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Multiplicative noise can lead to the collapse of dissipative solitons

Orazio Descalzi,[1,2,*] Carlos Cartes,[1] and Helmut R. Brand[2]

[1]*Complex Systems Group, Facultad de Ingeniería y Ciencias Aplicadas, Universidad de los Andes, Avenida Monseñor Álvaro del Portillo 12.455, Las Condes, Santiago, Chile*

[2]*Department of Physics, University of Bayreuth, 95440 Bayreuth, Germany*

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We investigate the influence of spatially homogeneous multiplicative noise on the formation of localized patterns in the framework of the cubic-quintic complex Ginzburg-Landau equation. We find that for sufficiently large multiplicative noise the formation of stationary and temporally periodic dissipative solitons is suppressed. This result is characterized by a linear relation between the bifurcation parameter and the noise amplitude required for suppression. For the regime associated with exploding dissipative solitons we find a reduction in the number of explosions for larger noise strength as well as a conversion to other types of dissipative solitons or to filling-in and eventually a collapse to the zero solution.

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Noise, a phenomenon well known and extensively studied in physics and chemistry [1,2], is ubiquitous in nature. More recently interest in noise and its possible effects is also growing in biology [3].

Although most studies addressed the effect of noise added to a deterministic equation, multiplicative noise for which the stochastic force multiplies a function of the stochastic variables, has also attracted a considerable amount of attention over the years. Originally this interest was driven by experiments on spatially homogeneous systems, such as electronic circuits [4] and optical systems, namely, the dye laser [5]. This in turn led to theoretical investigations of these zero-dimensional problems [6,7].

For spatially extended pattern-forming systems early work focused on the effect of spatially homogeneous multiplicative noise on the onset of pattern formation in electroconvection in nematic liquid crystals. It was demonstrated that the onset of spatial patterns could be postponed by a substantial amount by superposing noise on the driving voltage and that relaxation rates showed a strong linear dependence on the noise strength [8,9]. Even until today quantitative experimental studies on the interaction of noise and pattern formation in spatially extended nonequilibrium systems are quite infrequent. When it comes to the influence of noise, whose amplitude is a substantial fraction (say ∼10% or so) of the deterministic amplitude most of the recent experimental studies have been carried out on surface reactions under the influence of combined additive and multiplicative noise [10–14] or purely multiplicative noise [15].

For pattern formation in cubic Ginzburg-Landau and Swift-Hohenberg-type equations with real coefficients under the influence of multiplicative noise there is a large body of modeling literature including Refs. [16,17] (and references cited therein) focusing in particular on applications to domain growth processes and on modeling Benard convection in simple fluids.

We note that the supercritical Ginzburg-Landau equation with complex coefficients and without spatial dependence has in all cases one attractor (zero for $\mu < 0$) and a limit cycle (for $\mu > 0$). For $\mu < 0$ multiplicative noise cannot be amplified starting from the stable solution. For $\mu > 0$ the limit cycle cannot jump to the zero solution because this is unstable. For the subcritical case (the case in the following study), we have, without spatial degrees of freedom, two attractors: the zero solution and a limit cycle which coexist. Again, starting from the stable zero solution multiplicative noise does not become amplified. Starting from the limit cycle one needs a very large amount of noise to jump to the zero stable solution. In this sense the bifurcation points can be shifted a bit. We emphasize that in the study described below we investigate the influence of a small amount of multiplicative noise (less than 10% compared to the pulse amplitude) on stable localized solutions in the cubic-quintic complex Ginzburg-Landau equation, that is, including spatial degrees of freedom.

A field for which the influence of noise on pattern-forming nonequilibrium systems has been addressed only recently is stable spatially localized solutions (also denoted as dissipative solitons (DSs) [18]). These spatially localized solutions, which have been studied in particular for the complex cubic-quintic Ginzburg-Landau equation (CQGLE), can be stationary [19–23], breathing with one frequency, two frequencies, or even chaotically [24]. For the case of anomalous linear dispersion exploding dissipative solitons have been found [25] and studied experimentally [26,27] and theoretically [28–34].

