any $z > 1$: (i) The definition of the effective temperature θ conjugated to the entropy s_2. (ii) The construction of a Carnot cycle, whose efficiency is shown to be $\eta = 1 - (\theta_2/\theta_1)$, where θ_1 and θ_2 are the effective temperatures associated with two isothermal transformations, with $\theta_1 > \theta_2$. The special character of the Carnot cycle is indicated by analyzing another cycle that presents an efficiency depending on z. (iii) Applying Legendre transformations for a distinct pair of variables, different thermodynamic potentials are obtained, and furthermore, Maxwell relations and response functions are derived. The present approach shows a consistent thermodynamic framework, suggesting that these results should hold for a general confining potential $\phi(x)$, increasing the possibility of experimental verifications.

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I. INTRODUCTION

The connection of the entropy concept with dynamics represents an outstanding result of nonequilibrium statistical mechanics [1–3]. Statistical entropy is defined as a functional depending only on the probabilities of a physical system [2], i.e., $s \equiv s\{P_i(t)\}$ [for a discrete set of states, where $P_i(t)$ represents the probability for finding the system in a state i, at time t], or $s \equiv s[P(x,t)]$ (for continuous states, where x usually denotes the position in one-dimensional space). The association with dynamics appears through the following properties: (i) it may be extremized under certain constraints in order to yield an equilibrium probability that coincides with the stationary-state distribution obtained from some equations describing the time evolution of the probabilities (e.g., the Fokker-Planck equation [4]); and (ii) it follows an H-theorem, which is proven by considering the statistical entropy and a given equation for the time evolution of the probabilities [1–4].

These associations have been extended recently for generalized entropic forms, mostly by making use of nonlinear Fokker-Planck equations (NLFPEs) [5–16], relating many nonlinear phenomena occurring in complex systems, e.g., anomalous diffusion, as well as entropy production [17], with generalized entropies. A particular interest has been dedicated to the NLFPE leading to anomalous diffusion [18,19], which is associated with Tsallis entropy [20], and it has produced an appropriate framework for dealing with a wide range of natural phenomena [21–23].

Recently, a system of particles interacting repulsively, under overdamped motion, has appeared as a potential physical application for nonextensive statistical mechanics, being associated with Tsallis entropy with an entropic index $q = 2$ [24–33]. This connection has emerged by means of a coarse-graining procedure in the corresponding equations of motion, leading to the following NLFPE [24–28]:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + 2D\,\frac{\partial}{\partial x}\left\{[\lambda P(x,t)]\frac{\partial P(x,t)}{\partial x}\right\} + kT\,\frac{\partial^2 P(x,t)}{\partial x^2}, \qquad (1.1)$$

where λ is a characteristic length of the system, whereas D results from the coarse-graining procedure, being directly related to particle-particle interactions, and thus it depends on the physical system under investigation. The equation above also takes into account the effects of a heat bath at a temperature T (with k denoting the Boltzmann constant), whose contribution may be obtained in the standard way, through the introduction of thermal noise in the system [1]. Additionally, $A(x) = -d\phi(x)/dx$ corresponds to an external force derived from a confining potential $\phi(x)$, being fundamental for the approach to equilibrium, as well as for the resulting form of the probability distribution. For a system of N particles in a rectangular box, the parameter D is given by $D = aN/(2\lambda L_y)$, with L_y corresponding to the size of the system in its y direction, whereas a is defined, within the coarse-graining procedure, as an integral over space of the particle-particle interactions [24–28]. For repulsive particles with forces characterized by an amplitude f_0 ($f_0 > 0$) and a characteristic length λ, one has $a = 2\pi f_0 \lambda^3$ for vortices in type-II superconductors [26–28], as well as for charged particles, interacting through a Yukawa potential [27], these later systems being considered as relevant for describing dusty plasmas and colloidal suspensions [34–38]. Moreover,

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[*] ribeiro@cbpf.br

[†] Corresponding author: fdnobre@cbpf.br

for particles interacting through a Gaussian core potential, useful in complex fluids such as polymer chains dispersed in a solvent [38,39], one obtains $a = \pi f_0 \lambda^3/2$ [27].

In this work, we shall restrict ourselves to the particle-particle interactions considered in Refs. [25–27], typical of vortices in type-II superconductors. In this case, the vortex-vortex interactions are repulsive and radially symmetric, being given by [24,40–43]

$$\mathbf{F}_i^{\mathrm{pp}} = \frac{f_0}{2} \sum_{j \neq i} K_1(r_{ij}/\lambda)\, \hat{\mathbf{r}}_{ij}, \tag{1.2}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ stands for the distance between vortices i and j, and $\hat{\mathbf{r}}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)/r_{ij}$ is a vector defined along the axis joining them. Moreover, K_1 represents a modified Bessel function of the second kind of order 1; the quantities f_0 and λ present constant values for each type-II superconducting material [40], the latter corresponding to the London penetration length. The vortices penetrate a two-dimensional box of dimensions L_x and L_y, which are measured in units of length λ. Within the coarse-graining approach, the above interactions lead to the nonlinear diffusion term in Eq. (1.1), where its coefficient provides the definition of an effective temperature θ [29],

$$k\theta \equiv D = \frac{N \pi f_0 \lambda^2}{L_y} = n \pi f_0 \lambda^2. \tag{1.3}$$

One should notice that θ is proportional to the linear density $n = N/L_y$, as well as to the intensity of interactions between vortices f_0, being always positive; moreover, according to recent advances in experimental techniques, the density of vortices became a controllable quantity [44–47], leading to the desirable possibility of a variable effective temperature. The quantity $k\theta$ presents dimensions of energy, and it was shown recently to be much larger than typical thermal energies of a type-II superconducting phase, i.e., $k\theta \gg kT$ [29], so that Eq. (1.1) may be approximated by

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial [A(x)P(x,t)]}{\partial x} + 2D\, \frac{\partial}{\partial x}\left\{[\lambda P(x,t)] \frac{\partial P(x,t)}{\partial x}\right\}. \tag{1.4}$$

The stationary-state solution of the above equation may be found for any integrable confining potential $\phi(x)$,

$$P_{\mathrm{st}}(x) = \frac{1}{2k\theta\lambda}[\phi_0 - \phi(x)]_+, \tag{1.5}$$

where $[y]_+ = y$, for $y > 0$ (zero otherwise), and ϕ_0 is an integration constant [48]. In the particular case of a harmonic potential, $\phi(x) = \alpha x^2/2$, one has [25,26]

$$P_{\mathrm{st}}(x) = \frac{\alpha}{4k\theta\lambda}(x_e^2 - x^2) = \frac{\alpha\lambda}{4k\theta}\left[\left(\frac{x_e}{\lambda}\right)^2 - \left(\frac{x}{\lambda}\right)^2\right], \tag{1.6}$$

with $|x| < x_e$, where $x_e = (3k\theta\lambda/\alpha)^{1/3}$ is found from the normalization condition for $P_{\mathrm{st}}(x)$.

