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Dark soliton pair of ultracold Fermi gases for a generalized Gross-Pitaevskii equation model

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We present the theoretical investigation of dark soliton pair solutions for one-dimensional as well as three-dimensional generalized Gross-Pitaevskii equation (GGPE) which models the ultracold Fermi gas during Bardeen-Cooper-Schrieffer–Bose-Einstein condensates crossover. Without introducing any integrability constraint and via the self-similar approach, the three-dimensional solution of GGPE is derived based on the one-dimensional dark soliton pair solution, which is obtained through a modified F-expansion method combined with a coupled modulus-phase transformation technique. We discovered the oscillatory behavior of the dark soliton pair from the theoretical results obtained for the three-dimensional case. The calculated period agrees very well with the corresponding reported experimental result [Weller *et al.*, Phys. Rev. Lett. **101**, 130401 (2008)], demonstrating the applicability of the theoretical treatment presented in this work.

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I. INTRODUCTION

Solitons, arising from self-stabilization against dispersive effect, have emerged as fascinating shape-perserving phenomena in various nonlinear media, extending from solid-state physics to hydrodynamics and from nonlinear optics to cold atom physics. Dark solitons are typical fundamental excitations for systems modeled by the nonlinear Schrödinger equation (NLSE) [1]. Generally they appear as dips in the density wave background accompanied by a phase jump [2] and have been observed experimentally in diverse contexts, including liquids [3], optical media [4–6], and recently in Bose-Einstein condensates (BECs) [7–13]. The possibility to create more than one dark soliton has stimulated considerable interest for the particular dynamical behavior arising from the short-range repulsive interaction between solitons [14–16]. There is specific reported experimental work [17] on the creation and oscillatory evolution investigation for the very typical case of a dark soliton pair.

Here in this paper, we will investigate theoretically dark soliton pair dynamics for the three-dimensional ultracold Fermi gas in external harmonic trapping potential. As for cold Fermi gas, with the implementation of the Feshbach resonance experimental technique, the interparticle scattering length's sign ("+" for repulsive interaction and "-" for attractive interaction) and strength can be tuned continuously from $-\infty$ to ∞ so the long-pursued Bardeen-Cooper-Schrieffer (BCS) state to the BEC crossover is realized [18,19], we adopt for our study the 3D GGPE model, where the nonlinear term takes the form of polytropic approximation $\sim|\psi|^{2\gamma}\psi$ (ψ is the wave function) [20–29]. The parameterized polytropic index γ falls into the range $[2/3,1]$, reflecting the tunability of the nonlinear interaction with $\gamma = 1$ corresponding to the BEC limit and $\gamma = 2/3$ corresponding to the BCS and unitary limit. It is well known that the one-dimensional Gross-Pitaevskii equation (GPE) possesses a dark (bright) soliton solution and there are many prior works on the one-dimensional setting case with specific $\gamma = 1$ [30–35], relatively few works focus on the generalized model (3D GGPE) with parameterized nonlinear interaction and harmonic trapping terms. Here in this paper, we utilize the modified F-expansion method [36,37], identifying the dark soliton pair solution for the 1D GGPE first, and then adopt the self-similar approach to obtain the dark soliton pair solution for the three-dimensional case. We find from our approach that the dark soliton pair obtained evolves with a very similar pattern and oscillatory period compared with what was reported in prior experimental work [17] regarding dark soliton pair dynamics, demonstrating the applicability of our theoretical approach.

This paper is arranged as follows. The next section presents the analytical dark-soliton pair solution findings for one-dimensional GGPE, followed by Sec. III, where the 3D dark soliton pair solution is obtained with comparison to actual experimental findings and we discuss results. The last section gives conclusive remarks.

