Perturbative expansion for the maximum of fractional Brownian motion

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Brownian motion is the only random process which is Gaussian, scale invariant, and Markovian. Dropping the Markovian property, i.e., allowing for memory, one obtains a class of processes called fractional Brownian motion, indexed by the Hurst exponent H. For $H = 1/2$, Brownian motion is recovered. We develop a perturbative approach to treat the nonlocality in time in an expansion in $\varepsilon = H - 1/2$. This allows us to derive analytic results beyond scaling exponents for various observables related to extreme value statistics: the maximum m of the process and the time $t_{\max}$ at which this maximum is reached, as well as their joint distribution. We test our analytical predictions with extensive numerical simulations for different values of H. They show excellent agreement, even for H far from $1/2$.

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I. INTRODUCTION

Random processes are ubiquitous in nature. Though many processes can successfully be modeled by Markov chains and are well analyzed with tools of statistical mechanics, there are also interesting and realistic systems which do not evolve with independent increments and, thus, are non-Markovian, i.e., history dependent. Dropping the Markov property, but demanding that a continuous process be scale-invariant and Gaussian with stationary increments, defines an enlarged class of random processes, known as *fractional Brownian motion* (fBm). Such processes appear in a broad range of contexts: anomalous diffusion [1], diffusion of a marked monomer inside a polymer [2,3], polymer translocation through a pore [3–6], single-file diffusion in ion channels [7,8], dynamics of a tagged monomer [9,10], finance (fractional Black-Scholes, fractional stochastic volatility models, and their limitations) [11–13], hydrology [14,15], and many more.

While averaged quantities have been studied extensively and are well characterized, it is often more important to understand the extremal behavior of a process, or the time when it satisfies a given criterion [16]. These quantities are associated with failure of fracture or earthquakes, stock market crashes, breakage of dams, the time when one has to heat, etc. The three *arcsine laws* of Brownian motion are well-studied examples. They state that for a Brownian process X_t, with $0 < t < 1$ and $X_0 = 0$, three observables Y have the same cumulative distribution function, (1), the arcsine law, equivalent to the probability density, (2):

$$\Pr(Y < y) = \frac{2}{\pi}\arcsin(\sqrt{y}) \quad (1)$$

$$\Leftrightarrow \; \mathcal{P}(y) = \frac{1}{\pi\sqrt{y(1-y)}}. \quad (2)$$

The observables in question are (see Fig. 1) as follows.

(1) First (Lévy's) arcsine law: The time when the process X_t is positive (horizontal red lines in Fig. 1):

$$t_+ := \int_0^1 \Theta(X_t)\,\mathrm{d}t. \quad (3)$$

(2) Second arcsine law: The last time the process is at its initial position (vertical blue line in Fig. 1):

$$t_{\text{last}} := \sup\{t \in [0,1], X_t = 0\}. \quad (4)$$

(3) Third arcsine law: The time at which the process X_t achieves its maximum (which is almost surely unique) (vertical green line in Fig. 1):

$$t_{\max} := t, \quad \text{s.t. } X_t = \sup\{X_s, s \in [0,1]\}. \quad (5)$$

While these laws are well studied for Brownian motion, little is known about their generalization to other random processes. In this article, we generalize the third arcsine law to fractional Brownian motion and obtain the distribution of the achieved maximum.

Fractional Brownian motion is a random process X_t characterized by the Hurst exponent H, which quantifies the growth of the two-point function in time:

$$\langle (X_t - X_s)^2 \rangle = 2|t-s|^{2H}. \quad (6)$$

Up to now, analytical tools to study its extreme-value statistics (EVS) were available only for Brownian motion, i.e., $H = 1/2$. In this article, we aim to extend this to $H \neq 1/2$. This is achieved by constructing a path integral and evaluating

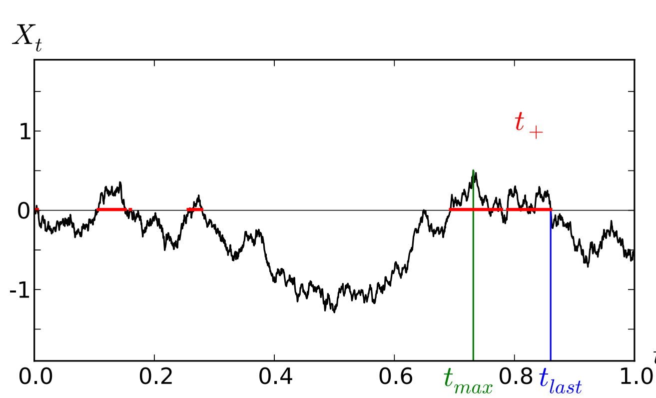

FIG. 1. The three arcsine laws discussed in the text. $t_{\max}$ (vertical green line) is the time at which the process achieves its maximum. t_{last} (vertical blue line) is the last time at which the process is at its starting value $X_0 = 0$. Finally, t_+(horizontal red lines) is the time spent in the positive half-space, which is the sum of the red intervals.

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it perturbatively around a Brownian, setting $H = 1/2 + \varepsilon$. This technique was introduced in Ref. [17]. We calculate the probability distribution of the maximum m of the process and the time $t_{\max}$ at which the maximum is reached, as well as their joint distribution. A short account of this work was published in Ref. [18].

The article is structured as follows: Section II defines the fBm, discusses its relation to anomalous diffusion, and defines the observables related to extremal value statistics we wish to study.

Section III introduces the path integral we need to calculate, followed by its perturbative expansion in $\varepsilon = H - 1/2$. This defines the main integrals to be calculated, for which we also give a diagrammatic representation. As the calculations are rather tedious, they are relegated to Appendix C.

Section IV presents our results: We start by recalling scaling relations in Sec. IV A, before introducing our most general formula, the probability of starting at $m_1 > 0$, of reaching the minimum $x_0 \approx 0$ at time t, and of finishing at time $T > t$ in m_2. This allows us to derive several simpler results: first, the distribution of times at which the maximum is achieved, for a Brownian known as the third arcsine law (Sec. IV C); second, the distribution of the value of this maximum; and third, the joint distribution of the maximum, and the time at which this maximum is taken.

Extensive numerical simulations for different values of H test these analytical predictions in Sec. V.

Conclusions are given in Sec. VI, followed by several appendices: Appendix A gives details on the perturbation expansion. Appendix B reviews results from [17], including a new derivation of the latter. Appendix C calculates the main new, and most difficult, contribution. Appendix D gives details on the corrections to the third arcsine law, while for the attained maximum and its cumulative distribution this is done in Appendixes E and F. Appendix G gives a list of inverse Laplace transforms used. Finally, in Appendix H it is verified that the second cumulant is correctly reproduced.

II. FRACTIONAL BROWNIAN MOTION AND OBSERVABLES

A. Definition of fBm

Fractional Brownian motion is a generalization of standard Brownian motion to other fractal dimensions, introduced in its final form by Mandelbrot and Van Ness [19]. It is a Gaussian process $(X_t)_{t\in\mathbb{R}}$, starting at 0, $X_0 = 0$, with mean $\langle X_t\rangle = 0$ and covariance function (variance)

$$\langle X_t X_s\rangle = s^{2H} + t^{2H} - |t-s|^{2H}. \tag{7}$$

A fBm Y_t starting at a nonzero value $y = Y_0$ is defined as $Y_t = X_t + y$, with X_t as above. The parameter $H \in (0,1)$ appearing in Eq. (7) is the Hurst exponent. Standard Brownian motion corresponds to $H = 1/2$; there the covariance function, (7), reduces to $\langle X_t X_s\rangle = 2\min(s,t)$. Unless $H = 1/2$, the process is non-Markovian, i.e., its increments are not independent. For $H > 1/2$ they are positively correlated, whereas for $H < 1/2$ they are anticorrelated:

$$\langle \partial_t X_t\, \partial_s X_s\rangle = 2H(2H-1)|t-s|^{2(H-1)}. \tag{8}$$

It is important to note that the process is stationary, as the second moment (and thus the whole distribution) of the increments is a function of the time difference $|t-s|$ only:

$$\langle (X_t - X_s)^2\rangle = 2|t-s|^{2H}. \tag{9}$$

The fact that a fBm process is non-Markovian makes its study difficult, as most of the standard stochastic-process tools (decomposing transition probabilities into products of propagators or writing the evolution of a density using a Fokker-Plank equation) rely on the Markov property.

B. Anomalous diffusion

Anomalous diffusion is another interesting property of fBm. It is characterized by nonlinear growth (for $H \neq 0.5$) of the second moment of the process,

$$\langle X_t^2\rangle = 2t^{2H}. \tag{10}$$

For $H < 1/2$, fBm is a subdiffusive process, while for $H > 1/2$, it is superdiffusive.

Anomalous diffusion is usually implied by a stronger property (but equivalent in the case of a Gaussian process): self-similarity to exponent H. This means that rescaling time by $\lambda > 0$ and space by λ^{-H} leaves every averaged observable $\langle O[X_t]\rangle$ defined on the process invariant,

$$\langle O[\lambda^{-H} X_{\lambda t}]\rangle = \langle O[X_t]\rangle. \tag{11}$$

This property is stronger in the sense that the growth of every moment, and not just the second one, is governed by the same exponent H: $\langle X_t^n\rangle \sim t^{nH}$.

It is well known that standard Brownian motion is the only *continuous* process with *stationary*, independent (*Markovian*), and *Gaussian* increments. As a consequence, every process in this class is $\frac{1}{2}$-self-similar, i.e., exhibits normal diffusion. To obtain an anomalous diffusive process, one of these three hypotheses has to be removed. This gives three main classes of anomalous diffusion:

(i) Heavy tails of the increments (Levy-flight process) or heavy tails in the waiting time between increments (CTRW processes); these processes are *non-Gaussian*.

(ii) Time dependence of the diffusive constant: the distribution of the increments is time dependent, i.e., the process has *nonstationary* increments.

(iii) Correlations between increments: the process is *non-Markovian*.

fBm is the only process which is Gaussian, has stationary increments, and *statistically self-similar*. As the first two hypotheses are natural in a large class of processes appearing in nature, and self-similarity to exponent $H \neq 1/2$ is equivalent to anomalous diffusion for a Gaussian process, fBm appears to be an important representative of anomalous diffusion.

Interestingly, several processes commonly used in physics, mathematics, and computer science belong to the fBm class. For example, it was recently proven that the dynamics of a tagged particle in single-file diffusion (cf. [8,20–22]) has, at long times, the fBm covariance function, (7), with Hurst exponent $H = 1/4$.

C. Extreme-value statistics

The objective of this article is to study fBm in the context of what is now called *extreme-value statistics*. While the knowledge of averages or of the typical behavior is an important step in understanding and comparing stochastic models to experiments or data, there are situations where the interest lies in the extremes or rare events. For example, the physics of disordered systems at low temperatures is governed by states with a (close to) minimal energy in the random energy landscape. Extreme weather conditions are of importance in the dimensioning of infrastructures such as dams and bridges. More generally, extreme-value questions appear naturally in many optimization problems.

The simplest and first case studied for these EVS was the distribution of the maximum of a large number N of independent and identically distributed random variables, which is now well understood in the large-N limit thanks to the classification of the Fisher-Tippett-Gnedenko theorem: Depending on the initial distribution of the variables, the rescaled maximum follows either a Weibull, a Gumbel, or a Fréchet distribution [16,24]. This is the equivalent of the central-limit theorem, which classifies the sums, or equivalently averages, of a large number of independent identically distributed (i.i.d.) variables.

The case of strongly correlated variables was a natural extension to this problem, as many physically relevant situations present significant deviations from the i.i.d. case. Many results were derived for random walks and Brownian motion [25,26]. The distribution of the largest eigenvalue is also a central question in random matrix theory [27]. Finally, some previous studies of the context of non-Markovian processes can be found in Refs. [28–30].

In this article we study the extremal properties of an fBm X_t. The main observables are the maximum $m = \max_{t\in[0,T]} X_t$ and the time $t_{\max}$ at which this maximum is reached. Figure 2 shows an illustration for different values of H, using the same random numbers for the Fourier modes. We denote

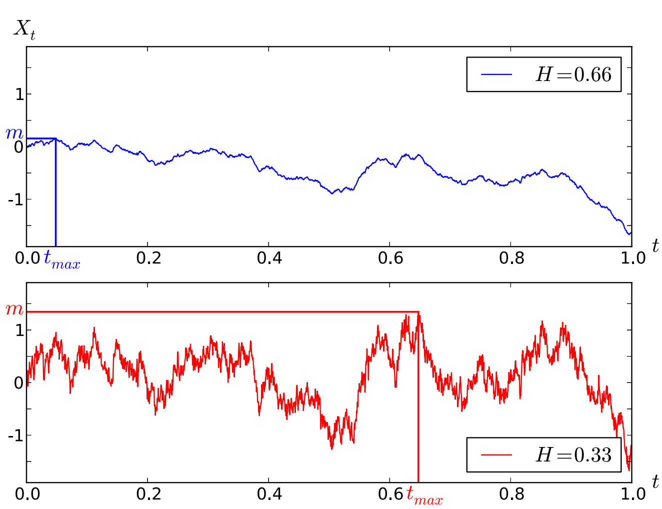

FIG. 2. Two realizations of fBm paths for different values of H, generated using the same random numbers for Fourier modes in the Davis and Harte procedure [23]. The observables m and $t_{\max}$ are shown.

their respective probability distributions $P_H^T(m)$ and $P_H^T(t)$. Previous studies of these distributions, focusing on the small-scale behavior, can be found in Refs. [31,32].

These observables are closely linked to other quantities of interest, such as the first-return time, the survival probability, the persistence exponent, and the statistics of records.

III. THE PERTURBATIVE APPROACH

A. Path-integral formulation and the action

Following the ideas in Refs. [17,33,34] we start with the path integral,

$$Z^+(m_1,t_1;x_0;m_2,t_2) = \int_{X_0=m_1}^{X_{t_1+t_2}=m_2} \mathcal{D}[X]\,\Theta[X]\,\delta\big(X_{t_1}-x_0\big)\,e^{-S[X]}. \quad (12)$$

It sums over all paths X_t, weighted by their probability $e^{-S[X]}$, starting at $X_0 = m_1 > 0$, passing through x_0 (close to 0) at time t_1, and ending in $X_{t_1+t_2} = m_2 > 0$, while staying positive for all $t \in [0,T = t_1 + t_2]$. The latter is enforced by the product of Heaviside functions $\Theta[X] := \prod_{s=0}^{t_1+t_2} \Theta(X_s)$. This path integral depends on the Hurst exponent H through the action. Since X_t is a Gaussian process, the action S can (at least formally) be constructed from the covariance function of X_t,

$$S[X] = \frac{1}{2}\int_{t_1,t_2} X_{t_1} G(t_1,t_2) X_{t_2}. \quad (13)$$

Here $\langle X_{t_1} X_{t_2}\rangle = G^{-1}(t_1,t_2)$. This, however, is not enough to evaluate the path integral, (12), since it is not evident how to implement the product of Θ functions. Following the formalism in Ref. [17], we use standard Brownian motion as a starting point for a perturbative expansion, setting $H = \frac{1}{2} + \varepsilon$, with ε a small parameter; then the action at first order in ε is (we refer to the Appendix in Ref. [17] for the derivation)

$$S[X] = \frac{1}{4D_{\varepsilon,\tau}}\int_0^T \dot{X}_{\tau_1}^2\, d\tau_1 - \frac{\varepsilon}{2}\int_0^{T-\tau} d\tau_1 \int_{\tau_1+\tau}^{T} d\tau_2\, \frac{\dot{X}_{\tau_1}\dot{X}_{\tau_2}}{|\tau_2-\tau_1|} + O(\varepsilon^2). \quad (14)$$

The time τ is a regularization cutoff for coinciding times (a UV cutoff). We will see that it has no impact on the distribution of observables which can be extracted from the path integral. (One can also introduce discrete times spaced by τ [17].)

The first line of Eq. (14), which we denote $S_0[X]$, is the action for standard Brownian motion, with a rescaled diffusion constant

$$D_{\varepsilon,\tau} = 1 + 2\varepsilon[1+\ln(\tau)] + O(\varepsilon^2) \simeq (e\tau)^{2\varepsilon}. \quad (15)$$

It is a dimensionful constant, as fBm and standard Brownian motion do not have the same time dimension. The second line, which we denote $S_1[X]$, is the first correction to the action. It is nonlocal in time, which implies that the process is non-Markovian [even if we neglect $O(\varepsilon^2)$ terms]. We check this expansion of the action in Appendix H, where we compute the covariance of the process from a path integral and recover Eq. (7) at first order in ε.

As we see in Sec. IV, this path integral $Z^+(m_1,t_1;x_0;m_2,t_2)$, in the limit of $x_0 \to 0$, encodes a plethora of information about the maximum of the process: both distributions, $P_H^T(m)$ and $P_H^T(t)$, can be extracted from it, as well as the joint distribution. Further, the distributions are the same as in the case of an fBm bridge.

It is important to note that the limit of $x_0 \to 0$ is nontrivial, as it forces the process to go close to an absorbing boundary which leads to nontrivial scaling involving the persistence exponent θ defined in Sec. IV A, below.