Regarding the influence of noise on dissipative solitons, the focus has been so far on additive noise of various strengths. It has been demonstrated that weak additive noise can lead to the partial annihilation of counterpropagating pulses [35], a phenomenon that had been observed before experimentally near the onset of binary fluid convection [36,37] and for surface reactions [38,39]. For single DSs weak noise was shown to induce explosions via various routes [40], whereas large noise can induce noisy localized structures for values of the bifurcation parameter for which DSs no longer stably exist [41].

Here we study the question of localized solutions in the framework of the CQGLE under the influence of spatially homogeneous multiplicative noise. We find that spatially homogeneous multiplicative noise has qualitatively different effects on dissipative solitons from the previously studied influence of additive noise δ correlated in space and time.

*Corresponding author: odescalzi@miuandes.cl

The motivation to study spatially homogeneous noise in the multiplicative case is due to the fact that previously well-controlled experiments studying the influence of multiplicative noise on pattern formation in spatially extended systems have been performed for this type of noise [8,9,15].

As the most striking result we find a linear suppression of the onset of the formation of dissipative solitons by multiplicative noise. This result is reminiscent of previous experimental results obtained for the effect of multiplicative noise on spatially extended regular patterns, for example, for the suppression of the onset of electroconvection in liquid crystals [8,9].

The CQGLE with multiplicative noise we investigate here is of the form

$$\partial_t A = \mu A + (\beta_r + i\beta_i)|A|^2 A + (\gamma_r + i\gamma_i)|A|^4 A + (D_r + i D_i)\partial_{xx} A + A\eta\xi, \tag{1}$$

where $A(x,t)$ is a complex field, β_r is positive, and γ_r is negative in order to guarantee that the bifurcation is subcritical but saturates to quintic order. The stochastic force $\xi(t)$ denotes white noise with the properties $\langle \xi \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$. That means we consider multiplicative noise, which is real and homogeneous in space.

In our numerical simulations we keep all parameters fixed except for μ, the distance from linear onset, and η, the noise strength. The parameter values are $\beta_r = 1$, $\beta_i = 0.8$, $\gamma_r = -0.1$, $\gamma_i = -0.6$, $D_r = 0.125$, and $D_i = 0.5$ (positive) corresponding to an anomalous dispersion regime. Stable pulses can only exist when the CQGLE becomes nonvariational. Thus, at least one of the parameters (β_i, γ_i, D_i) must be different from zero [19,20,34,42].

For deterministic dissipative solitons it has been shown that stationary DS, oscillatory DS with one and two frequencies, chaotically breathing DSs, as well as exploding DSs stably exist over a range of parameters. Although stationary dissipative solitons are known to exist for one nonvanishing imaginary coefficient (compare, for example, Ref. [42]), exploding dissipative solitons have only been observed for at least three nonvanishing imaginary parts of the coefficients. This can be traced back to the fact that exploding dissipative solitons are characterized in their time evolution by several stages, which are all unstable. One must have an instability in the side wings, one must be able to generate a collapse at maximum amplitude, and they have to be wide enough to show the instability in the wings but not too wide to prevent filling-in. Therefore to study the influence of multiplicative noise on the sequence of transitions from stationary DS to exploding DSs we have chosen three nonvanishing values for the imaginary parts of the coefficients in the cubic-quintic CGL equation.

The bifurcation parameter μ is varied from ~ -1.23 to -0.07. This range is chosen in a way to guarantee that there are stable dissipative solitons for the whole parameter range considered, including stationary, periodic, doubly periodic (two frequencies), as well as exploding dissipative solitons. For $\mu < -1.23$ stable stationary DSs no longer exist deterministically. One is below the "saddle node" for localized solutions.

In the discretized problem the stochastic force $\xi(x,t)$ is replaced by $\chi_r/\sqrt{dt}$, where χ_r corresponds to uncorrelated random numbers obeying a standard normal distribution.