In fact, for the harmonic external potential, and considering an initial condition $P(x,0) = \delta(x)$, one can show that the time-dependent solution of Eq. (1.4) falls in the class of q-Gaussian distributions, typical of nonextensive statistical mechanics [21–23],

$$P(x,t) = B(t)[1 - b(t)x^2]_+. \tag{1.7}$$

In the equation above, the time-dependent parameters $B(t)$ and $b(t)$ are related to each other to yield a normalized distribution for all times [18,19]. Hence, $P(x,t)$ presents a compact support in the interval $[-\bar{x}(t),\bar{x}(t)]$, where $\bar{x}(t)$ increases in time following $\bar{x}(t) = b^{-1/2}(t)$, so that $\lim_{t\to\infty} \bar{x}(t) = x_e$.

In addition to this, one may prove the H-theorem using Eq. (1.4) and imposing a well-defined sign for the time derivative of the one-vortex free-energy functional [25–27],

$$f = u - \theta s_2, \quad u = \int_{-\bar{x}(t)}^{\bar{x}(t)} dx\, \phi(x) P(x,t), \tag{1.8}$$

where θ has to be identified precisely as the effective temperature of Eq. (1.3). As usual, the internal energy is defined solely in terms of the external potential; the contribution of the interactions among vortices appears through the parameter θ. To satisfy the H-theorem, the associated entropy should be given by [25–27]

$$s_2[P] = k\left\{1 - \lambda \int_{-\bar{x}(t)}^{\bar{x}(t)} dx [P(x,t)]^2\right\}, \tag{1.9}$$

which corresponds to the particular case $q = 2$ of Tsallis entropy s_q [21–23]. It is important to remember that the distribution in Eq. (1.7) coincides with the one obtained through an extremization of the above entropy, considering the constraint for probability normalization, in addition to the constraint of Eq. (1.8) for the internal energy.

Recent works have indicated the entropy $s_2[P]$ as deeply associated with the present system [29–33], and herein we will reinforce its relevance. It should be mentioned that this entropic form has already been useful in other systems, being directly related to the enstrophy $\mathcal{Z}_2[P]$ [49,50] by means of

$$s_2[P] = k\,\{1 - 2\lambda \mathcal{Z}_2[P]\}. \tag{1.10}$$

From the above relation, one sees that the enstrophy $\mathcal{Z}_2[P]$ tends to decrease as $s_2[P]$ increases in time (considering an isolated system); the former quantity has shown its importance for fluids, where it is defined as an integral of the square of the vorticity. Within the context of relative entropies (also known as mutual information measures, or f divergences), the corresponding generalization of the Kullback-Leibler measure of information [defined as the Boltzmann-Gibbs entropy for two distributions $P(x,t)$ and $P_0(x,t)$] has led to the mutual information measure $\mathcal{I}_q[P_0(x,t),P(x,t)]$ [51], expressed in terms of the entropy s_q. The particular case $\mathcal{I}_2[P_0(x,t),P(x,t)]$ is also very important within the context of information theory, being directly related to well-known quantities, such as the Pearson and Neyman chi-squared divergences, χ^2_{P} and χ^2_{N}, respectively [52–54].

Due to the H-theorem, the stationary-state distribution of Eq. (1.6) is considered as an equilibrium distribution. Based on this, a consistent thermodynamic framework has been developed recently, using the effective temperature introduced in Eq. (1.3) as the parameter thermodynamically conjugated to the entropy of Eq. (1.9) [29–33]. However, the general form of the H-theorem proven in Refs. [12–14] guarantees that the stationary-state solution of Eq. (1.5) is an equilibrium distribution for any confining potential $\phi(x)$. The purpose of the present work is to show that such a thermodynamic framework is valid for a quite general class of powerlike, nonharmonic potentials; the present results suggest that this scheme should hold for any integrable and differentiable (at least once) confining potential. In the next section, we define the confining potential, present the corresponding equilibrium distribution, and calculate the internal energy and entropy in Eqs. (1.8) and (1.9), showing the consistency of the effective-temperature definition of Eq. (1.3). In Sec. III, we propose a form for the first law of thermodynamics, within the present nonextensive context, and derive the equation of state. In Sec. IV, we define physical transformations and study two thermodynamic cycles, namely the Carnot and Otto cycles. It is shown that the Carnot cycle is very special, preserving the well-known form of standard thermodynamics for its efficiency, contrary to the Otto cycle, whose efficiency presents a dependence on both entropy and external potential parameters. In Sec. V, we explore further the first law by applying Legendre transformations, deriving thermodynamic potentials and Maxwell relations. For completeness, we show that the response functions are very similar to those found in standard thermodynamics. One should emphasize that all results obtained herein recover the previous ones, obtained for the confining harmonic potential as a particular case [29–32]. Finally, in Sec. VI, we present our conclusions.

II. EXTERNAL POTENTIAL AND EQUILIBRIUM STATE

Herein we will extend the harmonic potential employed in Refs. [25,26,29–33] to a quite general form; the potential, and associated confining force that appears in Eq. (1.4), will be considered as

$$\phi(x) = \frac{\alpha|x|^z}{z},$$
$$A(x) = -\frac{d\phi(x)}{dx} = -\alpha|x|^{z-1}\,\mathrm{sgn}(x) \quad (z>1). \tag{2.1}$$

The motivation for the above potential is to amplify the range of validity of the recently introduced thermodynamics framework, recovering the previous studies in the particular case $z=2$. The positive quantity α is defined in such a way that $\alpha\lambda^z$ presents dimensions of energy for each $z>1$ [55].