II. DARK SOLITON PAIR SOLUTION FOR ONE-DIMENSIONAL GENERALIZED GROSS-PITAEVSKII EQUATION (GGPE)

A. Problem formulation

The one-dimensional case for the GGPE is relatively easier to handle compared with its higher-dimensional cases. Certain experimental scenarios, the system in an elongated harmonic trap, for example, can be modeled by 1D GGPE. The 1D generalized GPE with harmonic potential takes the following form:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{1}{2}k(t)m\hbar\omega^2x^2\psi(x,t) + g(t)|\psi(x,t)|^{2\gamma}\psi(x,t), \quad (1)$$

where γ is a polytropic real index in the nonlinear term, determined by experiment as discussed in Refs. [20–29]. The variation of γ corresponds to different BCS-BEC crossover regimes. Equation (1)'s precise description for the BCS-BEC crossover regime ($2/3 < \gamma < 1$) relies on the precise determination of γ, corresponding to the scattering length a_f signaling the crossover regime, these are elaborated in Refs. [38,39]. The effectiveness of Eq. (1) in the BCS-BEC crossover regime based on the polytropic index γ calibration is validated in some prior work [20,40]. The first term on the right-hand side of (1) models the dispersion effect, the second term on the right-hand side comes from external harmonic trapping, and the third term arises from interaction with the Landau coefficient $g(t) < 0$, corresponding to attractive interaction, and $g(t) > 0$, corresponding to repulsive interaction. Equation (1) is derived from its 3D analog in certain specific settings like the elongated external harmonic trapping potential $[V(\mathbf{r}) = \frac{1}{2}m(\omega_\rho^2\rho^2 + \omega_x^2x^2)$ with $\omega_\rho \gg \omega_x]$ as discussed in some prior work [41,42]; ω_x^2 is replaced by $k(t)$ in our problem.

To find the analytical solution of 1D GGPE (1) without introducing an additional integrability constraint, that is, making sure that there is no constraint formula connecting $k(t)$ and $g(t)$ and that they are allowed to vary freely without any interdependence, we introduce a parametric function $\sigma(t)$ with the following coupled modulus-phase transformation:

$$x' = \sqrt{\frac{2m\omega}{\hbar}}\sigma(t')x, \quad (2a)$$

$$t' = \omega t, \quad (2b)$$

$$\psi(x,t) = \sigma^{1/2}(t')\exp\left[i\left(\frac{m\omega}{\hbar}\frac{\sigma_{t'}(t')}{\sigma(t')}x^2\right)\right]\varphi(x',t'), \quad (3)$$

Substituting (25) and (2) into Eq. (1) and switching notation from (x',t') to (x,t) we get the transformed 1D GGPE with transformed coefficients as

$$i\varphi_t + \sigma^2(t)\varphi_{xx} + \left\{\frac{k(t)}{4\sigma^2(t)} - \left[\frac{\sigma_t(t)}{\sigma(t)}\right]^2 - \frac{1}{4}\left[\frac{\sigma_t(t)}{\sigma(t)}\right]_t\right\}x^2\varphi + \frac{g(t)}{\hbar\omega}\sigma^\gamma(t)|\varphi|^{2\gamma}\phi = 0. \quad (4)$$

Assume that the wave function takes the following form:

$$\varphi(x,t) = v^{1/2\gamma}(x,t)e^{i\theta(x,t)}. \quad (5)$$

Substituting (5) into Eq. (4), we reach the equations for $v(x,t)$ and $\theta(x,t)$ as follows:

$$v^2\theta_t + \sigma^2(t)\left(a_0vv_{xx} + b_0v_x^2 + v^2\theta_x^2\right) + \alpha(t)x^2v^2 + \beta(t)v^3 = 0, \quad (6a)$$

$$v_t + \sigma^2(t)(2v_x\theta_x + \gamma v\theta_{xx}) = 0, \quad (6b)$$

where $\alpha(t) = k(t)/4\sigma^2(t) - [\sigma_t(t)/\sigma(t)]^2 - [\sigma_t(t)/4\sigma(t)]_t$, $\beta(t) = g(t)\sigma^\gamma(t)/\hbar\omega$, $a_0 = -1/\gamma$, and $b_0 = -(1-\gamma)/\gamma$ are constants. Equations (6) takes the the format from which the F-expansion method can be utilized in concrete problem-solving steps. These are elaborated in the following sections.