To perform the numerical simulations for Eq. (1) we implemented a split-step pseudospectral method where the differential operator is computed in Fourier space and the nonlinear terms are computed in the time step by using a fourth order Runge-Kutta algorithm. The simulations were performed using 1024 Fourier modes ensuring that even small scales are well solved. This fact was verified by measuring the spectral convergence during explosions. To further ensure that our simulations were performed, using the right numerical parameters, we tried different values for the number of Fourier modes from 256 to 2048, and the results were always consistent. In parallel we carried out extensive numerical calculations using finite differencing, a box of size $L = 50$ and $N = 625$, leading to a grid spacing of $dx = 0.08$ and typically a time step dt of $dt = 0.005$. In both cases we varied the time step and N to make sure that none of our results is sensitively dependent on this choice.

We make use of spatially homogeneous multiplicative noise since it is the type of noise one can apply most easily to experimental systems showing spatiotemporal pattern formation. This applies to systems, such as the onset of pattern formation and/or higher instabilities in electroconvection in nematic liquid crystals [8,9], as well as to concentration and temperature noise applied to the catalytic oxidation of CO under ultrahigh vacuum conditions [10–15]. In all experimentally studied cases in spatially extended systems the multiplicative noise applied externally was spatially homogeneous in nature.

To clarify the qualitatively different noise effects of additive noise δ correlated in space and time and multiplicative noise homogeneous in space and δ correlated in time only, we have plotted in Figs. 1(a) and 1(b) snapshots of a stationary DS under the influence of additive noise δ correlated in space and time [Fig. 1(a)] and multiplicative noise only δ correlated in time [Fig. 1(b)]. Inspection of these snapshots reveals immediately this qualitatively different behavior. Whereas in the additive case noise acts mainly as a perturbation on short length and time scales giving the state a noisy appearance, multiplicative noise of the type considered here leads to a collective enhancement and depression of the amplitude as a function of space. We focus here, inspired by previous experimental work, on spatially homogeneous multiplicative noise. It is this spatially homogeneous nature which leads to the collective enhancement and suppression (in space) of the amplitude. Correspondingly the effects of multiplicative spatially homogeneous noise will tend to favor spatially homogeneous solutions for large enough noise strength leading to suppression of a spatial pattern or to filling-in.

In Fig. 2 we have plotted the transition from a noisy dissipative soliton to collapse (spatially homogeneous state) where we have plotted on the ordinate the noise strength η. For the range of the bifurcation parameter μ plotted in Fig. 2 we have deterministically stationary spatially localized solutions ($-1.23 < \mu < -0.227$), a temporally periodic localized solution with one frequency (f_1) ($-0.227 < \mu < -0.202$), and temporally localized solutions with two frequencies (f_1, f_2) ($-0.202 < \mu < -0.183$). For $-0.183 = \mu_c < \mu$