B. The order-0 term

Having expressed the perturbative expansion of the action, the main task is to compute the path integral, (12), at first order in ε and in the limit of small x_0. Expanding the exponential of the action in Eq. (12),

$$e^{-S[X]} = e^{-S_0[X]}(1 - S_1[X] + \cdots), \tag{16}$$

allows us to compute the path integral perturbatively in the nonlocal interaction $S_1[X]$, defined as the second line of Eq. (14):

$$S_1[X] = -\frac{\varepsilon}{2}\int_0^{T-\tau} \mathrm{d}\tau_1 \int_{\tau_1+\tau}^{T} \mathrm{d}\tau_2\, \frac{\dot X_{\tau_1}\dot X_{\tau_2}}{|\tau_2-\tau_1|}. \tag{17}$$

This gives

$$\begin{aligned} &Z^+(m_1,t_1;x_0;m_2,t_2) \\ &\quad= Z_0^+(m_1,t_1;x_0;m_2,t_2) + \varepsilon Z_1^+(m_1,t_1;x_0;m_2,t_2) + O(\varepsilon^2). \end{aligned} \tag{18}$$

Z_0^+ is the term with no nonlocal interaction, while εZ_1^+ is the term with one interaction (it is proportional to ε because the nonlocal interaction itself has an amplitude of order ε). Formally, the order-0 term is

$$\begin{aligned} &Z_0^+(m_1,t_1;x_0;m_2,t_2) \\ &\quad= \int_{X_0=m_1}^{X_{t_1+t_2}=m_2} \mathcal{D}[X]\,\Theta[X]\,\delta\big(X_{t_1}-x_0\big)\,e^{-S_0[X]}, \end{aligned} \tag{19}$$

where S_0 is the action of a standard Brownian motion,

$$S_0[X] = \frac{1}{4D_{\varepsilon,\tau}}\int_0^t \dot X_{\tau_1}^2\,\mathrm{d}\tau_1. \tag{20}$$

Since Brownian motion is a Markov process, this action is local in time. It allows us to write the path integral as a product:

$$\begin{aligned} &Z_0^+(m_1,t_1;x_0;m_2,t_2) \\ &\quad= \int_{X_0=m_1}^{X_{t_1}=x_0} \mathcal{D}[X]\Theta[X]e^{-S_0[X]} \int_{X_{t_1}=x_0}^{X_T=m_2} \mathcal{D}[X]\Theta[X]e^{-S_0[X]} \\ &\quad= P_0^+(m_1,x_0,t_1)P_0^+(x_0,m_2,t_2). \end{aligned} \tag{21}$$

In the second line, the constraint $\delta(X_{t_1}-x_0)$ is enforced by the boundary conditions of the path integral. In the last line, we have expressed each path integral in terms of the propagator $P_0^+(x_1,x_2,t)$ of standard Brownian motion, constrained to the positive half-space. It is obtained via the method of images,

$$\begin{aligned} P_0^+(x_1,x_2,t) &= \frac{1}{\sqrt{4\pi Dt}}\big(e^{-\frac{(x_1-x_2)^2}{4Dt}} - e^{-\frac{(x_1+x_2)^2}{4Dt}}\big) \\ &\underset{x_1\to 0}{\simeq} x_1x_2\frac{e^{-\frac{x_2^2}{4Dt}}}{\sqrt{4\pi D^3t^3}}, \end{aligned} \tag{22}$$

for an arbitrary diffusive constant D. We now use that the diffusive constant is $D_{\varepsilon,\tau} = 1 + O(\varepsilon)$. This allows us to express the path integral, (12), at leading order in ε, and in the limit of small x_0, as

$$Z_0^+(m_1,t_1;x_0;m_2,t_2) \underset{x_0\to 0}{\simeq} x_0^2\frac{m_1m_2e^{-\frac{m_1^2}{4t_1}-\frac{m_2^2}{4t_2}}}{4\pi t_1^{3/2}t_2^{3/2}} + O(\varepsilon). \tag{23}$$

To include the order-ε term in the diffusive constant to get the full result for Z_+ at order ε, we use Eq. (15) expanded in ε:

$$\begin{aligned} Z_0^+ &\underset{x_0\to 0}{\simeq} x_0^2\frac{m_1m_2e^{-\frac{m_1^2}{4t_1}-\frac{m_2^2}{4t_2}}}{4\pi t_1^{3/2}t_2^{3/2}}\left\{1+\varepsilon[1+\ln(\tau)]\left(\frac{m_1^2}{2t_1}+\frac{m_2^2}{2t_2}-6\right)\right\} \\ &\quad + O(\varepsilon^2). \end{aligned} \tag{24}$$

It is interesting to note that the order-ε term appearing here can also be computed from the result, (23), as

$$2(1+\ln(\tau))\big(t_1\partial_{t_1} + t_2\partial_{t_2}\big)Z_0^+. \tag{25}$$

C. The first-order terms

To go beyond Brownian motion and include non-Markovian effects, i.e., interactions nonlocal in time, we need to compute the first-order correction in the expansion, (18), which is called Z_1^+ and reads

$$\begin{aligned} &Z_1^+(m_1,t_1;x_0;m_2,t_2) \\ &\quad= \frac{1}{2}\int_0^{T-\tau}\mathrm{d}\tau_1\int_{\tau_1+\tau}^{T}\mathrm{d}\tau_2\int_{X_0=m_1}^{X_T=m_2}\mathcal{D}[X]\frac{\dot X_{\tau_1}\dot X_{\tau_2}}{|\tau_2-\tau_1|} \\ &\qquad\times\delta\big(X_{t_1}-x_0\big)\,\Theta[X]\,e^{-S_0[X]}. \end{aligned} \tag{26}$$

As before, we denote $T = t_1 + t_2$. To compute Z_1^+, we decompose it into three terms, distinguished by their time ordering. Denote Z_α^+ the part where $\tau_1 < \tau_2 < t_1$, Z_β^+ the part where $t_1 < \tau_1 < \tau_2$, and Z_γ^+ the term where $\tau_1 < t_1 < \tau_2$. Then

$$\begin{aligned} Z_1^+(m_1,t_1;x_0;m_2,t_2) = {} & Z_\alpha^+(m_1,t_1;x_0;m_2,t_2) \\ &+ Z_\beta^+(m_1,t_1;x_0;m_2,t_2) \\ &+ Z_\gamma^+(m_1,t_1;x_0;m_2,t_2). \end{aligned} \tag{27}$$

In the first term, the *interaction* affects only the process in the time interval $[0,t_1]$, and there is no coupling with the process in the time interval $[t_1,t_1+t_2]$. This leads, as shown in Appendix A, to a factorized expression for Z_α^+:

$$Z_\alpha^+(m_1,t_1;x_0;m_2,t_2) = P_1^+(m_1,x_0,t_1)P_0^+(x_0,m_2,t_2). \tag{28}$$

Here $P_1^+(m,x_0,t)$ is the order-ε correction to the propagator of fBm in the half-space (i.e., constrained to remain positive). This object, which we need in the limit of $x_0 \to 0$, was studied and computed in Ref. [17]. The result is recalled in Appendix B and recalculated using more efficient technology developed here. The second term is similar to the first, swapping the two time intervals:

$$Z_\beta^+(m_1,t_1;x_0;m_2,t_2) = P_0^+(m_1,x_0,t_1)P_1^+(x_0,m_2,t_2). \tag{29}$$

The third term, Z_γ^+, is more complicated, as the interaction couples the two time intervals $[0,t_1]$ and $[t_1, T = t_1 + t_2]$. We can still take advantage of locality in time of the action S_0 to

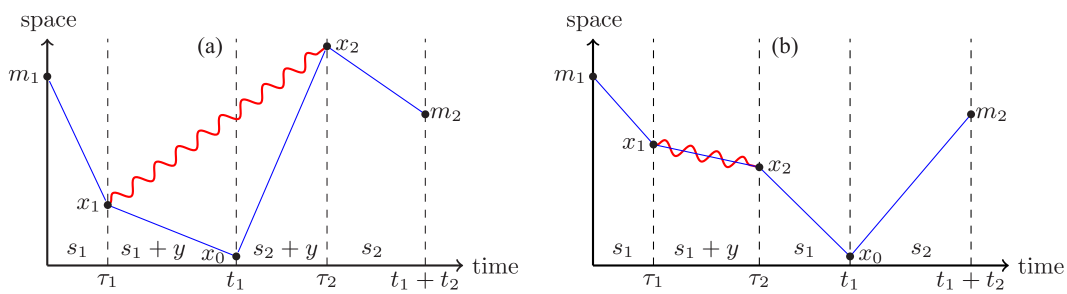

FIG. 3. (a) Graphical representation of the contribution Z_γ^+ to the path integral $Z^+(m_1,t_1;x_0;m_2,t_2)$ given in Eq. (12). The red curve represents the nonlocal interaction in the action [second line of Eq. (14)], while blue lines are bare propagators. We also indicate the Laplace variable which appears in each time slice in Eq. (32). (b) Graphical representation of Z_α^+.

write the path integral, (26), with time integrals restricted to $0 < \tau_1 < t_1 < \tau_2 < T$, as a product of simpler path integrals:

$$Z_\gamma^+(m_1,t_1;x_0;m_2,t_2) = \frac{1}{2}\int_0^{t_1} \mathrm{d}\tau_1 \int_{t_1}^{T} \frac{\mathrm{d}\tau_2}{\tau_2-\tau_1} \int_{x_1,x_2>0} \int_{X_0=m_1}^{X_{\tau_1}=x_1} \mathcal{D}[X]\Theta[X]e^{-S_0[X]} \int_{X_{\tau_1}=x_1}^{X_{t_1}=x_0} \mathcal{D}[X]\Theta[X]\dot{X}_{\tau_1} e^{-S_0[X]}$$
$$\times \int_{X_{t_1}=x_0}^{X_{\tau_2}=x_2} \mathcal{D}[X]\Theta[X]e^{-S_0[X]} \int_{X_{\tau_2}=x_2}^{X_T=m_2} \mathcal{D}[X]\Theta[X]\dot{X}_{\tau_2} e^{-S_0[X]}. \tag{30}$$

In this expression, all path integrals can be expressed in terms of the bare propagator P_0^+; we refer to Appendix A for how to deal with the terms containing $\dot{X}$. We have not written the cutoff τ, as there are no short-time divergences that need to be regularized (contrary to the terms Z_α^+ and Z_β^+). The structure of the time integrals, which are products of convolutions, suggests the use of Laplace transforms (with respect to the time variables: $t_1 \to s_1$, $t_2 \to s_2$). This and the identity

$$\frac{1}{\tau_2-\tau_1} = \int_{y>0} e^{-y(\tau_2-\tau_1)} \tag{31}$$

give us a simple form for the double Laplace transform of Z_γ^+, which we denote with a tilde (for details see Appendix A):

$$\tilde{Z}_\gamma^+(m_1,s_1;x_0;m_2,s_2) = 2\int_{x_1,x_2,y>0} \tilde{P}_0^+(m_1,x_1;s_1)\,\partial_{x_1}\tilde{P}_0^+(x_1,x_0;s_1+y)\tilde{P}_0^+(x_0,x_2;s_2+y)\,\partial_{x_2}\tilde{P}_0^+(x_2,m_2;s_2). \tag{32}$$

The Laplace-transformed constrained propagator appearing in this expression is

$$\begin{aligned}\tilde{P}_0^+(x_1,x_2;s) &= \int_0^\infty \mathrm{d}t\, e^{-st} P_0^+(x_1,x_2,t)\\ &= \frac{e^{-\sqrt{s}|x_1-x_2|} - e^{-\sqrt{s}(x_1+x_2)}}{2\sqrt{s}}\\ &\underset{x_1\to 0}{\simeq} x_1 e^{-\sqrt{s}x_2}.\end{aligned} \tag{33}$$

The Laplace transformation gives another simplification: the space dependence is now exponential, compared to the Gaussian form of $P_0^+(x_1,x_2,t)$, which renders the space integrations elementary. (Without the Laplace transform, already the first space integration gives an error function, and the remaining integrations are highly nontrivial.) Nevertheless, the final result for $Z_\gamma^+(m_1,t_1;x_0;m_2,t_2)$ is complicated and requires us to compute the three integrals in Eq. (32) and two inverse Laplace transformations. These steps are performed in Appendix C.

D. Graphical representation

It is useful to give a diagrammatic representation of the terms of the perturbative expansion; see Fig. 3. We denote bare propagators, (33), with solid blue lines. The interaction between two points, (τ_1,x_1) and (τ_2,x_2), is represented by the red curves. As can be seen from Eq. (32), it acts as $2\partial_{x_1}$ on the propagator starting at x_1 and $2\partial_{x_2}$ on the propagator starting at x_2; it also translates the Laplace variable of each time slice between these two points by $+y$. The space variables x_1 and x_2 and the *interaction* variable y (which has the inverse dimension of time) have to be integrated from 0 to ∞. In case of divergences, the integration has to be cut off with a large-y cutoff (cf. Appendix G for the link between the short-time cutoff τ and the large-y cutoff).

The contribution of Z_γ^+ is computed in detail in Appendix C and represented in Fig. 3(a), together with the contribution of Z_α^+, in Fig. 3(b).

IV. ANALYTICAL RESULTS

We present here some known scaling results about extremal properties of fBm. We then show how our perturbative expansion, and the computation of $Z^+(m_1,t_1;x_0;m_2,t_2)$, allows us to obtain analytical results on the distributions beyond these scaling arguments. Some of our results have been presented in a Letter [18].

A. Scaling results

Let us start with the the survival probability $S(T,x)$, and the persistence exponent θ, defined for any random process X_t

with $X_0 = x > 0$ as

$$S(T,x) := \text{prob}(X_t \geqslant 0 \text{ for all } t \in [0,T]) \underset{T\to\infty}{\sim} T^{-\theta_x + o(1)}. \tag{34}$$

For a review of these concepts in the context of theoretical physics, we refer to [35]. In a large class of processes the exponent θ is independent of x and characterizes the power-law decay for the probability of long positive excursions. For fBm with Hurst exponent H it was shown that $\theta_x = \theta = 1 - H$ [32,36]. To understand the link of $S(T,x)$ with the maximum distribution for fBm, we use the self-affinity of the process X_t to write $P_H^T(m)$ as

$$P_H^T(m) = \frac{1}{\sqrt{2}T^H} f_H\left(y = \frac{m}{\sqrt{2}T^H}\right). \tag{35}$$

Here f is a scaling function depending on H. The survival probability is related to the maximum distribution by

$$S(T,x) = \int_0^x P^T(m)\,dm = \int_0^{\frac{x}{\sqrt{2}T^H}} f_H(y)\,dy. \tag{36}$$

This states that due to translational invariance a realization of a fBm starting at x and remaining positive is the same as a realization starting at 0 and having a minimum larger than $-x$. Finally, the symmetry $x \to -x$ (for an fBm starting at $X_0 = 0$) gives the correspondence between minima and maxima. These considerations allow us to predict the scaling behavior of $P_H^T(m)$ at small m from the large-T behavior of $S(T,x)$ [32],

$$f(y) \underset{y\to 0}{\sim} y^\alpha \Leftrightarrow S(T) \sim T^{-(\alpha+1)H}, \tag{37}$$

and, finally,

$$P_H^T(m) \underset{m\to 0}{\sim} m^{\frac{\theta}{H}-1} = m^{\frac{1}{H}-2}. \tag{38}$$

For the distribution of the time at which the maximum is achieved we can estimate the behavior close to the origin by assuming that small values of the maximum are reached close to the origin. Starting with

$$P_H^T(m)\text{d}m = P_H^T(t)\text{d}t \tag{39}$$

and using scaling, $m \sim t^H$, we obtain

$$P_H^T(t) \sim P_H^T(m)\frac{\text{d}m}{\text{d}t} \sim (t^H)^{\frac{1}{H}-2} t^{H-1} \sim t^{-H}. \tag{40}$$

This should be valid when $t \to 0$ (or $m \to 0$). By time-reversal symmetry $t \to T - t$, we also have

$$P_H^T(t) \underset{t\to T}{\sim} (T-t)^{-H}. \tag{41}$$

B. The complete result for $Z^+(m_1,t_1;x_0;m_2,t_2)$

We present here the final result for Z^+, defined in Eq. (12), at order ε. This path integral was first expanded [cf. Eq. (18)] by treating the nonlocal term in the action, (14), perturbatively. The first term Z_0^+ of this expansion is given in Eq. (24), while the second term Z_1^+ was split into three contributions: Z_α^+, Z_β^+, and Z_γ^+ [see Eq. (27)]. The first two terms can be obtained explicitly from (B8), while the third one is computed in Appendix C, the result being split among (C13), (C29), and (C45).