B. *F*-expansion method

The F-expansion method [36,37] can be utilized to solve nonlinear partial differential equations of the form

$$G(u,u_t,u_x,u_{xx},\ldots) = 0, \quad (7)$$

where the terms of G are unknown function $u(x,t)$ and its partial derivatives of various order. The F-expansion is implemented by expressing the unknown function $u(x)$ as a polynomial of $F(\xi)$, with $F(\xi)$ defined as function of $\xi = p(t)x + q(t)$ through

$$\frac{d^2}{d\xi^2}F(\xi) = c_0\left(2F^3(\xi) + \frac{3}{2}\lambda F^2(\xi) + \mu F(\xi) + \frac{1}{2}\eta\right) \quad (8)$$

or

$$\frac{dF(\xi)}{d\xi} = \pm\sqrt{c_0(F^4(\xi) + \lambda F^3(\xi) + \mu F^2(\xi) + \eta F(\xi) + \varsigma)}, \quad (9)$$

where λ, μ, and η are certain constants. We express the unknown function $u(x,t)$ in polynomial form as

$$u(x,t) = \sum_{i=0}^{m} h_i(t)F^i(\xi), \;\; h_m(t) \neq 0. \quad (10)$$

We express G as a polynomial of $F(\xi)$ plus another polynomial of $F(\xi)$ times $dF(\xi)/d\xi$ after substituting (10) into the original nonlinear partial differential equation (7) via making use of (8), and m is determined by balancing between the nonlinear term and highest differential term. Equation (7) is solved by setting the coefficient formula of all terms $[F^i(\xi)$ and $F^j(\xi)dF(\xi)/d\xi]$ of G to zero. This will end with a set of ODEs for $h_i(t)$, which will put the unknown function $u(x,t)$ (10) in an explicit form if the ODEs can be solved consistently.

C. Analytical dark soliton pair solution of 1D GGPE based on the *F*-expansion method

Prior work on the 1D GGPE identifies the single soliton solution [43] here in order to obtain multiple soliton solution we will adopt different strategy. To search for the possible dark soliton pair solution, we set $c_0 = 0$ in the definition (9) of $F(\xi)$, so the balancing formula for m in (10) is $3m = 2(m+1)$ [order of highest differential term and nonlinear term of (6a), when differential operation increases the order of the polynomial by 1], which gives $m = 2$, or $3m = 2(m+2)$ (when differential operation increases the order of the polynomial by 2), which gives $m = 4$. Since we are searching for a soliton pair solution, we choose $m = 4$ and assume

$$v(x,t) = h(t)F(\xi) + f(t), \quad (11)$$

$$\theta(x,t) = \Phi(t)x^2 + \Gamma(t)x + \Omega(t), \quad (12)$$

with

$$F(\xi) = ag^4(\xi) - bg^2(\xi), \tag{13a}$$

$$\frac{g(\xi)dg(\xi)}{d\xi} = \alpha_g g^4(\xi) - \beta_g g^2(\xi) + \gamma_g, \tag{13b}$$

$$\left[\frac{dF(\xi)}{d\xi}\right]^2 = \alpha_3 F^3 + \alpha_2 F^2 + \alpha_1 F + \alpha_0, \tag{13c}$$

where α_i are dependent on $a,b,\alpha_g,\beta_g,\gamma_g$, which are determined by the consistency requirement for Eqs. (13). Substituting (13a) and (13b) into Eq. (13c), we get

$$b/a = 1, \tag{14a}$$

$$\beta_g/\alpha_g = 1, \tag{14b}$$

$$\gamma_g/\alpha_g = 1/8. \tag{14c}$$

It is not hard to obtain, from Eq. (13b),

$$g(\xi) = \sqrt{0.5 + 0.6112\tanh(\xi)}, \xi > \xi_0, g(\xi_0) = 0, \tag{15a}$$

$$g(\xi) = -\sqrt{0.5 + 0.6112\tanh(-\xi + 2\xi_0)},\ \xi \leqslant \xi_0. \tag{15b}$$

We can see from Eq. (11) and Eq. (13a) that $v(x,t)$ (or $|\psi(x,t)|$) has one maximum at $\xi = \xi_0$ and two minimum at $\xi = 0$ and $\xi = 2\xi_0$ [roots of $\frac{dF(g)}{dg} = 0$] which correspond to a double soliton (dark) type solution. The precise formula for $v(x,t)$ requires information on various parametric functions on t [including $h(t), f(t), p(t), q(t)$] which are determined by the following set of ODEs [by substituting the ansatzes (11) and (12) into Eq. (6), making use of formulas (8) and (9)]:

$$x^2F^2(\xi): \quad h^2(t)[\Phi'(t) + 4\sigma^2(t)\Phi^2(t) + \alpha(t)] = 0, \tag{16a}$$

