Reynolds analogy for the Rayleigh problem at various flow modes

A. A. Abramov* and A. V. Butkovskii†
Central Aerohydrodynamic Institute (TsAGI) 1, Zhukovsky street, 140180 Zhukovsky, Russia
(Received 14 October 2015; revised manuscript received 26 May 2016; published 18 July 2016)

The Reynolds analogy and the extended Reynolds analogy for the Rayleigh problem are considered. For a viscous incompressible fluid we derive the Reynolds analogy as a function of the Prandtl number and the Eckert number. We show that for any positive Eckert number, the Reynolds analogy as a function of the Prandtl number has a maximum. For a monatomic gas in the transitional flow regime, using the direct simulation Monte Carlo method, we investigate the extended Reynolds analogy, i.e., the relation between the shear stress and the energy flux transferred to the boundary surface, at different velocities and temperatures. We find that the extended Reynolds analogy for a rarefied monatomic gas flow with the temperature of the undisturbed gas equal to the surface temperature depends weakly on time and is close to 0.5. We show that at any fixed dimensionless time the extended Reynolds analogy depends on the plate velocity and temperature and undisturbed gas temperature mainly via the Eckert number. For Eckert numbers of the order of unity or less we generalize an extended Reynolds analogy. The generalized Reynolds analogy depends mainly only on dimensionless time for all considered Eckert numbers of the order of unity or less.

DOI: 10.1103/PhysRevE.94.013110

I. INTRODUCTION

One of the most interesting results of fluid dynamics is the Reynolds analogy, i.e., the proportionality of the shear stress and heat flux in the boundary layer flow problem [1].

The extended Reynolds analogy along a sharp leading-edge flat plate at zero angle of attack in a hypersonic flow was proved in [2] using the Burnett equation in the near-continuum flow regime and direct simulation Monte Carlo (DSMC) method for the transitional regime. Similar results for blunt-nosed bodies in hypersonic flows were obtained in [3]. The extended Reynolds analogy near the leading-edge region has been observed for the transitional regime in experimental data [4,5]. Recall that the extended Reynolds analogy is the relation between the momentum flux and energy flux transferred to the plate. In the limit of small Knudsen numbers the Reynolds analogy and the extended Reynolds analogy coincide.

There exists the Reynolds analogy for the Navier-Stokes plane Couette problem for arbitrary plate temperatures [6]. In the case of equal plate temperatures this result leads to the fact that the energy fluxes transferred to the plate are equal to each other and are equal to half the product of the shear stress and the relative velocity of the plates [7].

In this article we study the Reynolds analogy and the extended Reynolds analogy for another classical problem of gas dynamics namely, the Rayleigh problem.

This paper is organized as follows. In Sec. II we state the problem. In Sec. III we study the problem for the viscous incompressible fluid. In Sec. IV we consider the rarefied gas flow. In Sec. V the results of the study are summarized.

II. PROBLEM STATEMENT

We consider a semi-infinite space $y > 0$ filled with an undisturbed fluid of temperature T_∞. Let a plane $y = 0$ acquire temperature T_s at time $\bar{t} = 0$ and begin to move in its own plane with velocity $\mathbf{U}_s$. Velocity $\mathbf{U}_s$ is directed opposite to the x axis. The axes x and y form a Cartesian coordinate system and the y axis is normal to the plate. In Sec. III we consider the case of a viscous incompressible fluid of dynamic viscosity μ, thermal conductivity k, and specific heat c. The fluid flow satisfies the Navier-Stokes equations and the no-slip boundary conditions on the plane. In Sec. IV we consider the case of a monatomic gas of numerical density n_∞. The gas flow satisfies the Boltzmann equation for the molecule distribution function f and the conditions of diffuse reflection at a plane. The initial condition for f is the Maxwell distribution for undisturbed gas flow. Note that in the limit of small $\mathbf{U}_s$ and large $\bar{t}$ the formulations of the problems for fluid and gas are equivalent [8].