In order to display a compact form, we choose $T \equiv t_1 + t_2 = 1$ (which is equivalent to rescaling m_1 and m_2 by T^{-H} and t_1 and t_2 by T^{-1}) and introduce new rescaled (dimensionless) variables:

$$y_1 = \frac{m_1}{\sqrt{2}t_1^H}, \quad y_2 = \frac{m_2}{\sqrt{2}t_2^H}, \tag{42}$$

$$t_1 = \vartheta, \quad t_2 = 1 - \vartheta. \tag{43}$$

With these new variables, the final result is

$$\begin{aligned} Z^+(m_1,t_1;x_0;m_2,t_2) \underset{x_0\to 0}{\simeq} x_0^{2-4\varepsilon} \frac{y_1 y_2 \exp\left(-\frac{1}{2}y_1^2 - \frac{1}{2}y_2^2\right)}{2\pi[\vartheta(1-\vartheta)]^{2H}} \Bigg\{ 1 + \varepsilon\Bigg[\mathcal{I}(y_1)\left(1 + \sqrt{\frac{1-\vartheta}{\vartheta}}\frac{y_2}{y_1}\right) + \mathcal{I}(y_2)\left(1 + \sqrt{\frac{\vartheta}{1-\vartheta}}\frac{y_1}{y_2}\right) \\ + \frac{(1-y_2^2)\mathcal{I}(\sqrt{1-\vartheta}\,y_1)}{\sqrt{\vartheta(1-\vartheta)}y_1 y_2} + \frac{(1-y_1^2)\mathcal{I}(\sqrt{\vartheta}\,y_2)}{\sqrt{\vartheta(1-\vartheta)}y_1 y_2} - \frac{\mathcal{I}(\sqrt{1-\vartheta}\,y_1 + \sqrt{\vartheta}\,y_2)}{\sqrt{\vartheta(1-\vartheta)}y_1 y_2} + 2\frac{(1-\vartheta)y_1^2 + \vartheta y_2^2 - 1}{\sqrt{\vartheta(1-\vartheta)}y_1 y_2} \\ + (y_1^2 - 2)\left(\ln(2y_1^2) + \gamma_\text{E}\right) + (y_2^2 - 2)\left(\ln(2y_2^2) + \gamma_\text{E}\right) - 4 - 2\gamma_\text{E} \Bigg]\Bigg\} + O(\varepsilon^2). \end{aligned} \tag{44}$$

The special function $\mathcal{I}$ appearing in this expression is

$$\mathcal{I}(z) = \frac{z^4}{6}\,{}_2F_2\left(1,1;\frac{5}{2},3;\frac{z^2}{2}\right) + \pi(1-z^2)\,\text{erfi}\left(\frac{z}{\sqrt{2}}\right) - 3z^2 + \sqrt{2\pi}e^{\frac{z^2}{2}}z + 2\,. \tag{45}$$

C. The third arcsine law: Distribution of the time when the maximum is reached

To simplify the result, (44), we can extract from it the distribution of a single observable. We start with the probability distribution $P_H(t)$ of t_max, the time when the fBm achieves its maximum. For Brownian motion ($H = 1/2$), this distribution is well known as the third arcsine law, because the cumulative distribution involves the arcsin function [cf. Eq. (1)],

$$P_{\frac{1}{2}}^T(t) = \frac{1}{\pi\sqrt{t(T-t)}} \quad \text{for } t \in [0,T]. \tag{46}$$

Until now, only scaling properties were known for this distribution in the general case [37], as recalled in Eq. (40).

The path integral, (12), in the limit of $x_0 \to 0$, selects paths which go through $x_0 \approx 0^+$ at time t_1 while staying positive. This means that we sum over paths reaching their minimum (in the interval $[0,t_1+t_2]$, which is almost surely unique) at t_1, starting at m_1, and ending at m_2. This is equivalent to summing over paths starting at 0, reaching their minimum with value

$-m_1$ at time t_1, and ending at $m_2 - m_1$. Integrating over m_1 and m_2 finally gives the sum over all paths reaching their minimum in t_1, independent of the value of this minimum, and the end point. Up to a normalization, this is the probability distribution of $t_{\min}$. By symmetry, this is the same as the distribution of $t_{\max}$. Formally, it reads

$$P_H^T(t) = \lim_{x_0\to 0} \frac{1}{Z^N} \int_{m_1,m_2>0} Z^+(m_1,t;x_0;m_2,T-t). \quad (47)$$

The normalization Z^N depends on x_0 and T. It ensures that $P_H^T(t)$ is normalized; it can be expressed in terms of Z^+ as

$$Z^N(x_0,T) = \int_0^T \mathrm{d}t \int_{m_1,m_2>0} Z^+(m_1,t;x_0;m_2,T-t). \quad (48)$$

At order 0, starting from Eq. (23) and integrating over m_1 and m_2 allows us to recover Eq. (46) with normalization $Z_N = x_0^2$.

For the order-ε correction, the integrations over m_1 and m_2 are lengthy. This is done in Appendix D. It allows us to write an ε expansion for the distribution of $t_{\max}$ in the form

$$P_H^T(t) = P_{\frac{1}{2}}^T(t) + \varepsilon\, \delta P^T(t) + O(\varepsilon^2). \quad (49)$$

The result, (D13), reads

$$\delta P^T(t) = \frac{1}{\pi\sqrt{t_1 t_2}}\left\{\sqrt{\frac{t_1}{t_2}}\left[\pi - 2\arctan\left(\sqrt{\frac{t_1}{t_2}}\right)\right] + \sqrt{\frac{t_2}{t_1}}\left[\pi - 2\arctan\left(\sqrt{\frac{t_2}{t_1}}\right)\right] - \ln(t_1 t_2) + \mathrm{cst}\right\}, \quad (50)$$

where $t_1 = t$ and $t_2 = T - t$. It takes a simple form if we exponentiate this order-ε correction:

$$P_H^T(t) = \frac{1}{\pi[t(T-t)]^H} \exp\left(\varepsilon\,\mathcal{F}\left(\frac{t}{T-t}\right)\right) + O(\varepsilon^2). \quad (51)$$

The term $\ln(t_1 t_2) = \ln(t(T-t))$ in $\delta P^T(t)$ gives the expected change, from Eqs. (40) and (41), in the scaling form of the arcsine law, $\sqrt{t(T-t)} \to [t(T-t)]^H$. The regular part induces a nontrivial change in the shape,

$$\mathcal{F}(u) = \sqrt{u}[\pi - 2\arctan(\sqrt{u})] + \frac{1}{\sqrt{u}}\left[\pi - 2\arctan\left(\frac{1}{\sqrt{u}}\right)\right] + \mathrm{cst}. \quad (52)$$

The time-reversal symmetry $t \to T - t$ (corresponding to $u \to u^{-1}$) is explicit and the constant ensures normalization. The contribution of $\mathcal{F}(u)$ to the probability that the maximum is attained at time t is quite noticeable, as shown in Fig. 4.

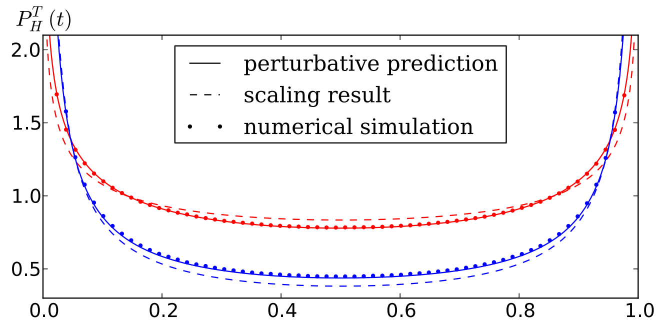

FIG. 4. Distribution of $t_{\max}$ for $T = 1$ and $H = 0.25$ (red curves) or $H = 0.75$ (blue curves) given in Eq. (51) (solid lines) compared to the scaling ansatz, i.e., $\mathcal{F} = $ cst. (dashed lines) and numerical simulations (dotted lines). For $H < 0.5$ realizations with $t_{\max} \approx T/2$ are less probable (by about 10%) than expected from scaling. For $H > 0.5$ the correction has the opposite sign.

D. The distribution of the maximum

We now present results for the distribution of the maximum $P_H^T(m)$. For standard Brownian motion

$$P_{\frac{1}{2}}^T(m) = \frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}, \quad m > 0. \quad (53)$$

On the other hand, the scaling results presented in Sec. IV A predict that for any H, $P_H^T(m)$ behaves at a small scale as $m^{1/H-2}$, as given in Eq. (38).

Using our path integral, we can go farther. Similarly to the distribution of $t_{\max}$, the distribution of the maximum m itself can be extracted from Z^+, defined in Eq. (12):

$$P_H^T(m) = \lim_{x_0\to 0} \frac{1}{Z^N} \int_0^T \mathrm{d}t \int_{m_2>0} Z^+(m,t;x_0;m_2,T-t). \quad (54)$$

The details of these computations (integrations over t and m_2) are given in Appendix E. Its ε expansion, recast in exponential form, leads to the scaling form of Eq. (35), with

$$f_H(y) = \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}}\, e^{\varepsilon[\mathcal{G}(y)+\mathrm{cst}]} + O(\varepsilon^2). \quad (55)$$

The constant term ensures normalization. Figure 5 shows the form of this scaling function for different values of H, as well as a *first comparison* to numerical simulations. The function $\mathcal{G}$ involves a combination of special functions denoted $\mathcal{I}$ in

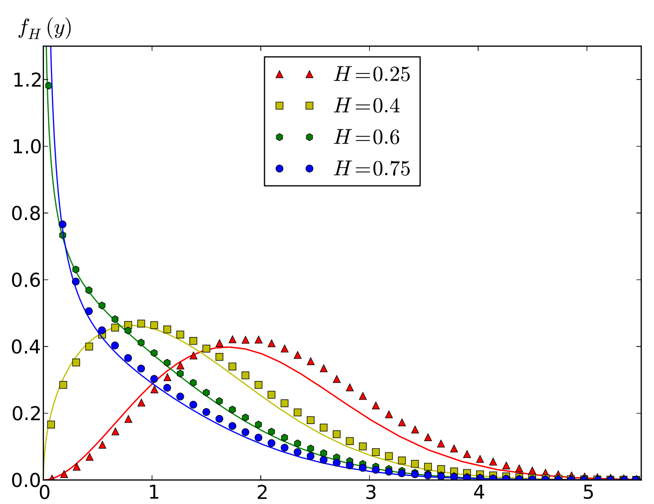

FIG. 5. Scaling function $f_H(y)$ for the distribution of the maximum, as defined in Eq. (35), for different values of H: $H = 0.25$ (red curve), $H = 0.4$ (yellow curve), $H = 0.6$ (green curve), and $H = 0.75$ (blue curve). Solid lines represent the analytic prediction from our perturbative theory (at first order in ε) given in Eq. (55); symbols are results from numerical simulations (cf. Sec. V).

Eq. (45) and logarithmic terms,

$$\mathcal{G}(y) = \mathcal{I}(y) + (y^2 - 2)[\gamma_{\rm E} + \ln(2y^2)]. \tag{56}$$

It has different asymptotics for small and large y:

$$\mathcal{G}(y) \sim \begin{cases} -2\ln(y) & \text{for } y \to \infty, \\ -4\ln(y) & \text{for } y \to 0. \end{cases} \tag{57}$$

The second line implies that $P_H^T(m) \sim m^{-4\varepsilon}$ when $m \to 0$, which is consistent (at order ε) with the scaling result, (38), $\frac{1}{H} - 2 = -4\varepsilon + O(\varepsilon^2)$. Formulas (55)–(57) also predict the distribution at large m. It is known that the leading behavior of $P_H^T(m)$ is Gaussian, which can be formalized as

$$\lim_{y\to\infty} \frac{\ln(f_H(y))}{y^2} = -\frac{1}{2}. \tag{58}$$

This is a direct consequence of an important theorem in the theory of Gaussian processes, the *Borrel inequality*. It states that for any Gaussian process X_t the cumulative distribution of its maximum value over the interval $[0,T]$, $m = \sup_{t\in[0,T]} X_t$, verifies

$$\text{Prob}(m > u) \leqslant \exp\left(-\frac{(u - \langle m\rangle)^2}{2\sigma^2}\right), \tag{59}$$

where $\langle m\rangle$ and $\sigma^2 = \sup_{t\in[0,T]}\langle X_t^2\rangle$ are assumed to be finite. Specifying this to fBm with $T = 1$ allows us to derive Eq. (58). A proof of this theorem and a derivation of its implications for fBm can be found in Ref. [38].

Our result, (55), goes farther and gives the subleading term in the large-m (and, equivalently, large-y) regime, a power law with exponent $-2\varepsilon + O(\varepsilon^2)$. It can be written as

$$\lim_{y\to\infty} \frac{\ln\left(f_H(y)\exp\left(\frac{y^2}{2}\right)\right)}{\ln(y)} = -2\varepsilon + O(\varepsilon^2). \tag{60}$$

Comparison of our full prediction (i.e., not only the asymptotics) with numerical simulations of the fBm are presented in Sec. V.

E. Survival probability

The survival probability $S(x,T)$ is defined as the probability of a process X_t's staying positive up to time t, while starting at $X_0 = x$:

$$S(x,t) := \text{prob}(X_t > 0, \forall t \in [0,T] \,|\, X_0 = x). \tag{61}$$

As before, the scaling properties of fBm allow us to write this as a function of $y = \frac{x}{\sqrt{2}T^H}$. As mentioned, the survival probability is the cumulative distribution of the maximum value and reads

$$S(y) = \int_0^y \mathrm{d}u\, f_H(u), \tag{62}$$

with f_H defined in Eq. (35). Similarly to the other distributions, we can compute its ε expansion and recast it into an exponential form to get

$$S(y) = \text{erf}\left(\frac{y}{\sqrt{2}}\right)\exp\left(\varepsilon\frac{\mathcal{M}(y)}{\text{erf}\left(\frac{y}{\sqrt{2}}\right)}\right) + O(\varepsilon^2). \tag{63}$$

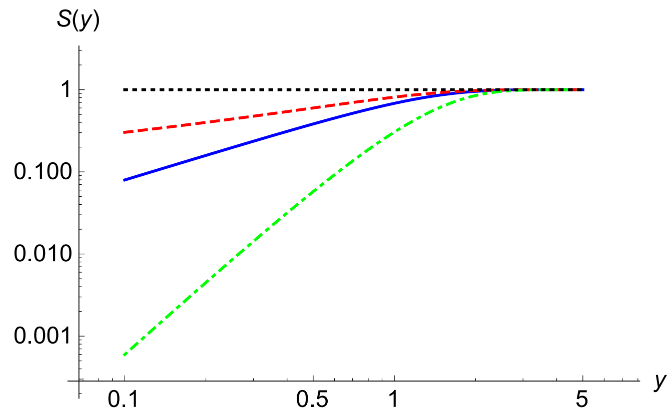

FIG. 6. Survival probability $S(y)$ for $H = 1/2$ (solid blue line), $H = 0.75$ (dashed red line), $H = 0.25$ (dot-dashed green line), and asymptotics $S(y) = 1$ (dotted black line), in a log-log plot.

The function $\mathcal{M}(y)$ is

$$\begin{aligned}\mathcal{M}(y) = {}& \sqrt{\frac{8}{\pi}}\, y\, {}_2F_2\left(\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};-\frac{y^2}{2}\right) \\ & - \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} y^3\, {}_2F_2\left(1,1;\frac{3}{2},2;\frac{y^2}{2}\right) \\ & + \sqrt{2\pi}\, e^{-\frac{y^2}{2}}\, y\, \text{erfi}\left(\frac{y}{\sqrt{2}}\right) \\ & - \left[\text{erf}\left(\frac{y}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} y\right][\ln(2y^2) + \gamma_{\rm E}]. \end{aligned} \tag{64}$$

Some details of its derivation are given in Appendix F and this result is plotted on Fig. 6.

F. The joint distribution for $t_{\max}$ and m

The result, (44), was obtained by considering paths starting at $X_0 = m_1 > 0$, with an absorbing boundary at $x = 0$ constraining the process to stay positive, as can be seen from the path-integral definition, (12). Using translational invariance and the symmetry $x \leftrightarrow -x$ of fBm, we can reinterpret this as the sum over paths starting at $X_0 = 0$, reaching their maximum (over the interval $[0, T = t_1 + t_2]$) of value m_1 at time t_1, and ending in $X_T = m_1 - m_2 < m_1$.

The integral over m_2 is then, in the limit $x_0 \to 0$ and up to a normalization factor Z^N, the joint probability density for a fBm's having a maximum value $m = m_1$ at a time $t = t_{\max} = t_1$ over the interval $[0,T]$; this we can write as

$$P_H^T(m,t) = \lim_{x_0\to 0}\frac{1}{Z^N}\int_0^\infty \mathrm{d}m_2\, Z^+(m,t;x_0;m_2,T-t). \tag{65}$$

We recall the result for Brownian motion that we recover for $\varepsilon = 0$:

$$P_{\frac{1}{2}}^T(m,t) = \frac{m e^{-\frac{m^2}{4t}}}{2\pi t^{3/2}\sqrt{T-t}}. \tag{66}$$

To simplify the ensuing discussion, we now consider the conditional probability

$$P_H^T(m|t) := \frac{P_H^T(m,t)}{\int_{m>0} P_H^T(t,m)} = \frac{P_H^T(m,t)}{P_H^T(t)}. \tag{67}$$

Interestingly, in the case of Brownian motion, we can make a change of variables $m \to y := m/\sqrt{2t}$ such that this conditional distribution function becomes independent of t (or, equivalently, independent of $\vartheta = t/T$):

$$P^T_{\frac{1}{2}}(m|t) = m\frac{e^{-\frac{m^2}{4t}}}{2t} = \frac{1}{\sqrt{2t}}ye^{-\frac{y^2}{2}} = \frac{\mathrm{d}y}{\mathrm{d}m}P_{\frac{1}{2}}(y|\vartheta), \quad (68)$$

with

$$P_{\frac{1}{2}}(y|\vartheta) = ye^{-\frac{y^2}{2}}. \quad (69)$$

For $H \neq \frac{1}{2}$, this independence is broken, and the result at order ε can be written as

$$P_H(y|\vartheta) = ye^{-\frac{y^2}{2}}\mathrm{e}^{\varepsilon\mathcal{G}(y|\vartheta)} + O(\varepsilon^2), \quad (70)$$

where now $y = \frac{m}{\sqrt{2}t^H}$ (to keep y a dimensionless variable). It is important to note that the variable y here is not the same as in Eq. (55), as the maximum m is rescaled by t (the time at which the maximum is reached), and not by T (the total time of the process).

The nontrivial correction $\mathcal{G}(y|\vartheta)$ is obtained from the result, (44), as

$$\mathcal{G}(y_1|\vartheta) = \int_{y_2>0} y_2e^{-\frac{y_2^2}{2}}[\dots], \quad (71)$$

where $[\dots]$ are the terms in brackets in Eq. (44).