Time-dependent solutions of Eq. (1.4) are not known for nonharmonic potentials, like the ones of Eq. (2.1) for $z \neq 2$; however, approximate approaches have been proposed in the literature, e.g., the ansatz of Ref. [27] or the maximum-entropy procedure discussed in Ref. [56]. In the former case, an approximate solution has considered two known limits: (i) In the short-time regime, since the system starts with all particles confined in a narrow region around the origin, the effects of the external potential become negligible. In the absence of an external force, Eq. (1.4) describes an anomalous diffusion, and hence for the initial condition $P(x,0)=\delta(x)$ its solution should be precisely the one for the external harmonic potential, given by Eq. (1.7). (ii) In the long-time limit, one should approach the stationary distribution of Eq. (1.5). Taking into account these two limiting behaviors, the following approximate time-dependent solution was proposed in Ref. [27]:

$$P(x,t) = \bar{B}(t)[1-\bar{\beta}(t)x^2-\bar{\gamma}(t)|x|^z]_+, \tag{2.2}$$

where $\bar{B}(t)$, $\bar{\beta}(t)$, and $\bar{\gamma}(t)$ are arbitrary, positive, time-dependent coefficients. To recover the above limits, one should have $\bar{\gamma}(0)=0$, whereas in the large-time limit, $\bar{\beta}(t)\to 0$, with $\bar{B}(t)$ and $\bar{\gamma}(t)$ approaching well-defined values. By adjusting numerically the time-dependent coefficients, $\bar{B}(t)$, $\bar{\beta}(t)$, and $\bar{\gamma}(t)$, good approximations were found for the time-dependent solution $P(x,t)$, by analyzing typical values of z.

As mentioned above, since the NLFPE of Eq. (1.4), together with its associated entropy in Eq. (1.9), satisfies an H-theorem [25–27], from now on we will refer to the time-independent solution of Eq. (1.4) as an equilibrium state, so that the corresponding distribution will be denoted by $P_{\rm eq}(x)$. In this way, substituting the above potential in Eq. (1.5), one gets

$$P_{\rm eq}(x) = \frac{1}{2z\lambda^{z+1}}\frac{\alpha\lambda^z}{k\theta}\left[x_e^z-|x|^z\right]_+ \tag{2.3}$$

with $|x|<x_e$, where x_e is found from the normalization condition for $P_{\rm eq}(x)$,

$$x_e = \lambda\left[(z+1)\frac{k\theta}{\alpha\lambda^z}\right]^{1/(z+1)}, \tag{2.4}$$

representing a typical positive length, as expected. Hence, the distribution of Eq. (2.3) is characterized by a cutoff x_e, which increases with the effective temperature as $x_e \propto \theta^{1/(z+1)}$. Moreover, $P_{\rm eq}(x)$ presents a variance,

$$\langle x^2\rangle = \frac{(z+1)^{(z+3)/(z+1)}}{3(z+3)}\lambda^2\left(\frac{k\theta}{\alpha\lambda^z}\right)^{2/(z+1)} = \frac{z+1}{3(z+3)}x_e^2, \tag{2.5}$$

from which one obtains

$$k\theta = \alpha\lambda^z\left\{\frac{3(z+3)}{(z+1)^{(z+3)/(z+1)}}\frac{\langle x^2\rangle}{\lambda^2}\right\}^{(z+1)/2}. \tag{2.6}$$

Therefore, the effective temperature θ is related to the variance in particle positions, such that $\theta \propto \langle x^2\rangle^{(z+1)/2}$. This behavior is a signature of the NLFPE of Eq. (1.4), and it should be contrasted with the standard classical dilute gas [1,2], for which the temperature is related linearly to the second moment of the corresponding velocity probability distribution, i.e., $T \propto \langle v^2\rangle$.

In fact, the results above may be extended by calculating the mean value $\langle|x|^{z'}\rangle$, where in general one may have $z' \neq z$. One obtains

$$\langle|x|^{z'}\rangle = \frac{z+1}{(z'+1)(z+z'+1)}x_e^{z'}, \tag{2.7}$$

which recovers Eq. (2.5) in the case $z'=2$. Moreover, one also has

$$\langle|x|^{z'}\rangle = \frac{z+1}{(z'+1)(z+z'+1)}\lambda^{z'}\left[(z+1)\frac{k\theta}{\alpha\lambda^z}\right]^{z'/(z+1)}, \tag{2.8}$$

showing the relation $\theta \propto \langle|x|^{z'}\rangle^{(z+1)/z'}$, which presents a form typical of a generalized equipartition theorem for the present system.

Making use of the equilibrium probability distribution in Eq. (2.3), one may calculate average values corresponding to physical quantities, such as those defined in Eqs. (1.8) and (1.9),

$$u = \int_{-x_e}^{x_e} dx \frac{\alpha |x|^z}{z} P_{\rm eq}(x) = \frac{(z+1)^{z/(z+1)}}{z(2z+1)} \alpha \lambda^z \left(\frac{k\theta}{\alpha \lambda^z} \right)^{z/(z+1)}, \tag{2.9}$$

$$\begin{aligned} \frac{s_2}{k} &= 1 - \lambda \int_{-x_e}^{x_e} dx [P_{\rm eq}(x)]^2 \\ &= 1 - \frac{(z+1)^{z/(z+1)}}{2z+1} \left(\frac{\alpha \lambda^z}{k\theta} \right)^{1/(z+1)}. \end{aligned} \tag{2.10}$$

Using Eq. (2.6) the entropy above may also be written in the form

$$\frac{s_2}{k} = 1 - a(z) \left(\frac{\lambda^2}{\langle x^2 \rangle} \right)^{1/2}, \tag{2.11}$$

where $a(z)$ represents a positive number for any $z > 1$. Hence, for increasing values of $\langle x^2 \rangle$, the entropy grows with a power that does not depend on the parameter z. One should recall that the equilibrium state, for which the following thermodynamic framework will be developed, consists of a mechanical equilibrium, so that the present approach comes as a direct consequence of the fluctuations in particle positions (leading to a variance $\langle x^2 \rangle > 0$), as illustrated in the quantities of Eqs. (2.6) and (2.11).