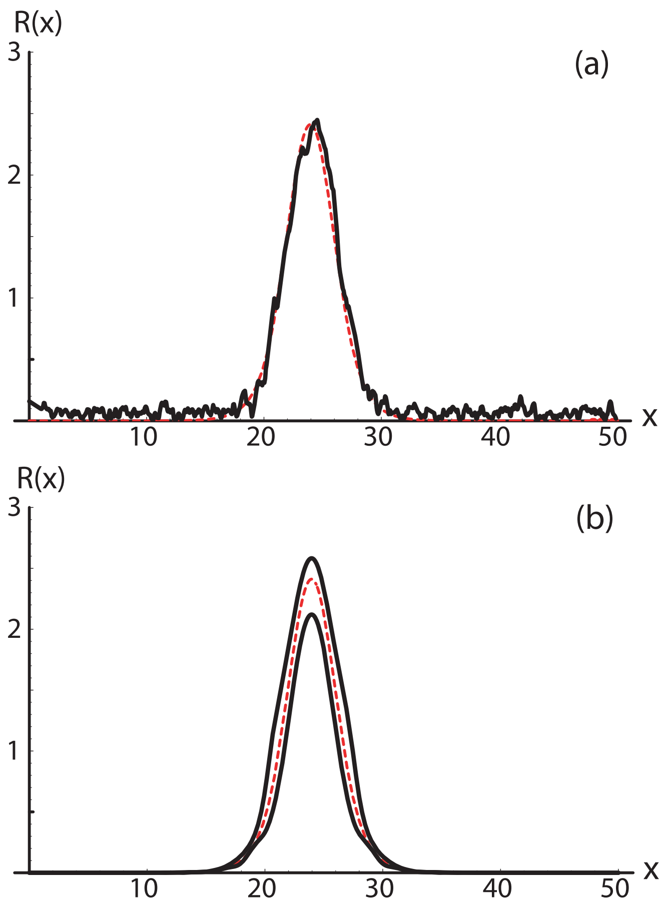

FIG. 1. The qualitative difference between the effects of additive noise δ correlated in space and time versus spatially homogeneous multiplicative noise δ correlated in time is shown for $\mu = -0.4$. While additive noise δ correlated in space and time ($\eta = 0.05$) leads to rapid spatial and temporal random oscillations superposed on a DS (a), spatially homogeneous multiplicative noise ($\eta = 0.15$) leads to a spatially correlated (collective) increase and decrease in the pattern amplitude (solid lines), which is random as a function of time (b). The dashed lines are the deterministic stationary DSs in both cases.

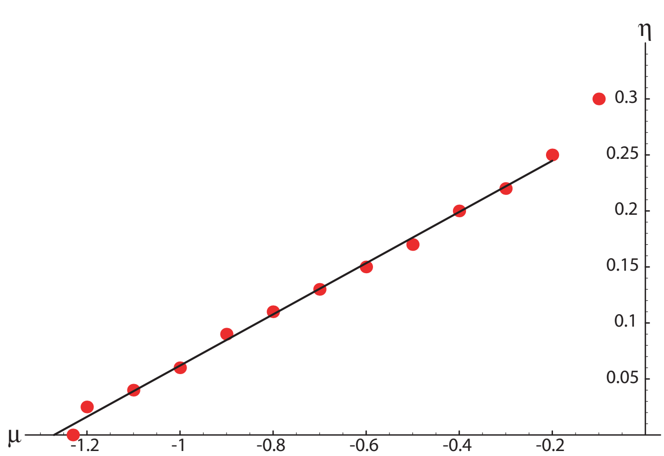

FIG. 2. The critical strength of multiplicative noise η to induce the transition from stationary and periodic dissipative solitons to collapse is plotted as a function of the distance from linear onset μ. We see that there is a linear relation over a large range of values for μ.

exploding dissipative solitons prevail. The data points denote the limit of collapse for $T_{\max} = 2000$.

Inspection of Fig. 2 reveals that there is a linear relation between the value of the noise strength necessary to induce collapse almost all the way for all values of the bifurcation parameter for which one has a transition to a state which is regular in space and time. Only close to the transition to exploding dissipative solitons does this simple relation break down. Thus we conclude that the zero solution can be stabilized by external multiplicative noise against the onset of formation of dissipative solitons by a significant magnitude brought out by Fig. 2. It is clear that by increasing $T_{\max}$ one would get a smaller slope. In the limit of infinite waiting time only the zero solution will be obtained since it corresponds to the global minimum for sufficiently negative values of μ.