$$x^2F(\xi): \quad 2h(t)f(t)[\Phi'(t) + 4\sigma^2(t)\Phi^2(t) + \alpha(t)] = 0, \tag{16b}$$

$$x^2: \quad f^2(t)[\Phi'(t) + 4\sigma^2(t)\Phi^2(t) + \alpha(t)] = 0, \tag{16c}$$

$$xF^2(\xi): \quad h^2(t)[\Gamma'(t) + 4\sigma^2(t)\Phi(t)\Gamma(t)] = 0, \tag{17a}$$

$$xF(\xi): \quad 2h(t)f(t)[\Gamma'(t) + 4\sigma^2(t)\Phi(t)\Gamma(t)] = 0, \tag{17b}$$

$$x: \quad f^2(t)[\Gamma'(t) + 4\sigma^2(t)\Phi(t)\Gamma(t)] = 0, \tag{17c}$$

and

$$F^3(\xi): \quad 1.5\alpha_3 a_0\sigma^2(t)h^2(t)p^2(t) + b_0\alpha_3\sigma^2(t)h^2(t)p^2(t) + g\sigma^\gamma(t)h^3(t) = 0, \tag{18a}$$

$$F^2(\xi): \quad h^2(t)[\Omega'(t) + \Gamma^2(t)\sigma^2(t)] + a_0[\alpha_2 h(t) + 1.5\alpha_3 f(t)]\sigma^2(t)h(t)p^2(t)$$
$$b_0\alpha_2\sigma^2(t)h^2(t)p^2(t) + 3g\sigma^\gamma f(t)h^2(t) = 0, \tag{18b}$$

$$F^1(\xi): \quad 2f(t)h(t)[\Omega'(t) + \Gamma^2(t)\sigma^2(t)] + a_0[0.5\alpha_1 h(t) + \alpha_2 f(t)]\sigma^2(t)h(t)p^2(t)$$
$$+b_0\alpha_1\sigma^2(t)h^2(t)p^2(t) + 3g\sigma^\gamma(t)f^2(t)h(t) = 0, \tag{18c}$$

$$F^0(\xi): \quad f^2(t)[\Omega'(t) + \Gamma^2(t)\sigma^2(t)] + 0.5a_0\alpha_1\sigma^2(t)f(t)h(t)p^2(t)$$
$$b_0\alpha_0\sigma^2(t)h^2(t)p^2(t) + g\sigma^\gamma(t)f^3(t) = 0, \tag{18d}$$

while for Eq. (6b) we have

$$xF'(\xi): \quad h(t)[p'(t) + 4\sigma^2(t)\Phi(t)p(t)] = 0, \tag{19a}$$

$$F'(\xi): \quad h(t)[q'(t) + 2\sigma^2(t)p(t)\Gamma(t)] = 0, \tag{19b}$$

and

$$F(\xi): \quad h'(t) + 2\gamma\sigma^2(t)\Phi(t)h(t) = 0, \tag{20a}$$

$$F^0(\xi): \quad f'(t) + 2\gamma\sigma^2(t)\Phi(t)f(t) = 0. \tag{20b}$$

From Eqs. (18), Eqs. (19), and Eqs. (20), we can see that

$$\sigma^\gamma(t)h(t) = C_1, \quad \sigma^\gamma(t)f(t) = C_2, \tag{21a}$$

$$\sigma(t)p(t) = C_3, \quad \sigma(t)\Gamma(t) = C_4, \tag{21b}$$

where C_1, C_2, C_3, and C_4 are constants determined by initial condition. For $g < 0$, combining (11), (13a), (15), and (21), we have

$$|\psi(x,t)|^{2\gamma} = \begin{cases} A_0(\{1 + 1.225\tanh[C_3x + \sigma(t)q(t)]\}^2 \\ \quad -\{1.225\tanh[C_3x + \sigma(t)q(t)] + 1\} + C_2/C_1), \quad \xi > \xi_0, & (22a) \\ A_0(\{1 + 1.225\tanh[-C_3x - \sigma(t)q(t) + \sigma(t)\xi_0]\}^2 \\ \quad -\{1.225\tanh[-C_3x - \sigma(t)q(t) + \sigma(t)\xi_0] + 1\} + C_2/C_1), \quad \xi < \xi_0, & (22b) \end{cases}$$

