Pseudo-Hermitian ensemble of random Gaussian matrices

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It is shown how pseudo-Hermiticity, a necessary condition satisfied by operators of *PT* symmetric systems can be introduced in the three Gaussian classes of random matrix theory. The model describes transitions from real eigenvalues to a situation in which, apart from a residual number, the eigenvalues are complex conjugate.

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I. INTRODUCTION

Hermitian operators play a major role in quantum mechanics and the reason for this is simple: at the end, the quantities of interest in physics, that is the observables, are real variables to be represented by operators with a real spectrum. What is less realized is that, in order to have real eigenvalues, the Hermiticity condition is not a necessary, only a sufficient condition. In other words, non-Hermitian operators also can have real spectra. This becomes evident when the spectrum of a complex non-Hermitian Hamiltonian such as

$$H = p^2 - (ix)^N \tag{1}$$

is fully analyzed [1,2]. It is found that for $N \geqslant 2$ all eigenvalues are real. Then, as progressively N decreases eigenvalues move into the complex in conjugate pairs. It has also been considered that this property is a consequence of the invariance this Hamiltonian has under the combined parity (P) and time-reversal (T) transformations,

$$P\psi(x) = \psi(-x) \quad T\psi(x) = \bar{\psi}(x), \tag{2}$$

as can be easily verified. This result lead to a proposal of an extension of quantum mechanics to include this special class of non-Hermitian *PT* symmetric operators [3] (see [4] for a review). From a more general approach, the main property of this class of non-Hermitian operators is to be connected to their respective adjoints by a similarity transformation

$$H^\dagger = \eta H \eta^{-1} \tag{3}$$

which defines the so-called pseudo-Hermitian operators and means that the operator and its adjoint share the same set of eigenvalues [5–7]. Of course, this necessarily implies that the eigenvalues are real or complex conjugate. Moreover, if the matrix $\eta^{\frac{1}{2}}$ and its inverse exist and are Hermitian then the matrix

$$K = \eta^{\frac{1}{2}} H \eta^{-\frac{1}{2}} = \eta^{-\frac{1}{2}}(\eta H \eta^{-1})\eta^{\frac{1}{2}} = \eta^{-\frac{1}{2}} H^\dagger \eta^{\frac{1}{2}} = K^\dagger \tag{4}$$

is Hermitian and share with H the same real spectrum.

The importance random matrix theory (RMT) has obtained since its introduction by Wigner in the late 1950s and, also, the role the time-reversal symmetry plays in it, naturally lead to attempts to find, in its context, ensembles of matrices satisfying the above Eq. (3). Initially, the case of 2×2 pseudo-unitary matrices was considered [8] and extended for full cyclic matrices [9,10]. Random matrix theory of absorbing and amplifying resonators for coupled interacting finite systems of arbitrary matrix size were also studied [11].

Another approach was based in the β-ensemble, obtained by performing a sequence of Householder transformations such that the Gaussian matrices are reduced to a tridiagonal matrices [12,13]. This finding defined the β-ensemble in which the Dyson index β can assume any real positive value, whereas in the standard RMT it may only assume the values 1, 2 and 4, for the three classes of Gaussian matrices, namely, the orthogonal (GOE), unitary (GUE), and symplectic (GSE) ensembles with real, complex, and quaternion elements, respectively [14]. The special properties tridiagonal matrices have made the β-ensemble a useful tool to discuss aspects of pseudo-Hermitian operators. Thus, by an appropriate removal of their Hermitian condition, the tridiagonal matrices become an ensemble of pseudo-Hermitian random matrices with real eigenvalues [15]. These eigenvalues occupy the same compact support of the Hermitian ones but repel each other within a different way. Extending this investigation, an ensemble of pseudo-Hermitian tridiagonal matrices isospectral with the β-ensemble matrices [16] has been constructed. The metric η however becomes singular when the size of the matrices increases, making them very sensitive to small perturbations. As a consequence, as it occurs with *PT*, eigenvalues start to move in conjugated pairs into the complex plane. This was further explored in the case in which the positive-definiteness is broken for which a qualitatively similar behavior is seen [17].

Despite this relative success, an ensemble of random pseudo-Hermitian Gaussian matrices is still lacking. The aforementioned ensembles do not describe full, random, pseudo-Hermitian matrices whose entries are entirely independent. A recent promising approach which tackles this problem involved the use of split-complex and split-quaternion random matrices [18].

In the present work, we use projections into separate subspaces to study the behavior of the eigenvalues as the interactions between these subspaces are varied. We thus propose a new distinct ensemble of pseudo-Hermitian matrices derived from the three classical cases of Gaussian matrices and based on the use of projection operators.

II. THE MODEL

Our starting point is the standard RMT matrices whose elements can be real, complex, or quaternion numbers such that they can be written as

$$H_{mn} = H_{mn}^{(0)} + i H_{mn}^{(1)} + j H_{mn}^{(2)} + k H_{mn}^{(3)} \tag{5}$$

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with $i^2 = j^2 = k^2 = ijk = -1$ and have the density joint distribution

$$P(H) = Z_N^{-1} \exp\left[-\frac{\beta}{2}\mathrm{tr}(H^\dagger H)\right], \tag{6}$$

where the number of non-zero elements in Eq. (5) denoted by β can be equal to 1, 2, or 4 [14]. Therefore, the elements are Gaussian distributed and can be real, complex, or quaternion. To introduce the pseudo-Hermiticty condition (3) to these matrices, we use the projection operator method in the way which will be discussed in the following subsections.