While we can integrate Eq. (44) over y_1 and y_2 to obtain the probability that the maximum is attained at time t, we were *in general* not able to analytically integrate it solely over y_2, due to the presence of the term $\mathcal{I}(\sqrt{1-\vartheta}\,y_1 + \sqrt{\vartheta}\,y_2)$. Exceptions are the two limiting cases $\vartheta = 0$ and $\vartheta = 1$, for which

$$\mathcal{G}(y|0) = (y^2-2)[\gamma_{\mathrm{E}} + \ln(2y^2)] + \frac{(3-y^2)[\mathcal{I}(y)-2]}{1-y^2} + \frac{2\sqrt{2\pi}}{y}\left[1 - y^2 - \frac{e^{\frac{y^2}{2}}\mathrm{erfc}\left(\frac{y}{\sqrt{2}}\right)}{1-y^2}\right], \quad (72)$$

$$\mathcal{G}(y|1) = (y^2-2)[\gamma_{\mathrm{E}} + \ln(2y^2)] + \mathcal{I}(y) - 2. \quad (73)$$

Note that $P_H(y|1)$ is also the conditional probability that an fBm path, starting at $x_0 \ll 1$ and having survived up to time T, has the final position $m = \sqrt{2}yT^H$. This reproduces Eqs. (9) and (10) in Ref. [17]. These results are represented in Fig. 7.

The asymptotic behaviors for small y are

$$P_H(y|\vartheta) \sim y^{\frac{1}{H}-1} \simeq y^{1-4\varepsilon} + O(\varepsilon^2). \quad (74)$$

For large y, the situation is more complicated. For the two limiting cases the behavior is consistent with

$$P_H(y|0) \sim y^{1+2\varepsilon}\mathrm{e}^{-y^2/2-\sqrt{8\pi}y\varepsilon} + O(\varepsilon^2), \quad (75)$$

$$P_H(y|1) \sim y^{1-2\varepsilon}\mathrm{e}^{-y^2/2} + O(\varepsilon^2). \quad (76)$$

It would be interesting to understand this behavior from scaling arguments.

The conditional probability, (70), is plotted in Fig. 7 for various value of H, supplemented by results obtained via numerical integration of Eq. (71) for $\vartheta = 0.1$, 0.5, and 0.9. It varies smoothly as a function of ϑ.

V. NUMERICAL RESULTS

To validate the perturbative approach used in this article, we tested our analytical results with direct numerical simulations of fBm paths. The discretized fBm paths are generated using the Davis and Harte procedure as described in Ref. [23] (and references therein). The idea is to take advantage of the stationarity of the increments and use fast Fourier transformations to compute efficiently the square root of its covariance function. This method is exact, i.e., the samples generated have exactly the covariance function given in Eq. (7), and is adapted to situations where the length of the path to generate is fixed. Other simulation techniques exist, reviewed in Ref. [39].

A. The third arcsine law

For the distribution of $t_{\max}$, we want to test our analytical results given in Eqs. (51) and (52). Figure 4 shows the good agreement between theory and numerics. To perform a more precise comparison, we extract from the numerically computed

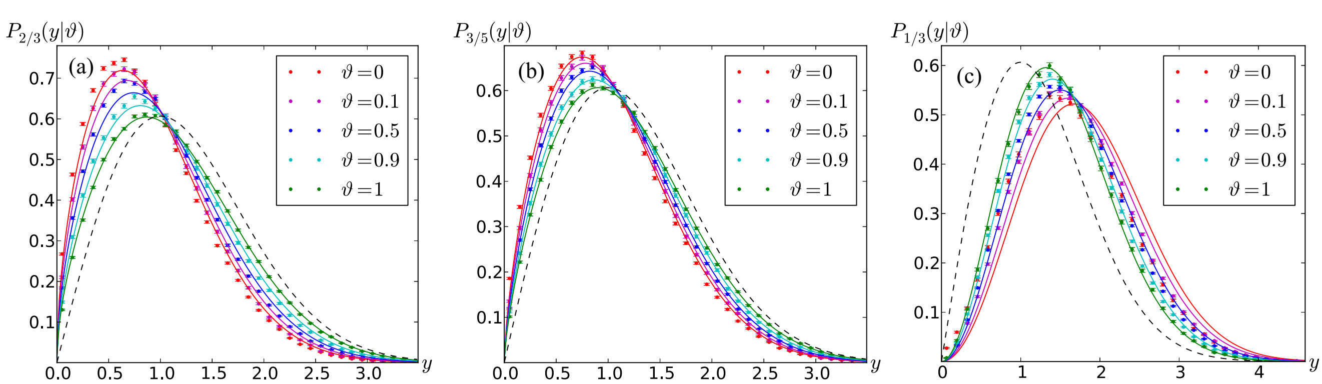

FIG. 7. (a) Conditional probability $P_H(y|\vartheta)$ for $H = \frac{2}{3}$ and various values of ϑ. (b), (c) Same as (a), for $H = \frac{3}{5}$ and $H = \frac{1}{3}$. Solid curves are the analytical prediction, (70), where the scaling functions are given analytically for the two extremal cases, $\vartheta = 0$ and $\vartheta = 1$ [Eqs. (72) and (73)]; for $0 < \vartheta < 1$ the curves are obtained via numerical integration. The predicted spread of the curves [which collapse for $H = \frac{1}{2}$ to Eq. (66), plotted with black circles) is well reproduced in the numerics, for both $\varepsilon > 0$ and $\varepsilon < 0$. For $\vartheta \to 1$ the agreement with numerics is remarkable, while for ϑ close to 0, we see significant deviations. These deviations may be due to both discretization effects and ε^2 corrections (they have the same sign for both $\varepsilon > 0$ and $\varepsilon < 0$).

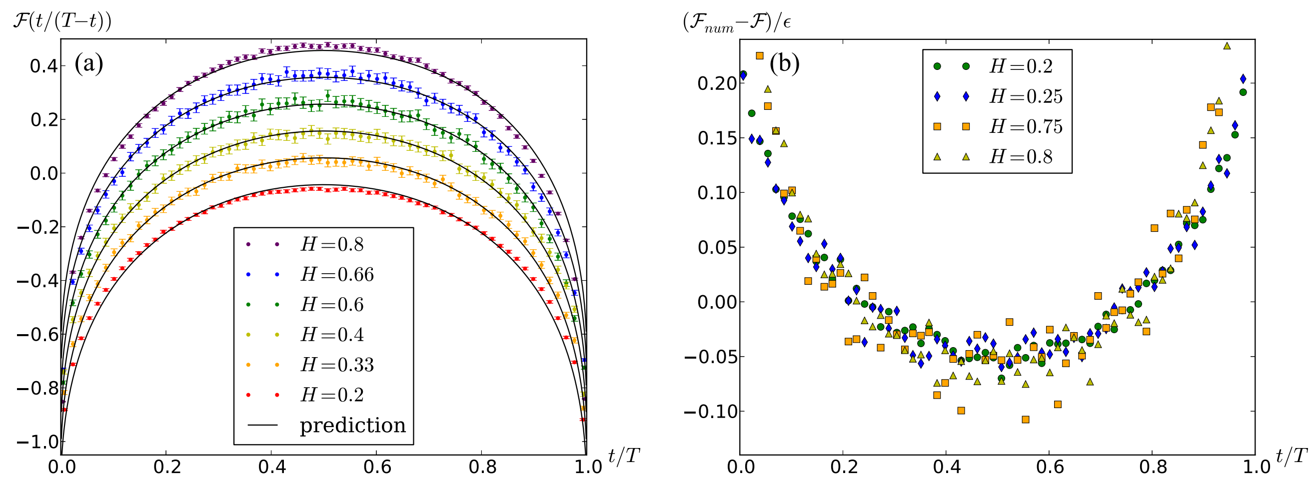

FIG. 8. (a) Numerical estimation of $\mathcal{F}$ for different values of H in a discrete system of size $N = 2^{12}$, using 10^8 realizations. Solid curves represent the theoretical prediction, (52), vertically translated for better visualization. Error bars are 2σ estimates. Note that for $H = 0.6$, $H = 0.66$, and $H = 0.8$ the expansion parameter ε is positive, while for $H = 0.4$, $H = 0.33$, and $H = 0.2$ it is negative. (b) Deviation for large $|\varepsilon|$ between the theoretical prediction, (52), and the numerical estimations, (77), rescaled by ε [cf. Eq. (78)]. These curves collapse for different values of H, allowing for an estimate of the $O(\varepsilon^2)$ correction to $P_H^T(t)$, as written in Eq. (79).

distribution $P^{T,H}_{\rm num}(t)$ an estimation $\mathcal{F}^{\varepsilon}_{\rm num}$ of the function $\mathcal{F}$ as

$$\mathcal{F}^{\varepsilon}_{\rm num}\left(\frac{t}{T-t}\right) := \frac{1}{\varepsilon}\ln\left(P^{T,H}_{\rm num}(t)\times[t(T-t)]^H\right). \qquad (77)$$

This function should converge, as $\varepsilon \to 0$, to the theoretical prediction, (52). Obviously, statistical errors become relevant in this limit due to the factor of ε^{-1}, while for larger ε we expect to see deviation due to order-ε^2 (and larger) corrections, which are not taken into account in our analytical computations. As shown in Fig. 8(a), our numerical and analytical results are in remarkable agreement for all values of H studied, both for ε positive and for ε negative. This means, in particular, that even for large values of ε ($H = 0.8$ or $H = 0.2$ in the cases studied here), the order-ε correction is large compared to higher-order corrections.

The precision of our simulations allows us to numerically investigate these subleading $O(\varepsilon^2)$ corrections, extracted as follows:

$$\begin{aligned}\mathcal{F}^{\varepsilon}_2(u) &= \frac{1}{\varepsilon}\left(\mathcal{F}^{\varepsilon}_{\rm num}(u)-\mathcal{F}(u)\right)\\ &= \frac{1}{\varepsilon^2}\ln\left(\frac{P^{T,H}_{\rm num}(t)\times[t(T-t)]^H}{e^{\varepsilon\mathcal{F}(u)}}\right).\end{aligned} \qquad (78)$$

This is shown in Fig. 8(b). The collapse of the curves for different values of ε (once rescaled by ε^{-1}) suggests that there exists a function $\mathcal{F}_2(u)$, which would be the limit of $\mathcal{F}^{\varepsilon}_2(u)$ as $\varepsilon \to 0$, such that the probability distribution can be written as

$$P^T_H(t) = \frac{e^{\varepsilon\mathcal{F}(u)+\varepsilon^2\mathcal{F}_2(u)}}{[t(T-t)]^H} + O(\varepsilon^3)\,. \qquad (79)$$

Our estimation of $\mathcal{F}_2$ is plotted in Fig. 8(b). Our perturbative approach and its diagrammatic representation allow us to write the integrals needed to compute $\mathcal{F}_2$ analytically; this, however, is left for future work [40].

B. The distribution of the maximum

For the distribution of the maximum we rewrite formula (55) such that the small-m behavior reproduces the exact scaling result, (38), without changing the result at ε order:

$$f_H(y) = \sqrt{\frac{2}{\pi}}\,y^{\frac{1}{H}-2}e^{-\frac{y^2}{2}}e^{\varepsilon[\mathcal{G}(y)+4\ln y+{\rm cst}]} + O(\varepsilon^2). \qquad (80)$$

To extract the nontrivial contribution from numerical simulations, we study, for $T = 1$ (see Fig. 9),

$$m^{2-\frac{1}{H}}e^{\frac{m^2}{4}}P^{1,H}_{\rm num}(m) = e^{\varepsilon[\mathcal{G}(\frac{m}{\sqrt{2}})+4\ln m+{\rm cst}]} + O(\varepsilon^2). \qquad (81)$$

The left-hand side is evaluated from the normalized binned distribution of the maximum for each fBm path, denoted $P^{1,H}_{\rm num}(m)$. The right-hand side is the analytical result; the constant term is evaluated by numerical integration such that $f_H(y)$, given in Eq. (80), is normalized to 1.

The sample size N (i.e., lattice spacing $dt = N^{-1}$) of the discretized fBm used for this numerical test is important, as the samples recover Brownian behavior for m smaller than a cutoff of order N^{-H}. This can be understood by assuming that the typical value of the first discretized point $X_{1/N}$ is of order N^{-H}; thus for $m \ll N^{-H}$,

$$P^{1,H}_{\rm num}(m) \sim {\rm prob}(X_{1/N} = m) \sim m^0. \qquad (82)$$

Far small H the system size necessary to obtain the asymptotic behavior at a small scale is very large, so we focus our tests on $H > 0.4$. Figures 9(a)–9(c) present results for $H = 0.4$, H=0.6, and $H = 0.75$, respectively, without any fitting parameter. As predicted, convergence to the small-scale behavior is quite slow. For example, in the $H = 0.6$ plot [Fig. 9(b)] the convergence to the small-scale behavior is somewhere between 10^{-1} and 10^{-2} (in dimensionless variables where we have rescaled the total time to $T = 1$). This might lead to an incorrect numerical estimation of the persistence exponent or other related quantities, if the crossover to the large-scale

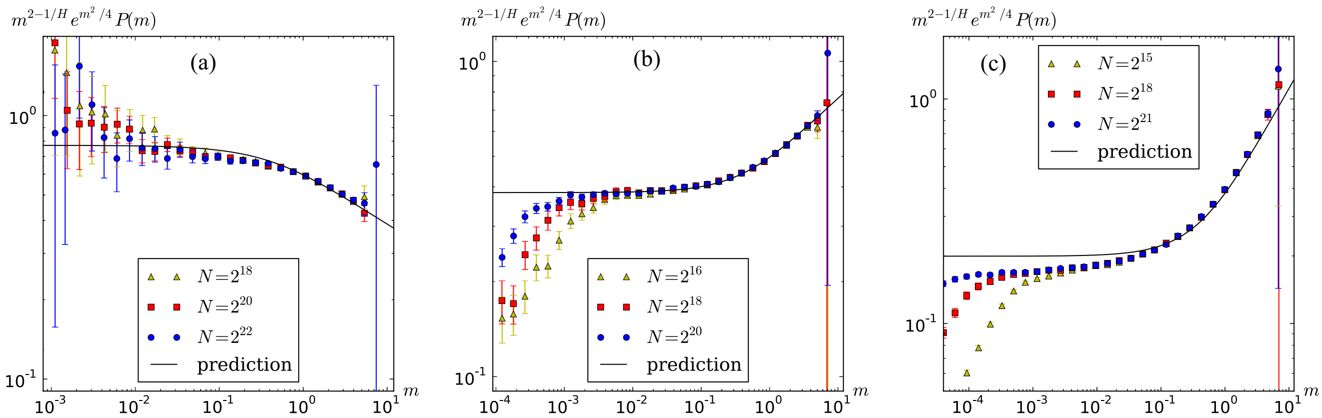

FIG. 9. (b) The combination, (81), for $H = 0.6$. The solid line is the analytical prediction $\exp(\varepsilon[\mathcal{G}(m/\sqrt{2}) + 4\ln m] + \text{cst})$ of the distribution of the maximum without its small-scale power law and large-scale Gaussian behavior. Symbols are numerical estimations for $T = 1$ of the same quantity, $m^{2-1/H}\exp(m^2/4)P_{\text{num}}^{T=1,H}(m)$, for different sample sizes. At small scales discretization errors appear. At large scales the statistics is poor due to the Gaussian prefactor. For the four decades in between, theory and numerics are in very good agreement. (a) Same as (b), for $H = 0.4$; (c) same as (b), for $H = 0.75$. In all cases, the large scale-behavior in both plots is consistent with $m^{2\varepsilon}$.

behavior is not properly taken into account. At large scales, the numerical data in Fig. 9 grow as $m^{2\varepsilon}$, consistent with the prediction, (60).

As stated, for $H < 0.5$ the numerical simulations do not allow us to investigate the small-scale behavior of the distribution, as can be seen for $H = 0.4$ in Fig. 9(a). Nevertheless, the agreement with the theoretical prediction is good in the crossover region and at the beginning of the tail. The numerical prefactor of the small-scale power law is also very sensitive to numerical errors (and probably to ε^2 corrections) due to a vanishing probability when $m \to 0$ for $H < 0.5$, as shown in Fig. 5.

VI. CONCLUSIONS

To conclude, we have developed a perturbative approach for the extreme-value statistics of fractional Brownian motion. This allows us to derive, to our knowledge, the first analytical results for generic values of H in the range $0 < H < 1$, beyond scaling relations. The main, and most general, result is the joint probability of the value of the maximum and the time when this maximum is reached, conditioned on the value of the end point, as given in Eq. (44). From this, we extracted a simpler result, as the unconditioned distribution of the value of the maximum, as well as the distribution of the time when this maximum is reached. These two distributions have nontrivial features, which we compared to numerical simulations. The remarkable agreement of the simulations with our predictions is a valuable check of our method. It also shows that the perturbative approach gives surprisingly good results, even far from the expansion point $H = \frac{1}{2}$.

The method can be generalized to other cases of interest, such as the other two arcsine laws, linear and nonlinear drift, and fractional Brownian bridges. Work in these directions is in progress.

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APPENDIX A: DETAILS ON THE PERTURBATIVE EXPANSION

We explicit here details on the steps transforming Eq. (30) into Eq. (32). We have to deal with terms of the form

$$\begin{aligned}
&\int_{X_0=x_1}^{X_t=x_2} \mathcal{D}[X]\Theta[X]\dot{X}_0 e^{-S_0[X]} \\
&= \lim_{\delta\to 0}\int_{X_0=x_1}^{X_t=x_2} \mathcal{D}[X]\Theta[X]\frac{X_\delta - x_1}{\delta}e^{-S_0[X]} \\
&= \lim_{\delta\to 0}\int_0^\infty \mathrm{d}x\,\frac{x-x_1}{\delta}P_0^+(x_1,x,\delta)P_0^+(x,x_2,t-\delta) \\
&= \lim_{\delta\to 0}\int_0^\infty \mathrm{d}x\; 2\partial_x P_0^+(x_1,x,\delta)P_0^+(x,x_2,t-\delta) \\
&= \int_0^\infty \mathrm{d}x\;\delta(x-x_1)2\partial_x P_0^+(x,x_2,t) \\
&= 2\partial_{x_1}P_0^+(x_1,x_2,t). \qquad \text{(A1)}
\end{aligned}$$

We first introduced a discretized version of the derivative, then expressed the path integral in terms of propagators, did an integration by parts, and, finally, took the limit of $\delta \to 0$.