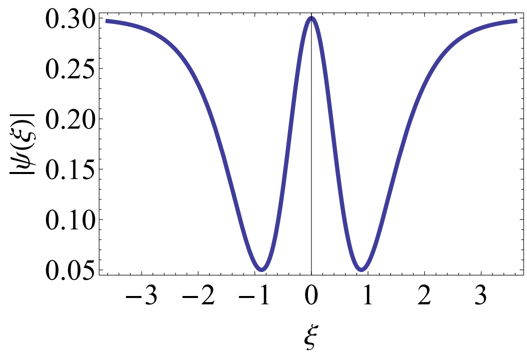

FIG. 1. Plot of modulus of wave function $|\psi(\xi)|$ (with $A_0 = 1.0, C_2/C_1 = 0.3$).

where A_0 is normalization constant. Figure 1 shows the plot of modulus of wave function $|\psi(\xi)|$ which possesses the dark soliton pair characteristic. Now we can identify the clear qualitative property of our problem without writing out explicitly the ordinary differential equations for $\sigma(t),\Phi(t),\Omega(t),q(t)$. We can see from Eqs. (22) that the factor in front of the space variable x in modulus $|\psi(x,t)|$ is constant (independent of t), which means that the spatial distance between the two dark solitons is not varying with time t in our theoretical treatment of the pure one-dimensional setting. This is in contrast to the dynamical feature of the single dark soliton case as elaborated on in Ref. [44]. It is found both theoretically and experimentally that a 1D single dark soliton is unstable in the framework of a regular Gross-Pitaevskii equation model. The exception is when nonlocal interaction is incorporated which will accommodate stable single dark soliton solution, as stated in Ref. [44]. We will see in the following section that the three-dimensional case regarding the dynamical feature of a dark soliton pair differ from that of the one-dimensional case.

III. OSCILLATING DARK SOLITON PAIR IN A THREE-DIMENSIONAL SETTING

A. Self-similar solution for three-dimensional generalized Gross-Pitaevskii equation

Based on the analytical solution derived for one-dimensional GGPE in the previous section, we search for the same type of solution for three-dimensional GGPE, which takes the following general form:

$$i\psi_t + \rho_0(t)\Delta\psi + g_0(t)|\psi|^{2\gamma}\psi + \sum_j k_{0j}(t)x_j^2\psi = i\Gamma_0(t)\psi. \tag{23}$$

Equation (23) is the 3D generalization of Eq. (1) in dimensionless format. The parametric functions of time $\rho_0(t),g_0(t),k_{0j}(t),\Gamma_0(t)$ are supplied, reflecting the adjustable external experimental setting. For example, the $g_0(t)$ model's time-dependent nonlinear interaction strength which could be tuned by Feshbach resonance, $k_{0j}(t)(j = x,y,z)$, accommodate the tunable external harmonic trapping strength in three spatial directions. We derive the analytical solution of (23) based on the self-similar approach that is developed in Refs. [45,46]. But in order to obtain a practical solution without introducing any additional integrability constraint, we also introduce a parametric function $\chi(t)$ and a coupled modulus-phase transformation as follows:

$$\boldsymbol{r}' = \chi(t')\boldsymbol{r}, \tag{24a}$$

$$t' = \int \rho_0(t)\chi^2(t)dt. \tag{24b}$$

The wave function is transformed as

$$\psi(\boldsymbol{r},t) = \chi^{3/2}(t')\exp\left[\frac{\chi_{t'}(t')}{\chi^3(t')}\sum_j x_j'^2\right]\varphi(\boldsymbol{r}',t'). \tag{25}$$

Changing notations from $(\boldsymbol{r}',t')$ to $(\boldsymbol{r},t)$, Eq. (23) is transformed into the following form:

$$i\varphi_t + \Delta\varphi + g(t)|\varphi|^{2\gamma}\varphi + \sum_j k_j(t)x_j^2\varphi = i\Gamma(t)\varphi, \tag{26}$$

where $g(t) = g_0(t)\chi^{3\gamma-2}(t)/\rho_0(t)$, $\Gamma(t) = \Gamma_0(t)/[\rho_0(t)\chi^2]$, and $k_i(t) = \{4k_{0j}(t) + \chi_t^2(t)/\chi^2(t) - [\chi_t(t)/\chi(t)]_t\}/[4\rho_0(t)\chi^4(t)]$. The self-similar projective equation for the three-dimensional GGPE (23) is

$$iu_\tau + \varepsilon u_{\varsigma\varsigma} + \delta|u|^{2\gamma}u = 0, \tag{27}$$

which is the specialized case of Eq. (1), with $k(t) = 0$ and $\delta < 0$, and Eq. (27) possesses a dark soliton pair solution as shown in the previous section.