The projection operator method consists in considering $m \geqslant 1$ projection operators $\{P_k\}$ such that $P_k^2 = P_k$, their defining property, and also $P_k P_{k'} = \delta_{k,k'} P_k$ and $\sum_{k=1}^{m} P_k = \mathbf{1}$. We may provide a physical picture of the interest in this method if we consider the operator representation of a system's Hamiltonian, H. We may then consider the time-dependent Schrödinger equation

$$H\Psi = i\partial_t \Psi$$

which describes the complete wave function Ψ. The time evolution of the occupation probabilities for a given basis state Ψ will be a function of the Hamiltonian H, and the projection $P_k H P_k$ describes the transitions of H on the elements of the subspace tied to P_k within that subspace. Analogously, the projection $P_k H P_{k'}$ for $k \neq k'$ describes how H affects elements of the subspace tied to $P_{k'}$ as they transition into the subspace P_k. Therefore, by splitting the matrix in blocks through the use of projection operators, we may introduce a form of mixing in the system, which allows the description of systems with symmetries of physical interest.

In the present paper, we are interested in discussing the cases in which this mixing occurs between two blocks and the case in which it occurs between several blocks. This is related to the decomposition of the space in subspaces labeled by a quantum number such as spin and here we discuss how it may also be extended to complex interactions as it occurs in PT-symmetric Hamiltonians.

A. Projection into two subspaces

To do this we start by recalling the symmetry case which has been treated in Ref. [19] using projection operators P and Q to write

$$A = PHP + QHQ + \xi_1 PHQ + \xi_2 QHP, \tag{7}$$

where

$$P = \sum_{i=1}^{M} |i\rangle\langle i| \tag{8}$$

with $Q = \mathbf{1}_N - P$ [20] and H is a Hermitian $N \times N$ random matrix taken from one of the classes of the Gaussian ensemble.

The two first terms of Eq. (7) are Hermitian diagonal blocks while the two other ones represent the off-diagonal blocks which become non-Hermitian if $\xi_1 \neq \bar{\xi}_2$. With real parameters, the above equation describes partial conservation of a quantum number like isospin [21] if $0 < \xi_1 = \xi_2 < 1$, conservation if they vanish and no conservation if they are equal to one.

To apply this structure to the PT symmetric case, we allow the parameters to be complex in such a way that the adjoint of the operator A is

$$A^\dagger = PHP + QHQ + \bar{\xi}_1 QHP + \bar{\xi}_2 PHQ \tag{9}$$

with the overbar indicating complex conjugation. We then write $\xi_1 = r_1 \exp(i\alpha_1)$ and $\xi_2 = r_2 \exp(i\alpha_2)$ such that

$$A = PHP + QHQ + r_1 \exp(i\alpha_1) PHQ + r_2 \exp(i\alpha_2) QHP. \tag{10}$$

For this operator to verify the pseudo-Hermitian condition (3), η must be of the form

$$\eta = zP + wQ \tag{11}$$

as no higher order terms are possible [20]. It is then straightforward to show that imposing Eq. (3) with Eq. (11) on Eqs. (7) and (9) implies that

$$\alpha_1 + \alpha_2 = k\pi, \quad k = 0,1, \quad r_2 w = (-1)^k r_1 z. \tag{12}$$

The case for $k = 0$ defines an operator η which is positive definite, as it consists of the scaling by a positive number of two interacting blocks. Thus η has a positive square root, and following Eq. (4), the eigenvalues of H are real [22].

In this work, we are interested in the case of $k = 1$, for which said operator is not positive-definite, and it may be written as

$$\eta = r_2 P - r_1 Q. \tag{13}$$

Noting that α_1 and α_2 may differ up to a phase of opposite sign, we may then express A for the case of interest, $k = 1$, as

$$A(\alpha) = PHP + QHQ + r_1 \exp(i\alpha) PHQ - r_2 \exp(-i\alpha) QHP. \tag{14}$$

We observe that, if we had taken $\alpha_1 = \alpha_2$, then we would have $\alpha_1 = \alpha_2 = \pi/2$. This case may however be obtained from a similarity transformation from Eq. (14).

In principle, the parameters of the above model are the integer M which defines the size of the blocks, the phase α, and r_1 and r_2 which weights the importance of the pseudo-Hermitian elements. However, defining the operator

$$\mu(\alpha) = P \exp\left(\frac{i\alpha}{2}\right) + Q \exp\left(-\frac{i\alpha}{2}\right), \tag{15}$$

we find that the gauge transformation

$$A(\alpha) = \mu A(0) \mu^\dagger \tag{16}$$

determines an isospectral family of matrices parametrized by the phase α. On the other hand, in terms of the eigenvectors of the matrix $A(0)$, the eigenvectors can be expressed for a generic value of α as

$$|\Psi(\alpha)\rangle = \mu(\alpha)|\Psi(0)\rangle. \tag{17}$$

Therefore, when α goes from zero to 2π, the matrix returns to its original value but the eigenvectors change sign. It is also interesting to observe that for $\alpha = \frac{\pi}{2}$ elements of the interacting blocks are multiplied by the same imaginary unit. In particular, in the real symmetric case the matrix becomes invariant under the transpose operation, that is they have the symmetry $A^T = A$.