With this result we can express every path integral in Eq. (30) in terms of the bare propagator $P_0^+(x_1,x_2,t)$:

$$Z_\gamma^+(m_1,t_1,x_0,t_2,m_2) = \frac{1}{2}\int_{t_1}^T d\tau_2\int_0^{t_1} d\tau_1\int_{x_1,x_2>0}\frac{P_0^+(m_1,x_1,\tau_1)\,2\partial_{x_1}P_0^+(x_1,x_0,t_1-\tau_1)\,P_0^+(x_0,x_2,\tau_2-t_1)\,2\partial_{x_2}P_0^+(x_2,m_2,T-\tau_2)}{\tau_2-\tau_1}. \qquad \text{(A2)}$$

We now use the identity $\frac{1}{\tau_2-\tau_1} = \int_{y>0} e^{-y(\tau_2-\tau_1)}$ and perform two Laplace transformations ($t_1 \to s_1$ and $t_2 \to s_2$). It is important to note that the time integrals are, in general, divergent at small times, thus we introduced a short-time cutoff τ in the action [cf. Eq. (14)]. The short-time cutoff τ corresponds to a large-y cutoff $\Lambda = e^{-\gamma_E}/\tau$. This value is imposed by the following equality, valid for all $T > 0$, in the limit of $\Lambda \to \infty$ and $\tau \to 0$:

$$\int_0^T \mathrm{d}t \int_0^\Lambda e^{-yt}\mathrm{d}y = \ln(T\Lambda) + \gamma_E + O(e^{-T\Lambda}) \overset{!}{=} \ln\left(\frac{T}{\tau}\right) = \int_\tau^T \frac{1}{t}\,\mathrm{d}t. \tag{A3}$$

To simplify the computations, we introduce new time variables:

$$T_1 = \tau_1, \quad T_2 = t_1 - \tau_1, \quad T_3 = \tau_2 - t_1, \quad T_4 = t_1 + t_2 - \tau_2. \tag{A4}$$

This gives

$$\begin{aligned}\tilde{Z}_\gamma^+(s_1,s_2) &= 2\int_{t_1,t_2>0} e^{-s_1t_1-s_2t_2}\int_{t_1}^{t_1+t_2}\mathrm{d}\tau_2\int_0^{t_1}\mathrm{d}\tau_1\int_0^\Lambda \mathrm{d}y\, e^{-y(\tau_2-\tau_1)}\, P_0^+(t_1)\,\partial P_0^+(\tau_1-t_1)\, P_0^+(\tau_2-t_1)\,\partial P_0^+(t_1+t_2-\tau_2)\\ &= 2\int_0^\Lambda \mathrm{d}y\int_{T_i>0} e^{-(T_1+T_2)s_1}e^{-(T_3+T_4)s_2}e^{-(T_2+T_3)y}\, P_0^+(T_1)\,\partial P_0^+(T_2)\, P_0^+(T_3)\,\partial P_0^+(T_4)\,.\end{aligned} \tag{A5}$$

The space dependence (i.e., x_0, x_1, x_2 dependence) is omitted for notational clarity. The successive integrations over time variables transform this expression into a product of Laplace-transformed propagators with different Laplace variables:

$$\tilde{Z}_\gamma^+(m_1,s_1;x_0;m_2,s_2) = 2\int_0^\Lambda \mathrm{d}y\int_{x_1,x_2>0}\tilde{P}_0^+(m_1,x_1,s_1)\,\partial_{x_1}\tilde{P}_0^+(x_1,x_0,s_1+y)\tilde{P}_0^+(x_0,x_2,s_2+y)\,\partial_{x_2}\tilde{P}_0^+(x_2,m_2,s_2). \tag{A6}$$

This is the formula given in in Eq. (32) in the text, except that here we made explicit the large-y cutoff. As we will see, there is no large-y divergence here, which renders the cutoff irrelevant. The other time orderings, corresponding to Z_α^+ and Z_β^+, have a similar structure. For Z_α, this gives

$$\begin{aligned}&Z_\alpha^+(m_1,t_1,x_0,t_2,m_2)\\ &= \frac{1}{2}\int_{\tau_1}^{t_1} d\tau_2\int_0^{t_1} d\tau_1\int_{x_1,x_2>0}\frac{P_0^+(m_1,x_1,\tau_1)\,2\partial_{x_1}P_0^+(x_1,x_2,\tau_2-\tau_1)\,2\partial_{x_2}P_0^+(x_2,x_0,t_1-\tau_2)\,P_0^+(x_0,m_2,t_2)}{\tau_2-\tau_1}.\end{aligned} \tag{A7}$$

This term is represented diagrammatically in Fig. 3(b); computing the double Laplace transform gives

$$\tilde{Z}_\alpha^+(m_1,s_1;x_0;m_2,s_2) = \left[2\int_0^\Lambda \mathrm{d}y\int_{x_1,x_2>0}\tilde{P}_0^+(m_1,x_1,s_1)\,\partial_{x_1}\tilde{P}_0^+(x_1,x_2,s_1+y)\,\partial_{x_2}\tilde{P}_0^+(x_2,x_0,s_1)\right]\tilde{P}_0^+(x_0,m_2,s_2). \tag{A8}$$

In this case, the integrations affect only the first three propagators. The term in brackets is the correction to the constrained propagator from m_1 to x_0, with Laplace variable s_1. This object was at the center of Ref. [17]; the results are recalled in the next Appendix. Similarly for Z_β, after the Laplace transformations, the integrations affect only the last three propagators, giving

$$\tilde{Z}_\beta^+(x_0,s_1;x_0;m_2,s_2) = \tilde{P}_0^+(m_1,x_0,s_1)\left[2\int_0^\Lambda \mathrm{d}y\int_{x_1,x_2>0}\tilde{P}_0^+(x_0,x_1,s_2)\,\partial_{x_1}\tilde{P}_0^+(x_1,x_2,s_2+y)\,\partial_{x_2}\tilde{P}_0^+(x_2,x_0,s_2)\right]. \tag{A9}$$

APPENDIX B: RECALL OF THE RESULTS FOR $Z_1^+(m,t)$

In Ref. [17], the propagator $Z^+(m,t)$ for fBm, conditioned to start at $x_0 \approx 0^+$, to remain positive, and to finish in m at time t was computed at order ε. For standard Brownian motion, this conditioned propagator is

$$Z_0^+(m,t) = \lim_{x_0\to 0}\frac{1}{x_0}P_0^+(x_0,m,t) = \frac{me^{-\frac{m^2}{4t}}}{2\sqrt{\pi}t^{3/2}}. \tag{B1}$$

The term x_0^{-1} is the normalization (i.e., one divides by the conditional probability). The order-ε correction of this propagator is given in Eq. (51) of [17]:

$$\begin{aligned}Z_1^+(m,t) &= Z_0^+(m,t)\left[\left(\frac{m^2}{2t}-2\right)(\ln(m^2)+\gamma_E)+\mathcal{I}\left(\frac{m}{\sqrt{2t}}\right)+\ln(t)-2\gamma_E\right]\\ &= Z_0^+(m,t)[\mathcal{I}(z)+z^2(\ln(2z^2)+\gamma_E)+(z^2-1)\ln(t)-4\ln(z)-4\gamma_E].\end{aligned} \tag{B2}$$

This result assumes a proper normalization of Z_1^+ such that x_0 and $\ln(x_0)$ terms cancel, i.e., the limit $x_0 \to 0$ is well defined, and the integral over m is equal to unity. We introduced $z := m/\sqrt{2t}$, and $\mathcal{I}$ is the combination of special functions defined in Eq. (45) and recalled in Eq. (G1).

We can also use the diagrammatic rules introduced in this article to compute the Laplace-transformed correction to this propagator (without conditioning). This corresponds to the diagram represented in Fig. 3(b) without the slice on the right:

$$\tilde{P}_1^+(x_0,m,s) = 2\int_0^\Lambda \mathrm{d}y \int_{x_1,x_2>0} \tilde{P}_0^+(x_0,x_1,s)\partial_{x_1}\tilde{P}_0^+(x_1,x_2,s+y)\partial_{x_2}\tilde{P}_0^+(x_2,m,s). \quad \text{(B3)}$$

This is the term appearing in brackets in Eqs. (A8) and (A9). The integrations over space can be done, giving the following integral, rescaling $y \to us$, and setting $m=1$ for simplicity:

$$\tilde{P}_1^+(x_0,1,s) = \frac{1}{\sqrt{s}}\int_0^{\Lambda/s}\frac{\mathrm{d}u}{u^2}\{[(\sqrt{s}-1)u-2]e^{-\sqrt{s}}\sinh(\sqrt{s}x_0) - x_0u\sqrt{s}\,e^{-\sqrt{s}}\cosh(\sqrt{s}x_0)$$
$$+\sqrt{u+1}[e^{-\sqrt{s}\sqrt{u+1}(1-x_0)} + e^{-\sqrt{s}\sqrt{u+1}(x_0+1)} - 2e^{-\sqrt{s}(\sqrt{u+1}+x_0)} - 2e^{-\sqrt{s}(x_0\sqrt{u+1}+1)} + 2e^{-\sqrt{s}(x_0+1)}]\}. \quad \text{(B4)}$$

This is a logarithmically diverging integral at large u, which makes the UV cutoff necessary (cf. Appendix A, where we detail the link between the y cutoff Λ and the time cutoff τ). Doing the integration over u, and then taking the limit $x_0 \to 0$ as well as expressing the cutoff Λ in terms of τ gives

$$\frac{1}{x_0}\tilde{P}_1^+(x_0,m,s) \underset{x_0\to 0}{\simeq} e^{m\sqrt{s}}(m\sqrt{s}+1)\mathrm{Ei}(-2m\sqrt{s}) - e^{-m\sqrt{s}}(m\sqrt{s}+1)\ln(m\sqrt{s})$$
$$+m\sqrt{s}e^{-m\sqrt{s}}\left[\ln\left(\frac{m^2}{2\tau}\right)-1\right] + e^{-m\sqrt{s}}\left[\ln\left(\frac{\tau^2}{2x_0^4}\right) - 3\gamma_{\mathrm{E}} + 4\right]. \quad \text{(B5)}$$

This expression in Laplace variables for the correction to the propagator is a new result [in Ref. [17] a more complicated transformation was used to derive Eq. (B2)]. The inverse Laplace transform can be done, using Eqs. (G8)–(G11) for the complicated terms,

$$\frac{P_1^+(x_0,m,t)}{P_0^+(x_0,m,t)} \underset{x_0\to 0}{\simeq} \mathcal{I}(z) + z^2[\ln(2z^2)+\gamma_{\mathrm{E}}] + (z^2-1)\left[\ln\left(\frac{t}{\tau}\right)-1\right] + \ln\left(\frac{\tau^2}{4x_0^4z^4}\right) - 4\gamma_{\mathrm{E}} + 2. \quad \text{(B6)}$$

We still need to correct this with the rescaling of the diffusion constant, i.e., taking into account the order-ε correction in Eq. (22) given the expression of the diffusive constant, (15). This gives

$$2t\,\partial_t P_0^+(x_0,m,t)(1+\ln(\tau)) = P_0^+(x_0,m,t)(z^2-3)[1+\ln(\tau)]. \quad \text{(B7)}$$

A check of consistency is that this cancels all dependence on τ, and we find, for the propagator at order ε,

$$P^+(x_0,m,t) \underset{x_0\to 0}{\simeq} P_0^+(x_0,m,t)\{1+\varepsilon[\mathcal{I}(z) + z^2(\ln(2z^2)+\gamma_{\mathrm{E}}) + (z^2-1)\ln(t) - \ln(4x_0^4z^4) - 4\gamma_{\mathrm{E}}]\} + O(\varepsilon^2). \quad \text{(B8)}$$

This propagator, integrated over m, reads, in both time and Laplace variables,

$$\int_0^\infty \mathrm{d}m\,\tilde{P}_1^+(x_0,m,s) \underset{x_0\to 0}{\simeq} \frac{x_0}{\sqrt{s}}\left(3-3\gamma_{\mathrm{E}} - \ln(4s\tau) + \ln\left(\frac{\tau^2}{x_0^4}\right)\right),$$
$$\int_0^\infty \mathrm{d}m\,P_1^+(x_0,m,t) \underset{x_0\to 0}{\simeq} \frac{x_0}{\sqrt{\pi t}}\left(3-2\gamma_{\mathrm{E}} + \ln\left(\frac{t\tau}{x_0^4}\right)\right). \quad \text{(B9)}$$

APPENDIX C: COMPUTATION OF Z_γ^+

1. Outline of the calculation

We present here details of the calculation of Z_γ^+, starting from its expression in Laplace variables, (32), graphically represented in Fig. 3. First, we introduce the notation

$$\mathcal{S}(m,x_0,s,y) := \frac{1}{x_0}\int_0^\infty \mathrm{d}x\,\tilde{P}_0^+(m,x,s)\,\partial_x\tilde{P}_0^+(x,x_0,s+y)$$
$$= \frac{1}{x_0}\frac{e^{-(m-x_0)\sqrt{s+y}} - e^{-(m+x_0)\sqrt{s+y}} + 2e^{-x_0\sqrt{s+y}-m\sqrt{s}} - e^{-(m-x_0)\sqrt{s}} - e^{-(m+x_0)\sqrt{s}}}{2y}. \quad \text{(C1)}$$

The expression of $\tilde{P}_0^+$ is given in Eq. (33). We see from Eq. (32) that one can write $\tilde{Z}_\gamma^+(m_1,s_1;x_0;m_2,s_2)$ as

$$\tilde{Z}_\gamma^+(m_1,s_1;x_0;m_2,s_2) = -2x_0^2\int_{y>0}\mathcal{S}(m_1,x_0,s_1,y)\mathcal{S}(m_2,x_0,s_2,y). \quad \text{(C2)}$$

The minus sign comes from an integration by parts. It is interesting to look at the asymptotics of $\mathcal{S}$ in the limit of $x_0 \to 0$:

$$\mathcal{S}(m,x_0,s,y) \underset{x_0\to 0}{\simeq} \frac{1}{y}(e^{-m\sqrt{s+y}}\sqrt{s+y} - e^{-m\sqrt{s}}\sqrt{s+y}) \underset{y\to\infty}{\sim} \frac{e^{-m\sqrt{s}}}{\sqrt{y}}. \tag{C3}$$

This implies that the $x_0 \to 0$ limit *cannot* be taken before integrating over y, as this induces a new large-y, i.e., short-time, divergence. Taking this limit before integration and regularizing the new divergence with the large-y cutoff Λ would lead to an incorrect result. This is expected, as the scaling of the result in terms of x_0 depends on H, thus inducing a $\ln(x_0)$ term at order ε.

In the following, we note $\mathcal{S} = \bar{\mathcal{S}} + \delta\mathcal{S}$ with

$$\bar{\mathcal{S}}(m,x_0,s,y) := \frac{1}{x_0}\frac{e^{-(m-x_0)\sqrt{s+y}} - e^{-(m+x_0)\sqrt{s+y}} + 2e^{-(x_0+m)\sqrt{s}} - e^{-(m-x_0)\sqrt{s}} - e^{-(m+x_0)\sqrt{s}}}{2y}, \tag{C4}$$

$$\delta\mathcal{S}(m,x_0,s,y) := \frac{1}{x_0}\frac{e^{-x_0\sqrt{s+y}-m\sqrt{s}} - e^{-x_0\sqrt{s}-m\sqrt{s}}}{y}. \tag{C5}$$

Denoting $\mathcal{S}_i := \mathcal{S}(m_i,x_0,s_i,y)$, the integration over y is a sum of four terms (with the last two related by exchanging point 1 and point 2):

$$\int_{y>0}\mathcal{S}_1\mathcal{S}_2 = \int_{y>0}\bar{\mathcal{S}}_1\bar{\mathcal{S}}_2 + \int_{y>0}\delta\mathcal{S}_1\delta\mathcal{S}_2 + \int_{y>0}\bar{\mathcal{S}}_1\delta\mathcal{S}_2 + \int_{y>0}\bar{\mathcal{S}}_2\delta\mathcal{S}. \tag{C6}$$

This leads to the following decomposition of $Z_\gamma^+(m_1,t_1;x_0;m_2,t_2)$:

$$Z_\gamma^+ = x_0^2[Z_A(m_1,t_1;m_2,t_2) + Z_B(m_1,t_1;x_0;m_2,t_2) + Z_C(m_1,t_1;m_2,t_2) + Z_C(m_2,t_2;m_1,t_1)], \tag{C7}$$

with

$$\begin{aligned}
Z_A(m_1,s_1;m_2,s_2) &= -2\mathcal{L}^{-1}_{s_2\to t_2}\circ\mathcal{L}^{-1}_{s_1\to t_1}\left[\lim_{x_0\to 0}\int_{y>0}\bar{\mathcal{S}}(m_1,x_0,s_1,y)\bar{\mathcal{S}}(m_2,x_0,s_2,y)\right],\\
Z_B(m_1,s_1;x_0;m_2,s_2) &= -2\mathcal{L}^{-1}_{s_2\to t_2}\circ\mathcal{L}^{-1}_{s_1\to t_1}\left[\lim_{x_0\to 0}\int_{y>0}\delta\mathcal{S}(m_1,x_0,s_1,y)\delta\mathcal{S}(m_2,x_0,s_2,y)\right],\\
Z_C(m_1,s_1;m_2,s_2) &= -2\mathcal{L}^{-1}_{s_2\to t_2}\circ\mathcal{L}^{-1}_{s_1\to t_1}\left[\lim_{x_0\to 0}\int_{y>0}\bar{\mathcal{S}}(m_1,x_0,s_1,y)\delta\mathcal{S}(m_2,x_0,s_2,y)\right].
\end{aligned} \tag{C8}$$

We anticipate here that Z_A and Z_C have a well-defined $x_0 \to 0$ limit, and only Z_B has a divergence (as shown later). The next step consists of computing these three integrals over y, taking the limit of small x_0, and performing the inverse Laplace transforms with regard to (w.r.t.) s_1 and s_2. The order of these manipulations can sometimes be inverted to simplify the calculations.

