Optimization of a relativistic quantum mechanical engine

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We present an optimal analysis for a quantum mechanical engine working between two energy baths within the framework of relativistic quantum mechanics, adopting a first-order correction. This quantum mechanical engine, with the direct energy leakage between the energy baths, consists of two adiabatic and two isoenergetic processes and uses a three-level system of two noninteracting fermions as its working substance. Assuming that the potential wall moves at a finite speed, we derive the expression of power output and, in particular, reproduce the expression for the efficiency at maximum power.

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I. INTRODUCTION

The concept of a quantum mechanical engine was introduced by Scovil and Schultz- Dubois [1] and has been discussed extensively in the literature [2–25]. The principal difference between the classical engine cycles and the quantum version resides in the quantum mechanical nature of the working substance, which has exotic properties [2,3]. Several theoretical implementations for a quantum mechanical engine have been reported, such as entangled states in a qubit [4], quantum mechanical versions of the Otto cycle [5,6,8], photocells [7,9], and a strained single-layer graphene flake [10]. In recent years, it has been proposed that if the reservoirs are also of a quantum mechanical nature, these could be engineered into quantum coherent states [7,9] or into squeezed thermal states [6], thus allowing for a theoretical enhancement of the engine efficiency beyond the classical Carnot limit [6,7,9].

One of the simplest theoretical implementations for a quantum mechanical engine is a system composed of one or more particles trapped in a one-dimensional potential well [2,3,11–14,19–22]. The different processes can be driven by a quasistatic deformation of the potential well by applying an external force. The case of the Schrödinger spectrum for two levels and one particle in an isoenergetic cycle originally proposed by Bender *et al.* [2] lead to many studies and publications that considered replacing the heat baths with energy baths. The basic idea of this possibility is that the expectation value of the energy is a quantity well defined in quantum mechanics [2]. One of the most interesting studies about isoenergetic cycles is the optimization scheme proposed by Abe [12], which consists of the possibility that the well width's movement speed is finite, in analogy to making the speed of the piston finite in the context of the finite-time thermodynamics [12–14]. This study is extended in the publication of Wang *et al.* [13] for two particles and three levels, showing an enhanced value for the power output and includes the possibility to have an energy leakage $\dot{Q}_r$ between the two energy baths. The generalization of this problem, N fermions in M levels, is presented in Ref. [14], which includes an excellent discussion of the power-law energy spectrum.

The case of a relativistic regime of the work of Bender *et al.* [2] was studied in Ref. [19], which found an analytical and exact solution for the efficiency and showed a lower value for the case of the ultrarelativistic particles. Unfortunately, the extension for the case of more than one particle is difficult due to the structure of the energy spectrum, reported in Ref. [19], which complicates optimization studies. In the present work, we study the possibility of using a Taylor series to the power of (λ/L), in which λ is the Compton wave length and L is the width of the potential, to find a solution to the contribution for the first relativistic order correction for two particles and three levels for a one-dimensional box and to show how it affects the calculation of the optimization region. It is important to emphasize that this work is the first attempt to combine two power-law spectrum in the literature in the context of optimal analysis for a quantum mechanical engine. In spite of that, several approximations must be made to obtain relevant physical information. Another important limit of theoretical interest is the ultrarelativistic case whose spectrum energy is proportional to L^{-1} in contrast to the Schrödinger spectrum, which is proportional to L^{-2}. The discussion of this last case will allow us to enrich the results and conclusions we obtain for the first-order correction.

II. A DIRAC PARTICLE TRAPPED IN A ONE-DIMENSIONAL INFINITE POTENTIAL WELL

The problem of a Dirac particle in the presence of a one-dimensional, finite potential well $V(x)$ is expressed by the Dirac Hamiltonian operator [26–28],

$$\hat{H} = -i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\hat{\beta} + V(x)\hat{\mathbf{1}} = \hat{H}_0 + V(x)\hat{\mathbf{1}}. \tag{1}$$

Here,

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

are Dirac matrices in four dimensions, with $\hat{\sigma}_i$ the Pauli matrices, and $\hat{H}_0$ corresponds to the *free* Hamiltonian particle,

$$\hat{H}_0 = -i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\hat{\beta}. \tag{2}$$

The domain of this operator is $\mathcal{D}(\hat{H}_0) = \mathcal{H}^1(\mathbb{R},\mathbb{C}^4)$, and corresponds to the first Sobolev space of (complex-valued) four-component spinors $\hat{\psi}(x) = (\phi_1,\phi_2,\chi_3,\chi_4)$, which is a natural domain for first-order differential operators [26]. The mathematical and physical pictures are given by considering

the singular limit of an *infinite* potential well,

$$V(x) = \begin{cases} 0\,, & |x| \leqslant L/2 \\ +\infty\,, & |x| > L/2 \end{cases}. \tag{3}$$

The singular character of the infinite potential well, which is the same as that in the more familiar Schrödinger case [29], requires a different mathematical statement of the problem. One must define a self-adjoint extension [26,29–31] of the *free* Hamiltonian particle, whose domain $\mathcal{D}(\hat{H}_0) \subset \mathcal{H}_\Omega$ is a dense proper subset of the Hilbert space $\mathcal{H}_\Omega = L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega) \oplus L^2(\Omega) \equiv L^2(\Omega, \mathbb{C}^4)$ of square-integrable (complex-valued) four-component spinors in the closed interval $x \in \Omega = [-L/2, L/2]$. In general, the domain of $\hat{H}_0$ and its adjoint $\hat{H}_0^\dagger$ verify $\mathcal{D}(\hat{H}_0) \subseteq \mathcal{D}(\hat{H}_0^\dagger)$ [26]. However, physics requires that $\hat{H}_0$ be self-adjoint. The self-adjoint extension is obtained by imposing appropriate boundary conditions [26,29–31] on the spinors at the boundary $\partial\Omega$ of the finite domain Ω and in the use of a fundamental discrete symmetry of the Dirac Hamiltonian (parity), as discussed in detail in Ref. [19]. This approach physically provides acceptable spinor-eigenfunctions, given by

$$\hat{\psi}_n(x) = A \begin{pmatrix} \sin(n\pi(x - L/2)/L) \\ 0 \\ 0 \\ -\frac{in\lambda/(2L)}{\sqrt{1+n^2(\lambda/2L)^2}} \cos(n\pi(x - L/2)/L) \end{pmatrix}, \tag{4}$$

with associated discrete energy eigenvalues,

$$E_n^D(L) = mc^2(\sqrt{1 + (n\lambda/2L)^2} - 1), \tag{5}$$

where $\lambda = 2\pi\hbar/(mc)$ is the Compton wavelength. The positive sign corresponds to the particle solution [27]. Two important limits can be obtained for this spectrum; one corresponds to the case when $\lambda/L \ll 1$,

$$E_n^D(L) \to \frac{mc^2}{2}(n\lambda/2L)^2 = E_n^S(L), \tag{6}$$

with $E_n^S(L) = n^2\pi^2\hbar^2/2mL^2$ being the solution to the well-known Schrödinger problem. The other important limit of Eq. (5) corresponds to a massless Dirac particle with $\lambda \to \infty$, where the spectrum reduces to the expression

$$E_n^D(L)\Big|_{m=0} = \frac{n\pi\hbar c}{L}. \tag{7}$$

This situation may be of interest in graphene systems, where conduction electrons in the vicinity of the so-called Dirac point can be described as effective massless chiral particles, satisfying Dirac's equation in two dimensions [32–35].