Taking the case in which $\xi_1 = -\xi_2^* = r\exp(i\alpha)$, and defining

$$A_1 = PHP + QHQ = \frac{A + A^\dagger}{2}, \tag{18}$$

$$A_2 = e^{i\alpha}PHQ - e^{-i\alpha}QHP = \frac{A - A^\dagger}{2r} \tag{19}$$

we have $A_1^\dagger = A_1$ and $A_2^\dagger = -A_2$ and (cf. Appendix A)

$$\mathrm{tr}\left(A_1^2 - A_2^2\right) = \mathrm{tr}H^2 = \mathrm{tr}\left(H^\dagger H\right), \tag{20}$$

where the Hermiticity of H was used. Therefore we may write the trace of $H^\dagger H$ in terms of A using the explicit forms of A_1 and A_2, Eqs. (18) and (19):

$$\mathrm{tr}(H^\dagger H) = \mathrm{tr}\left[\left(1 - \frac{1}{r^2}\right)\frac{A^2 + A^{\dagger 2}}{4} + \left(1 + \frac{1}{r^2}\right)\frac{AA^\dagger + A^\dagger A}{4}\right]. \tag{21}$$

This implies that we may obtain the density of A directly from the joint probability density function of H, that is

$$\begin{aligned} P(A) &= Z_N\left|\frac{\partial H}{\partial A}\right|\exp\left\{-\frac{\beta}{2}\mathrm{tr}[H^\dagger(A,A^\dagger)H(A,A^\dagger)]\right\} \\ &= \zeta_N(r)\exp\left\{-\frac{\beta}{2}\mathrm{tr}\left[\left(1 - \frac{1}{r^2}\right)\frac{AA + A^\dagger A^\dagger}{4}\right.\right. \\ &\quad \left.\left. + \left(1 + \frac{1}{r^2}\right)\frac{AA^\dagger + A^\dagger A}{4}\right]\right\} \end{aligned} \tag{22}$$

since the Jacobian $|\frac{\partial H}{\partial A}|$ is linear in all elements of H and is, thus, a polynomial in r. This distribution describes a transition from a situation in which the eigenvalues lie in the real axis, for $r = 0$, into one in which the eigenvalues move into the complex plane, as r goes to 1.

The similarity of Eq. (22) to that of the transition from the Hermitian to the Ginibre case [23] suggests the ansatz that the resulting distribution of eigenvalues are likely to follow an elliptic law [24–26]. This is a known property for the general case of $n \times n$ complex matrices [27,28] for which the axes are known:

$$a = \sqrt{\frac{2N}{1 + r^2}}, \tag{23}$$

$$b = r^2\sqrt{\frac{2N}{1 + r^2}}, \tag{24}$$

and for which the eccentricity may be calculated to give

$$\begin{aligned} \epsilon &= \sqrt{1 - r^4} \quad 0 \leqslant r \leqslant 1 \\ \epsilon &= \sqrt{1 - \frac{1}{r^4}} \quad r > 1. \end{aligned} \tag{25}$$

Let us now consider the case in which both r_1 and r_2 are equal to 1. By direct substitution in Eq. (22), we have that

$$P(A) = Z_N^{-1}\exp\left[-\frac{\beta}{2}\mathrm{tr}(A^\dagger A)\right]. \tag{26}$$

This implies that we have a particular class of the non-Hermitian Gaussian matrices, resembling those of the Ginibre ensemble, since the elements are all independent and identically distributed Gaussian variables, with the exception of the diagonal elements.

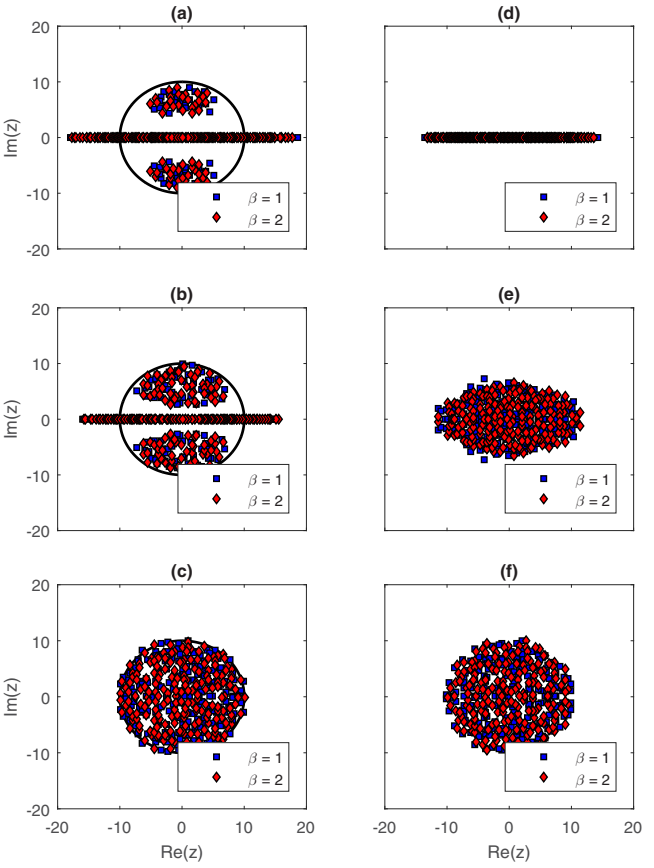

FIG. 1. Two transitions observed in this system. The left shows the transition for the size $N = 200$ block pseudo-Hermitian matrix into the circle for block sizes 25, 50, and 100 for (a)–(c), respectively. The right column shows the transition for the size $N = 200$ unit block pseudo-Hermitian matrix with the parameter $r = 0, 0.75, 1.00$ for (d)–(f), respectively.