III. REYNOLDS ANALOGY FOR THE RAYLEIGH PROBLEM: INCOMPRESSIBLE FLUID FLOW

First of all, it should be noted that due to a self-similar solution of the Rayleigh problem for an incompressible fluid, the Reynolds analogy is not time dependent. Using [9], we obtain for the incompressible Rayleigh flow at any time,

$$\frac{E}{\tau U_s} \equiv \mathrm{R}(\mathrm{Pr}\,,\mathrm{Ec}) = \frac{2}{\sqrt{\pi}\,\mathrm{Pr}}\int_0^\infty \mathrm{erfc}(x)\exp\left[\left(1-\frac{2}{\mathrm{Pr}}\right)x^2\right]dx - \frac{1}{\mathrm{Ec}\sqrt{\mathrm{Pr}}}. \tag{3.1}$$

Here τ is the shear stress at the plate, E is the energy flux transferred to the plate (heat flux), $\mathrm{Pr} = \mu c/k$ is the Prandtl number, and $\mathrm{Ec} = U_s^2/c(T_s - T_\infty)$ is the Eckert number.

The dependence of the Reynolds analogy on the Prandtl number for the incompressible Rayleigh flow at $T_s = T_\infty$ is shown in Fig. 1.

It is easy to see that if $T_s = T_\infty$, then

$$\lim \mathrm{R}(\mathrm{Pr}) = \frac{\sqrt{2}}{2},$$
$$\mathrm{Pr} \to 0.$$

*alabr54@mail.ru
†albutkov@mail.ru

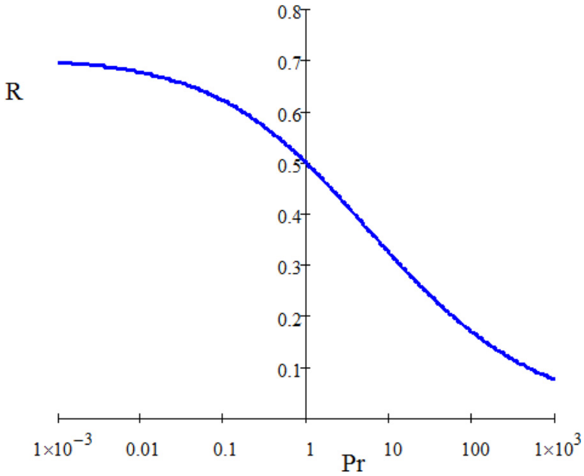

FIG. 1. The Reynolds analogy for the incompressible Rayleigh flow at plate temperature equal to the undisturbed gas temperature.

In the important special case when Pr = 1, we have

$$\mathrm{R}(1,\mathrm{Ec}) = \frac{1}{2} - \frac{1}{\mathrm{Ec}}. \tag{3.2}$$

If Ec > 0 (i.e., $T_s > T_\infty$) and the Prandtl number is small enough, the energy flux is negative (that is, the flux is directed away from the plate). However, viscosity increase and/or thermal conductivity decrease change the direction of the energy flux at some value of the Prandtl number. On the other hand R(Pr,Ec) → 0 as Pr → ∞. Therefore, at a certain Prandtl number R(Pr,Ec > 0) reaches the positive maximum.

The value of the Eckert number at which the energy flux transferred to the plate equals zero is

$$\mathrm{Ec}_0(\mathrm{Pr}) = \frac{\sqrt{\pi}}{2\int_0^\infty \mathrm{erfc}(x)\exp\left[\left(1-\frac{2}{\mathrm{Pr}}\right)x^2\right]dx},$$

$$\begin{matrix} \mathrm{Ec}_0(\mathrm{Pr}) \to \infty; & \mathrm{Ec}_0(\mathrm{Pr}) \to 0 \\ \mathrm{Pr} \to 0 & \mathrm{Pr} \to \infty \end{matrix}. \tag{3.3}$$

Figure 2 shows this dependence. The dependencies R(Pr ,Ec) at different values of the Eckert number are shown in Fig. 3.