Manipulating Eqs. (2.9) and (2.10), one can rewrite the entropy in terms of the internal energy in the following way:

$$\frac{s_2(u,\alpha)}{k} = 1 - \frac{z+1}{2z+1} \left[\frac{\alpha \lambda^z}{z(2z+1)u} \right]^{1/z}, \tag{2.12}$$

where we have expressed the dependence $s_2 = s_2(u,\alpha)$ for reasons that will become clear in the next section.

From Eq. (2.12) one obtains the fundamental thermodynamic relation,

$$\left(\frac{\partial s_2}{\partial u} \right)_\alpha = \frac{1}{\theta}, \tag{2.13}$$

which is analogous to the temperature definition of standard thermodynamics [1,2,57], showing that the parameter θ introduced in Eq. (1.3) represents an appropriate effective-temperature definition for any $z > 1$.

The result of Eq. (2.13) suggests a definition of a type of energy exchange, $\delta Q = \theta ds_2$, to be referred to hereafter as heat exchange [30–32]. In what follows, we will use these results to construct a thermodynamic framework for this system.

III. FIRST-LAW PROPOSAL

In the previous studies [30–32], the work contribution related to the external potential was associated with variations in the parameter α, which controls directly the volume occupied by the vortices in the equilibrium state. In this way, an infinitesimal change in α, modifying the external potential acting on each particle, leads to an infinitesimal work defined as $\delta W = \sigma d\alpha$, where σ represents a parameter thermodynamically conjugated to α, to be determined. This definition, together with the one for the infinitesimal amount of heat, $\delta Q = \theta ds_2$, yields a proposal equivalent to the first law [30–32],

$$du = \delta Q + \delta W = \theta ds_2 + \sigma d\alpha, \tag{3.1}$$

where δW corresponds to the work done *on* the system, and σ should present dimensions $[L]^z$. In this way, the dependence $s_2 = s_2(u,\alpha)$ [cf. Eq. (2.12)] becomes clear; the consistency of this proposal for the first law will be shown throughout the next sections.

From Eq. (3.1), one gets that

$$\left(\frac{\partial s_2}{\partial \alpha} \right)_u = -\frac{\sigma}{\theta}, \tag{3.2}$$

and deriving Eq. (2.12) with respect to α, using the internal energy of Eq. (2.9), one obtains the following equation of state:

$$\sigma = \frac{(z+1)^{z/(z+1)}}{z(2z+1)} \lambda^z \left(\frac{k\theta}{\alpha \lambda^z} \right)^{z/(z+1)}. \tag{3.3}$$

Therefore, the parameter σ increases with θ (for α fixed), whereas for a fixed θ, an increase in σ yields a decrease in α. Moreover, comparing Eq. (3.3) with Eq. (2.9), one notices the interesting relation $u = \sigma\alpha$, which will have implications on the thermodynamic potentials, particularly on the enthalpy, to be discussed later on.

Using Eq. (2.6) in the equation of state above, one obtains

$$\sigma = \frac{\lambda^z}{z(2z+1)} \left[\frac{3(z+3)}{z+1} \frac{\langle x^2 \rangle}{\lambda^2} \right]^{z/2}, \tag{3.4}$$

relating σ and the variance $\langle x^2 \rangle$, showing that the work term acts directly on the equilibrium distribution.

IV. TRANSFORMATIONS AND CYCLES

From the proposal of Eq. (3.1) for the first law, one can study the possible physical transformations and reversible cycles, like those investigated in Refs. [30–32]. In analogy with the standard thermodynamic processes [1,57], one can define the isothermic process ($\theta =$ const), the adiabatic process ($s_2 =$ const), and two additional ones, corresponding to constant values of the parameters in the work contribution of Eq. (3.1), namely iso-α and iso-σ. The adiabatic process corresponds to one of the conditions [cf. Eqs. (2.9)–(2.12) and (3.3)]

$$\frac{\alpha}{\theta} = \text{const}, \quad \sigma = \frac{u}{\alpha} = \text{const}, \tag{4.1}$$

showing that the adiabatic and iso-σ transformations correspond precisely to the same process. Below, we describe the three possible transformations associated with Eq. (3.1), taking the system from an initial equilibrium state i to a final equilibrium state f.

A. Adiabatic (or iso-σ) transformations

The total work done *on* the system in an adiabatic transformation, from an initial state characterized by (θ_i,α_i) to a final one, with (θ_f,α_f), is given by

$$u_f - u_i = W = \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \sigma(\alpha_f - \alpha_i), \tag{4.2}$$

so that work is positive (negative) for $\alpha_f > \alpha_i$ ($\alpha_f < \alpha_i$). Along this transformation, the parameter σ is given by

$$\sigma = \frac{(z+1)^{z/(z+1)}}{z(2z+1)}\lambda^z\left(\frac{k\theta_i}{\alpha_i\lambda^z}\right)^{z/(z+1)} = \frac{(z+1)^{z/(z+1)}}{z(2z+1)}\lambda^z\left(\frac{k\theta_f}{\alpha_f\lambda^z}\right)^{z/(z+1)}. \tag{4.3}$$

B. Isothermal transformations

From the equation of state [cf. Eq. (3.3)], one may obtain isothermal transformations, which present a decay $(\sigma/\lambda^z) \sim (\alpha\lambda^z)^{-[z/(z+1)]}$, as exhibited in Fig. 1(a) for different values of z. For an isothermal process at a temperature θ, one has for the heat and work contributions, respectively,

$$Q = \int_{s_{2,1}}^{s_{2,f}} \theta ds_2 = \theta(s_{2,f} - s_{2,i}) = \frac{(z+1)^{z/(z+1)}}{2z+1}k\theta\left\{\left(\frac{\alpha_i\lambda^z}{k\theta}\right)^{1/(z+1)} - \left(\frac{\alpha_f\lambda^z}{k\theta}\right)^{1/(z+1)}\right\}, \tag{4.4}$$

$$W = \int_{\alpha_i}^{\alpha_f} \sigma d\alpha = \frac{\lambda^z}{z(2z+1)}\left[(z+1)\frac{k\theta}{\lambda^z}\right]^{z/(z+1)}\int_{\alpha_i}^{\alpha_f}\alpha^{-z/(z+1)}d\alpha = \frac{(z+1)^{(2z+1)/(z+1)}}{z(2z+1)}k\theta\left\{\left(\frac{\alpha_f\lambda^z}{k\theta}\right)^{1/(z+1)} - \left(\frac{\alpha_i\lambda^z}{k\theta}\right)^{1/(z+1)}\right\}. \tag{4.5}$$

Consistently, one notices that the sum of the two forms of energy above coincides with the internal energy difference [using Eq. (2.9)],

$$u_f - u_i = Q + W = \frac{(z+1)^{z/(z+1)}}{z(2z+1)}k\theta\left\{\left(\frac{\alpha_f\lambda^z}{k\theta}\right)^{1/(z+1)} - \left(\frac{\alpha_i\lambda^z}{k\theta}\right)^{1/(z+1)}\right\}. \tag{4.6}$$

Then, the work and variation of internal energy are positive (negative), whereas the system releases (absorbs) heat if $\alpha_f > \alpha_i$ ($\alpha_f < \alpha_i$).