For arbitrary matrix size, the only remaining parameter is the size M of the blocks. In Figs. 1(a), 1(b), and 1(c), the evolution of the spectra of a matrix of size $N = 200$ for sizes $M = 25, 50, 100$, respectively, for $\beta = 1, 2$ are shown. It is seen that starting from $M = 25$, which is almost real spectra, pairs of eigenvalues progressively evaporate forming at some distance from the real axis two symmetric clouds above and below the axis which repel each other. As the transition proceeds the clouds fill a disk of radius $\sqrt{N/2}$ in such a way that for $M = 50$ it becomes completely full with only a residual number of eigenvalues at the real axis. At the final value $M = N/2$, the situation is similar to the unitary class of the Ginibre ensemble.

It is instructive to work out the case of 2×2 matrices, in terms of the Hermitian elements their eigenvalues are given by

$$\lambda_{1,2} = \frac{H_{11} + H_{22}}{2} \pm \sqrt{\left(\frac{H_{22} - H_{11}}{2}\right)^2 - |H_{12}|^2} \tag{27}$$

such that they can be real or complex conjugate depending on the discriminant. For the real ones the eigenvectors can expressed as

$$|\lambda_1\rangle = \exp\left(\frac{i\theta}{2}\right)\sin\frac{\gamma}{2}|1\rangle + \exp\left(-\frac{i\theta}{2}\right)\cos\frac{\gamma}{2}|2\rangle \tag{28}$$

and

$$|\lambda_2\rangle = \exp\left(\frac{i\theta}{2}\right)\cos\frac{\gamma}{2}|1\rangle + \exp\left(-\frac{i\theta}{2}\right)\sin\frac{\gamma}{2}|2\rangle, \quad (29)$$

where $\sin\gamma = 2|H_{12}|/(H_{11} - H_{22})$ and θ is the phase of the element H_{12}. From the above we find that $\langle\lambda_2|\lambda_1\rangle = \sin\gamma$. For the complex eigenvalues, with $\cosh\delta = (H_{11} - H_{22})/(2|H_{12}|)$ the eigenvectors are

$$|\lambda_{1,2}\rangle = \frac{1}{\sqrt{2\cosh\gamma}}\left[\exp\left(\frac{i\theta \mp \gamma}{2}\right)|1\rangle + \exp\left(\frac{i\theta \pm \gamma}{2}\right)|2\rangle\right] \quad (30)$$

and $\cos^{-1}\gamma$ is the angle between them. The eigenvectors of the adjoint matrices are obtained as $|\lambda_{1,2}\rangle^\dagger = \eta|\lambda_{1,2}\rangle$ which changes the signs of the components in $|2\rangle$. The transition from real to complex eigenvalues corresponds to a spontaneous break of the PT symmetry.

It is noteworthy, however, that the fraction of the eigenvalues which remains in the real axis is dependent on the size M of the block. Figure 2 shows for a size $N = 400$ and $\beta = 2$ matrix H and multiple block sizes M the fraction of eigenvalues, up to floating point accuracy, with imaginary part greater than zero. That fraction increases as r increases in all but the $M = 0$ case, which is expected, as in this case the matrix is Hermitian. Our results also indicate that there is a limit to what fraction of the eigenvalues can become complex which depends on the size of the block considered. This contrasts with the results of [11] for which the fraction seemed to present an asymptotic tendency to $f_* = 1$.

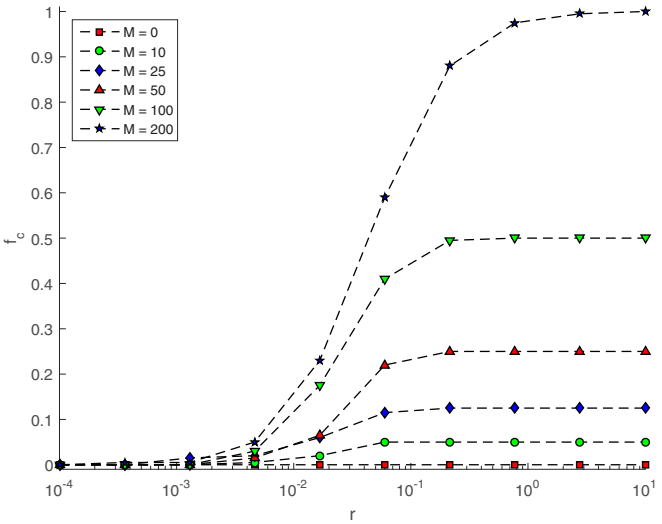

FIG. 2. Average fraction of eigenvalues with a complex part for a $\beta = 2$ matrix of size $N = 400$ and varying parameter r. The multiple lines denote varying block sizes, as indicated in the box in the top-left corner.