The situation becomes more complex when the influence of multiplicative noise on exploding dissipative solitons is investigated. In Fig. 3 we show two phase diagrams of the states occurring as a function of the bifurcation parameter μ and the noise strength η for the range of $-0.12 < \mu < -0.07$ and $0.28 < \eta < 0.33$. For Fig. 3(a) we have used $T_{\max} = 2000$ whereas simulations in Fig. 3(b) ran until $T_{\max} = 6000$. Naturally small changes in dx or dt lead to small shifts of the different behaviors but not to a significantly different phase diagram. Differences between both phase diagrams correspond to points which need more time (longer than $T = 2000$) to lead to filling-in. Most points remain unchanged. $T_{\max} = 6000$ corresponds to $\sim 10^6$ iterations. For Fig. 3 the initial conditions are exploding solitons corresponding to a certain μ and $\eta = 0$. Then we connect the noise. We used periodic boundary conditions. We note that first of all there is no simple linear relation between the value of collapse of exploding dissipative solitons and the bifurcation parameter anymore. But in addition other states intervene the transition collapse—exploding dissipative solitons. This includes—as is readily inferred from Fig. 3—a filling-in transition to a spatially homogeneous finite amplitude pattern as well as a two-phase region for which one can obtain—depending on initial conditions in the asymptotic limit in time—different outcomes, namely, filling-in or, alternatively, noisy exploding dissipative solitons.

As the bifurcation parameter μ is approaching zero the behavior is becoming increasingly complex. This can be traced back to the additional slow time scale coming into play as the linear onset is approached. As is already known from the purely deterministic behavior this region is characterized by rather long transients and associated with a pattern filling a fairly large part of the sample [31].

In Fig. 4 we have plotted the frequency of explosions in the range of noisy exploding dissipative solitons as a function of noise strength. Figures 4(a) and 4(b) show the frequency of explosions far from the transition exploding dissipative solitons to dissipative solitons with two frequencies (f_1, f_2) for four different values of the bifurcation parameter μ in a linear plot [Fig. 4(a)] and in a semilogarithmic plot [Fig. 4(b)]. We conclude that the number of explosions is reduced monotonically as the noise strength increases. From these two plots we conclude that there is no simple scaling behavior covering the entire range of noise strengths

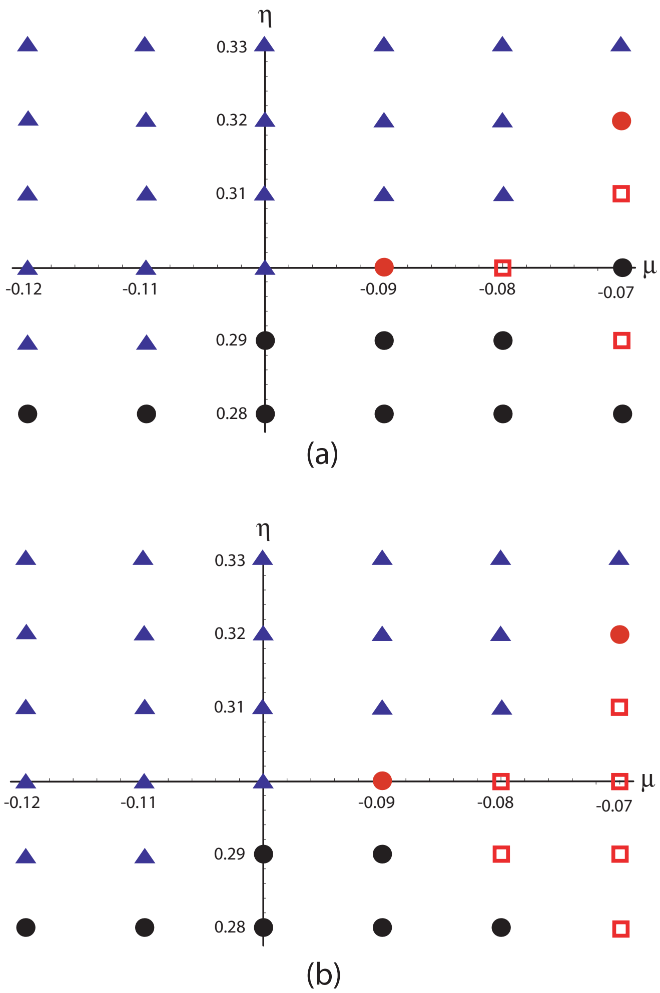

FIG. 3. Phase diagrams showing the observed patterns as a function of noise strength η on the ordinate versus the bifurcation parameter μ in the regime of exploding dissipative solitons for (a) $T_{\max} = 2000$ and for (b) $T_{\max} = 6000$. We show the parameter range μ from -0.12 to -0.07 and for $0.28 < \eta < 0.33$. Solid triangles (▲) are representing collapsed states (zero solution), black solid circles (•) correspond to exploding dissipative solitons that neither fill-in nor collapse, squares (□) are representing a "two-phase" behavior corresponding either to a filling-in or to an explosive state that is neither filling-in nor collapsing, depending on the chosen initial conditions. Finally the region denoted by the red solid circles (•) corresponds to filling-in independent of initial conditions.