III. THE FIRST LAW OF THERMODYNAMICS

Through this work, we describe a very special type of dynamics, where we shall assume that one or more physical parameters in the set $\{\mu_j\}$ such as geometrical dimensions in this case, on which the Hamiltonian $\hat{H}(\{\mu_j\})$ depends explicitly, can be varied at an arbitrary slow rate $\dot{\mu_j}$. To be more precise, let us assume that $|n;\{\mu_j\}\rangle$ constitutes the set of eigenvectors of $\hat{H}$,

$$\hat{H}|n;\{\mu_j\}\rangle = E_n(\{\mu_j\})|n;\{\mu_j\}\rangle, \tag{8}$$

where n represents a set of indexes that labels the spectrum of the Hamiltonian. The density matrix operator is diagonal in the energy eigenbasis,

$$\hat{\rho} = \sum_n p_n(\{\mu_j\})|n;\{\mu_j\}\rangle\langle n;\{\mu_j\}|, \tag{9}$$

where the coefficients $0 \leqslant p_n(\{\mu_j\}) \leqslant 1$ represent the probability for the system to be in the particular state $|n;\{\mu_j\}\rangle$. Therefore, due to the normalization condition $\mathrm{Tr}\hat{\rho} = 1$, we have

$$\sum_n p_n(\{\mu_j\}) = 1. \tag{10}$$

In this representation, the von Neumann entropy [36] adopts a simple expression in terms of the probability coefficients,

$$\begin{aligned} S(\{\mu_j\}) &= -k_B \mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) \\ &= -k_B \sum_n p_n(\{\mu_j\})\ln[p_n(\{\mu_j\})]. \end{aligned} \tag{11}$$

The ensemble-average energy $E = \langle\hat{H}\rangle$ of the system is given by

$$E = \mathrm{Tr}(\hat{\rho}\hat{H}) = \sum_n p_n(\{\mu_j\})E_n(\{\mu_j\}). \tag{12}$$

The statistical ensemble just described can be submitted to an arbitrary quasistatic process, involving the modulation of one or more of the parameters $\{\mu_j\}$, and hence the ensemble-average energy in Eq. (12) changes accordingly,

$$\begin{aligned} dE &= \mathrm{Tr}(\hat{H}\,\delta\hat{\rho}) + \mathrm{Tr}(\hat{\rho}\,\delta\hat{H}) \\ &= \sum_n \sum_j E_n(\{\mu_j\})\frac{\partial}{\partial\mu_j}p_n(\{\mu_j\})\delta\mu_j \\ &\quad + \sum_n \sum_j p_n(\{\mu_j\})\frac{\partial}{\partial\mu_j}E_n(\{\mu_j\})\delta\mu_j \\ &= \delta Q + \delta W, \end{aligned} \tag{13}$$

and corresponds to the first law of quantum thermodynamics [2–4,8,10–20,22–24]. The first term in Eq. (13) is associated with the energy exchange, while the second one represents the work done. That is, energy exchange between a quantum mechanical system and its surroundings is induced by transition between quantum states of the systems, in which the temperature (heat bath) is included or not, while the work is performed due to the variation of energy spectrum with fixed occupation probabilities. The quasistatic process described above via Eq. (13) can be considered as a very particular form of a dynamical process, provided two main assumptions are made: First, the dynamics is uniquely determined by the rate of change of the set parameters $\{\dot{\mu}_j\}$, such that in a given interval of time δt we have $\delta\mu_j = \dot{\mu}_j\delta t$. Second, the rates must be

slow enough in order to satisfy that the quantity $\delta\mu_j/\dot{\mu}_j$ must be considerably higher compared with the relaxation times of the system and reservoir [12–19].

As in a classical system, we define a generalized force $Y = -\delta W/\delta\mu_j$. For this case, the external force driving the change in the width of the potential well must be equal to the "internal pressure" of a one-dimensional system,

$$F = -\frac{\delta W}{\delta L} = -\sum_n p_n \frac{dE_n}{dL}. \tag{14}$$

IV. A RELATIVISTIC ENGINE OF TWO PARTICLES IN AN ISOENERGETIC CYCLE

In our case, we have only one parameter in the set $\{\mu_j\}$, corresponding to the case of the width of the potential well, L. An arbitrary state $|\Psi\rangle$ can be expanded in terms of the eigenstates $|\psi_n\rangle$ as $|\Psi\rangle = \sum_n a_n|\psi_n\rangle$, with the expansion coefficients satisfying $\sum_{n=1}^{\infty}|a_n|^2 = 1$. The working substance of our quantum mechanical engine consists of two noninteracting relativistic particles in a system of three possible levels operating under an isoenergetic cycle. So, throughout the paper we shall assume that there are only three states $|\psi_1\rangle$ with $n = 1$, $|\psi_2\rangle$ with $n = 2$, and $|\psi_3\rangle$ with $n = 3$ employed by the quantum mechanical engine model with two particles. The isoenergetic cycle, a scheme for a quantum mechanical engine originally proposed by Bender *et al.* [2,3], is composed of two isoentropic and two isoenergetic processes. In particular, during the isoenergetic processes, the "working substance" must exchange energy with an energy reservoir [11,13], keeping constant the expectation value of the Hamiltonian. Therefore, to realize this process of the work done by the external parameter μ, on which the Hamiltonian of the quantum system depends parametrically, it can be precisely counterbalanced. In the isoenergetic process the quantum system evolves from initial state $|\psi(0)\rangle$ to a final state $|\psi(t)\rangle$ through a unitary evolution [37]. Therefore, one possibility to satisfy the constancy of the expectation value of the Hamiltonian is given by $\frac{dH}{dt} = i\hbar[H(t),H(t')] + \frac{\partial H}{\partial t} = i\hbar[H(t),H(t')] + \frac{\partial H}{\partial \mu}\frac{\partial \mu}{\partial t} = 0$. A possible practical realization of this cycle was proposed in several works [2,8,11,13,37], where the working substance exchanges energy with an external field, which acts as an energy reservoir and plays the role of heat baths in a traditional quantum heat engine [8,13,37,38]. During the isoenergetic stage, the energy exchange between a quantum mechanical system and its surroundings induced transitions between the quantum states of the system. Currently, an isoenergetic process is not very easy to be realized in experiments, but this is not the case in numerical simulations [39,40]. Throughout this work, we assume that the final state after the isoenergetic process corresponds to the maximal expansion, or compression, that is, the particles end completely localized in the closest upper (lower) levels. On the other hand, during the isoentropic process, the occupation probabilities p_n do not change. Thus, no transition occurs between levels during this process, and no energy is exchanged between the system and the energy bath during this stage.

The scheme of this work is illustrated in Fig. 1. During the first stage, $1 \to 2$, the width of the potential, expands slowly, and the expectation of the Hamiltonian, $E_{12}^D(L) = E_H^D$,

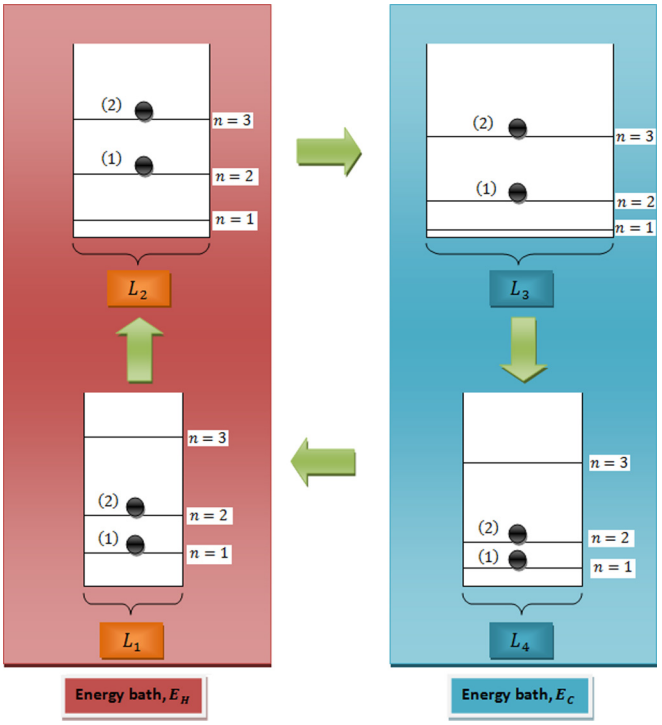

FIG. 1. The four processes of the isoenergetic cycle schematically represented. The initial configuration corresponds to the first particle (1) in the ground-state level and the second particle (2) in the first excited state. The first process correspond to isoenergetic expansion from $L_1 \to L_2$ and are coupled with an energy bath E_H. In the context of maximal expansion, when the system is in L_2, the first particle (1) is in the first excited state and the second particle (2) in the second excited state. During this process, the system absorbs energy from E_H. Similarly, the third process corresponds to isoenergetic compression from $L_3 \to L_4$ coupled with an energy bath E_C. For maximal compression, the particles return to the initial configuration, and the system releases energy to the energy bath E_C. For the two adiabatic processes ($L_2 \to L_3$ and $L_4 \to L_1$) the entropy remains constant, and the two particles stay in fixed states.