IV. EXTENDED REYNOLDS ANALOGY FOR THE RAYLEIGH PROBLEM: RAREFIED GAS FLOW

First, we consider the free molecular flow. Using [10], we easily obtain for the free molecular Rayleigh flow of polyatomic ideal gas the extended Reynolds analogy:

$$\frac{E_s}{p_{xy}U_s} = \frac{1}{2} - \frac{\kappa+1}{2\kappa}\frac{1}{\mathrm{Ec}}. \tag{4.1}$$

Here κ is the adiabatic index, E_s is the energy flux transferred to the plate, p_{xy} is shear stress, $\mathrm{Ec} = U_s^2/c_p(T_s - T_\infty)$, and c_p is the heat capacity at constant pressure.

Thus, the extended free molecular Reynolds analogy as well as the Reynolds analogy for viscous incompressible fluid depends only on the Prandtl number [$\kappa = \kappa(\mathrm{Pr})$] and on the Eckert number and does not depend on time.

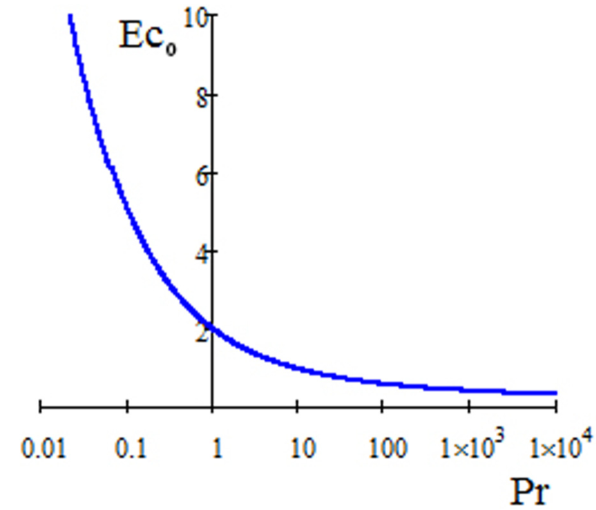

FIG. 2. The dependence of the Eckert number at which the energy flux transferred to the plate equals zero vs. the Prandtl number.

Now we move on to the Rayleigh problem for the transitional regime. This problem is solved in [10] using the DSMC method. The solution of the Rayleigh problem depends on the dimensionless variables $T_w = T_s/T_\infty$; $U_w = U_s/c_\infty$; $t = \bar{t}c_\infty/\lambda_\infty$. Here $c_\infty = \sqrt{2R_{\mathrm{gas}}T_\infty}$, R_{gas} is the gas constant, and λ_∞ is the mean free path of the molecules in the undisturbed gas. In the paper we use the "hard spheres" molecular collisional model. For this model $\lambda_\infty = (\sqrt{2}\sigma n_\infty)^{-1}$, where σ is the collision cross section of the molecules**.** The dimensionless time t plays the role of the inverse Knudsen number. Initially, at $t \ll 1$ the Rayleigh flow is close to the free molecular flow. As t increases, the regime of the flow changes. Namely, at $t \sim 1$ the regime becomes transitional and at $t \gg 1$ it becomes Navier-Stokes regime.