2. The term Z_A

In the first term in Eq. (C8) it is possible to take the $x_0 \to 0$ limit inside the integral, as this integrand converges rapidly enough for large y, given the asymptotic of $\bar{\mathcal{S}}$:

$$\bar{\mathcal{S}} \underset{x_0\to 0}{\simeq} \frac{e^{-m\sqrt{s+y}}\sqrt{s+y} - e^{-m\sqrt{s}}\sqrt{s}}{y}. \tag{C9}$$

This gives

$$\int_{y>0}\bar{\mathcal{S}}_1\bar{\mathcal{S}}_2 \underset{x_0\to 0}{\simeq} \int_{y>0}\frac{(e^{-m_1\sqrt{s_1+y}}\sqrt{s_1+y} - e^{-m_1\sqrt{s_1}}\sqrt{s_1})(e^{-m_2\sqrt{s_2+y}}\sqrt{s_2+y} - e^{-m_2\sqrt{s_2}}\sqrt{s_2})}{y^2}. \tag{C10}$$

We can do the inverse Laplace transformations $s_1 \to t_1$ and $s_2 \to t_2$ before integrating over y, using

$$\mathcal{L}^{-1}_{s\to t}[-e^{-m\sqrt{s+y}}\sqrt{s+y}] = \frac{e^{-\frac{m^2}{4t}}}{2\sqrt{\pi}t^{3/2}}\left(1 - \frac{m^2}{2t}\right)e^{-ty}. \tag{C11}$$

One thus finds

$$\mathcal{L}^{-1}_{s_2\to t_2}\circ\mathcal{L}^{-1}_{s_1\to t_1}\int_{y>0}\bar{\mathcal{S}}_1\bar{\mathcal{S}}_2 \underset{x_0\to 0}{\simeq} \frac{e^{-\frac{m_1^2}{4t_1}-\frac{m_2^2}{4t_2}}}{4\pi t_1^{3/2}t_2^{3/2}}\left(1 - \frac{m_1^2}{2t_1}\right)\left(1 - \frac{m_2^2}{2t_2}\right)\int_{y>0}\frac{(1-e^{-t_1 y})(1-e^{-t_2 y})}{y^2}. \tag{C12}$$

Integrating over y and using the definition of Z_A, the final result for this term is

$$Z_A(m_1,t_1;m_2,t_2) = \frac{e^{-\frac{m_1^2}{4t_1}-\frac{m_2^2}{4t_2}}}{2\pi(t_1t_2)^{3/2}}\left(1 - \frac{m_1^2}{2t_1}\right)\left(1 - \frac{m_2^2}{2t_2}\right)[t_1\ln(t_1) + t_2\ln(t_2) - (t_1+t_2)\ln(t_1+t_2)]. \tag{C13}$$

3. The term Z_B

For the second term in Eq. (C8), the limit $x_0 \to 0$ cannot be taken inside the integral, as

$$\delta\mathcal{S} = \frac{1}{x_0}\frac{e^{-x_0\sqrt{s+y}-m\sqrt{s}} - e^{-x_0\sqrt{s}-m\sqrt{s}}}{y} \underset{x_0\to 0}{\simeq} \frac{e^{-m\sqrt{s}}}{y}(\sqrt{s} - \sqrt{s+y}) \underset{y\to\infty}{\sim} -\frac{e^{-m\sqrt{s}}}{\sqrt{y}}. \tag{C14}$$

However, we can extract the diverging part by writing

$$\int_{y>0} \delta\mathcal{S}_1\delta\mathcal{S}_2 = e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}\ln\left(x_0^{-2}+1\right) + \int_{y>0}\left[\delta\mathcal{S}_1\delta\mathcal{S}_2 - \frac{e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}}{y+1}\Theta\left(y < x_0^{-2}\right)\right]. \tag{C15}$$

This expression is constructed such that for all $x_0 > 0$ the term added outside the integral and the term subtracted inside the integral cancel. The diverging part when $x_0 \to 0$ is now the term outside the integral and the integral has a finite limit when $x_0 \to 0$. To proceed, denote $\mathcal{K} := e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}$. We then decompose the integral as a sum of three terms:

$$\begin{aligned}\int_{y>0}\left[\delta\mathcal{S}_1\delta\mathcal{S}_2 - \frac{\mathcal{K}}{y+1}\Theta\left(y < x_0^{-2}\right)\right] &= \int_0^{x_0^{-2}} \mathrm{d}y\left[\delta\mathcal{S}_1\delta\mathcal{S}_2 - \mathcal{K}\frac{(\sqrt{s_1+y}-\sqrt{s_1})(\sqrt{s_2+y}-\sqrt{s_2})}{y^2}\right]\\ &\quad + \mathcal{K}\int_0^{x_0^{-2}}\mathrm{d}y\left[\frac{(\sqrt{s_1+y}-\sqrt{s_1})(\sqrt{s_2+y}-\sqrt{s_2})}{y^2} - \frac{1}{y+1}\right] + \int_{x_0^2}^{\infty}\mathrm{d}y\,\delta\mathcal{S}_1\delta\mathcal{S}_2.\end{aligned} \tag{C16}$$

In the second term we can take the limit of $x_0 \to 0$ to obtain (without the $\mathcal{K}$ factor in front)

$$\begin{aligned}&\int_{y>0}\left[\frac{(\sqrt{s_1+y}-\sqrt{s_1})(\sqrt{s_2+y}-\sqrt{s_2})}{y^2} - \frac{1}{y+1}\right]\\ &= -\left(2+\sqrt{\frac{s_1}{s_2}}+\sqrt{\frac{s_2}{s_1}}\right)\ln(\sqrt{s_1}+\sqrt{s_1}) + \frac{1}{2}\sqrt{\frac{s_1}{s_2}}\ln(s_1) + \frac{1}{2}\sqrt{\frac{s_2}{s_1}}\ln(s_2) - 1 + \ln(4).\end{aligned} \tag{C17}$$

For the first and third terms, we first perform a rescaling of the integration variable ($y \to x_0^{-2}v$) and then take the limit of $x_0 \to 0$:

$$\int_0^{x_0^{-2}}\mathrm{d}y\left[\delta\mathcal{S}_1\delta\mathcal{S}_2 - \mathcal{K}\frac{(\sqrt{s_1+y}-\sqrt{s_1})(\sqrt{s_2+y}-\sqrt{s_2})}{y^2}\right] \underset{x_0\to 0}{\simeq} \mathcal{K}\int_0^1 \mathrm{d}v\left[\frac{(e^{-\sqrt{v}}-1)^2}{v^2} - \frac{1}{v}\right], \tag{C18}$$

$$\int_{x_0^2}^{\infty}\mathrm{d}u\,\delta\mathcal{S}_1\delta\mathcal{S}_2 \underset{x_0\to 0}{\simeq} \mathcal{K}\int_1^{\infty}\mathrm{d}v\frac{(e^{-\sqrt{v}}-1)^2}{v^2}. \tag{C19}$$

The sum of the last two contributions in the limit of $x_0 \to 0$ is

$$\mathcal{K}\int_1^{\infty}\mathrm{d}v\frac{(e^{-\sqrt{v}}-1)^2}{v^2} + \mathcal{K}\int_0^1\mathrm{d}v\left[\frac{(e^{-\sqrt{v}}-1)^2}{v^2} - \frac{1}{v}\right] = \mathcal{K}[3 - 2\gamma_{\mathrm{E}} - 2\ln(4)]. \tag{C20}$$

Summing all these contributions gives

$$\int_{y>0}\delta\mathcal{S}_1\delta\mathcal{S}_2 \underset{x_0\to 0}{\simeq} e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}\left[-\left(2+\sqrt{\frac{s_1}{s_2}}+\sqrt{\frac{s_2}{s_1}}\right)\ln(\sqrt{s_1}+\sqrt{s_2}) + \sqrt{\frac{s_1}{s_2}}\ln(\sqrt{s_1}) + \sqrt{\frac{s_2}{s_1}}\ln(\sqrt{s_2}) - 2\ln(2x_0) + 2 - 2\gamma_{\mathrm{E}}\right]. \tag{C21}$$

We now need a series of inverse Laplace transforms obtained in Appendix G. To deal with the double Laplace inversion, we start with formula (G6) and use the special function $\mathcal{J}$ defined in Eq. (G2). Using commutativity of derivation and integration with the Laplace transform, we can use the identity

$$\left(2+\sqrt{\frac{s_1}{s_2}}+\sqrt{\frac{s_2}{s_1}}\right)e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}} = \left(\partial_{m_1}+\partial_{m_2}\right)\left(\int_{m_1}+\int_{m_2}\right)e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}} \tag{C22}$$

to obtain

$$\begin{aligned}&\mathcal{L}^{-1}_{s_2\to t_2}\circ\mathcal{L}^{-1}_{s_1\to t_1}\left[e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}\left(2+\sqrt{\frac{s_1}{s_2}}+\sqrt{\frac{s_2}{s_1}}\right)\ln(\sqrt{s_1}+\sqrt{s_2})\right]\\ &= \left(\partial_{m_1}+\partial_{m_2}\right)^2\left\{\frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{\pi\sqrt{t_1t_2}}\left[\mathcal{J}\left(\frac{(m_2t_1+m_1t_2)^2}{4t_1t_2(t_1+t_2)}\right) + \frac{1}{2}\ln\left(\frac{1}{4t_1}+\frac{1}{4t_2}\right) - \frac{\gamma_{\mathrm{E}}}{2}\right]\right\}.\end{aligned} \tag{C23}$$

For the other terms, the inverse Laplace transforms are decoupled and can be computed from Eq. (G7). We get

$$\mathcal{L}^{-1}_{s_2\to t_2}\circ\mathcal{L}^{-1}_{s_1\to t_1}\left[e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}\sqrt{\frac{s_1}{s_2}}\ln(\sqrt{s_1})\right]=\partial^2_{m_1}\left\{\frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{\pi\sqrt{t_1t_2}}\left[\mathcal{J}\left(\frac{m_1^2}{4t_1}\right)+\frac{1}{2}\ln\left(\frac{1}{4t_1}\right)-\frac{\gamma_{\rm E}}{2}\right]\right\}. \tag{C24}$$

The sum of all terms, with a prefactor of -2 coming from the definition of Z_B, is

$$\begin{aligned}Z_B(m_1,t_1;x_0;m_2,t_2)=&\frac{m_1m_2e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{2\pi(t_1t_2)^{3/2}}[2\ln(2x_0)-2+2\gamma_{\rm E}]\\&+2(\partial_{m_1}+\partial_{m_2})^2\left\{\frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{\pi\sqrt{t_1t_2}}\left[\mathcal{J}\left(\frac{(m_2t_1+m_1t_2)^2}{4t_1t_2(t_1+t_2)}\right)+\frac{1}{2}\ln\left(\frac{1}{4t_1}+\frac{1}{4t_2}\right)-\frac{\gamma_{\rm E}}{2}\right]\right\}\\&-2\,\partial^2_{m_1}\left\{\frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{\pi\sqrt{t_1t_2}}\left[\mathcal{J}\left(\frac{m_1^2}{4t_1}\right)+\frac{1}{2}\ln\left(\frac{1}{4t_1}\right)-\frac{\gamma_{\rm E}}{2}\right]\right\}+(1\leftrightarrow 2).\end{aligned} \tag{C25}$$

The derivatives can be computed explicitly, using the relation between $\mathcal{I}$ and $\mathcal{J}$ given in Eq. (G3):

$$\partial^2_{m_1}\left\{\frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{\pi\sqrt{t_1t_2}}\left[\mathcal{J}\left(\frac{m_1^2}{4t_1}\right)+\frac{1}{2}\ln\left(\frac{1}{4t_1}\right)-\frac{\gamma_{\rm E}}{2}\right]\right\}=-\frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{4\pi(t_1t_2)^{3/2}}t_2\left[\mathcal{I}\left(\frac{m_1}{\sqrt{2t_1}}\right)+\left(\frac{m_1^2}{2t_1}-1\right)(\ln(4t_1)+\gamma_{\rm E})\right]. \tag{C26}$$

The same result holds for the term involving $\partial^2_{m_2}$. For the term involving m_1 and m_2 simultaneously, we can use almost the same trick:

$$\begin{aligned}(\partial_{m_1}+\partial_{m_2})^2\left[e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}\mathcal{J}\left(\frac{(m_2t_1+m_1t_2)^2}{4t_1t_2(t_1+t_2)}\right)\right]&=\frac{t_1+t_2}{4t_1t_2}e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}\left[2(z^2-1)\mathcal{J}\left(\frac{z^2}{2}\right)-2(2z^2-1)\mathcal{J}'\left(\frac{z^2}{2}\right)+2z^2\mathcal{J}''\left(\frac{z^2}{2}\right)\right]\\&=-\frac{t_1+t_2}{4t_1t_2}e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}\mathcal{I}\left(\frac{m_1t_2+m_2t_1}{\sqrt{2t_1t_2(t_1+t_2)}}\right).\end{aligned} \tag{C27}$$

The second line is the explicit derivative of the first line, expressed, for simplicity, in terms of the variable

$$z=\frac{m_1t_2+m_2t_1}{\sqrt{2t_1t_2(t_1+t_2)}}. \tag{C28}$$

The combination of $\mathcal{J}$ and its derivatives appearing in the second line is exactly the function $\mathcal{I}$, as can be checked from Eq. (G3). After these simplifications,

$$\begin{aligned}Z_B\underset{x_0\to0}{\simeq}\frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{2\pi(t_1t_2)^{3/2}}&\left\{2m_1m_2[\ln(2x_0)+\gamma_{\rm E}-1]-(t_1+t_2)\left[\mathcal{I}(z)+(z^2-1)\left(\ln\left(\frac{4t_1t_2}{t_1+t_2}\right)+\gamma_{\rm E}\right)\right]\right.\\&\left.+t_2\left[\mathcal{I}\left(\frac{m_1}{\sqrt{2t_1}}\right)+\left(\frac{m_1^2}{2t_1}-1\right)(\ln(4t_1)+\gamma_{\rm E})\right]+t_1\left[\mathcal{I}\left(\frac{m_2}{\sqrt{2t_2}}\right)+\left(\frac{m_2^2}{2t_2}-1\right)(\ln(4t_2)+\gamma_{\rm E})\right]\right\}.\end{aligned} \tag{C29}$$

4. The term Z_C

For this term, we can take the limit $x_0\to0$ inside the integral, as it converges for large y using asymptotics (C9) and (C14), giving

$$\int_{y>0}\bar{\mathcal{S}}_1\delta\mathcal{S}_2\underset{x_0\to0}{\simeq}e^{-m_2\sqrt{s_2}}\int_{y>0}\frac{e^{-m_1\sqrt{s_1+y}}\sqrt{s_1+y}-e^{-m_1\sqrt{s_1}}\sqrt{s_1}}{y}\frac{\sqrt{s_2}-\sqrt{s_2+y}}{y}. \tag{C30}$$

To compute the Laplace inversion $s_1\to t_1$, we use Eq. (C11):

$$\begin{aligned}\mathcal{L}^{-1}_{s_1\to t_1}\left[\int_{y>0}\tilde{\mathcal{S}}_1\delta\mathcal{S}_2\right]&=\frac{e^{-\frac{m_1^2}{4t_1}}}{2\sqrt{\pi}t_1^{3/2}}\left(\frac{m_1^2}{2t_1}-1\right)e^{-m_2\sqrt{s_2}}\int_{y>0}\frac{(1-e^{-t_1y})(\sqrt{s_2+y}-\sqrt{s_2})}{y^2}\\&=\frac{e^{-\frac{m_1^2}{4t_1}}}{2\sqrt{\pi}t_1^{3/2}}\left(\frac{m_1^2}{2t_1}-1\right)\frac{e^{-m_2\sqrt{s_2}}}{\sqrt{s_2}}\int_{v>0}\frac{(1-e^{-t_1s_2v})(\sqrt{v+1}-1)}{v^2}.\end{aligned} \tag{C31}$$

We changed variables $y \to s_2 v$ between the two lines. To perform the inverse Laplace transform w.r.t. s_2, we need

$$\mathcal{L}^{-1}_{s_2\to t_2}\left[\frac{e^{-m_2\sqrt{s_2}}}{\sqrt{s_2}}e^{-t_1 s_2 v}\right] = \theta(t_2 - vt_1)\frac{e^{-\frac{m_2{}^2}{4(t_2-vt_1)}}}{\sqrt{\pi(t_2 - vt_1)}}. \tag{C32}$$