B. Projection into $K \geqslant 2$ subspaces

This pseudo-Hermitian ensemble introduced using the P and Q projectors is just a particular case which can be generalized. To see this, with $k = 1,2,\ldots,K$ let us introduce the K projectors P_K

$$P_k = \sum_{i=1}^{M_k} |i_k\rangle\langle i_k|, \quad (31)$$

where $\sum_{k=1}^{K} M_k = N$ and $\{i_k\}_{1,\ldots,M_1} \cap \{i_k\}_{1,\ldots,M_2} \cdots \cap \{i_k\}_{1,\ldots,M_K} = \{1,2,\ldots,N\}$, such that

$$\sum_{k=1}^{K} P_k = 1, \quad P_k P_l = \delta_{k,l} P_k. \quad (32)$$

Using this set of projection operators and taking a matrix H of the Gaussian ensemble, an operator A can then be constructed as

$$A = \sum_{k=1}^{K} P_k H P_k + \sum_{j>i}(r^{s_{ij}} P_i H P_j + r^{s_{ij}}\cos[(i-j)\pi]P_j H P_i), \quad (33)$$

where $s_{ij} = 1/2 - \cos[(j-i)\pi]/2$. The corresponding matrix A is made of diagonal Hermitian blocks that come from the first terms in the right-hand side. Then, the neighboring subdiagonal is made of pseudo-Hermitian blocks, the next Hermitian followed by pseudo-Hermitian, and so on. Therefore, the matrix has a kind of Toeplitz structure in which inside each subdiagonal the blocks are of the same nature, namely Hermitian or pseudo-Hermitian. The parameter r controls the importance of the pseudo-Hermitian terms.

The metric operator is

$$\eta = r\sum_{k=1}^{K}(-1)^{k+1} P_k \quad (34)$$

which may be seen by direct application of Eq. (34) and its inverse to Eq. (33):

$$\eta A \eta^{-1} = \left(r\sum_{l=1}^{K}(-1)^{l+1}P_l\right) A \left(r^{-1}\sum_{p=1}^{K}(-1)^{p+1}P_p\right) = \sum_{l,p=1}^{K}(-1)^{l+p} P_l A P_p$$
$$= \sum_{l,p=1}^{K}\left[\sum_{k}\delta_{l,k}\delta_{p,k}P_k H P_k + \sum_{j>i}(-1)^{l+p} r^{s_{ij}}\{\delta_{l,i}\delta_{p,j}P_i H P_j + \delta_{l,j}\delta_{p,i}\cos[(i-j)\pi]P_j H P_i\}\right] = A^\dagger \quad (35)$$

since $\cos[(i-j)\pi] = (-1)^{i+j}$.

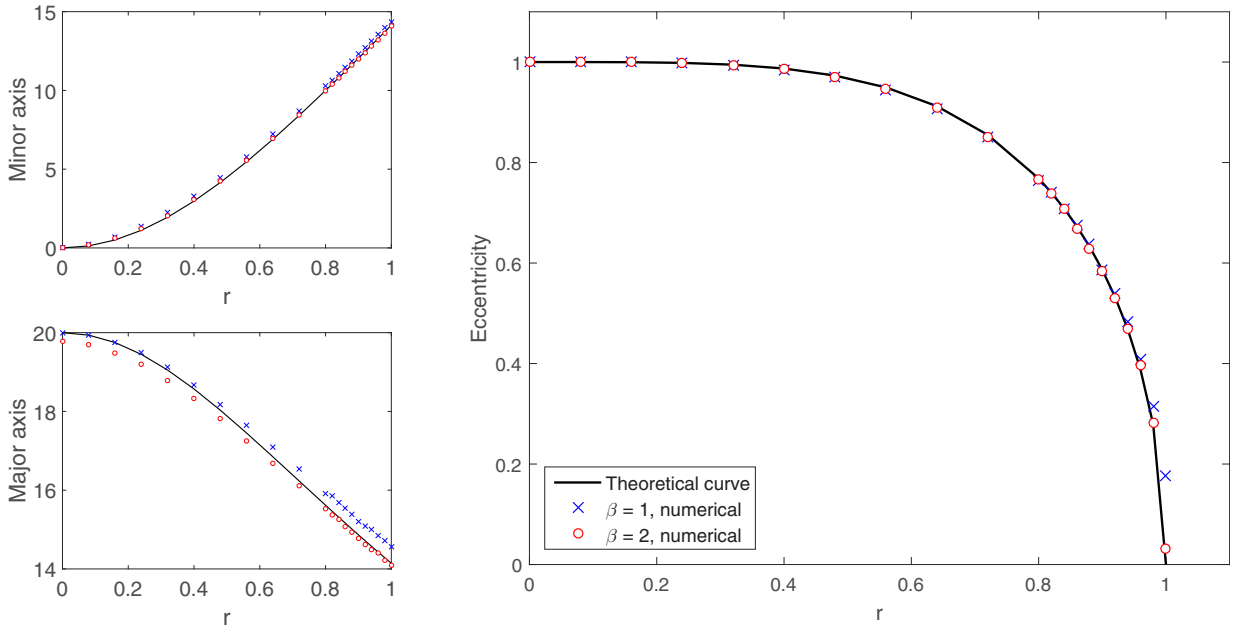

FIG. 3. Eccentricity of the observed ellipse for $\beta = 1$ and $\beta = 2$ matrices of size $N = 400$. The eccentricity is averaged in a sample of 200 matrices for each parameter r step.