It should be noted that the properties of the Rayleigh flow are not investigated in [10], since the purpose of [10] is only to demonstrate the effectiveness of the DSMC method. We use the programming code [10] to investigate a relation between the energy flux transferred to the plate, shear stress on the

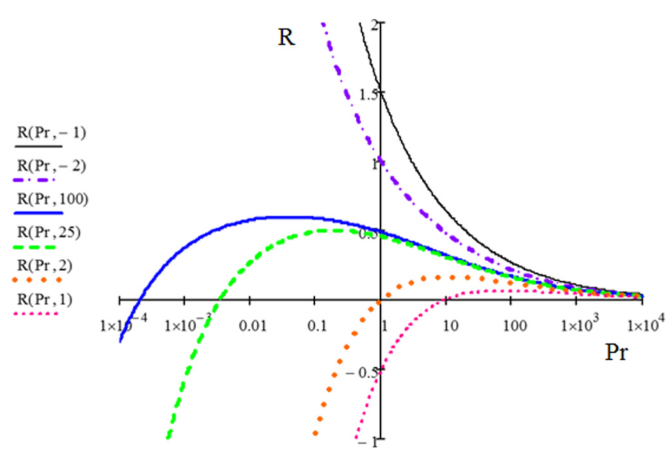

FIG. 3. The dependence of the Reynolds analogy on the Prandtl number for the incompressible Rayleigh flow at different values of the Eckert number.

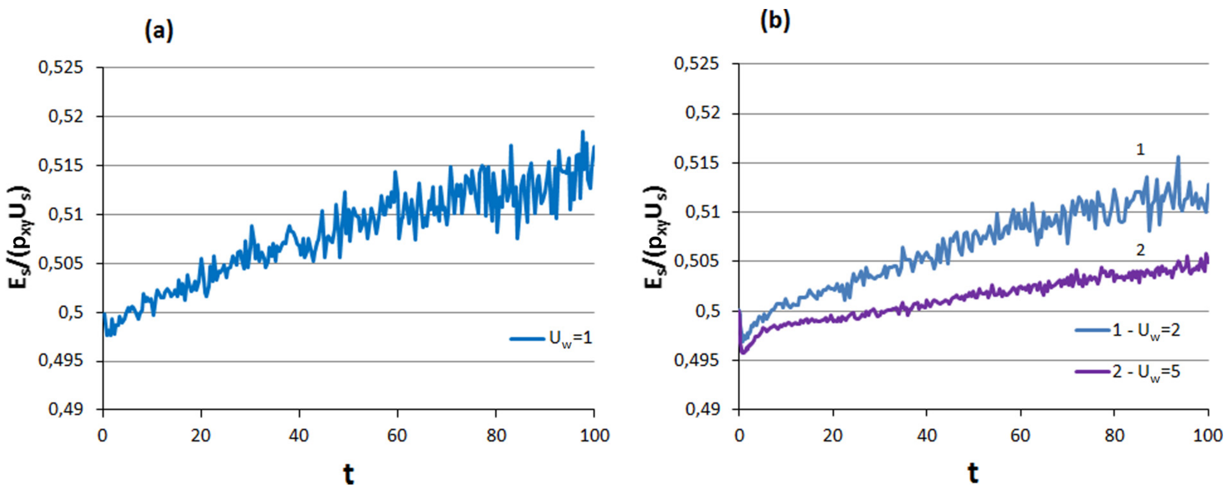

FIG. 4. The time dependence of the extended Reynolds analogy at different velocities for $T_w = 1$.

plate, and the velocity of the plate in the transitional mode of the Rayleigh problem for monatomic gas.

First, we consider rarefied gas flow with the temperature of the undisturbed gas equal to the plate surface temperature.

For the free molecular mode at $T_s = T_\infty$ we have

$$\frac{E_s}{p_{xy}U_s} = \frac{1}{2}. \tag{4.2}$$

On the other hand, using (3.1), for incompressible fluid at Pr = 2/3 (Prandtl number of a monatomic gas) we obtain R(2/3, 0) = 0.527. Thus, it is natural to assume that for the Rayleigh gas flow of monatomic gas in the transitional mode as well as in the compressible Navier-Stokes mode we have at $T_w = 1$,