Finally, to compute Z_C, only the integration over v remains to be done:

$$\begin{aligned} Z_C(m_1,t_1;t_2,m_2) &= -\frac{e^{-\frac{m_1^2}{4t_1}}}{\sqrt{\pi}t_1^{3/2}}\left(\frac{m_1^2}{2t_1}-1\right)\int_{v>0}\left(\frac{e^{-\frac{m_2{}^2}{4t_2}}}{\sqrt{\pi t_2}} - \Theta(t_2 - vt_1)\frac{e^{-\frac{m_2{}^2}{4(t_2-vt_1)}}}{\sqrt{\pi(t_2 - vt_1)}}\right)\frac{\sqrt{v+1}-1}{v^2} \\ &= -\frac{e^{-\frac{m_1^2}{4t_1}}e^{-\frac{m_2^2}{4t_2}}}{2\pi(t_1t_2)^{3/2}}(m_1^2 - 2t_1)\frac{t_2}{t_1}\int_{v>0}\left(1 - \Theta\left(\frac{t_2}{t_1} - v\right)\frac{e^{-\frac{m_2^2}{4t_2}\left(\frac{1}{1-vt_1/t_2}-1\right)}}{\sqrt{1 - v\frac{t_1}{t_2}}}\right)\frac{\sqrt{v+1}-1}{v^2} \\ &= -\frac{e^{-\frac{m_1^2}{4t_1}}e^{-\frac{m_2^2}{4t_2}}}{2\pi(t_1t_2)^{3/2}}(m_1^2 - 2t_1)\left[\nu\int_0^\nu \mathrm{d}v\left(1 - \frac{e^{-a\left(\frac{1}{1-v/\nu}-1\right)}}{\sqrt{1 - v/\nu}}\right)\frac{\sqrt{v+1}-1}{v^2} + \nu\int_\nu^\infty \mathrm{d}v\,\frac{\sqrt{v+1}-1}{v^2}\right]. \end{aligned} \tag{C33}$$

Here we have introduced $\nu = t_2/t_1$ and $a = m_2^2/(4t_2)$. Thus the following integrals needs to be computed:

$$I_1(a,\nu) = \nu\int_0^\nu \mathrm{d}v\left(1 - \frac{e^{-a\left(\frac{1}{1-v/\nu}-1\right)}}{\sqrt{1 - v/\nu}}\right)\frac{\sqrt{v+1}-1}{v^2} \quad \text{and} \quad I_2(\nu) = \nu\int_\nu^\infty \mathrm{d}v\,\frac{\sqrt{v+1}-1}{v^2}. \tag{C34}$$

The term I_2 is easy:

$$I_2(\nu) = \nu\int_\nu^\infty \mathrm{d}v\,\frac{\sqrt{v+1}-1}{v^2} = \sqrt{\nu+1} - 1 + \nu\,\mathrm{asinh}\left(\frac{1}{\sqrt{\nu}}\right) = \sqrt{\frac{t_1+t_2}{t_1}} - 1 + \frac{t_2}{t_1}\mathrm{asinh}\left(\sqrt{\frac{t_1}{t_2}}\right). \tag{C35}$$

The other integral is more involved. To evaluate it, we perform a change of variables:

$$I_1(a,\nu) = \nu\int_0^\nu \mathrm{d}v\left(1 - \frac{e^{-a\left(\frac{1}{1-v/\nu}-1\right)}}{\sqrt{1-\frac{v}{\nu}}}\right)\frac{\sqrt{v+1}-1}{v^2} = \int_0^\infty \mathrm{d}x\left(\frac{1}{\sqrt{x+1}} - e^{-ax}\right)\frac{\sqrt{(\nu+1)x+1} - \sqrt{x+1}}{x^2}. \tag{C36}$$

To simplify the integrand, we then take its second derivative w.r.t. a:

$$\partial_a^2 I_1(a,\nu) = -\int_0^\infty \mathrm{d}x\, e^{-ax}\left(\sqrt{(\nu+1)x+1} - \sqrt{x+1}\right) = -\frac{\sqrt{\pi}\left(\sqrt{\nu+1}e^{\frac{a}{\nu+1}}\mathrm{erfc}\left(\sqrt{\frac{a}{\nu+1}}\right) - e^a\mathrm{erfc}(\sqrt{a})\right)}{2a^{3/2}}. \tag{C37}$$

The function

$$f(a) = \tfrac{1}{2}\mathcal{I}(\sqrt{2a}) + 3a - 1 + a\ln(a), \tag{C38}$$

where $\mathcal{I}$ is defined in Eq. (G1) and satisfies

$$f''(a) = -\frac{\sqrt{\pi}}{2}\frac{e^a}{a^{3/2}}\mathrm{erfc}(\sqrt{a}). \tag{C39}$$

We can then express the second derivative of I_1 in terms of f:

$$\partial_a^2 I_1(a,\nu) = \frac{1}{1+\nu}f''\left(\frac{a}{1+\nu}\right) - f''(a). \tag{C40}$$

After two integrations over a we obtain, with yet unknown functions $A(\nu)$ and $B(\nu)$,

$$I_1(a,\nu) = (\nu+1)\,f\left(\frac{a}{\nu+1}\right) - f(a) + B(\nu)a + A(\nu). \tag{C41}$$

The small-a behavior of f can be obtained as

$$f(a) = 2\sqrt{\pi}\sqrt{a} + a\ln(a) - \frac{2\sqrt{\pi}}{3}a^{3/2} + \frac{a^2}{3} + O(a^{5/2}). \tag{C42}$$

We can compare this to the limit when a goes to 0 of the initial integral to determinate the integration constants A and B. The limit is computed by taking the limit inside the integral, with the result

$$\lim_{a\to 0} I_1(a,\nu) = 1 - \sqrt{\nu+1} + \tfrac{1}{2}(\nu+1)\ln(\nu+1) - \vartheta\ln(\sqrt{\nu+1}+1). \tag{C43}$$

Finally, we get

$$I_1\left(\frac{m_2^2}{4t_2},\frac{t_2}{t_1}\right) = \left(1+\frac{t_2}{t_1}\right) f\left(\frac{{m_2}^2}{4t_2}\frac{t_1}{t_2+t_1}\right) - f\left(\frac{{m_2}^2}{4t_2}\right) + 1 - \sqrt{\frac{t_2+t_1}{t_1}} + \frac{t_2+t_1}{t_1}\ln\left(\sqrt{\frac{t_2+t_1}{t_1}}\right) - \frac{t_2}{t_1}\ln\left(\sqrt{\frac{t_2+t_1}{t_1}}+1\right). \tag{C44}$$

This has been checked numerically with excellent precision.

There are a few terms that cancel between I_1 and I_2, and expressing asinh in terms of ln and f in terms of $\mathcal{I}$, finally, gives

$$Z_C(m_1,t_1;m_2,t_2) = \frac{e^{-\frac{m_1^2}{4t_1}}e^{-\frac{m_2^2}{4t_2}}}{2\pi(t_1t_2)^{3/2}}\left(1-\frac{m_1^2}{2t_1}\right)\left[(t_1+t_2)\mathcal{I}\left(\frac{m_2}{\sqrt{2t_2}}\sqrt{\frac{t_1}{t_2+t_1}}\right) - t_1\mathcal{I}\left(\frac{m_2}{\sqrt{2t_2}}\right) - 2t_2 + t_1\left(\frac{m_2^2}{2t_2}-1\right)\ln\left(\frac{t_1}{t_2+t_1}\right) + t_2\ln\left(\frac{t_1+t_2}{t_2}\right)\right]. \tag{C45}$$

We computed numerically the double Laplace transform of (C45) and checked with a high precision the agreement with (C30), where the integral over y is evaluated numerically.

APPENDIX D: CORRECTION TO THE THIRD ARCSINE LAW

As stated in the text, the distribution of $t_{\max}$ can be extracted from our path integral, (12), as follows:

$$P_H^T(t) = \lim_{x_0\to 0}\frac{1}{Z^N(T,x_0)}\int_{m_1,m_2>0} Z^+(m_1,t;x_0;m_2,T-t). \tag{D1}$$

The order-0 contribution, (23), gives for the normalization

$$Z^N = \int_0^T \mathrm{d}t \int_{m_1,m_2>0} Z_0^+(m_1,t;x_0;m_2,T-t) + O(\varepsilon) = x_0^2 + O(\varepsilon). \tag{D2}$$

We recover the well-known arcsine law distribution for standard Brownian motion:

$$P_{\frac{1}{2}}^T(t) = \lim_{x_0\to 0}\frac{\int_{m_1,m_2>0} Z_0^+(m_1,t;x_0;m_2,T-t)}{x_0^2} = \int_{m_1,m_2>0}\frac{m_1m_2e^{-\frac{m_1^2}{4t_1}-\frac{m_2^2}{4t_2}}}{4\pi t_1^{3/2}t_2^{3/2}} = \frac{1}{\pi\sqrt{t(T-t)}}. \tag{D3}$$

Let us now write every term in the ε expansion: $Z^N = Z^N_{(0)} + \varepsilon Z^N_{(1)} + O(\varepsilon^2)$ and $Z^+ = Z^+_{(0)} + \varepsilon Z^+_{(1)} + O(\varepsilon^2)$. It is important to note that these terms slightly differ from those in Eq. (18), where the expansion was done w.r.t. the nonlocal perturbation in the action. As computed in Eq. (24), the term Z_0^+ contains some order-ε correction, contrary to $Z^+_{(0)}$, which is defined as the constant part of Z^+ in its ε expansion.

Using these new notations, we have

$$P_H^T(t) = \lim_{x_0\to 0}\frac{\int Z^+_{(0)}}{Z^N_{(0)}}\left[1+\varepsilon\left(\frac{\int Z^+_{(1)}}{\int Z^+_{(0)}} - \frac{Z^N_{(1)}}{Z^N_{(0)}}\right)\right] + O(\varepsilon^2) = P_{\frac{1}{2}}^T(t)\lim_{x_0\to 0}\left[1+\varepsilon\left(\frac{\int Z^+_{(1)}}{\int Z^+_{(0)}} - \frac{Z^N_{(1)}}{Z^N_{(0)}}\right)\right] + O(\varepsilon^2), \tag{D4}$$

where the $\int$ symbol implicitly denotes integration over m_1 and m_2. The normalization ensures that the correction to the probability

$$\delta P^T(t) = P_{\frac{1}{2}}^T(t)\lim_{x_0\to 0}\left(\frac{\int Z^+_{(1)}}{\int Z^+_{(0)}} - \frac{Z^N_{(1)}}{Z^N_{(0)}}\right) \tag{D5}$$

does not change the normalization, i.e., its integral over t vanishes.

To compute the order-ε correction to the distribution, (D3), we have to compute the integral over m_1 and m_2 of Z_α^+, as well as Z_β^+ and $Z_\gamma^+(m_1,t_1;x_0;m_2,t_2)$. The last term, computed in Appendix C, was decomposed in four terms [see Eq. (C7)]. The expressions for these terms are given in Eqs. (C13), (C29), and (C45). Using the identity $\int_{z>0} e^{-\frac{z^2}{2}}(z^2-1) = 0$, we find the simplifications

$$\int_{m_1,m_2>0} Z_A = \int_{m_1,m_2>0} Z_C = 0. \tag{D6}$$

Thus, the only contribution of Z_γ^+ comes from Z_B, defined in Eq. (C8):

$$\frac{1}{x_0^2}\int_{m_1,m_2>0} Z_\gamma^+ = \int_{m_1,m_2>0} Z_B(m_1,t_1;x_0;m_2,t_2) = -\frac{2}{\pi\sqrt{t_1t_2}}\left(1+\ln\left(\frac{4t_1t_2}{t_1+t_2}\right)-2\ln(2x_0)+2\gamma_{\mathrm{E}}\right)+\frac{1}{t_1}+\frac{1}{t_2}$$
$$-\frac{t_1+t_2}{2\pi(t_1t_2)^{3/2}}\int_{m_1,m_2>0} e^{-\frac{m_1^2}{4t_1}-\frac{m_2^2}{4t_2}}\,\mathcal{I}\left(z=\frac{m_1t_2+m_2t_1}{\sqrt{2t_1t_2(t_1+t_2)}}\right). \tag{D7}$$

We have used the identity $\int_0^\infty \mathrm{d}z e^{-z^2/2}\mathcal{I}(z)=\sqrt{2\pi}$. To compute the last integral, we use relation (G3), which in this case gives

$$\frac{t_1+t_2}{2\pi(t_1t_2)^{3/2}}\int_{m_1,m_2>0} e^{-\frac{m_1^2}{4t_1}-\frac{m_2^2}{4t_2}}\mathcal{I}(z) = -\frac{2}{\pi\sqrt{t_1t_2}}\int_{m_1,m_2>0}(\partial_{m_1}+\partial_{m_2})^2\left[e^{-\frac{m_1^2}{4t_1}-\frac{m_2^2}{4t_2}}\mathcal{J}\left(\frac{(m_1t_2+m_2t_1)^2}{4t_1t_2(t_1+t_2)}\right)\right]. \tag{D8}$$

Only the cross term of the derivatives (i.e., the term with $2\partial_{m_1}\partial_{m_2}$) is not a total derivative and gives a nonzero contribution:

$$\frac{2}{\pi\sqrt{t_1t_2}}\int_{m_2>0} e^{-\frac{m_2^2}{4t_2}}\,\partial_{m_1}\mathcal{J}\left(\frac{(m_2t_1+m_1t_2)^2}{4t_1t_2(t_1+t_2)}\right)\bigg|_{m_1=0} = \frac{2}{\pi t_1}\arctan\left(\sqrt{\frac{t_2}{t_1}}\right). \tag{D9}$$

The final result for this correction is

$$\frac{1}{x_0^2}\int_{m_1,m_2>0} Z_\gamma^+ = \frac{-2}{\pi\sqrt{t_1t_2}}\left[\ln\left(\frac{4t_1t_2}{t_1+t_2}\right)-2\ln(2x_0)+1+2\gamma_{\mathrm{E}}\right]+\frac{1}{t_1}+\frac{1}{t_2}-\frac{2}{\pi t_1}\arctan\left(\sqrt{\frac{t_2}{t_1}}\right)-\frac{2}{\pi t_2}\arctan\left(\sqrt{\frac{t_1}{t_2}}\right). \tag{D10}$$

The contributions to the correction from Z_α^+ and Z_β^+ are easily computed from their expressions in terms of propagators given in the text [cf. Eqs. (28) and (29)], and then using formula (B9),

$$\frac{1}{x_0^2}\int_{m_1,m_2>0} P_0^+(x_0,m_1,t_1)P_1^+(x_0,m_2,t_2)+(1\leftrightarrow 2) \underset{x_0\to 0}{\simeq} \frac{1}{\pi\sqrt{t_1t_2}}\left[6-4\gamma_{\mathrm{E}}+\ln(t_1t_2)+\ln\left(\frac{\tau^2}{x_0^8}\right)\right]. \tag{D11}$$

The last term of order ε comes from the rescaling of the diffusive constant, which was made explicit in Eq. (24):

$$2[1+\ln(\tau)](t_1\partial_{t_1}+t_2\partial_{t_2})\frac{1}{x_0^2}\int_{m_1,m_2>0} Z_0^+ = -2\frac{[1+\ln(\tau)]}{\pi\sqrt{t_1t_2}}. \tag{D12}$$

Summing all these contributions at order ε and taking into account the correction from normalization gives the final result for the order-ε term of the probability,

$$\delta P^T(t) = \frac{1}{\pi\sqrt{t_1t_2}}\left\{-\ln(t_1t_2)+\sqrt{\frac{t_1}{t_2}}\left[\pi-2\arctan\left(\sqrt{\frac{t_1}{t_2}}\right)\right]+\sqrt{\frac{t_2}{t_1}}\left[\pi-2\arctan\left(\sqrt{\frac{t_2}{t_1}}\right)\right]\right.$$
$$\left.+2\ln(T)+4-6\gamma_{\mathrm{E}}+\ln\left(\frac{\tau^2}{x_0^4}\right)-\frac{Z_{(1)}^N(T,x_0)}{x_0^2}-2[1+\ln(\tau)]\right\}, \tag{D13}$$

with $t_1=t$ and $t_2=T-t$. As expected, the dependence in τ vanishes at the end of the computation, and the order ε of the normalization factor $Z_{(1)}^N$ is fixed by the condition $\int_0^T \mathrm{d}t\,\delta P^T(t)=0$, which gives

$$Z_{(1)}^N = x_0^2[8\ln(2)+2-6\gamma_{\mathrm{E}}-4\ln(x_0)]. \tag{D14}$$

Equivalently, the constant term, i.e., the second line of Eq. (D13) becomes $-8\ln(2)$. The interpretation of this result as well as a comparison to numerical simulations are presented in the text.

APPENDIX E: DISTRIBUTION OF THE MAXIMUM OF THE fBm

Similarly to the distribution of $t_{\max}$, the distribution of m can be computed from the path integral $Z^+(m_1,t_1,x_0,m_2,t_2)$. This is done by taking the limit of small x_0, the integral over m_2, and the integral over t_1 at $t_1+t_2=T$ fixed:

$$P_H^T(m) = \lim_{x_0\to 0}\frac{1}{Z^N(T,x_0)}\int_0^{m_2}\mathrm{d}m_2 \times\int_0^T \mathrm{d}t\,Z^+(m,t,x_0,m_2,T-t). \tag{E1}$$

It is useful to note that the integration over $t=t_1$ at fixed $T=t_1+t_2$ can be replaced by taking the Laplace transform of Z^+ at equal arguments ($s_1=s_2=s$) and then performing the inverse Laplace transform $s\to T$. The normalization $Z^N(T,x_0)$ is the same as the one for the distribution of $P_H^T(t)$; expanding in ε thus gives the same structure as (D4), with the $\int$ symbol now being the integrals over $m_2>0$ and $t_1\in[0,T]$.