remains constant. The total energy of the system can be rewritten as

$$E_H^D = mc^2 \sum_{j=1}^{N} \left(\sqrt{1 + \left(\frac{j\lambda}{2L_1}\right)^2} - 1 \right), \tag{15}$$

where we change the index n to j, and N represents the total number of particles in the quantum system. Unfortunately, this series does not have an analytical expression as found in the Schrödinger problem [14],

$$E_H^S = \frac{\pi^2\hbar^2}{2mL_1^2}\sum_{j=1}^{N} j^2 = \frac{\pi^2\hbar^2}{2mL_1^2} G_1, \tag{16}$$

where $G_1 \equiv \sum_{j=1}^{N} j^2 = \frac{1}{6}N(N+1)(2N+1)$. Using the notation of Ref. [14], the normalization condition for the particles can be written in the form $\sum |a_n^{(j)}|^2 = 1$, with $a_n^{(j)}$ being the expansion coefficients of jth particle occupying the nth eigenstate. We can then write the energy of the system as a function of L for the case of the relativistic spectrum in Eq. (5)

as follows:

$$E_H^D = mc^2 \sum_{j=1}^{N}\sum_{n=1}^{M} \left|a_n^{(j)}\right|^2 \left[\sqrt{1+\left(\frac{n\lambda}{2L}\right)^2}-1\right], \qquad (17)$$

where M represents the energy-level number. The condition of energy conservation along the isoenergetic process $1 \to 2$ (see Fig. 1) implies that Eq. (17) must be equal to Eq. (15). From this equality we obtain a relationship between the coefficients and the width of the potential well, which is used to simplify the expression

$$F_{12}(L) = \sum_{j=1}^{N}\sum_{n=1}^{M} \left|a_n^{(j)}\right|^2 \frac{(n\lambda)^2 mc^2}{4L^3\sqrt{1+\left(\frac{n\lambda}{2L}\right)^2}}, \qquad (18)$$

which corresponds to the force determined by Eqs. (14) and (17). In this case, we obtain a relationship between the coefficients and the width of the potential well, which is not as simple as in the case of a single power-law spectrum [37]. However, the physical interest of this work focuses on finding the relativistic correction of the work presented by Wang *et al.* [13]. To do that, we work with the first-order correction of the spectrum given in Eq. (5), and we develop the isoenergetic cycle using a combination of power-law spectrum [$E \propto (L^{-2} - L^{-4})$], considering the case of $N = 2$ and $M = 3$. Finally, we achieve interesting results when we take the ultrarelativistic limit and compare the results with the first-order relativistic correction previously developed.

A. First-order correction

1. Force and energy

For this case, we can use a Taylor series up to order $\mathcal{O}[(\lambda/L)^4]$ for the spectrum of Eq. (5), considering $\frac{N\lambda}{2L_1} \ll 1$. In all our calculations present in this subsection, the expression for the physical observables contain the expression $\mathcal{O}[(\lambda/L)^6]$, but for notational reasons we do not include this term in the manuscript. The initial condition for the cycle under this approach is given by

$$\begin{aligned} E_H &= \frac{mc^2}{2}\left(\frac{\lambda}{2L_1}\right)^2 \sum_{j=1}^{N} j^2 - \frac{mc^2}{8}\left(\frac{\lambda}{2L_1}\right)^4 \sum_{j=1}^{N} j^4 \\ &= \frac{mc^2}{2}\left(\frac{\lambda}{2L_1}\right)^2 G_1 - \frac{mc^2}{8}\left(\frac{\lambda}{2L_1}\right)^4 J_1, \end{aligned} \qquad (19)$$

where G_1 is given in Eq. (16) and $J_1 = \sum_{i=1}^{N} i^4 = (N^5/5) + (N^4/2) + (N^3/3) - (N/30)$. For $N = 2$ we obtain the values $G_1 = 5$ and $J_1 = 17$. Then, the initial energy for the cycle is given by

$$E_H = \frac{5}{2}mc^2\left(\frac{\lambda}{2L_1}\right)^2 - \frac{17}{8}mc^2\left(\frac{\lambda}{2L_1}\right)^4. \qquad (20)$$

Throughout the first process, the energy of the system as a function of L can be rewritten for our case as

$$\begin{aligned} E_{12}(L) &= \frac{mc^2}{2}\left(\frac{\lambda}{2L}\right)^2 \sum_{j=1}^{2}\sum_{n=1}^{3} \left|a_n^{(j)}\right|^2 n^2 \\ &\quad - \frac{mc^2}{8}\left(\frac{\lambda}{2L}\right)^4 \sum_{j=1}^{2}\sum_{n=1}^{3} \left|a_n^{(j)}\right|^2 n^4, \end{aligned} \qquad (21)$$

and must be equal to Eq. (20). On the other hand, the force to the first process is given by the expression

$$\begin{aligned} F_{12}(L) &= \frac{mc^2}{L}\left(\frac{\lambda}{2L}\right)^2 \sum_{j=1}^{2}\sum_{n=1}^{3} \left|a_n^{(j)}\right|^2 n^2 \\ &\quad - \frac{mc^2}{2L}\left(\frac{\lambda}{2L}\right)^4 \sum_{j=1}^{2}\sum_{n=1}^{3} \left|a_n^{(j)}\right|^2 n^4, \end{aligned} \qquad (22)$$

subject to restriction imposed by equating Eq. (20) with Eq. (21). For this restriction, we do not have a simple relation as one might expect between L and the coefficients. However, we found a solution of physical interest (see Appendix for details) for the force throughout the process which is given by

$$F_{12}(L) = \frac{5mc^2}{L}\left(\frac{\lambda}{2L_1}\right)^2 - \frac{mc^2}{L}\left(\frac{17}{4}+\frac{25K}{2D^2}\right)\left(\frac{\lambda}{2L_1}\right)^4, \qquad (23)$$

where we define for simplicity $D = \sum_{j=1}^{2}\sum_{n=1}^{3} |a_n^{(j)}|^2 n^2$ and $K = \sum_{j=1}^{2}\sum_{n=1}^{3} |a_n^{(j)}|^2 n^4$.

Under the context of maximal expansion, when $L_1 \to L_2$, the first particle is in the first excited state ($|a_2^{(1)}| = 1$), and the second particle is in the second excited state ($|a_3^{(2)}|^2 = 1$). The energy at that point can be rewritten as

$$E(L_2) = E_H = \frac{13}{2}mc^2\left(\frac{\lambda}{2L_2}\right)^2 - \frac{97}{8}mc^2\left(\frac{\lambda}{2L_2}\right)^4. \qquad (24)$$

The isoenergetic condition for a maximal expansion required to equalize Eq. (20) with Eq. (24) implies an equation in the form

$$\frac{97}{8}x_2^2 - \frac{13}{2}x_2 + c_1 = 0, \qquad (25)$$

where we define $x_2 = (\lambda/2L_2)^2$ and $c_1 = \frac{5}{2}(\lambda/2L_1)^2 - \frac{17}{8}(\lambda/2L_1)^4$. The physical solution of Eq. (25) is given by

$$x_2 = \frac{26}{97} - \frac{2}{97}\sqrt{169 - 194c_1}. \qquad (26)$$

Note that if we use the Taylor series for the last solution, we get $x_2 \sim \frac{2}{13}c_1$, and if we neglect the order $O(\lambda/L)^4$ and higher, we get

$$x_2 = \frac{5}{13}\left(\frac{\lambda}{2L_1}\right)^2 \to L_2 = L_1\sqrt{\frac{13}{5}}, \qquad (27)$$

which corresponds to the solution given by Wang *et al.* [13].

In the process $2 \to 3$, the system expands adiabatically from $L = L_2$ until L_3. No transition occurs during this stage. The energy of the system is given by $E_{23} = \frac{13}{2}mc^2(\frac{\lambda}{2L})^2 -$

$\frac{97}{8}mc^2(\frac{\lambda}{2L})^4$, and the force is $F_{23} = \frac{13mc^2}{L}(\frac{\lambda}{2L})^2 - \frac{97mc^2}{2L}(\frac{\lambda}{2L})^4$. The first terms in the force F_{23} are the nonrelativistic result as presented in Ref. [13].