Now we introduce the similarity ansatz for φ which is tailored to our needs as

$$\varphi(\boldsymbol{r},t) = A(t)u[\varsigma(\boldsymbol{r},t),\tau(t)]\exp[ia(\boldsymbol{r},t)], \tag{28}$$

where we choose $u(\varsigma,\tau)$ as the dark-soliton pair solution format of Eq. (27), which is the typical feature of the similarity approach adopted here, but the most important advantage is its combination with parametric function $\chi(t)$, which will eliminate all integrability constraints that are hard to avoid. These are to be shown in the following steps. In ansatz (28), $A(t)$, $\varsigma(\boldsymbol{r},t)$, $\tau(t)$, and $a(\boldsymbol{r},t)$ are to-be-determined functions differentiable with respect to the time and spatial coordinates. Substituting the solution ansatz (28) into Eq. (26) and putting the resultant equation in the same form as Eq. (27), we obtain the following relationship equations:

$$2gA^{2\gamma} - \delta\tau_t = 0, \tag{29a}$$

$$\sum_j \varsigma_{jj} = 0, \tag{29b}$$

$$\sum_j \varsigma_j^2 - \varepsilon\tau_t = 0, \tag{29c}$$

$$\varsigma_t + 2\sum_j \varsigma_j a_j = 0, \tag{29d}$$

$$2A_t - 2\Gamma A + 2A\sum_j a_{jj} = 0, \tag{29e}$$

$$a_t + a_x^2 + a_y^2 + a_z^2 - \sum_j k_j x_j^2 = 0, \tag{29f}$$

which are solvable with the following solutions:

$$A(t) = \sqrt{\frac{3\delta}{\varepsilon g}} G, \tag{30a}$$

$$\varsigma(\boldsymbol{r},t) = -6D_1 \int G^2 dt + G \sum_j x_j + D_2, \tag{30b}$$

$$\tau(t) = \frac{3}{\varepsilon} \int G^2 dt + D_3, \tag{30c}$$

$$a(\boldsymbol{r},t) = \frac{g_t + 2\Gamma g}{4g} \sum_j x_j^2 + D_1 G \sum_j x_j - 3D_1^2 \int G^2 dt + D_4, \tag{30d}$$

where $j = x,y,z$, $G = \exp[-\int(g_t/g + 2\Gamma)dt]$, and $D_{1,2,3,4}$ are integral constants. The resultant consistence equation reads

$$4\Gamma^2 + 4\frac{g_t}{g}\Gamma + 2\Gamma_t + \frac{g_{tt}}{g} - \frac{4}{3}\sum_j k_j(t) = 0. \tag{31}$$

Since the functions $\Gamma(t)$, $g(t)$, and $k_j(t)$ depend on $\chi(t)$, Eq. (31) is just an equation for $\chi(t)$, leaving $\rho_0(t)$, $g_0(t)$, $k_{0i}(t)$, and $\Gamma_0(t)$ as free varying functions. The solution (28) together with the Eq. (31) give the exact dark soliton pair solution for 3D GGPE (23).

B. Oscillatory behavior of the dark soliton pair

Considering the practical experimental setting without dissipation such that $\rho_0 = 2$, $k_{0i}(t) = k_{0i}$, $g_0(t) = g_0$, $\Gamma(t) = \Gamma_0(t) = 0$, $g(t) = \chi^{3\gamma-2}(t)g_0$, we can see from Eqs. (30),

$$G(t) = \chi^{2-3\gamma}(t), \tag{32a}$$

$$A(t) = A_0\chi^{2-3\gamma}(t), \tag{32b}$$

$$\varsigma(t) = \chi^{2-3\gamma}(t)(x+y+z) + Q(t), \quad Q(t) = -6D_1 \int G^2(t)dt + D_2, \tag{32c}$$

where A_0 is the normalization constant. So

$$|\varphi(\boldsymbol{r},t)| = |A_0\chi^{2-3\gamma}(t)u[\chi^{2-3\gamma}(t)(x+y+z) + Q(t),\tau(t)]|. \tag{33}$$