$$\frac{E_s}{p_{xy}U_s} \approx \frac{1}{2}. \tag{4.3}$$

The dependencies $E_s/(p_{xy}U_s)$ on t are shown in Fig. 4 for various U_w from 1 to 5. Note that at $U_w = 1$ the changes in the gas density are not large. At $U_w = 5$ the density of the gas varies quite significantly. In these graphs dimensionless time t increases from 0 to 100. This corresponds to a change in the flow mode from the free molecular to the Navier-Stokes one. As can be seen from Fig. 4, for t close to zero the value of $E_s/(p_{xy}U_s)$is very close to its free molecular value of 0.5. On the other hand, at $U_w = 1$ and $t = 100$ the value of $E_s/(p_{xy}U_s)$ is close to 0.527, i.e., to the corresponding value for the incompressible fluid at the Prandtl number equal to the Prandtl number of a monatomic gas. As seen from the graphs, the increase of U_w does not lead to the increase of the influence of t on $E_s/(p_{xy}U_s)$. Thus, as one can see from the graphs, the extended Reynolds analogy for the transitional regime at $T_w = 1$ weakly depends on time and plate velocity, so that the formula (4.2) remains valid to an accuracy of a few percent.

Certainly, the results of the numerical calculations presented in Fig. 4 do not prove the validity of (4.3) for any U_w and t. However, they are definitely a strong argument in favor of this conclusion as well as a stimulus for the further analysis. We assume that at $T_w = 1$ the extended Reynolds analogy for Rayleigh problem (4.3) is also true for polyatomic gases.

Let us consider now the influence of the energy flux that is created by the difference between the temperatures of the plate and undisturbed gas on the extended Reynolds analogy.

The dependencies $E_s/(p_{xy}U_s)$ on t are shown in Figs. 5 and 7 for various U_w and T_w. The free molecular values $E_s/(p_{xy}U_s)$ are shown in Figs. 5 and 7 by the solid line segments and the values for an incompressible fluid are shown as dashed line segments. As seen from Figs. 5 and 7, U_w and T_w affect the ratio $E_s/(p_{xy}U_s)$ mainly via the Eckert number. Thus, these results indicate that the extended Reynolds analogy in the transient mode is mainly a function of two independent parameters (t and Ec), rather than the three (t, U_w, and T_w).

The extended Reynolds analogy may be positive or negative, or it may even change the sign at some point (see Fig. 5). The latter is caused by the sign change effect of the energy flux in the rarefied Rayleigh problem [11].

As follows from Fig. 5, the values of the extended Reynolds analogy for all the considered parameters are located approximately in the range between its free molecular value and the value of the Reynolds analogy for an incompressible fluid,

$$\frac{1}{2} - \frac{4}{5}\frac{1}{\mathrm{Ec}} \lessapprox \frac{E_s}{p_{xy}U_s} \lessapprox \frac{1}{2} - \sqrt{\frac{3}{2}}\frac{1}{\mathrm{Ec}}. \tag{4.4}$$

The results presented in Fig. 5 allow us to generalize the extended Reynolds analogy for Ec ∼ 1 or less by normalizing.

$$R_g = \left(\frac{1}{2} - \frac{\kappa+1}{2\kappa}\frac{1}{\mathrm{Ec}} - \frac{E_s}{p_{xy}U_s}\right)\mathrm{Ec}.$$

The dependencies R_g on t are shown in Fig. 6 for various Ec ∼ 1 and Ec ≪ 1. As seen from Fig. 6 the generalized Reynolds analogy depends mainly only on t for all the Eckert numbers considered.