It is also noteworthy that, as in the case of a single size M block, we may obtain the *jpdf* of entries by considering the matrices

$$A_1 = \sum_{k=1}^{K} P_k H P_k + \sum_{\substack{j>i \\ \mathrm{mod}\,(i-j,2)=0}} (P_i H P_j + P_j H P_i) = \frac{A + A^\dagger}{2}, \tag{36}$$

$$A_2 = \sum_{\substack{j>i \\ \mathrm{mod}\,(i-j,2)=1}} (P_i H P_j - P_j H P_i) = \frac{A - A^\dagger}{2r} \tag{37}$$

such that cf. the Appendix,

$$\mathrm{tr}\left(A_1^2 - A_2^2\right) = \mathrm{tr}(H^\dagger H) \tag{38}$$

Therefore, as in the previous case, the *jpdf* will be given by Eq. (22).

As it happens with the $K = 2$ case, a phase can be introduced by the unitary transformation $\mu A(\alpha)\mu^\dagger$ with

$$\mu = \sum_{k=1}^{K} \exp\left[(-1)^{k+1} i \frac{\alpha}{2}\right] P_k. \tag{39}$$

A special case is that in which the operators are decomposed in square blocks, that is $M_1 = M_2 = \cdots = M_K = M$ with $M = N/K$. In particular, if $M = 1$ the matrix is decomposed in Hermitian and pseudo-Hermitian subdiagonals alternatively. In Figs. 1(d), 1(e), and 1(f), it is shown the evolution of the spectra of a matrix of size $N = 200$ when all blocks have size $M = 1$. Starting with no pseudo-Hermitian terms for $r = 0$ in Fig. 1(d), when the eigenvalues are real, the eigenvalues leave the real axis in conjugate pairs filling an ellipse for $r = 0.75$ as shown in Fig. 1(e), and eventually fill what becomes a circle for $r = 1$, depicted in Fig. 1(f).

We may compare the observed eccentricity of a sample matrix by taking the maximum absolute value of the real (imaginary) part, we may obtain the semi-major (minor) axis and thus compute the eccentricity of the resulting ellipse for $0 \leqslant r \leqslant 1$. In Fig. 3 we present the results for case of $N = 400$ projectors of dimension $m = 1$, which are shown compared to Eq. (25) for both $\beta = 1$ and $\beta = 2$. The model shows good agreement to the observed eccentricity, although some discrepancy from the model is seen for the major axis. Since the elliptic law is an asymptotic result, it is expected that this discrepancy should decrease as N increases. The numerical results in Fig. 4 show that the behavior of the major axis becomes closer to that of the elliptic law as N grows in Figs. 4(a) to 4(f) as $N = 10, 25, 50, 100, 200, 400$, respectively. The semi-minor axis follows the same qualitative behavior.

III. DISCUSSION

Decomposing the Hamiltonian with projectors as in Eq. (33) is not the only choice. In fact, our model allows many different choices of signs among the K blocks which, save for a global phase, amount to 2^{K-1} distinct sign compositions. However, our choice in the present paper is motivated by the regular chessboard-like structure, but it also leads to the interpretation as complex interactions between the blocks. To see the relation between this block decomposition and a Hamiltonian like