plotted. Nevertheless we conclude from (a) that there is a regime of values for η sufficiently far from collapse for which the frequency decreases approximately linearly as a function of η and from (b) that there is a linear scaling in a logarithmic plot close to the collapse of exploding dissipative solitons.

Figure 4(c) shows the behavior as the critical value μ_c for the appearance of exploding dissipative solitons is approached. As we can see from Fig. 4(c) the frequency of explosions goes through a maximum as μ_c is approached. This maximum could be associated with two observations. Purely deterministically it has been shown before that the frequency for explosions decreases as the bifurcation parameter μ is reduced [31].

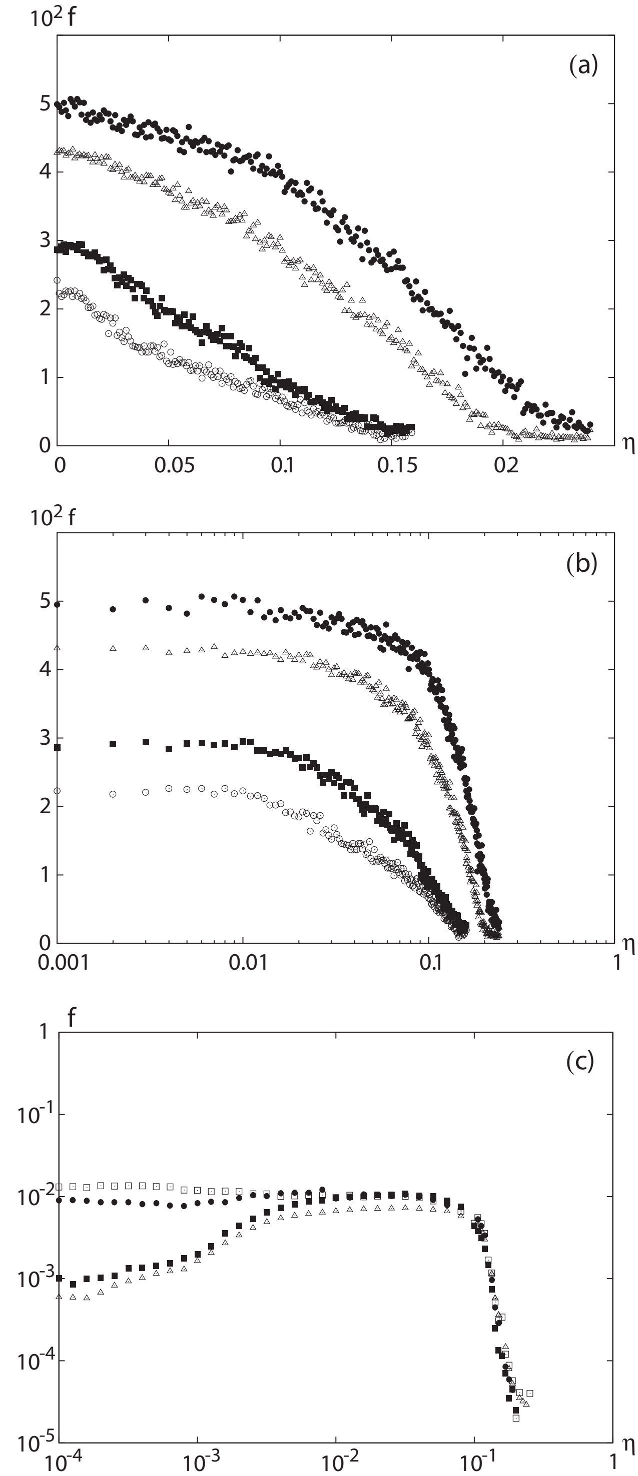

FIG. 4. The frequency of explosions as a function of the multiplicative noise strength is plotted in (a) on a linear scale and in (b) on a logarithmic scale for four values of the bifurcation parameter: $\mu = -0.12$ denoted as black solid circles (•), $\mu = -0.14$ denoted as triangles (△), $\mu = -0.17$ denoted as solid squares (■), and $\mu = -0.175$ denoted as open circles (○). (c) shows the behavior in the vicinity of the transition to exploding dissipative solitons, μ_c: $\varepsilon = 9\times10^{-4}$ denoted as squares (□), $\varepsilon = 4\times10^{-4}$ denoted as solid circles (•), $\varepsilon = 7\times10^{-6}$ denoted as solid squares (■), and $\varepsilon = 10^{-6}$ denoted as triangles (△), where ε denotes the distance from μ_c: $\varepsilon = \mu - \mu_c$.

Combining this feature with the fact that the number of explosions goes down with noise strength as the critical noise strength for collapse is approached, we conclude sufficiently close to μ_c the number of explosions can go through a maximum as it is indeed observed.

In Fig. 5 we show the possible types of behavior in response to multiplicative noise applied to exploding dissipative solitons for two points belonging to Fig. 3(b), namely, $\mu = -0.08$, $\eta = 0.31$ and $\mu = 0.08$, $\eta = 0.29$. For the former we show the collapse to the zero stable solution at $T \sim 1480$, and for the latter we show the two possible types of behavior: filling-in after a long time ($T \sim 3000$) and neither filling-in nor collapse. We have plotted in all cases [(a)–(c)] a global variable, namely, an averaged (in space) $R(x,t)$:[$\int R(x,t)dx]/L$. We note that the applied multiplicative noise shows no special features in the temporal vicinity of collapse or filling-in.