For |Ec| ≫ 1 the dependencies $E_s/(p_{xy}U_s)$ on t are shown in Fig. 7. As seen from Fig. 7, in this case the extended

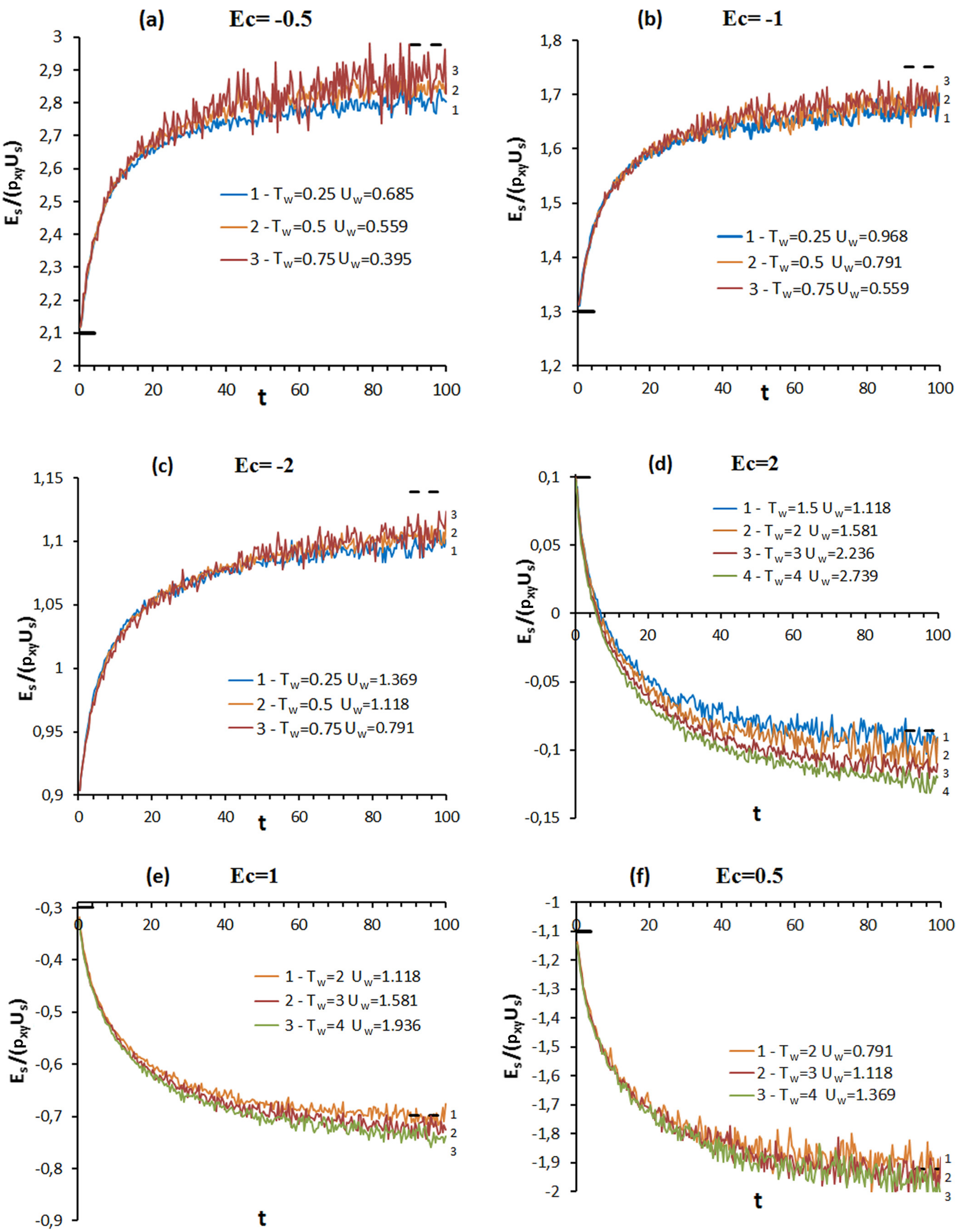

FIG. 5. The time dependence of the extended Reynolds analogy at different values of the Eckert number.