C. Iso-α transformations

For a transformation from an initial state (θ_i,α) to a final state (θ_f,α), the work done *on* the system is zero, since $\delta W = \sigma d\alpha$. The process is characterized by $du = \delta Q = \theta ds_2$, so that

$$u_f - u_i = Q = \int_{\theta_i}^{\theta_f} \theta ds_2 = \frac{\alpha\lambda^z}{z(2z+1)}\left\{\left[(z+1)\frac{k\theta_f}{\alpha\lambda^z}\right]^{z/(z+1)} - \left[(z+1)\frac{k\theta_i}{\alpha\lambda^z}\right]^{z/(z+1)}\right\}. \tag{4.7}$$

D. Carnot cycle

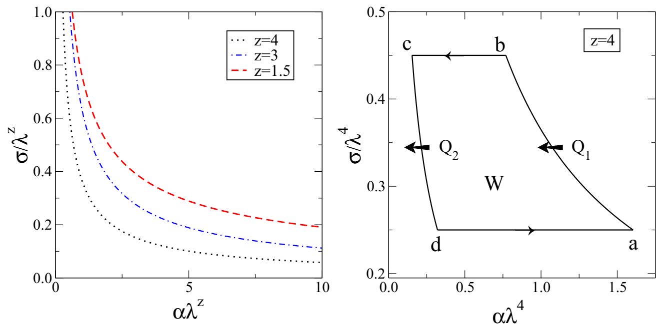
From the above transformations, one can define a set of transformations analogous to the Carnot cycle by considering two isothermal and two adiabatic processes, intercalated, as illustrated in Fig. 1(b) for the particular case $z = 4$. The quantities σ and α present dimensions that vary with z, being defined such that the ordinate σ/λ^4 is dimensionless, whereas the abscissa $\alpha\lambda^4$ presents dimensions of energy. Some of the properties of the cycle shown in Fig. 1(b) are described next. (i) An amount of heat Q_1 is absorbed in the isothermal process at the higher temperature θ_1, whereas the system releases heat Q_2 in the isothermal process at the lower temperature θ_2. (ii) In a plot σ versus α [or equivalently σ/λ^4 versus $\alpha\lambda^4$, as in Fig. 1(b)], the work associated with a given process corresponds to the area below such a transformation. As shown previously, work is positive (negative) for transformations that increase (decrease) α. Therefore, the total work done *on* the system, calculated as $W = W_{ab} + W_{bc} + W_{cd} + W_{da}$, is given by the area enclosed in the cycle of Fig. 1(b), being negative, as expected from Eq. (3.1). If one defines $\mathcal{W} = -W$ as the work done *by* the system, considering that the variation of internal energy is zero for the complete cycle, one has $Q_1 = \mathcal{W} + Q_2$ (conventionalizing all three quantities as positive). (iii) Manipulating Eqs. (3.3) and (4.4), one obtains the well-known result relating the two isothermal processes, $(Q_1/Q_2) = (\theta_1/\theta_2)$, leading to the celebrated efficiency of the Carnot cycle,

$$\eta = \frac{\mathcal{W}}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{\theta_2}{\theta_1} \quad (0 \leqslant \eta \leqslant 1). \tag{4.8}$$

This reinforces the fact that the Carnot cycle is very special within thermodynamics, so that its efficiency does not depend on the system under study, the external potential, or the particular entropic form considered. As shown in Refs. [30,32], where a harmonic external potential was considered, by following the cycle in the reverse way one obtains the corresponding Carnot refrigerator; this property holds also for the cycle in Fig. 1(b), as well as for those defined in terms of the general external potential of Eq. (2.1), for any $z > 1$. These results give further support for the fundamental relation of Eq. (3.1), as well as for the effective-temperature definition of Eq. (1.3), as appropriate for a consistent thermodynamic framework of the present system. Next, we analyze another cycle to show that its efficiency depends on the external potential considered, i.e., on the exponent z.

E. Otto cycle

The standard Otto cycle represents a rough approximation of a gasoline engine, consisting of two adiabatic and two isovolumetric transformations, intercalated [57,58]. Herein we will investigate a cycle analogous to the Otto cycle, consisting of two adiabatic and two iso-α transformations, as considered



(a) (b)

FIG. 1. (a) Isothermal transformations are exhibited in the plane σ/λ^z (dimensionless) vs $\alpha\lambda^z$ (dimensions of energy) for typical values of z; accordingly, the dimensions of both quantities σ and α vary with z in such a way that the product $\sigma\alpha$ presents dimensions of energy. (b) The Carnot cycle $a \to b \to c \to d \to a$ is represented in the particular case $z = 4$. The transformations for σ constant are adiabatic, and herein they were chosen to occur for $(\sigma/\lambda^4) = 0.45$ ($b \to c$) and $(\sigma/\lambda^4) = 0.25$ ($d \to a$). The isothermal transformations are characterized by $\sigma \sim \alpha^{-4/5}$ [cf. Eq. (3.3)], and they occur for $k\theta_1 = 5$ (units of energy) in $a \to b$, and $k\theta_2 = 1$ (units of energy) in $c \to d$, i.e., $\theta_1 > \theta_2$. The area inside the cycle represents the total work W done *on* the system, which is negative, as expected from Eq. (3.1). The cycle above holds for any system of units, e.g., one may consider all quantities with dimensions of energy in Joules.