Here $u(\varsigma,\tau)$ is just in the function form of the dark soliton pair solution (22) for the 1D GGPE as mentioned before. We can see that because of the time-dependent factor $\chi^{2-3\gamma}(t)$ in front of the spatial variables (x,y,z), the distance between the two dark solitons is varying arising from the modulation of function $\chi(t)$. We will investigate the explicit format of $\chi(t)$ for two cases:

(I) For the isotropic harmonic trapping potential $k_{0x} = k_{0y} = k_{0z} = k_0$, we obtain the equation for $\chi(t)$ from Eq. (31) or Eq. (29f),

$$h_1\chi(t)\chi_{tt}(t) + h_2\chi_t^2(t) - 2k_0\chi^2(t) = 0, \tag{34}$$

where

$$h_1 = -(6\gamma-5)/2$$
$$h_2 = 3(\gamma-1)(6\gamma-5).$$

Since we are interested in the periodic solution of $\chi(t)$, we can assume that the solution is of the following form:

$$\chi(t) = [\sin(\omega t)]^{\nu}. \tag{35}$$

Substituting Eq. (35) into Eq. (34), we get

$$[h_1\nu(\nu-1)\omega^2 + h_2\nu^2\omega^2]\cos^2(\omega t) - [h_1\nu\omega^2 - 2k_0]\sin^2(\omega t) = 0 \tag{36}$$

from which we can easily get

$$\nu = \frac{h_1}{h_1+h_2} = \frac{1}{7-6\gamma}$$
$$\omega = \sqrt{\frac{4(7-6\gamma)k_0}{6\gamma-5}}. \tag{37}$$

We can see that at the BEC limit ($\gamma = 1$), with $k_0 = \omega_0^2/4$, $\omega = \omega_0$, $\chi(t)$ oscillates with period $T = T_0$ ($T_0 = 2\pi/\omega_0$). The two dark solitons perform relative motion between them. Their spacing distance varies with period T_0 between 0 and a maximum value determined by the initial condition.

(II) For the quasi-one-dimensional setting with elongated harmonic trapping potential $k_{0x} = k_{0y} \gg k_{0z}$, we can see from Eqs. (29) that, in $\varsigma(x,y,z,t)$, the coefficients of x,y are much larger than that of z so the motion of the dark solitons are confined to move in the z direction. The equation for $\chi(t)$ only needs to count the contribution of the z component trapping parameters. So in Eq. (34), we only need to replace k_0 with k_z and obtain the same expression for $\chi(t)$ as (35) but with $\omega = \omega_z$ for the BEC limit case ($\gamma = 1$). The two dark solitons execute periodic motion with period T_z. Figure 2 shows the evolution of the dark soliton pair's position (z coordinates) with time t for elongated harmonic trapping at the BEC limit. We can see

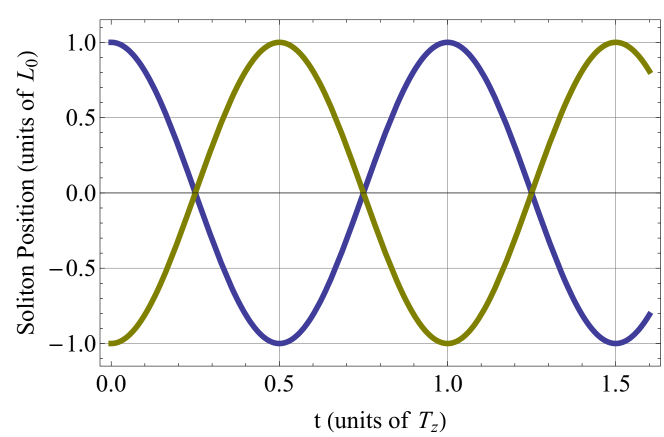

FIG. 2. Simulated evolution of the dark soliton pair's peak positions with time t (in unit T_z) for elongated harmonic trapping potential at the BEC limit ($\gamma = 1$). (The dark soliton pair's traces are plotted in blue and green-gray, respectively. The vertical coordinate is in the unit of $L_0 = \frac{x_0}{C_3}\sqrt{\frac{\hbar}{m\omega_z}}$; x_0 is the root of equation $\tanh(x_0) = 1/1.225$.)