The third process corresponds to isoenergetic compression from L_3 until L_4. As with the first process, the key point is the fact that the expectation value of the Hamiltonian is constant along the trajectory and is given by

$$E_C = \frac{13}{2}mc^2\left(\frac{\lambda}{2L_3}\right)^2 - \frac{97}{8}mc^2\left(\frac{\lambda}{2L_3}\right)^4. \quad (28)$$

Using the same treatment to constrain the force as we used in the first isoenergetic process (see Appendix for details), we found that the force throughout this process can be expressed as follows:

$$F_{34}(L) = \frac{13mc^2}{L}\left(\frac{\lambda}{2L_3}\right)^2 - \frac{mc^2}{L}\left(\frac{97}{4} + \frac{169K}{2D^2}\right)\left(\frac{\lambda}{2L_3}\right)^4. \quad (29)$$

In the context of maximal compression, the first particle now returns to the ground state ($|a_1^{(1)}|^2 = 1$) and the second one goes to the first excited state ($|a_2^{(2)}|^2 = 1$). The energy at that point is

$$E(L_4) = E_C = \frac{5}{2}mc^2\left(\frac{\lambda}{2L_4}\right)^2 - \frac{17}{8}mc^2\left(\frac{\lambda}{2L_4}\right)^4. \quad (30)$$

In order to match Eq. (30) with Eq. (28), we use an equation in the form of

$$\frac{17}{8}x_4^2 - \frac{5}{2}x_4 + c_3 = 0, \quad (31)$$

where we define $x_4 = (\lambda/2L_4)^2$ and $c_3 = \frac{13}{2}(\lambda/2L_3)^2 - \frac{97}{8}(\lambda/2L_3)^4$. The physical solution for this equation is given by

$$x_4 = \frac{10}{17} - \frac{2}{17}\sqrt{25 - 34c_3}. \quad (32)$$

Using a Taylor series, the first order in the series expansion is given by $x_4 \sim \frac{2}{5}c_3$ and if we neglect the $O(\lambda/L)^4$ we get

$$x_4 = \frac{13}{5}\left(\frac{\lambda}{2L_3}\right)^2 \rightarrow L_4 = \sqrt{\frac{5}{13}}L_3, \quad (33)$$

corresponding to the nonrelativistic case presented in the work of Wang *et al.* [13].

Finally, the fourth process corresponds to the last adiabatic trajectory and goes from L_4 to L_1, returning to the starting point. During the compression, the energy of the system as a function of L is $E_{41} = \frac{5}{2}mc^2(\frac{\lambda}{2L})^2 - \frac{17}{8}mc^2(\frac{\lambda}{2L})^4$, and the force applied to the wall of the potential is $F_{41} = \frac{5mc^2}{L}(\frac{\lambda}{2L})^2 - \frac{17mc^2}{2L}(\frac{\lambda}{2L})^4$.

2. *Energy exchange and total work*

In the two isoenergetic processes, the system is coupled with energy baths E_H and E_C. Since these energy baths are sufficiently large and their internal relaxation is very fast, we can assume the existence of an energy leakage $\dot{Q}_r$ between the two energy baths [12–17]; moreover, it can be considered constant [13,15]. On the other hand, the study of optimization of quantum engines has been discussed in another approximation called "low dissipation scheme" proposed by Esposito *et al.* [18] and generalized in Refs. [16,17] for the case of the so-called "Carnot cycles with external leakage losses." One of the points treated in the works [16,17] is the study of the efficiency at maximum power for the case of different working substances operating between two energy baths under isoenergetic conditions with a constant leakage between the baths. Inspired by these works, we therefore assume that the rate of this escape is a constant, so the energy Q_H and the absolute value of Q_C is given by

$$Q_H = \int_{L_1}^{L_2} F_{12}(L)dL + \dot{Q}_r\tau = 5mc^2\left(\frac{\lambda}{L_1}\right)^2 \ln\left(\frac{L_2}{L_1}\right) - mc^2\left(\frac{17}{4} + \frac{25K}{2D^2}\right)\left(\frac{\lambda}{2L_1}\right)^4 \ln\left(\frac{L_2}{L_1}\right) + \dot{Q}_r\tau, \quad (34)$$

$$|Q_C| = \int_{L_4}^{L_3} F_{34}(L)dL + \dot{Q}_r\tau = 13mc^2\left(\frac{\lambda}{L_3}\right)^2 \ln\left(\frac{L_3}{L_4}\right) - mc^2\left(\frac{97}{4} + \frac{169K}{2D^2}\right)\left(\frac{\lambda}{2L_3}\right)^4 \ln\left(\frac{L_3}{L_4}\right) + \dot{Q}_r\tau, \quad (35)$$

where we use the approximation $K/D^2 \sim$ const in the force expression. In general the fraction K/D^2 is a function of L, but unfortunately the complete analytical dependence of L cannot be obtained. We use the known results for the case of power-law potentials [12–14] predicting, for a power law of the type L^{-2} for two particles and three levels in an isoenergetic expansion, a relation in the form

$$5\left(\frac{L}{L_1}\right)^2 = \sum_{j=1}^{2}\sum_{n=1}^{3}\left|a_n^{(j)}\right|^2 n^2 \equiv D. \quad (36)$$

For the case of a spectrum of the type L^{-4}, for two particles and three levels in the isoenergetic expansion, the following relationship is obtained:

$$17\left(\frac{L}{L_1}\right)^4 = \sum_{j=1}^{2}\sum_{n=1}^{3}\left|a_n^{(j)}\right|^2 n^4 \equiv K. \quad (37)$$

Then, for the first approximation to the quotient K/D^2 there must be a constant given by the value $17/25$. The same analysis is done for the case of isoenergetic compression, where the relation found is $K/D^2 = 97/169$.

Equations (34) and (35) can be simplified using the first-order correction for the ratio between the different widths of the wall and using an approximation of the type $\ln(1 \pm x) \sim \pm x$. The two ln of L_2/L_1 and L_3/L_4 can be approximated to

$$\ln\left(\frac{L_2}{L_1}\right) \simeq \frac{1}{2}\ln\left(\frac{13}{5}\right) + \frac{17}{40}\left(\frac{\lambda}{2L_1}\right)^2 + \mathcal{O}[(\lambda/L_1)^4], \quad (38)$$

$$\ln\left(\frac{L_3}{L_4}\right) \simeq \frac{1}{2}\ln\left(\frac{13}{5}\right) - \frac{97}{104}\left(\frac{\lambda}{2L_3}\right)^2 + \mathcal{O}[(\lambda/L_1)^4]. \quad (39)$$

Using these equations in combination with Eqs. (34) and (35), we obtain the following equations for Q_H and Q_C:

$$Q_H = \frac{5}{2}mc^2\left(\frac{\lambda}{2L_1}\right)^2 \ln\left(\frac{13}{5}\right) - \frac{17}{8}mc^2\left(\frac{\lambda}{2L_1}\right)^4 \mathcal{A} + \dot{Q}_r\tau, \tag{40}$$

$$|Q_C| = \frac{13}{2}mc^2\left(\frac{\lambda}{2L_3}\right)^2 \ln\left(\frac{13}{5}\right) - \frac{97}{8}mc^2\left(\frac{\lambda}{2L_3}\right)^4 \mathcal{B} + \dot{Q}_r\tau, \tag{41}$$

where the expression for $\mathcal{A}$ and $\mathcal{B}$ are

$$\mathcal{A} = \ln\left(\frac{13}{5}\right) + \frac{50}{17}\frac{K}{D^2} - 1, \tag{42}$$

$$\mathcal{B} = \ln\left(\frac{13}{5}\right) + \frac{338}{97}\frac{K}{D^2} + 1. \tag{43}$$

Finally, the total mechanical work defined by

$$W = Q_H - Q_C, \tag{44}$$

can be rewritten for this case as

$$W = \frac{mc^2}{2}\ln\left(\frac{13}{5}\right)\left(\frac{\lambda}{2L_1}\right)^2\left[5 - 13\left(\frac{L_1}{L_3}\right)^2\right] - \frac{17mc^2}{8}\left(\frac{\lambda}{2L_1}\right)^4\mathcal{A}\left[1 - \frac{97\mathcal{B}}{17\mathcal{A}}\left(\frac{L_1}{L_3}\right)^4\right]. \tag{45}$$

Note that if we neglect the term $(\lambda/L_1)^4$ we obtain the result

$$W = \frac{\hbar^2\pi^2}{2m}\left(\frac{5}{L_1^2} - \frac{13}{L_3^2}\right)\ln\left(\frac{13}{5}\right), \tag{46}$$

which corresponds to the expression for the nonrelativistic case presented in the work of Wang *et al.* [13].