in Ref. [32]. The motivation for analyzing this particular cycle consists in the fact that its efficiency will depend on the parameter z, characteristic of the external potential. The main proprieties of this cycle are described next. (i) In the plane σ/λ^z versus $\alpha\lambda^z$, this cycle is represented by a rectangle, as illustrated in Fig. 2 for the particular case $z = 4$. The effective temperature changes along all four transformations, presenting the set of values $(\theta_a,\theta_b,\theta_c,\theta_d)$, at the corresponding vertices of the rectangle. (ii) An amount of heat Q_1 is absorbed in the iso-α transformation $a \to b$, whereas the system releases heat Q_2 in the iso-α transformation $c \to d$. The total work done *on* the system is given by the area enclosed in the cycle of Fig. 2. Similarly to the procedure carried out for the Carnot cycle, if one defines $\mathcal{W} = -W$ as the work done *by* the system, the variation of internal energy is zero for the complete cycle, and one has $Q_1 = \mathcal{W} + Q_2$. (iii) From the equation of state and the condition for the adiabatic transformation, $(\alpha/\theta) = \text{const}$, one finds that $\theta_b > \theta_a > \theta_d$ and $\theta_b > \theta_c > \theta_d$, so that θ_b and θ_d represent, respectively, the highest and lowest temperatures of the cycle. (iv) Using Eq. (4.7), one can calculate Q_1 and Q_2,

$$Q_1 = \frac{\alpha_a\lambda^z}{z(2z+1)} \times \left\{\left[(z+1)\frac{k\theta_b}{\alpha_a\lambda^z}\right]^{z/(z+1)} - \left[(z+1)\frac{k\theta_a}{\alpha_a\lambda^z}\right]^{z/(z+1)}\right\}, \tag{4.9}$$

$$Q_2 = \frac{\alpha_c\lambda^z}{z(2z+1)} \times \left\{\left[(z+1)\frac{k\theta_c}{\alpha_c\lambda^z}\right]^{z/(z+1)} - \left[(z+1)\frac{k\theta_d}{\alpha_c\lambda^z}\right]^{z/(z+1)}\right\}, \tag{4.10}$$

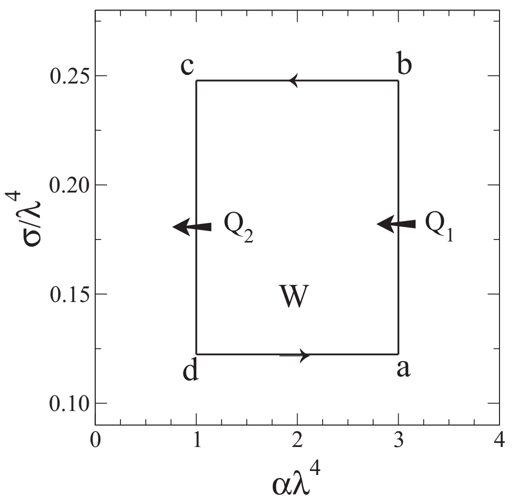


FIG. 2. The Otto cycle is represented for the particular case $z = 4$; the cycle is composed of two adiabatic (σ constant) transformations and two iso-α (α constant) transformations, intercalated. The abscissa $\alpha\lambda^4$ presents dimensions of energy (e.g., Joules), whereas the ordinate σ/λ^4 is dimensionless. An amount of heat Q_1 is absorbed at transformation $a \to b$, whereas Q_2 is released at $c \to d$. The area inside the cycle represents the total work W done *on* the system, which is negative, as expected from Eq. (3.1). The cycle above holds for any system of units, e.g., one may consider all quantities with dimensions of energy in Joules.

which are both positive quantities, with $Q_1 > Q_2$. In the adiabatic transformation $b \to c$, one has that $(\alpha_c/\alpha_b) = (\theta_c/\theta_b)$, and using that $\alpha_b = \alpha_a$, the efficiency of the cycle follows,

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{\theta_d}{\theta_a}\left[\frac{(\theta_c/\theta_d)^{z/(z+1)} - 1}{(\theta_b/\theta_a)^{z/(z+1)} - 1}\right]. \tag{4.11}$$

It is important to mention that for the standard Otto cycle, with an ideal gas as the working substance, the efficiency η differs from the expression above through a replacement of the exponents $z/(z+1)$ by unit [58]. This result generalizes the previous one presented in Ref. [32], recovered as the particular case $z = 2$. Comparing Eqs. (4.8) and (4.11), one notices the special nature of the Carnot cycle, which presents an efficiency that depends only on the two temperatures of the isothermal transformations.

V. THERMODYNAMIC POTENTIALS AND RESPONSE FUNCTIONS

Now we will explore the first-law proposal of Eq. (3.1) by following the usual procedures [1,2,57], e.g., performing Legendre transformations, in order to introduce additional thermodynamic potentials. Additionally, we define quantities analogous to the response functions of standard thermodynamics.

A. Internal energy

From Eq. (3.1), one has that the internal energy depends on the pair of independent variables (s_2,α), i.e., $u \equiv u(s_2,\alpha)$.

Inverting Eq. (2.12), one gets

$$u(s_2,\alpha) = \frac{\alpha\lambda^z}{z(2z+1)^{z+1}}\left(\frac{z+1}{1-s_2/k}\right)^z. \tag{5.1}$$

From the above equation, one obtains the equivalent to the effective-temperature definition [Eq. (2.13)], as well as the equation of state [Eq. (3.3)], respectively,

$$\left(\frac{\partial u}{\partial s_2}\right)_\alpha = \theta, \quad \left(\frac{\partial u}{\partial \alpha}\right)_{s_2} = \sigma. \tag{5.2}$$

Considering $u(s_2,\alpha)$ as a state function, its second derivatives should be independent of the order of differentiation, leading to the following Maxwell relation:

$$\frac{\partial^2 u}{\partial\alpha\partial s_2} = \frac{\partial^2 u}{\partial s_2\partial\alpha} \quad\Rightarrow\quad \left(\frac{\partial \sigma}{\partial s_2}\right)_\alpha = \left(\frac{\partial \theta}{\partial \alpha}\right)_{s_2}. \tag{5.3}$$

B. Helmholtz free energy

Now we deal with the Helmholtz free energy, $f(\theta,\alpha)$, defined in Eq. (1.8),

$$f(\theta,\alpha) = u - \theta s_2 \quad\Rightarrow\quad df = -s_2 d\theta + \sigma d\alpha. \tag{5.4}$$