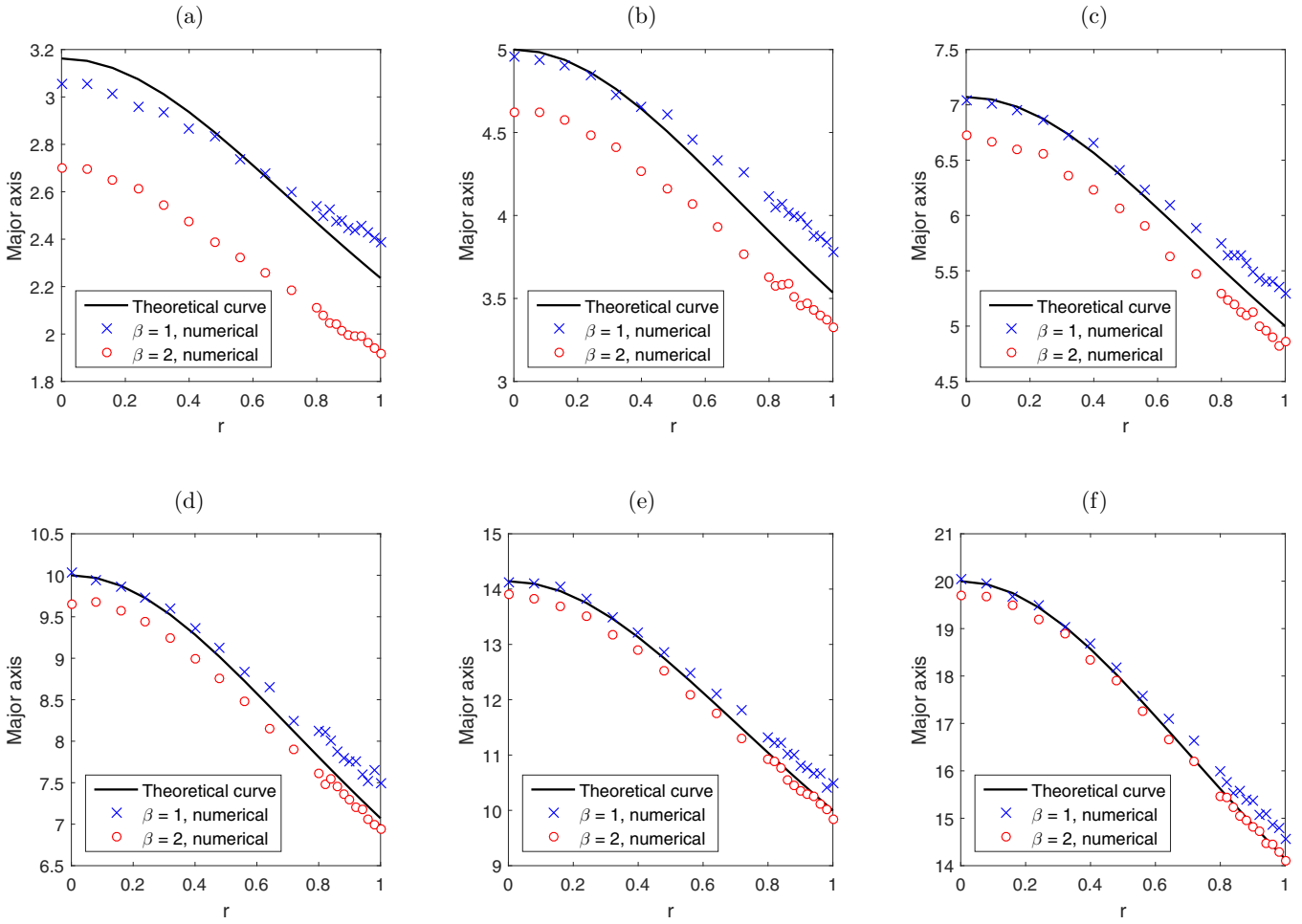

FIG. 4. Major axis of the observed ellipse for $\beta = 1$ and $\beta = 2$ matrices of varying size. The major axis is averaged in a sample of 12000 eigenvalues, corresponding to (a) 1200 samples of 10×10 matrices, (b) 480 samples of 25×25 matrices, (c) 240 samples of 50×50 matrices, (d) 120 samples of 100×100 matrices, (e) 60 samples of 200×200 matrices, and (f) 30 samples of 400×400 matrices.

that of Eq. (1), consider to transform it to a matrix form by using a complete set of eigenfunctions, for instance, the quantum harmonic oscillator wave functions. The second term in Eq. (1), gives rise to the element

$$V_{ij} = -\langle \Psi_i(x)|(ix)^N|\Psi_j(x)\rangle. \tag{40}$$

Taking the complex conjugate and changing x to $-x$ then as a consequence of the parity of the eigenfunctions we obtain

$$\bar{V}_{ij} = (-1)^{i+j} V_{ji}, \tag{41}$$

which shows that the elements follow a structure similar to the one of our model. As a matter of fact, Hamiltonians with similar structure have already appeared in the discussion of the quantum brachistochrone problem in the context of pseudo-Hermitian quantum mechanics [22].

We also note that the discussion pertaining to the fraction of eigenvalues with complex part has focused on the case of $0 \leqslant r \leqslant 1$. Although no less interesting, the case for $r > 1$ presents qualitatively similar behavior to that seen in Fig. 2, as for this case the semi-axes of the ellipse switch places. It is noteworthy, however, that the fully imaginary case is only expected to be recovered, as Eq. (25) shows, when r tends to infinity. Thus, besides being qualitatively similar, there is an asymmetry which although beyond the scope of this study, may interest further focus.

We also remark that the Hermitian ensembles of random matrices have been related to symmetry spaces and have been classified using the Cartan scheme. This kind of classification has been extended to the case of non-Hermitian matrices according to their discrete symmetries [29]. In this classification, the above Eq. (3) corresponds to the symmetry denoted by the Q symmetry. Our matrices with $r_1 = r_2 = 1$ have the structure described in the extension of this scheme made in Ref. [30].

IV. CONCLUSION

We have constructed families of Gaussian matrices which satisfy the pseudo-Hermitian condition for the orthogonal, unitary, and symplectic classes of RMT. In this model, the pseudo-Hermitian nature of each of the matrices' blocks may vary, and the connection between blocks is controlled through coupling parameters in the off-diagonal blocks as well as the size of the blocks.

The element density for a matrix A of the model was obtained for the case in which there is a single $M \times M$ projection operator P, and in the special case of a 2×2 matrix, the

eigenvalues and eigenvectors were described. The model was then expanded for K arbitrary projection operators, and it was shown that the *jpdf* remains as described for the previous case.