B. Ultrarelativistic case

Now we discuss the case of the asymptotic limit of vanishing mass for the spectrum in Eq. (5). In this case, the initial energy of the cycle described previously is given by

$$E_H = \frac{3\pi\hbar c}{L_1}. \tag{47}$$

Throughout the first process, the energy as a function of L can be rewritten as

$$E_{12}(L) = \frac{\pi\hbar c}{L}\left(\left|a_1^{(1)}\right|^2 + 2\left|a_2^{(1)}\right|^2 + 3\left|a_3^{(1)}\right|^2 + \left|a_1^{(2)}\right|^2 + 2\left|a_2^{(2)}\right|^2 + 3\left|a_3^{(2)}\right|^2\right). \tag{48}$$

The force is then given by

$$F_{12}(L) = \frac{\hbar\pi c}{L^2}\left(\left|a_1^{(1)}\right|^2 + 2\left|a_2^{(1)}\right|^2 + 3\left|a_3^{(1)}\right|^2 + \left|a_1^{(2)}\right|^2 + 2\left|a_2^{(2)}\right|^2 + 3\left|a_3^{(2)}\right|^2\right), \tag{49}$$

subject to restriction that Eqs. (47) and (48) remain equal,

$$L = \frac{L_1}{3}\left(\left|a_1^{(1)}\right|^2 + 2\left|a_1^{(2)}\right|^2 + 3\left|a_3^{(1)}\right|^2 + \left|a_1^{(2)}\right|^2 + 2\left|a_2^{(2)}\right|^2 + 3\left|a_3^{(2)}\right|^2\right). \tag{50}$$

Then, we can compact the force as follows:

$$F_{12}(L) = \frac{3\pi\hbar c}{L_1 L}. \tag{51}$$

In the context of maximal expansion, when the system is in L_2, we obtain from Eqs. (48) and (49), $L_2 = \frac{5}{3}L_1$.

For the process $2 \to 3$ the system expands adiabatically from $L = L_2$ to $L = L_3$. The system remains in the initial configuration before this process begins; therefore, that means $|a_2^{(1)}| = 1$, $|a_3^{(2)}| = 1$, and all other coefficients are equal to zero. The expected value for the energy throughout the process is given by $E_{23} = \frac{5\pi\hbar c}{L}$, and the force is given by $F_{23} = \frac{5\pi\hbar c}{L^2}$.

For the isoenergetic compression, the expectation value of the Hamiltonian is kept constant as

$$E_C = \frac{5\pi\hbar c}{L_3}. \tag{52}$$

Using the same treatment that was presented earlier, it is easy to show that force is given by

$$F_{34}(L) = \frac{5\pi\hbar c}{L_3 L}, \tag{53}$$

and under the maximal compression we obtain the relation $L_4 = \frac{3}{5}L_3$.

For the last process, adiabatic compression, from L_4 to L_1, the energy of the system as a function of L is given by $E_{41} = \frac{3\pi\hbar c}{L}$ and the force applied to the wall of the potential is $F_{41} = \frac{3\pi\hbar c}{L^2}$.

For this case, the energy absorbed Q_H and the energy released Q_C are, respectively,

$$Q_H = \frac{3\pi\hbar c}{L_1}\ln\left(\frac{5}{3}\right) + \dot{Q}_r\tau, \tag{54}$$

$$Q_C = \frac{5\pi\hbar c}{L_1}\ln\left(\frac{5}{3}\right) + \dot{Q}_r\tau. \tag{55}$$

The mechanical work W per cycle is given by

$$W = \pi\hbar c\left(\frac{3}{L_1} - \frac{5}{L_3}\right)\ln\left(\frac{5}{3}\right). \tag{56}$$

V. OPTIMIZATION OF THE PERFORMANCE OF THE HEAT ENGINE

To obtain a finite power in our heat engine, we use the approach proposed by Abe [12]. Therefore, we define finite average speed of the variation of L as $\bar{v}(t)$ and the total length variations along one cycle as L_0. Therefore,

$$L_0 = |L_1 - L_2| + |L_2 - L_3| + |L_3 - L_4| + |L_4 - L_1| = 2(L_3 - L_1). \tag{57}$$

It is important to recall that in order for the adiabatic theorem to apply, the timescale associated with the variation of the state must be assumed to be much larger than that of the dynamical one, $\sim\hbar/E$ [12–14]. We define the total time of the cycle τ as a function of average speed by the expression

$$\tau = \frac{L_0}{\bar{v}} = \frac{2(L_3 - L_1)}{\bar{v}}, \tag{58}$$

and this time has to be much larger than $\hbar/E$ in order to fulfill the adiabatic regime in the cycle.

Now, we discuss the optimization scheme as follows. First, we use a definition of power output, given by $P = W/\tau$, where W corresponds to the total work along one cycle discussed in the last section. Second, we define a dimensionless parameter $r = L_3/L_1$ to obtain the power output and the efficiency of the quantum engine as a function of r. It is convenient to define the dimensionless power output $P^*(r) = \frac{W}{s\tau}$, with s as a constant for the model which has units of power. For P^*, we can calculate the value $r = r_{\rm mp}$ that corresponds to the point given by the maximization condition $\frac{\partial P^*}{\partial r}|_{r=r_{\rm mp}} = 0$. On the other hand, the efficiency ($\eta = W/Q_H$) depends on the energy leakage $\dot{Q}_r$, which can be rewritten in the form of $\dot{Q}_r = \alpha\dot{q}$, where $\dot{q}$ it is an expression that depends on the model, and the parameter α is assumed to be constant. Therefore, the maximization condition for the efficiency given by $\frac{\partial \eta}{\partial r}|_{r=r_{m\eta}} = 0$ is strongly affected by the value of the parameter α.

Finally, we present the two cases discussed in the last section, the first-order correction and the ultrarelativistic case, and we find the characteristic curve of P^* versus η, which describes the two maximum points previously mentioned. The engine optimization is defined as

$$\eta_{\rm mp} \leqslant \eta \leqslant \eta_{\rm max}, \quad P_{m\eta} \leqslant P \leqslant P_{\rm max}. \tag{59}$$

For the first-order correction, we present a table of values for $r = r_{\rm mp}$ for the different cases of $(\lambda/2L_1)^2$ (fixed to do the optimization scheme). We compare these values with those of a nonrelativistic engine and see the effect of correction in the different graphics of interest. For the ultrarelativistic case, we obtain an analytical result in line with that presented in Refs. [14,37] for power-law potentials.

A. First-order correction

The power output after a single cycle for this case is given by

$$P = \frac{W}{\tau} = \frac{2\bar{v}mc^2}{\lambda}\left(\frac{\lambda}{2L_1}\right)^3\left[\frac{1}{4}\frac{5r^2-13}{r^3-r^2}\ln\frac{13}{5} - \mathcal{A}\left(\frac{\lambda}{2L_1}\right)^2\frac{17}{16}\frac{r^4-\frac{97\mathcal{B}}{17\mathcal{A}}}{r^5-r^4}\right], \tag{60}$$

and we can define the dimensionless power output as

$$P^* = \frac{W}{s\tau} = \frac{1}{4}\frac{5r^2-13}{r^3-r^2}\ln\frac{13}{5} - \mathcal{A}\left(\frac{\lambda}{2L_1}\right)^2\frac{17}{16}\frac{r^4-\frac{97\mathcal{B}}{17\mathcal{A}}}{r^5-r^4}, \tag{61}$$

where $s = \frac{2\bar{v}mc^2}{\lambda}(\frac{\lambda}{2L_1})^3$. This constant can be rewritten as $s = \frac{\hbar^2\pi^2\bar{v}}{mL_1^3}$, which exactly corresponds to the constant defined in Ref. [13]. However, for our case it is more convenient to define it in its first form, because in our optimization study we fixed $\bar{v}$ and $(\frac{\lambda}{2L_1})$ to control r. The value of $\mathcal{A}$ and $\mathcal{B}$ are subject to the value of the quotient $\frac{K}{D^2}$ and the possible values of this fraction are in the range $\frac{17}{25} \leqslant \frac{K}{D^2} \leqslant \frac{97}{169}$ as demonstrated before. We take the average between the two extreme values for our calculations and approximate to $(K/D^2) \sim 0.63$ to simplify the discussion.

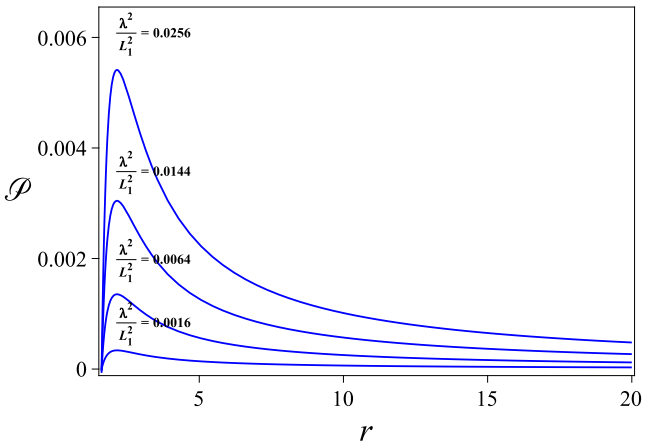

FIG. 2. The different curves for the correction term $\mathscr{P}$ for different values of $(\frac{\lambda}{L_1})^2$. This figure represents the difference for P^* between the value reported in Ref. [13] and our calculations in the first-order correction.