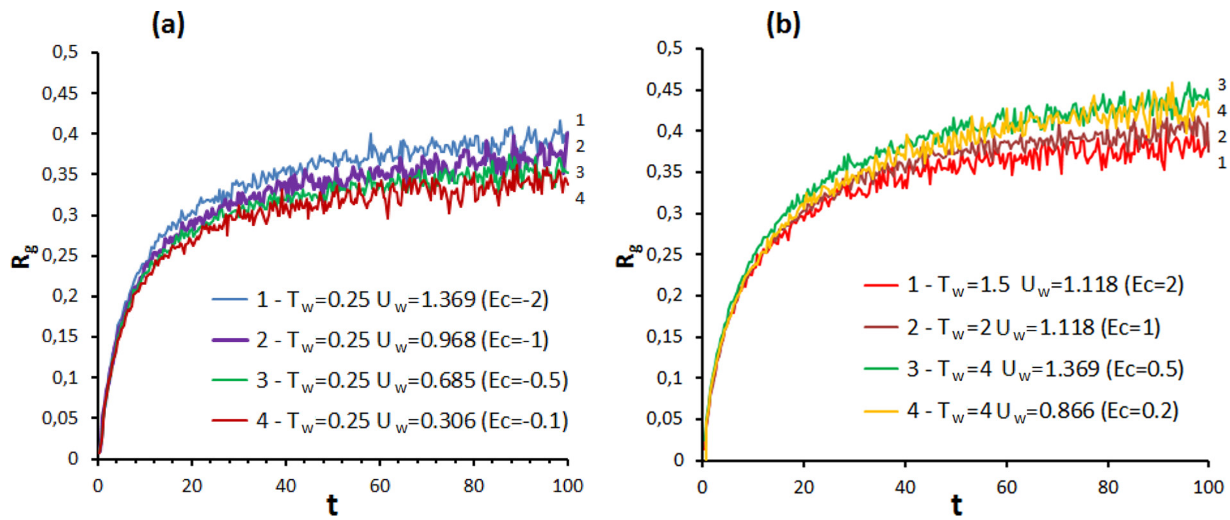

FIG. 6. The time dependence of the generalized Reynolds analogy at different values of the Eckert number.

Reynolds analogy also satisfies (4.4) and may be written as

$$\frac{E_s}{p_{xy}U_s} \approx \frac{1}{2} - \frac{1}{\mathrm{Ec}}. \tag{4.5}$$

Note that for the hypersonic velocities of the plate at $|T_w - 1|$ of the order of unity or less we have $|\mathrm{Ec}| \gg 1$ and consequently relation (4.5).

V. CONCLUSION

In this paper the Reynolds analogy for the Rayleigh problem has been considered for the incompressible, free molecular, and transitional regimes at arbitrary plate temperatures and arbitrary undisturbed fluid temperatures. For

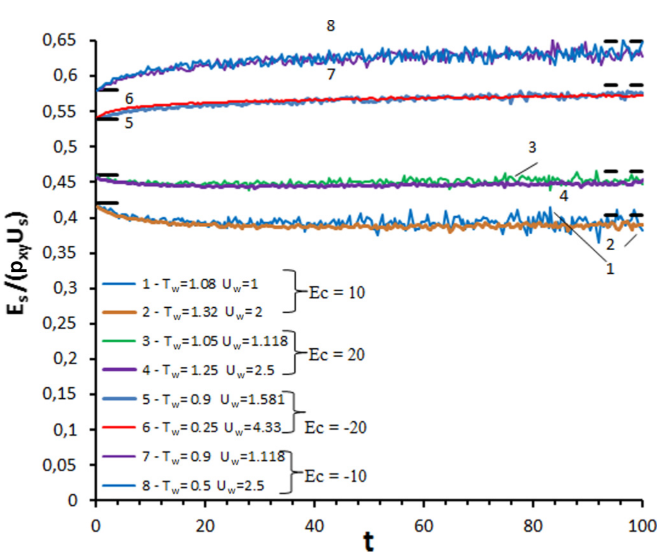

FIG. 7. The time dependence of the extended Reynolds analogy at large absolute values of the Eckert number.