The analytical results were compared to numerical simulations. Our findings show that in this model the block size and coupling parameters influence how the eigenvalues distribute themselves in the complex plane, albeit each does so in a different manner. In Fig. 1, both possible variations were presented for the case in which the coupling parameters ξ_1 and ξ_2 have the same modulus, which displays the role of block size and coupling parameter modulus.

Further properties of the model were obtained by studying the fraction of the eigenvalues which become complex as the coupling parameter is increased. The results of Fig. 2 were distinct from those in the previous literature, as the maximum fraction of eigenvalues which become complex is tied to the block size. This seems to be a distinctive feature of our model.

The comparison of the numerical experiments to the properties predicted for the ellipse occupied by the eigenvalues was made in Fig. 3. There was marked agreement between the predicted eccentricity and that of the sample's average, although some deviations were seen for the average calculated semi-major and semi-minor axes.

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APPENDIX

Proposition 1. Let A_1 and A_2 be $N \times N$ matrices such that $A_1^\dagger = A_1$ and $A_2^\dagger = -A_2$. Define

$$A(r) = A_1 + rA_2. \tag{A1}$$

Then for $r = 0$,

$$\mathrm{tr}\left(A_1^2\right) = \mathrm{tr}(A^\dagger A) = \mathrm{tr}(AA^\dagger) = \mathrm{tr}(A^2) \tag{A2}$$

and for $r \neq 0$

$$\mathrm{tr}\left(A_1^2 - A_2^2\right) = \frac{1}{4}\mathrm{tr}\left[\left(1 - \frac{1}{r^2}\right)(A(r)^2 + A(r)^{\dagger 2}) + \left(1 + \frac{1}{r^2}\right)(A(r)A(r)^\dagger + A(r)^\dagger A(r))\right]. \tag{A3}$$

Proof. The case in which $r = 0$ follows immediately from the definition of A. For $r \neq 0$, we write A_1 and A_2 in terms of $A(r)$:

$$A_1 = \frac{A + A^\dagger}{2}, \quad A_2 = \frac{A - A^\dagger}{2r}, \tag{A4}$$

from which we may write directly

$$A_1^2 = \frac{A^2 + A^{\dagger 2} + AA^\dagger + A^\dagger A}{4} = \frac{1}{4}[(A^2 + A^{\dagger 2}) + (AA^\dagger + A^\dagger A)] \tag{A5}$$

and

$$A_2^2 = \frac{A^2 + A^{\dagger 2} - AA^\dagger - A^\dagger A}{4} = \frac{1}{4}\left[\frac{1}{r^2}(A^2 + A^{\dagger 2}) - \frac{1}{r^2}(AA^\dagger + A^\dagger A)\right], \tag{A6}$$

from which the result directly follows. ■

Remark 1. Because the trace of a commutator is zero, we have that for any two $N \times N$ matrices A_1 and A_2,

$$\mathrm{tr}\left(A_1^2 - A_2^2\right) = \mathrm{tr}[(A_1 + A_2)(A_1 - A_2)]. \tag{A7}$$

In the case that $A_1^\dagger = A_1$ and $A_2^\dagger = -A_2$ we also have that

$$\mathrm{tr}\left(A_1^2 - A_2^2\right) = ||A_1 + A_2||_2^2 = \sum_{k,l=1}^{N} \left|(A_1 + A_2)_{k,l}\right|^2. \tag{A8}$$

Remark 2. Proposition 1 and Remark 1 imply that for $r = 1$, we have

$$\mathrm{tr}\left(A_1^2 - A_2^2\right) = \mathrm{tr}(A(1)A^\dagger(1)). \tag{A9}$$

Corollary 1. Let $H = \{h_{i,j}\}$ be a Hermitian $N \times N$ matrix and let $A = \{a_{i,j}\} = \{h_{i,j}e^{i\alpha_{i,j}}\}$ be a matrix whose elements differ from those of H by at most a complex phase. Also, define

$$A_1 = \tfrac{1}{2}(A + A^\dagger), \quad A_2 = \tfrac{1}{2}(A - A^\dagger). \tag{A10}$$

Then $\mathrm{tr}(A_1^2 - A_2^2) = \mathrm{tr}(H^2)$ and moreover, for any

$$A(r) = A_1 + rA_2 \tag{A11}$$

with $r \neq 0$ the same parametrization holds.

Proof. First note that

$$(A_1 + A_2)^\dagger = A_1 + A_2 = A \tag{A12}$$

and therefore, from Remark 1,

$$\mathrm{tr}\left(A_1^2 - A_2^2\right) = \sum_{k,l=1}^{N} |a_{l,l}|^2 = \sum_{k,l=1}^{N} |e^{i\alpha_{k,l}}h_{k,l}|^2 = \sum_{k,l=1}^{N} |h_{k,l}|^2 = \mathrm{tr}(HH^\dagger) = \mathrm{tr}(H^2). \tag{A13}$$

Since, for all $r \neq 0$ and $\mathrm{tr}(A_1^2 - A_2^2)$ does not depend on the parameter r, it follows from Proposition 1 that the right-hand side of Eq. (A3) is also constant. Therefore, for all $r \neq 0$

$$\mathrm{tr}\left(A_1^2 - A_2^2\right) = \mathrm{tr}(H^2). \tag{A14}$$

■

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