The following free energy is obtained:

$$f(\theta,\alpha) = \alpha\lambda^z\left\{\frac{(z+1)^{(2z+1)/(z+1)}}{z(2z+1)}\left(\frac{k\theta}{\alpha\lambda^z}\right)^{z/(z+1)} - \frac{k\theta}{\alpha\lambda^z}\right\}, \tag{5.5}$$

which satisfies the relations

$$\left(\frac{\partial f}{\partial \theta}\right)_\alpha = -s_2, \quad \left(\frac{\partial f}{\partial \alpha}\right)_\theta = \sigma. \tag{5.6}$$

Furthermore, the corresponding Maxwell relation appears,

$$\frac{\partial^2 f}{\partial\alpha\partial\theta} = \frac{\partial^2 f}{\partial\theta\partial\alpha} \quad\Rightarrow\quad \left(\frac{\partial s_2}{\partial \alpha}\right)_\theta = -\left(\frac{\partial \sigma}{\partial \theta}\right)_\alpha. \tag{5.7}$$

C. Gibbs free energy

We define the Gibbs potential $g(\theta,\sigma)$ through

$$\begin{aligned} g(\theta,\sigma) &= f - \sigma\alpha = u - \theta s_2 - \sigma\alpha \\ &\Rightarrow \quad dg = -s_2 d\theta - \alpha d\sigma. \end{aligned} \tag{5.8}$$

Using the above potentials, one gets

$$g(\theta,\sigma) = k\theta\left\{\frac{z+1}{(2z+1)^{(z+1)/z}}\left(\frac{\lambda^z}{z\sigma}\right)^{1/z} - 1\right\}, \tag{5.9}$$

which satisfies

$$\left(\frac{\partial g}{\partial \theta}\right)_\sigma = -s_2, \quad \left(\frac{\partial g}{\partial \sigma}\right)_\theta = -\alpha. \tag{5.10}$$

The Maxwell relation associated with this pair of variables is obtained as

$$\frac{\partial^2 g}{\partial\sigma\partial\theta} = \frac{\partial^2 g}{\partial\theta\partial\sigma} \quad\Rightarrow\quad \left(\frac{\partial s_2}{\partial \sigma}\right)_\theta = \left(\frac{\partial \alpha}{\partial \theta}\right)_\sigma. \tag{5.11}$$

One curious property that follows by manipulating Eqs. (5.9), (2.10), and (3.3) is the relation $g = -\theta s_2$, whose consequences will be discussed next.

D. Enthalpy

Previously, we have seen that by comparing Eqs. (3.3) and (2.9), the relation $u = \sigma\alpha$ follows; moreover, the Gibbs potential above satisfies $g = -\theta s_2$. These relations imply a trivial enthalpy, defined as

$$h(s_2,\sigma) = u - \sigma\alpha = f + \theta s_2 - \sigma\alpha = g + \theta s_2 = 0. \tag{5.12}$$

Hence,

$$dh = \theta ds_2 - \alpha d\sigma = 0 \quad\Rightarrow\quad ds_2 = \frac{\alpha}{\theta}\,d\sigma, \tag{5.13}$$

showing that variations in σ are directly related to variations in the entropy s_2, reinforcing the previous result that for an adiabatic process, for which the conditions of Eq. (4.1) apply, the present system cannot exchange "heat" (i.e., it cannot vary its entropy) for σ fixed.

Therefore, a complete thermodynamic framework is given in terms of the three previously defined potentials, namely internal energy, $u(s_2,\alpha)$, and free energies, $f(\theta,\alpha)$ and $g(\theta,\sigma)$; the enthalpy $h(s_2,\sigma)$ should not contain any new information. Similar behavior is also found in other well-known systems, as in the three-dimensional ideal gas, for which $pv = 2u/3$ (valid for the classic case, as well as in both quantum statistics [1]), leading to an enthalpy $h = 5u/3$, showing that in this case, the enthalpy does not represent an independent thermodynamic potential.

In spite of this, Eq. (5.13), which results from such a trivial enthalpy, may still be useful, e.g., it yields an equivalent way to calculate entropy changes in given transformations, by using only the equation of state [Eq. (3.3)]. As an example, for iso-α transformations, one has

$$s_{2,f} - s_{2,i} = \int_{\theta_i}^{\theta_f}\frac{\alpha}{\theta}\left(\frac{\partial\sigma}{\partial\theta}\right)_\alpha d\theta, \tag{5.14}$$

which leads to

$$\begin{aligned} s_{2,f} - s_{2,i} = {} & \frac{(z+1)^{z/(z+1)}}{2z+1}\,k \\ & \times\left\{\left(\frac{\alpha\lambda^z}{k\theta_i}\right)^{1/(z+1)} - \left(\frac{\alpha\lambda^z}{k\theta_f}\right)^{1/(z+1)}\right\}. \end{aligned} \tag{5.15}$$

E. Response functions

A quantity analogous to the specific heat was introduced in Refs. [29,30,32] for a fixed α, in such a way that it may be calculated in three different ways,

$$\begin{aligned} c_\alpha &= \left(\frac{\partial u}{\partial\theta}\right)_\alpha = \theta\left(\frac{\partial s_2}{\partial\theta}\right)_\alpha = -\theta\left(\frac{\partial^2 f}{\partial^2\theta}\right)_\alpha \\ &= \frac{k}{(2z+1)}\left[\frac{1}{(z+1)}\frac{\alpha\lambda^z}{k\theta}\right]^{1/(z+1)}, \end{aligned} \tag{5.16}$$

leading to $c_\alpha \geqslant 0$. In a similar way, one can define c_σ,

$$c_\sigma = \theta\left(\frac{\partial s_2}{\partial\theta}\right)_\sigma = \left(\frac{\partial h}{\partial\theta}\right)_\sigma = 0, \tag{5.17}$$

following trivially as a consequence of the fact that $h = 0$, and expected, since the system cannot exchange heat for σ fixed.