In conclusion, we have demonstrated that the influence of spatially homogeneous multiplicative noise can induce a transition to a spatially homogeneous state over a large range of the bifurcation parameter. For stationary DSs as well as for DSs with regular temporal behavior we find a linear relation between the critical value of the noise amplitude for the suppression of DSs and the bifurcation parameter. For exploding DSs the emerging picture as a function of the bifurcation parameter is more complex since multiplicative noise can also induce a transition to filling-in as the linear threshold is approached. In general we find that multiplicative noise leads to a reduction of the number of explosions with growing noise strength. This reduction can be understood using the explosion formation mechanism described in Ref. [31]. There explosions are generated by the growth of small perturbations around the DS. Due to the multiplicative noise the linear loss is changed, and the perturbations can be slowed down in their growth or even suppressed thus delaying the formation of explosions.

The present study opens the door to several areas of investigation. Clearly a key direction to go into is to study the influence of spatial dimensionality on the phenomena described here; this includes dissipative solitons localized in two dimensions [21,30,33] as well as quasi-one-dimensional DSs [22,23,34,43]. Second it will be important to examine to what extent the results presented here can be carried over to other models, such as reaction diffusion systems for which one has shown recently that they can support exploding DSs [44].

It has been noted in the context of modeling a Hopf bifurcation with multiplicative noise in the framework of a Brusselator model [45] that the coupling to fast variables must be modeled appropriately. Early experimental tests of these predictions have been carried out using analog simulator experiments with regard to postponement or advancement of the critical bifurcation parameter in the presence of colored multiplicative noise [46]. Quite recently the subcritical Hopf bifurcation in the presence of multiplicative noise in the framework of a Landau model without spatial degrees of freedom has been investigated [47]. When interpreting experiments one should therefore keep in mind a possible coupling to fast modes in the presence of multiplicative noise.