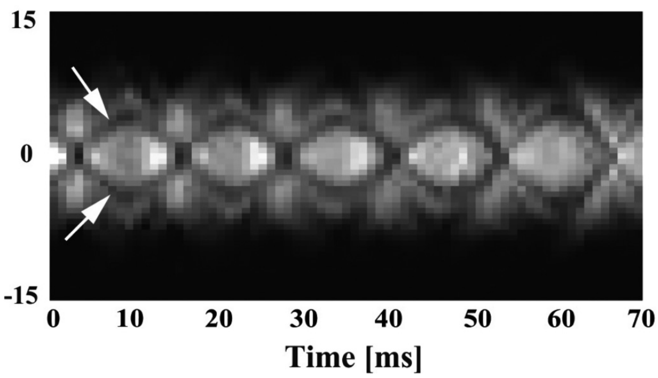

FIG. 3. Experimental observation of the dark soliton pair's oscillation with harmonic trapping frequency $\nu_z = 53$ Hz (from Fig. 1 in Ref. [17], courtesy of M. K. Oberthaler and APS).

that the theoretical period T_z presented here agrees fairly well with the experimental result in Ref. [17] (dark soliton pair's oscillating period approximately equals 20 ms $\simeq 1/\nu_z = 1/53$ s as shown in Fig. 1 in that article). Figure 3, which is cited from Fig. 1 in Ref. [17] for comparison, shows the experimental observation of the dynamics of the longitudinal atomic density showing the dark soliton pair's oscillation with harmonic trapping frequency $(\nu_z, \nu_\perp) = (53\,\text{Hz}, 890\,\text{Hz})$.

C. Discussion and prospect

The comparison we just made between the theoretical and experimental results for a three-dimensional dark soliton pair lies at the BEC side ($\gamma = 1$) of the whole BCS-BEC crossover regime, where we choose $\Gamma_0(t) = 0$ in the 3D GGPE (23). Far from the BEC limit, zero $\Gamma_0(t)$ is no longer a good approximation, and the so-called snake instability for the dark soliton is found both experimentally and theoretically [47,48]. Based on the polytropic approximation and numerical methods, such an instability is investigated in Ref. [48]. An analytical study of the snake instability based on a GGPE significantly far from the BEC limit is a possible extension of the work presented here.

It is true that we identify the oscillatory behavior of the dark soliton pair, especially in the elongated harmonic trap case where the oscillatory period agrees fairly well with the experimental value. It is worth mentioning that $\chi(t)$ can accommodate exponential decay (rise) solutions in addition to the periodic solution, which means that the spacing between the two solitons monotonously increases (decreases) from a certain initial time. This kind of phenomenon is corroborated in the numerical simulation that is elaborated on in Ref. [49], which shows that when the initial spacings of the soliton pair are short, moderate, and long range, they will approach, oscillate mutually, and move away from each other, respectively. One possible extension (application) of the theoretical approach adopted here is to study the collision of the dark soliton pair and investigate the variations (or conservation phenomena) before and after collision. But, for this case, we may need to adopt a modified model incorporating nonlocal interaction, as discussed in prior work [44]. This is another topic for future work.

IV. CONCLUSION

In this paper, based on the modified F-expansion method and the coupled modulus-phase transformation methodology, we first derived the dark soliton pair solution for the one-dimensional GGPE. Then, based on the one-dimensional results and through a self-similar approach, for the three-dimensional GGPE that models the ultracold Fermi gas in harmonic trapping potential during BCS-BEC crossover, we derive the analytical dark soliton pair solution without introducing any integrability constraint. For the three-dimensional setting with an elongated harmonic trapping potential at the BEC limit, our 3D theoretical results demonstrate clear oscillatory behavior for the dark soliton pair identified, and the oscillatory period value matches very well with that reported in the experimental observations [17], indicating the applicability of the mean-field theory-based 3D GGPE in modeling ultracold Fermi gas with harmonic trapping potential.

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