In order to show the relativistic correction for the power output from the no relativistic case, we can write Eq. (61) in the form $P^* = P_S^* - \mathscr{P}$, where P_S^* is the first term in Eq. (61) and $\mathscr{P}$ the first-order correction for the power output given by

$$\mathscr{P} = \mathcal{A}\left(\frac{\lambda}{2L_1}\right)^2\frac{17}{16}\frac{r^4-\frac{97\mathcal{B}}{17\mathcal{A}}}{r^5-r^4}, \tag{62}$$

which is presented in Fig. 2 for different values of $(\lambda/L_1)^2$. In Fig. 3 we present the scheme of the P^* of nonrelativistic particles and the first-order correction. The physical effect is clear: the power output of this engine decreases when considering the first-order correction. The total work under one cycle, given by Eq. (45), is lower than that reported in Ref. [13]. These results are coherent with those reported in Ref. [19], which demonstrated that the efficiency is smaller

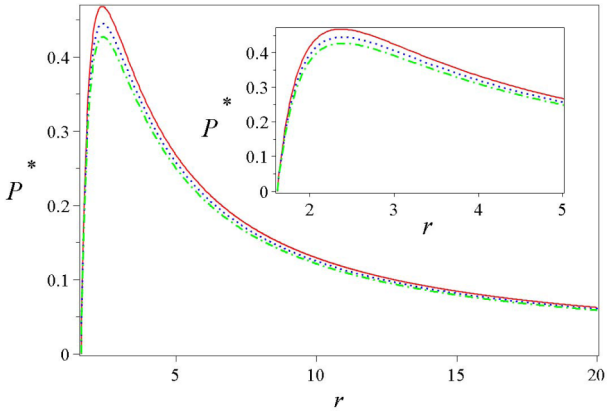

FIG. 3. The power output P^* vs. r for the nonrelativistic case and different corrections of interest. The solid line represents the no relativistic calculation (P_S^*). For this graphic we consider $(\lambda/2L_1)^2 = 0.0225$ (blue, dotted line) and $(\lambda/2L_1)^2 = 0.04$ (green, dash-dotted line) with the approximation $\frac{K}{D^2} \sim 0.63$ for simplicity. The inset depicts the zoomed region of the same figure.

TABLE I. The values of $r_{\rm mp}$ and $P^*_{\rm max}$ for different values of $(\frac{\lambda}{2L_1})^2$ for the first-order correction.

$(\frac{\lambda}{2L_1})^2$	$r_{\rm mp}$	$P^*_{\rm max}$
~ 0	2.367114902	0.4682644969
10^{-4}	2.367171434	0.4681567564
4×10^{-4}	2.367341230	0.4678335433
9×10^{-4}	2.367624884	0.4672948485
1.6×10^{-3}	2.368023392	0.4655713187
2.5×10^{-3}	2.368538162	0.4643867228
3.6×10^{-3}	2.369171021	0.4643867228
4.9×10^{-3}	2.3699924236	0.4629867035
6.4×10^{-3}	2.370800527	0.4613719506
8.1×10^{-3}	2.371803094	0.4595421083
10^{-2}	2.372935642	0.4574974961

in the relativistic particles as compared with the nonrelativistic particles. In Table I, we show the different values for $r_{\rm mp}$ and $P^*_{\rm max}$ starting with the values obtained in Ref. [13] and then for different values of $(\frac{\lambda}{2L_1})^2$.

Now, we study the efficiency η, which is given by

$$\eta = \frac{W}{Q_H}, \tag{63}$$

and for our case we obtain the expression

$$\eta = \frac{\left(1-\frac{13}{5r^2}\right) - \frac{34}{40}\left(\frac{\lambda}{2L_1}\right)^2\tilde{\mathcal{A}}\left(1-\frac{97}{17r^4}\frac{\mathcal{B}}{\mathcal{A}}\right)}{1+\alpha(r-1)-\frac{34}{40}\left(\frac{\lambda}{2L_1}\right)^2\tilde{\mathcal{A}}}, \tag{64}$$

where we have defined $\tilde{\mathcal{A}} = A/\ln(\frac{13}{5})$ and the energy leakage in the form $\dot{Q}_r = \alpha\dot{q}$ with $\dot{q} = \frac{5}{2}\frac{mc^2\bar{v}}{\lambda}(\frac{\lambda}{2L_1})^3\ln(\frac{13}{5})$. These results for the efficiency in Eq. (64) are different from $\eta = 1-\frac{E_C}{E_H}$ because this expression can only be obtained for a single power-law potential [13,14,37]. This is also because in quantum mechanics there is no analog of the second law of thermodynamics [2,41].

We remark when $(\frac{\lambda}{2L_1})^2 \ll 1$, Eqs. (61) and (64) converge toward the nonrelativistic case presented in Ref. [13]. On the other hand, when the engine reaches maximum efficiency $\eta_{\rm max}$, we obtain the general equation,

$$\left[\frac{26}{5r_{m\eta}^3}-\frac{97\tilde{\mathcal{B}}}{5r_{m\eta}^5}\left(\frac{\lambda}{2L_1}\right)^2\right]\left[1+\alpha(r_{m\eta}-1)-\frac{17\tilde{\mathcal{A}}}{20}\left(\frac{\lambda}{2L_1}\right)^2\right] - \left[1-\frac{13}{5r_{m\eta}^2}-\frac{17\tilde{\mathcal{A}}}{20}\left(1-\frac{97}{17r_{m\eta}^4}\frac{\mathcal{B}}{\mathcal{A}}\right)\left(\frac{\lambda}{2L_1}\right)^2\right]\alpha = 0, \tag{65}$$

where we have defined $\tilde{\mathcal{B}} = B/\ln(\frac{13}{5})$. The last equation shows the dependency of $r_{m\eta}$ on the value of the parameter α and the initial value of λ/L_1. When $(\frac{\lambda}{2L_1})^2 \to 0$, we obtain the following equation:

$$(5r_{m\eta}^3 - 39r_{m\eta} + 26)\alpha - 26 = 0, \tag{66}$$

which was reported for the nonrelativistic case in Ref. [13]. In Fig. 4, we show the dimensionless power output P^* as a function of the efficiency. This graphic displays the two characteristic points for the efficiency, $\eta_{\rm mp}$ and $\eta_{\rm max}$, and the two corresponding critical points for the dimensionless power output, $P^*_{\rm max}$ and $P^*_{m\eta}$. Therefore, if λ/L_1 is fixed, and the value of $r_{m\eta}$ can be obtained from Eq. (65) and used to replace its value in Eq. (61) to obtain $P^*_{m\eta}$. For the same value of λ/L_1 the point $r_{\rm mp}$ can be obtained by calculating the derivative of Eq. (61), then replacing the value in Eq. (64) to obtain the value of $\eta_{\rm mp}$. Thus, we have a family of loop-shaped curves

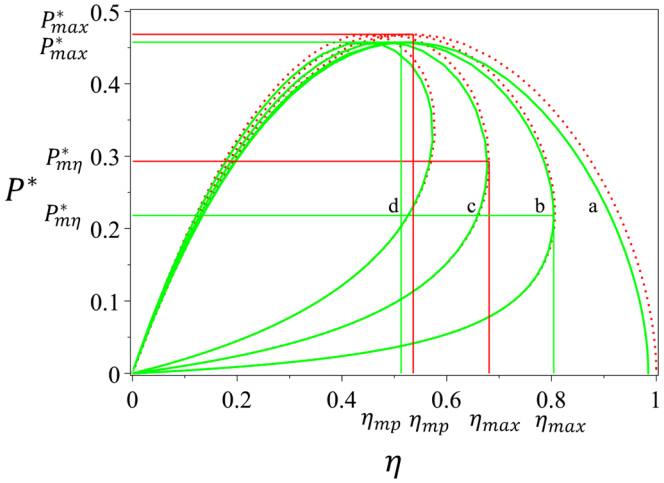

FIG. 4. The dimensionless power output versus the efficiency for the nonrelativistic case [13] (red, dotted line) and example of the first-order correction for $(\lambda/2L_1)^2 = 10^{-2}$ (green, solid line) showing the two critical points for the efficiency $(\eta_{\rm mp},\eta_{\rm max})$ and the two critical points for the power output $(P^*_{\rm max},P_{m\eta})$. The curves are: *a* ($\alpha = 0$), *b* ($\alpha = 0.03$), *c* ($\alpha = 0.08$), and *d* ($\alpha = 0.15$).