incompressible fluid we have obtained the Reynolds analogy as a function of the Prandtl number and Eckert number. We have shown that the Reynolds analogy reaches the maximum at a certain Prandtl number if the plate temperature is greater than undisturbed fluid temperature. We have investigated the relationship between the energy flux transferred to the surface and the shear stress on it in the free molecular and transitional regimes (the extended Reynolds analogy) for the various plate temperatures and undisturbed gas temperatures. The extended Reynolds analogy for the free molecular Rayleigh flow as well as the Reynolds analogy for incompressible fluid depend on the Prandtl number and Eckert number and do not depend on time. For the transitional regime the extended Reynolds analogy is unsteady. It depends on dimensionless time, plate velocity, plate temperature, and undisturbed gas temperature. We have numerically investigated the extended Reynolds analogy for the transitional regime of monatomic gas at various plate velocities and temperatures. We have shown that at fixed dimensionless time the extended Reynolds analogy for the transitional regime depends on plate velocity, plate temperature, and undisturbed gas temperature mainly via the Eckert number. Thus, for all three flow modes, the Eckert number is the basic parameter that determines the influence of the energy flux caused by the difference of temperatures of the plate and the undisturbed gas on both Reynolds and extended Reynolds analogies. For Eckert numbers of the order of unity or less we generalize an extended Reynolds analogy. The generalized Reynolds analogy is a combination of the extended Reynolds analogy and the Eckert number. This combination for all considered Eckert numbers of the order of unity and less depends mainly only on dimensionless time.

If the plate temperature coincides with the undisturbed gas temperature, the extended Reynolds analogy for the transitional regime is steady up to an accuracy of a few percent. In this case the extended Reynolds analogy is close to 0.5, and is close to the corresponding values for the cases of viscous incompressible fluid and free molecular flow.

[1] H. Schlichting and K. Gersten, *Boundary-Layer Theory*, 8th ed. (McGraw Hill, New York, 2003).

[2] X. X. Chen, Z. H. Wang, and Y. L. Yu, Nonlinear shear and heat transfer in hypersonic rarefied flows past flat plates, AIAA J. **53**, 413 (2015).

[3] X. X. Chen, Z. H. Wang, and Y. L. Yu, General Reynolds analogy for blunt-nosed bodies in hypersonic flows, AIAA J. **53**, 2410 (2015).

[4] J. E. Wallace and A. F. Burke, An experiment study of surface and flow field effects in hypersonic low density flow over a flat plate, in *Rarefied Gas Dynamics: Proceedings of the Fourth International Symposium on Rarefied Gas Dynamics*, edited by J. H. de Leeuw (Academic Press, New York, 1965), p. 487.

[5] R. J. Vidal and J. A. Bartz, Surface measurements on sharp flat plates and wedges in low-density hypersonic flow, AIAA J. **7**, 1099 (1969).

[6] H. W. Liepmann and A. Roshko, *Elements of Gasdynamics* (Wiley, New York, 1957).

[7] M. Perlmutter, Analysis of Couette flow and heat transfer between parallel plates enclosing rarefied gas by Monte Carlo, in *Proceedings of 5th International Symposium on Rarefied Gas Dynamics*, edited by C. L. Brundin (Academic Press, New York, 1967), Vol. 1, pp. 455–480.

[8] M. N. Kogan, *Rarefied Gas Dynamics* (New York, Plenum Press, 1969).

[9] G. G. Stokes, On the effect of the internal friction of fluids on the motion of pendulums, Trans. Cambridge Philos. Soc. **9**, 8 (1851).

[10] G. A. Bird, *Molecular Gas Dynamics and Direct Simulation of Gas Flows* (Oxford University Press, Oxford, 1994).

[11] A. A. Abramov and A. V. Butkovskii, On the Rayleigh problem in the transitional regime: The sign change effect of the energy flux and other effects, Phys. Fluids **26**, 077101 (2014).