Perhaps the most clear-cut candidate to study the effects predicted here experimentally are the stable localized convective patterns which have been observed near the

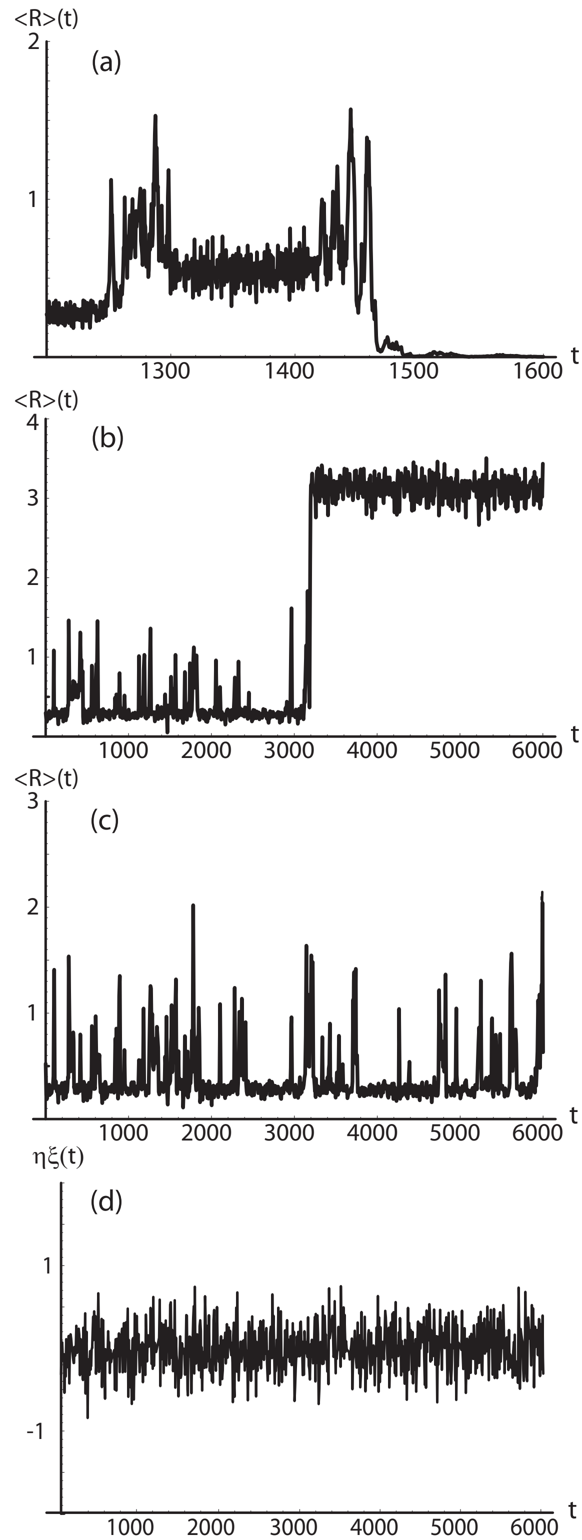

FIG. 5. Possible types of behavior in response to multiplicative noise applied to exploding dissipative solitons (a) $\mu = -0.08$, $\eta = 0.31$ and (b) and (c) $\mu = 0.08$, $\eta = 0.29$. (a) shows the collapse to zero at $T \sim 1480$, and (b) and (c) show two possible types of behavior: filling-in for long times ($T \sim 3000$) or persistent explosions. We have plotted in all cases [(a)–(c)] a global variable, namely, an averaged (in space) $R(x,t)$. We note that the applied multiplicative noise (d) shows no special features in the temporal vicinity of collapse or filling-in.

onset of thermal convection in binary fluid mixtures in an annulus [36,37,48,49]. Other candidates to investigate the results of our study experimentally are surface reactions, such as the catalytic oxidation of CO for which one has already a considerable amount of experience [10–15] for the externally controlled superposition of noise on parameters, such as partial pressure, flow rate, and temperature. To find experimentally a well-controllable chemical or bioinspired system to show the type of behavior described here is certainly a challenge. Candidates include systems for which solitonlike structures and their collisions have been observed [14,38,39,50].

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