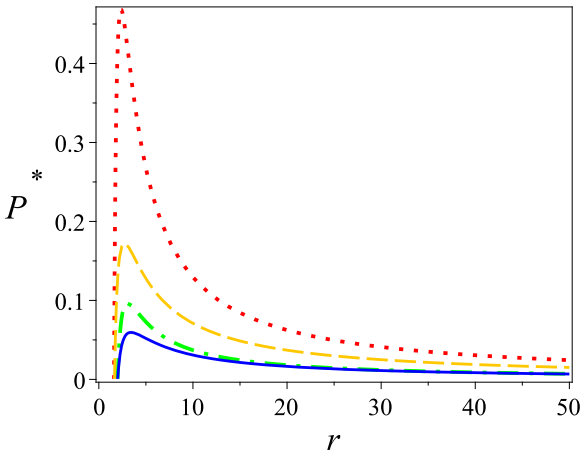

FIG. 5. The dimensionless power output P^* vs. the parameter r for $r \leqslant 50$, for different cases of interest. The dotted line (red) and the dot-dash line (green) represent the work of Wang *et al.* [13] and Abe [12], respectively. The dash line (orange) and the solid line (blue) represents the ultrarelativistic case for two particles in three levels and two particles in two levels presented in Eqs. (56) and (57), respectively.

always limited by the values presented in the work of Wang *et al.* [13].

B. Ultrarelativistic case

For this case, it is easy to show that the power output is

$$P = \frac{W}{\tau} = \frac{\hbar\pi c\bar{v}}{2L_1^2}\left(\frac{3r-5}{r^2-r}\right)\ln\left(\frac{5}{3}\right), \tag{67}$$

and the dimensionless power output is then

$$P^* = \frac{W}{s\tau} = \frac{1}{2}\left(\frac{3r-5}{r^2-r}\right)\ln\left(\frac{5}{3}\right), \tag{68}$$

with $s \equiv \frac{\hbar\pi c\bar{v}}{L_1^2}$. We can show without difficulty using the work of Abe [12] that the dimensionless power output for the case of the two ultrarelativistic particles and the two levels of energy are given by

$$P^* = \frac{1}{2}\left(\frac{r-2}{r^2-r}\right)\ln(2). \tag{69}$$

The fourth power output is presented in Fig. 5, outlining the case for two nonrelativistic particles in two levels and three levels versus the ultrarelativistic case presented in this work for the same cases. Remarkably, our results are consistent with those presented in Ref. [14,37] for this type of power-law trap.

Then, the efficiency for the ultrarelativistic engine is given by

$$\eta = \frac{W}{Q_H} = \frac{\hbar\pi c\left(\frac{3}{L_1}-\frac{5}{L_3}\right)\ln\left(\frac{5}{3}\right)}{\frac{3\hbar\pi c}{L_1}\ln\left(\frac{5}{3}\right)+\dot{Q}_r\frac{2(L_3-L_1)}{\bar{v}}}. \tag{70}$$

As before, we can select $\dot{Q}_r$ as follows:

$$\dot{Q}_r = \alpha\frac{3\hbar\pi c}{2L_1^2}\ln\left(\frac{5}{3}\right)\bar{v}. \tag{71}$$

Therefore, we rewrite the efficiency in terms of r and α to get

$$\eta = \frac{\left(1-\frac{5}{3r}\right)}{1+\alpha(r-1)}, \tag{72}$$

and it is displayed in Fig. 6 for different cases of interest. As the efficiency is a non negative number, we find from Eq. (72) that the restriction for the r values is given by

$$r > \frac{5}{3}. \tag{73}$$

We can control r to maximize the dimensionless power output P^* based on the assumption that L_1 and $\bar{v}$ are fixed; hence, the maximization condition $\frac{\partial P^*}{\partial r}|_{r=r_{\rm mp}} = 0$ yields

$$3r_{\rm mp}^2 - 10r_{\rm mp} + 5 = 0, \tag{74}$$

which has one valid solution that satisfies the condition in Eq. (73), $r_{\rm mp} \sim 2.72$. For $\eta_{\rm max}$ ($\frac{\partial\eta}{\partial r}|_{r=r_{m\eta}} = 0$) we obtain an equation that depends on the parameter α given by

$$\left(3r_{m\eta}^2 - 10r_{m\eta} + 5\right)\alpha - 5 = 0, \tag{75}$$

whose solution is

$$r_{m\eta} = \frac{1}{3}\frac{5\alpha+\sqrt{10\alpha^2+15\alpha}}{\alpha}. \tag{76}$$

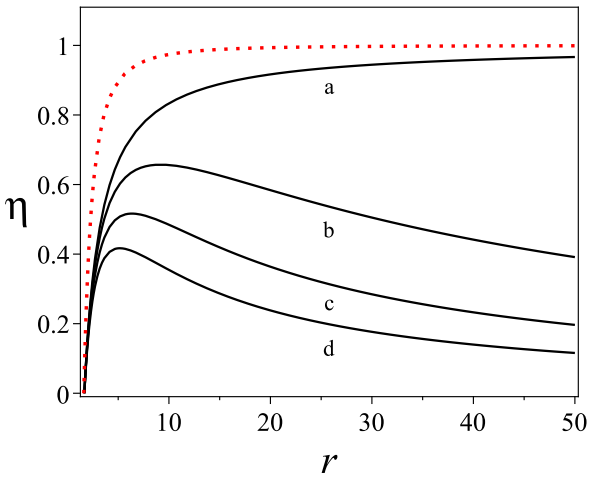

FIG. 6. Efficiency of the ultrarelativistic case for the case of two particles in three levels obtained in the Eq (60) for different values of α. The curves are: a ($\alpha = 0$), b ($\alpha = 0.03$), c ($\alpha = 0.08$), and d ($\alpha = 0.15$). The dotted line corresponds to the same case for nonrelativistic particles with $\alpha = 0$ obtained in the Ref. [13].

It is important to recall that the limits of Eq. (76) are obtained when $\alpha \to \infty, r_{m\eta} \to \frac{5+\sqrt{10}}{3} \sim 2.72$, and when $\alpha \to 0$, $r_{m\eta} \to \infty$, we get

$$2.72 \leqslant r \leqslant r_{m\eta}. \tag{77}$$

The values of $r_{m\eta}$, $\eta_{\rm mp}$, and $\eta_{\rm max}$ for given parameters α for the ultrarelativistic case is given in the Table II.

When $\alpha = 0$, which energy leakage $\dot{Q}_r = 0$, we obtain the following result for the efficiency

$$\eta = 1 - \frac{5}{3r} = 1 - \frac{E_C}{E_H}. \tag{78}$$

Using the fact that $r_{\rm mp} \sim 2.721$, we obtain for the efficiency the following value:

$$\eta_{\rm mp} = 1 - \frac{5}{3r_{\rm mp}} \simeq 0.387, \tag{79}$$

which can be compared with the nonrelativistic case $\eta_{\rm mp} \simeq 0.536$ presented in Ref. [13] and showed in Fig. 4. The efficiency at the maximum power output in the ultrarelativistic limit is lower (see Fig. 7) than that of the nonrelativistic case, and is in line with the results in the first-order correction presented in this work.

TABLE II. The values of $r_{m\eta}$, $\eta_{\rm mp}$, and $\eta_{\rm max}$ for given parameters α for the ultrarelativistic case.

α	$r_{m\eta}$	$\eta_{\rm mp}$	$\eta_{\rm max}$
0	∞	0.387	1
0.03	9.194	0.368	0.657
0.08	6.351	0.340	0.516
0.15	5.163	0.308	0.417

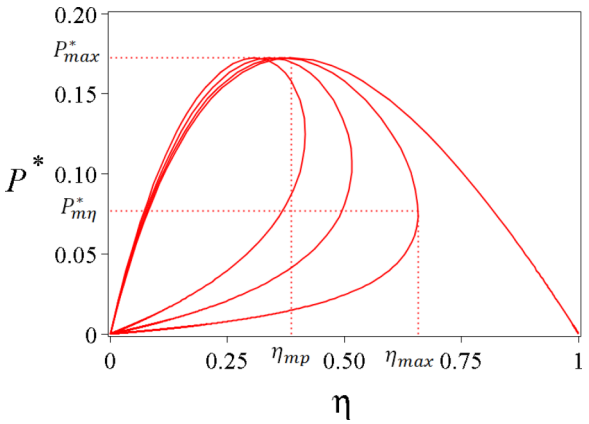

FIG. 7. The dimensionless power output P^* vs. the efficiency η for different values of α for the ultrarelativistic case of two particles in three levels. The curves a ($\alpha = 0$), b ($\alpha = 0.03$), c ($\alpha = 0.08$), and d ($\alpha = 0.15$) are presented in order to make a comparison with the work of Wang *et al.* [13].