In the Appendix, we show that important quantities, corresponding to the coefficient of expansion and isothermal compressibility of standard thermodynamics, can be expressed, respectively, as

$$\gamma = \frac{1}{\alpha}\left(\frac{\partial \alpha}{\partial \theta}\right)_{\sigma} = \frac{1}{\theta} \tag{5.18}$$

and

$$\kappa = -\frac{1}{\alpha}\left(\frac{\partial \alpha}{\partial \sigma}\right)_{\theta} = \frac{z+1}{z\sigma}. \tag{5.19}$$

Moreover, in the Appendix we also show that the response functions defined above are related by

$$c_{\alpha} = \alpha\theta\,\frac{\gamma^2}{\kappa}, \tag{5.20}$$

showing that γ, κ, and c_{α} are all positive for any $z > 1$. Therefore, these quantities behave very similarly to the corresponding ones of standard thermodynamics [57], including the relation between them [Eq. (5.20)], which is independent of z.

VI. CONCLUSIONS

The analyses of Refs. [29–32] were carried for an equilibrium state, reached after a sufficiently long-time evolution of a system of vortices interacting repulsively under overdamped motion, and in the presence of an external harmonic potential, $\phi(x) = \alpha x^2/2$ ($\alpha > 0$). Since the corresponding equilibrium probability distribution $P_{\rm eq}(x)$ depends on $\phi(x)$, and important thermodynamic quantities (e.g., internal energy and entropy) are calculated from $P_{\rm eq}(x)$, a natural question emerges concerning whether the main thermodynamic results obtained depend on the particular form of $\phi(x)$. To investigate this aspect, herein we have extended the former works to a wider class of potentials, of the form $\phi(x) = \alpha|x|^z/z$ ($z > 1$), so that the previous results are recovered in the particular case $z = 2$.

The approach is based on the definition of an effective temperature θ, conjugated to a generalized entropy s_q with $q = 2$, typical of nonextensive statistical mechanics. The physical system in mind corresponds to a model for type-II superconductors, for which the variable θ presents values much higher than those of typical room temperatures T, so that the thermal noise can be neglected ($T/\theta \simeq 0$). Moreover, θ is proportional to the linear density of vortices, which, according to recent advances in experimental techniques, became a controllable quantity [44–47], leading to the desirable possibility of a variable effective temperature. Consequently, definitions analogous to those of standard thermodynamics were introduced, such as an infinitesimal amount of heat, $\delta Q = \theta ds_2$, and a first-law proposal, $du = \delta Q + \delta W$. From this we have constructed a Carnot cycle, and we have shown its efficiency to be $\eta = 1 - (\theta_2/\theta_1)$, where θ_1 and θ_2 represent the effective temperatures associated with two isothermal transformations, with $\theta_1 > \theta_2$. In addition to this, we have also studied an additional cycle, the Otto cycle, which presented an efficiency depending on z, illustrating the special character of the Carnot cycle.

We have consolidated the first-law proposal by applying Legendre transformations for distinct pairs of variables, obtaining different potentials, such as internal energy, Helmholtz free energy, Gibbs free energy, and enthalpy. From these potentials, we have derived the equation of state, Maxwell relations, and response functions analogous to those of standard thermodynamics, such as specific heat, the coefficient of thermal expansion, and isothermal compressibility. All results obtained were shown to be consistent with usual thermodynamic concepts for any $z > 1$.

The present results reinforce the idea that a system of interacting vortices, commonly used for modeling type-II superconductors, represents an important physical application for nonextensive statistical mechanics. Herein, we have shown the robustness of the whole thermodynamic framework, suggesting that a similar scheme should hold for a general confining potential $\phi(x)$, thus increasing significantly the potentiality of experimental observations.

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APPENDIX

In this appendix, we will show important results for the response functions, including relations between them. We will consider standard calculations (see, e.g., Ref. [57]) in order to introduce these response functions for this system. In the equation of state [Eq. (3.3)] one has $\sigma = \sigma(\theta,\alpha)$, so that

$$d\sigma = \left(\frac{\partial \sigma}{\partial \theta}\right)_{\alpha} d\theta + \left(\frac{\partial \sigma}{\partial \alpha}\right)_{\theta} d\alpha, \tag{A1}$$

or, rewriting the equation of state as $\alpha = \alpha(\theta,\sigma)$, one gets

$$d\alpha = \left(\frac{\partial \alpha}{\partial \theta}\right)_{\sigma} d\theta + \left(\frac{\partial \alpha}{\partial \sigma}\right)_{\theta} d\sigma. \tag{A2}$$

Let us then define quantities, corresponding, respectively, to the coefficient of expansion and isothermal compressibility of standard thermodynamics,

$$\gamma = \frac{1}{\alpha}\left(\frac{\partial \alpha}{\partial \theta}\right)_{\sigma}, \quad \kappa = -\frac{1}{\alpha}\left(\frac{\partial \alpha}{\partial \sigma}\right)_{\theta}, \tag{A3}$$

which measure changes in α with respect to variations in θ (σ fixed) and σ (θ fixed). Considering the pair of independent variables (θ,σ), one has

$$\theta ds_2 = \theta\left[\left(\frac{\partial s_2}{\partial \theta}\right)_{\sigma} d\theta + \left(\frac{\partial s_2}{\partial \sigma}\right)_{\theta} d\sigma\right] = \theta\left(\frac{\partial s_2}{\partial \sigma}\right)_{\theta} d\sigma, \tag{A4}$$

where we have used $c_{\sigma} = 0$ [cf. Eq. (5.17)] in the last equality. Substituting Eq. (A1) for α constant in the above equation, the quantity c_{α} in Eq. (5.16) may be written as

$$c_{\alpha} = \theta\left(\frac{\partial s_2}{\partial \sigma}\right)_{\theta}\left(\frac{\partial \sigma}{\partial \theta}\right)_{\alpha}. \tag{A5}$$

Now, using Eq. (A1) for α constant and the definitions of Eq. (A3), one obtains

$$\left(\frac{\partial \sigma}{\partial \theta}\right)_{\alpha} = \frac{\gamma}{\kappa}. \tag{A6}$$

The above equation, together with the Maxwell relation of Eq. (5.11), allows us to write Eq. (A5) as

$$c_\alpha = \alpha\theta \, \frac{\gamma^2}{\kappa}, \tag{A7}$$

and since the variables θ and α are positive, c_α and κ should present the same sign.

Considering the equation of state, one may calculate the quantities of Eq. (A3),

$$\gamma = \frac{1}{\theta}, \quad \kappa = \frac{z+1}{z\sigma}, \tag{A8}$$

corresponding to the results in Eqs. (5.18) and (5.19), respectively. These equations, together with Eq. (A7), show that γ, κ, and c_α are all positive for the present system.

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