VI. CONCLUSIONS

In this work, we found the first-order relativistic correction for the calculations presented in Ref. [13] and the ultrarelativistic case that used the power-law spectrum presented in Refs. [14,37]. We have shown that the power output decreases as compared with the nonrelativistic case, and this is in agreement with the results for ultrarelativistic calculations. For the case of the first-order correction, we have a family of functions that can be plotted in the characteristic curve P^* versus η and we provide the values $P^*_{\max}$, $P_{m\eta}$, $\eta_{\rm mp}$, and $\eta_{\max}$. From Table I, we can see that the dimensionless power output decreases and the value of $r_{\rm mp}$ increases from the value $r_{\rm mp} \sim 2.367$ to $r_{\rm mp} \sim 2.373$, yielding an expected result if we see the value of the ultrarelativistic case, which is given by $r_{\rm mp} \sim 2.721$. The combination of different power-law spectra provides a nontrivial relationship for the force along the isoenergetic cycle. We believe that by exploring some small parameters of the model, a simplified version could be obtained. This could be used to study its effects on known models or to address a new problem of interest. The different approximations used in this work makes the correction of power output small, but we think they are interesting and relevant. One possibility to improve these kinds of corrections, when having two power-law spectra, is to think of a weight factor. This weight factor must depend on L and needs to have the correct asymptotic behavior. This, of course, is beyond our present work and discussion. Finally, we completed the study with the ultrarelativistic case, and we plotted our results with the curves for the case of two particles and two levels studied by Abe [12] and Wang *et al.* [13], which are the limiting cases of our more general approach.

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APPENDIX

We shall use the approximation to obtain the force in compact form, present in Eqs. (23) and (29). For the first isoenergetic process, we have a restriction in the form of

$$\frac{5}{2}mc^2\left(\frac{\lambda}{2L_1}\right)^2 - \frac{17}{8}mc^2\left(\frac{\lambda}{2L_1}\right)^4 = \frac{mc^2}{2}\left(\frac{\lambda}{2L}\right)^2 D - \frac{mc^2}{8}\left(\frac{\lambda}{2L}\right)^4 K, \tag{A1}$$

where

$$D = \sum_{j=1}^{2}\sum_{n=1}^{3}\left|a_n^{(j)}\right|^2 n^2 \tag{A2}$$

and

$$K = \sum_{j=1}^{2}\sum_{n=1}^{3}\left|a_n^{(j)}\right|^2 n^4. \tag{A3}$$

On the other hand, the force throughout the process in terms of these definitions can be rewritten as

$$F_{12}(L) = \frac{mc^2}{L}\left(\frac{\lambda}{2L}\right)^2 D - \frac{mc^2}{2L}\left(\frac{\lambda}{2L}\right)^4 K, \tag{A4}$$

subject to restriction imposed by Eq. (A1). To solve Eq. (A1), we can define the variables

$$x = \left(\frac{\lambda}{2L}\right)^2 c_1 = \frac{5}{2}mc^2\left(\frac{\lambda}{2L_1}\right)^2 - \frac{17}{8}mc^2\left(\frac{\lambda}{2L_1}\right)^4 \tag{A5}$$

and easily find the quadratic equation

$$x^2 K - 4xD + 8c_1 = 0, \tag{A6}$$

whose solution is

$$x = \frac{2D}{K}\left(1 \pm \sqrt{1 - 2c_1\frac{K}{D^2}}\right). \tag{A7}$$

The solution of physical interest is

$$x = \frac{2D}{K}\left(1 - \sqrt{1 - 2c_1\frac{K}{D^2}}\right), \tag{A8}$$

and as we work under the condition $(\frac{\lambda}{2L_1})^2 \ll 1$, we can use a Taylor expansion of the root and easily find

$$x \simeq 2\frac{c_1}{D}. \tag{A9}$$

It is important to check our approximation considering the case of maximal expansion when $L_1 \to L_2$, so $x \to x_2$. The first particle is in the first excited state, and the second particle is in the second excited sate. Under this condition, D is fixed in the value $D = 13$, and we get for $x_2 = (\frac{\lambda}{2L_2})^2$ the expression

$$x_2 \simeq \frac{2c_1}{13} = \frac{2}{13}\left[\frac{5}{2}mc^2\left(\frac{\lambda}{2L_1}\right)^2 - \frac{17}{8}mc^2\left(\frac{\lambda}{2L_1}\right)^4\right]. \tag{A10}$$

Note that, neglecting the term $(\frac{\lambda}{L_1})^4$, we obtain the result of a nonrelativistic case [13] as we comment in the work. Therefore, to find an elegant and physical solution, we can work to order $(\frac{\lambda}{L})^4$ without losing important information. So, we can replace the approximate solution given by Eq. (A9) in the expression of the force, to obtain

$$F_{12}(L) = \frac{2mc^2}{L}\left[\frac{5}{2}\left(\frac{\lambda}{2L_1}\right)^2 - \frac{17}{8}\left(\frac{\lambda}{2L_1}\right)^4\right] - \frac{2mc^2}{L}\frac{K}{D^2}\left[\frac{5}{2}\left(\frac{\lambda}{2L_1}\right)^2 - \frac{17}{8}\left(\frac{\lambda}{2L_1}\right)^4\right]^2, \tag{A11}$$

and if we work to order $(\lambda/L_1)^4$, we can take the first term in the second term of the force

$$\left[\frac{5}{2}\left(\frac{\lambda}{2L_1}\right)^2 - \frac{17}{8}\left(\frac{\lambda}{2L_1}\right)^4\right]^2 \sim \frac{25}{4}\left(\frac{\lambda}{2L_1}\right)^4, \tag{A12}$$

to get

$$F_{12}(L) = \frac{5mc^2}{L}\left(\frac{\lambda}{2L_1}\right)^2 - \frac{mc^2}{L}\left(\frac{17}{4} + \frac{25K}{2D^2}\right)\left(\frac{\lambda}{2L_1}\right)^4. \tag{A13}$$

Using the same method, we obtain for the isoenergetic compression the quadratic equation

$$x^2K - 4xD + 8c_3 = 0, \tag{A14}$$

with

$$c_3 = \frac{13}{2}\left(\frac{\lambda}{2L_3}\right)^2 - \frac{97}{8}\left(\frac{\lambda}{2L_3}\right)^4. \tag{A15}$$

Under the same conditions discussed before, we obtain

$$x \simeq \frac{2c_3}{D}. \tag{A16}$$

In the context of maximal compression when $L_3 \to L_4$ ($x \to x_4$), the first particle returns to the ground state, and the second particle returns to the first excited state, and then $D = 5$. The solution of physical interest is then

$$x_4 \simeq \frac{2}{5}\left[\frac{13}{2}\left(\frac{\lambda}{2L_3}\right)^2 - \frac{97}{8}\left(\frac{\lambda}{2L_3}\right)^4\right]. \tag{A17}$$

Neglecting the order $(\lambda/L_3)^4$, we obtain the result presented in the nonrelativistic case as discussed in the work. On the other hand, the force during the compression phase is given by the same Eq. (A4) subject to different constraint imposed by Eq. (A16). Then, we obtain for the force

$$F_{34}(L) = \frac{2mc^2}{L}\left[\frac{13}{2}\left(\frac{\lambda}{2L_3}\right)^2 - \frac{97}{8}\left(\frac{\lambda}{2L_3}\right)^4\right] - \frac{2mc^2}{L}\left[\frac{13}{2}\left(\frac{\lambda}{2L_3}\right)^2 - \frac{97}{8}\left(\frac{\lambda}{2L_3}\right)^4\right]^2, \tag{A18}$$

and if we work to order $(\lambda/L_3)^4$, we can take the first term in the second term of the force,

$$\left[\frac{13}{2}\left(\frac{\lambda}{2L_3}\right)^2 - \frac{97}{8}\left(\frac{\lambda}{2L_3}\right)^4\right]^2 \sim \frac{169}{4}\left(\frac{\lambda}{2L_3}\right)^4, \tag{A19}$$

and finally we get

$$F_{34}(L) = \frac{13mc^2}{L}\left(\frac{\lambda}{2L_3}\right)^2 - \frac{mc^2}{L}\left(\frac{97}{4} + \frac{169K}{2D^2}\right)\left(\frac{\lambda}{2L_3}\right)^4. \tag{A20}$$

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