We start with the contribution of Z_γ. As before, the integral over m_2 of Z_A vanishes, so this term does not contribute. The correction from Z_B can be computed starting with Eq. (C21), taken at equal Laplace variables (i.e., $s_1 = s_2 = s$),

$$\int_{m_2}\int_t Z_B = 4\frac{e^{-m\sqrt{s}}}{\sqrt{s}}[\ln(x_0) - 1 + \gamma_E + 2\ln(2) + \ln(\sqrt{s})]. \tag{E2}$$

To take the inverse Laplace transform, we use Eq. (G8). This gives

$$\int_{m_2}\int_t Z_B = 4\frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left[\mathcal{J}\left(\frac{m^2}{4T}\right) + \ln\left(\frac{4x_0}{\sqrt{T}}\right) + \frac{\gamma_E}{2} - 1\right]. \tag{E3}$$

For the contribution of Z_C, it is easier to compute the inverse Laplace transform of Eq. (C30) ($s_1 = s_2 = s \to T$) before integrating over y. This gives

$$\int_{m_2}\int_t Z_C = -2\frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\int_0^\infty \frac{dy}{y^2}\left[e^{-\frac{m^2}{4T}y}(\sqrt{1+y} - 1 - y) + \sqrt{1+y} - 1\right]. \tag{E4}$$

Let us define

$$I_C(a) := \int_0^\infty \frac{du}{u^2}(e^{-au}(\sqrt{1+u} - 1 - u) + \sqrt{1+u} - 1). \tag{E5}$$

After deriving twice w.r.t. a, then integrating twice, and fixing the integration constants, we get

$$I_C(a) = \gamma_E + 1 + \ln(4) + a[3 - \gamma_E - \ln(4)] - \frac{a^2}{3}\,{}_2F_2\left(1,1;\frac{5}{2},3;a\right) + \frac{\pi}{2}(2a-1)\mathrm{erfi}(\sqrt{a}) - e^a\sqrt{\pi a} + (1-a)\ln(a). \tag{E6}$$

We can express this in terms of the special function $\mathcal{I}$:

$$I_C\left(\frac{z^2}{2}\right) = \gamma_E + 2 + \ln(4) - \frac{z^2}{2}[\gamma_E + \ln(4)] - \frac{1}{2}\mathcal{I}(z) + \left(1 - \frac{z^2}{2}\right)\ln\left(\frac{z^2}{2}\right). \tag{E7}$$

This has been checked numerically. The final result for this correction is (with $z := m/\sqrt{2T}$)

$$\int_{m_2}\int_t Z_C = \frac{e^{-\frac{z^2}{2}}}{\sqrt{\pi T}}[\mathcal{I}(z) + (z^2 - 2)(\gamma_E + \ln(2z^2)) - 4]. \tag{E8}$$

The last corrections are $x_0^{-2}\int_{m_2}\int_t Z_\alpha^+$ and $x_0^{-2}\int_{m_2}\int_t Z_\beta^+$. The first one is easy to compute using the results for the correction to the propagator recalled in Eq. (B9) and the inverse Laplace transform (G8):

$$\begin{aligned}\frac{1}{x_0^2}\int_0^T dt\int_0^\infty dm_2\, P_0^+(x_0,m,t)P_1^+(x_0,m_2,T-t) &\underset{x_0\to 0}{\simeq} \mathcal{L}^{-1}_{s\to T}\left[\frac{e^{-m\sqrt{s}}}{\sqrt{s}}\left(3 - \ln(4s\tau) - 3\gamma_E + \ln\left(\frac{\tau^2}{x_0^4}\right)\right)\right]\\ &\underset{x_0\to 0}{\simeq} \frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left[-2\mathcal{J}\left(\frac{m^2}{4T}\right) + \ln\left(\frac{T}{\tau}\right) + 2 - 2\gamma_E + \ln\left(\frac{\tau^2}{x_0^4}\right)\right].\end{aligned} \tag{E9}$$

For the correction from Z_β^+, we start with the Laplace expression of the correction to the propagator, (B5), where the integration over m_2 simplifies the last slice to $\frac{x_0}{\sqrt{s}}$. Then the needed inverse Laplace transform is

$$\begin{aligned}\frac{1}{x_0^2}\int_0^T dt\int_0^\infty dm_2\, P_1^+(x_0,m,t)P_0^+(x_0,m_2,T-t) &\underset{x_0\to 0}{\simeq} \frac{1}{x_0}\mathcal{L}^{-1}_{s\to T}\left[\frac{P_1^+(x_0,m,s)}{\sqrt{s}}\right]\\ &\underset{x_0\to 0}{\simeq} \frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left[-2\mathcal{J}\left(\frac{m^2}{4T}\right) + \frac{m^2}{2T}\ln\left(\frac{T}{\tau}\right) + 2 - 2\gamma_E + \ln\left(\frac{\tau^2}{x_0^4}\right)\right].\end{aligned} \tag{E10}$$

The final result for this is obtained using Eqs. (G8)–(G11).

We now give a summary of all corrections, in the limit of $x_0 \to 0$:

$$\begin{aligned}
&\frac{1}{x_0^2}\int_t\int_{m_2} P_1^+(x_0,m,t)P_0^+(x_0 m_2,T-t) \underset{x_0\to 0}{\simeq} \frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left[-2\mathcal{J}\left(\frac{m^2}{4T}\right)+\frac{m^2}{2T}\ln\left(\frac{T}{\tau}\right)+2-2\gamma_E+\ln\left(\frac{\tau^2}{x_0^4}\right)\right],\\
&\frac{1}{x_0^2}\int_t\int_{m_2} P_0^+(x_0,m,t)P_1^+(x_0,m_2,T-t) \underset{x_0\to 0}{\simeq} \frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left[-2\mathcal{J}\left(\frac{m^2}{4T}\right)+\ln\left(\frac{T}{\tau}\right)+2-2\gamma_E+\ln\left(\frac{\tau^2}{x_0^4}\right)\right],\\
&\int_t\int_{m_2} Z_C(m,t;m_2,T-t) \underset{x_0\to 0}{\simeq} \frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left[\mathcal{I}\left(\frac{m}{\sqrt{2T}}\right)+\left(\frac{m^2}{2T}-2\right)\left(\gamma_E+\ln\left(\frac{m^2}{T}\right)\right)-4\right],\\
&\int_t\int_{m_2} Z_B(m,t;m_2,T-t) \underset{x_0\to 0}{\simeq} \frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left[4\mathcal{J}\left(\frac{m^2}{4T}\right)+4\ln\left(\frac{4x_0}{\sqrt{T}}\right)+2\gamma_E-4\right],\\
&\frac{4(1+\ln(\tau))}{x_0^2}T\partial_T\int_t\int_{m_2} Z_0^+ \underset{x_0\to 0}{\simeq} \frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left[1+\ln(\tau)\left(\frac{m^2}{2T}-1\right)\right].
\end{aligned} \tag{E11}$$

The last line is the correction to the diffusion constant, i.e., the order-ε term appearing in Eq. (24). The final result at order ε is

$$\int_{m_2}\int_t Z^+ = \frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left\{1+\varepsilon\left[\mathcal{I}\left(\frac{m}{\sqrt{2T}}\right)+\left(\frac{m^2}{2T}-2\right)\left(\gamma_E+\ln\left(\frac{m^2}{T}\right)\right)+\left(\frac{m^2}{2T}-1\right)\ln(T)+\text{cst}\right]\right\}+O(\varepsilon^2). \tag{E12}$$

To better interpret the different terms, we recast the corrections, especially those as $\frac{m^2}{2T}\ln(T)$ and $\ln(T)$, into an exponential form:

$$\frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}\left[1+\varepsilon\left(\frac{m^2}{2T}-1\right)\ln(T)\right]+O(\varepsilon^2)=\frac{e^{-\frac{m^2}{4T}}}{\sqrt{\pi T}}e^{\varepsilon\frac{m^2}{2T}\ln(T)}T^{-\varepsilon}+O(\varepsilon^2)=\frac{e^{-\frac{m^2}{4T^{1+2\varepsilon}}}}{\sqrt{\pi}T^{1/2+\varepsilon}}+O(\varepsilon^2). \tag{E13}$$

This part of the correction gives the correct dimension to the variables in the order-0 result,

$$z=\frac{m}{\sqrt{2t}}\to y=\frac{m}{\sqrt{2}t^H}=\frac{m}{\sqrt{\langle x_t^2\rangle}}. \tag{E14}$$

The other parts of the correction, which are a function of $z=\frac{m}{\sqrt{2t}}$ and which we call $\mathcal{G}(z)$, give a nontrivial change to the scaling function of the distribution:

$$\begin{aligned}
P_H^T(m) &= \frac{e^{-\frac{m^2}{4T^{2H}}}}{\sqrt{\pi}T^H}e^{\varepsilon[\mathcal{G}(z=\frac{m}{\sqrt{2t}})+\text{cst}]}+O(\varepsilon^2)\\
&= \frac{e^{-\frac{y^2}{2}}}{\sqrt{\pi}T^H}e^{\varepsilon[\mathcal{G}(y)+\text{cst}]}+O(\varepsilon^2).
\end{aligned} \tag{E15}$$

We changed the variable in $\mathcal{G}$ from z to y, as it does not change the result at order ε and since it is more consistent in terms of dimensions. The function $\mathcal{G}$ is given by

$$\mathcal{G}(y)=\mathcal{I}(y)+(y^2-2)[\ln(2y^2)+\gamma_E]. \tag{E16}$$

The function $\mathcal{I}$ is regular at $y=0$, and its asymptotic behavior is given in Eq. (G5); this gives the asymptotics for $\mathcal{G}$ as

$$\mathcal{G}(y)\sim\begin{cases}-2\ln(y) & \text{for } y\to\infty,\\ -4\ln(y) & \text{for } y\to 0.\end{cases} \tag{E17}$$

Since these asymptotics are logarithmic, new power laws are obtained for the density distribution, at botht $m\to 0$ and $m\to\infty$, which multiply the Gaussian term, with

$$P^T_{\frac12+\varepsilon}(m)\times e^{\frac{m^2}{4T^{1+2\varepsilon}}}\sim\begin{cases}m^{-4\varepsilon} & \text{for } m\to 0,\\ m^{-2\varepsilon} & \text{for } m\to\infty.\end{cases} \tag{E18}$$

The constant term in Eq. (E12) is fixed by normalization. Instead of computing it at order ε, we can also evaluate it numerically such that (E15) is exactly normalized, and not only at order ε. This is appropriate for numerical checks and the procedure we adopted for the latter.

APPENDIX F: SURVIVAL DISTRIBUTION

To compute the survival probability up to time T of an fBm starting in m, we need to take the primitive function w.r.t. m of (E12). We can deal with the terms involving $\mathcal{I}$ using (G3); the difficult part comes from

$$\int_0^y \mathrm{d}m\, e^{-\frac{m^2}{2}}(2-m^2)\ln(m). \tag{F1}$$

To deal with this integration, we consider $e^{-\frac{m^2}{2}}m^a$, compute the primitive function w.r.t. m, and then take the derivative w.r.t. a, at $a=0$ and $a=2$.

The final result can be written as

$$S(y)=\text{erf}\left(\frac{y}{\sqrt{2}}\right)+\varepsilon\mathcal{M}(y)+O(\varepsilon^2). \tag{F2}$$

This is at leading order in ε equivalent to the exponentiated form given in the text, (63), with the function $\mathcal{M}$ given by Eq. (64).

APPENDIX G: SPECIAL FUNCTIONS AND SOME INVERSE LAPLACE TRANSFORMS

In our computations there are two combinations of special functions which appear frequently, and which we denote $\mathcal{I}$ and

$\mathcal{J}$. Their expressions in terms of hypergeometric functions and error functions are

$$\mathcal{I}(z) = \frac{z^4}{6}\,{}_2F_2\left(1,1;\frac{5}{2},3;\frac{z^2}{2}\right) + \pi(1-z^2)\operatorname{erfi}\left(\frac{z}{\sqrt{2}}\right) - 3z^2 + \sqrt{2\pi}e^{\frac{z^2}{2}}z + 2, \tag{G1}$$

$$\mathcal{J}(x) = \frac{\pi}{2}\operatorname{erfi}(\sqrt{x}) - x\,{}_2F_2\left(1,1;\frac{3}{2},2;x\right). \tag{G2}$$

These functions are linked by

$$\partial_z^2\left[e^{-\frac{z^2}{2}}\mathcal{J}\left(\frac{z^2}{2}\right)\right] = -\frac{1}{2}e^{-\frac{z^2}{2}}\mathcal{I}(z). \tag{G3}$$

It is useful to give their asymptotics, as their natural definition in terms of a series does not allow for an efficient evaluation at large arguments:

$$\mathcal{J}(x) \underset{x\to\infty}{\simeq} \frac{1}{2}[\ln(4x)+\gamma_{\rm E}] + \frac{1}{4x} - \frac{3}{16x^2} + \frac{5}{16x^3} - \frac{105}{128x^4} + O\left(\frac{1}{x^5}\right), \tag{G4}$$

$$\mathcal{I}(z) \underset{z\to\infty}{\simeq} -z^2[\ln(2z^2)+\gamma_{\rm E}] + \ln(2z^2) + \gamma_{\rm E} + 3 + \frac{1}{2z^2} - \frac{1}{2z^4} + O\left(\frac{1}{z^5}\right). \tag{G5}$$

These functions appear in the inverse Laplace transforms involving $\ln(x)$ or $\operatorname{Ei}(x)$ functions. We give here the main nontrivial formulas used to deal with Laplace inversions:

$$\mathcal{L}^{-1}_{s_2\to t_2}\circ\mathcal{L}^{-1}_{s_1\to t_1}[e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}\ln(\sqrt{s_1}+\sqrt{s_2})] = \partial_{m_1}\partial_{m_2}\left\{\frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{2\pi\sqrt{t_1t_2}}\left[2\mathcal{J}\left(\frac{(m_2t_1+m_1t_2)^2}{4t_1t_2(t_1+t_2)}\right) + \ln\left(\frac{1}{4t_1}+\frac{1}{4t_2}\right) - \gamma_{\rm E}\right]\right\}, \tag{G6}$$

$$\mathcal{L}^{-1}_{s_2\to t_2}\circ\mathcal{L}^{-1}_{s_1\to t_1}[e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}\ln(\sqrt{s_1})] = \partial_{m_1}\partial_{m_2}\left\{\frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{2\pi\sqrt{t_1t_2}}\left[2\mathcal{J}\left(\frac{m_1^2}{4t_1}\right) - \ln(4t_1) - \gamma_{\rm E}\right]\right\}, \tag{G7}$$

$$\mathcal{L}^{-1}_{s\to t}\left[\frac{e^{-m\sqrt{s}}}{m\sqrt{s}}\ln(m^2s)\right] = \frac{e^{-\frac{m^2}{4t}}}{m\sqrt{\pi t}}\left[2\mathcal{J}\left(\frac{m^2}{4t}\right) + \ln\left(\frac{m^2}{4t}\right) - \gamma_{\rm E}\right], \tag{G8}$$

$$\mathcal{L}^{-1}_{s\to t}[m\sqrt{s}e^{-m\sqrt{s}}\ln(m^2s)] = \frac{me^{-\frac{m^2}{4t}}}{2\sqrt{\pi}t^{3/2}}\left\{-\mathcal{I}\left(\frac{m}{\sqrt{2t}}\right) + \left(\frac{m^2}{2t}-1\right)\left[\ln\left(\frac{m^2}{4t}\right) - \gamma_{\rm E}\right]\right\}, \tag{G9}$$

$$\mathcal{L}^{-1}_{s\to t}\left[\frac{e^{m\sqrt{s}}}{m\sqrt{s}}\operatorname{Ei}(-2m\sqrt{s})\right] = \frac{e^{-\frac{m^2}{4t}}}{2m\sqrt{\pi t}}\left[-2\mathcal{J}\left(\frac{m^2}{4t}\right) + \ln\left(\frac{m^2}{t}\right) + \gamma_{\rm E}\right], \tag{G10}$$

$$\mathcal{L}^{-1}_{s\to t}[e^{m\sqrt{s}}\operatorname{Ei}(-2m\sqrt{s})] = \frac{me^{-\frac{m^2}{4t}}}{4\sqrt{\pi}t^{3/2}}\left[2\mathcal{J}\left(\frac{m^2}{4t}\right) - \ln\left(\frac{m^2}{t}\right) - \gamma_{\rm E} - \frac{2\sqrt{\pi t}}{m}e^{\frac{m^2}{4t}}\operatorname{erfc}\left(\frac{m}{2\sqrt{t}}\right)\right]. \tag{G11}$$

To derive Eq. (G6), we start with an integral representation of the logarithm:

$$\ln(\sqrt{s_1}+\sqrt{s_2}) = \int_0^\infty \frac{d\alpha}{\alpha}(e^{-\alpha} - e^{-\alpha(\sqrt{s_1}+\sqrt{s_2})}). \tag{G12}$$

We now compute the inverse Laplace transform of this integral representation, with the exponential prefactor

$$\mathcal{L}^{-1}_{s_2\to t_2}\circ\mathcal{L}^{-1}_{s_1\to t_1}[e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}(e^{-\alpha} - e^{-\alpha(\sqrt{s_1}+\sqrt{s_2})})] = \frac{m_1m_2e^{-\frac{m_1^2}{4t_1}-\frac{m_2^2}{4t_2}}}{4\pi(t_1t_2)^{3/2}}\left[e^{-\alpha} - \left(1+\frac{\alpha}{m_2}\right)\left(1+\frac{\alpha}{m_2}\right)e^{-\alpha^2(\frac{1}{4t_1}+\frac{1}{4t_2})-\alpha(\frac{m_1}{2t_1}+\frac{m_2}{2t_2})}\right]. \tag{G13}$$

To simplify this expression, it is useful to take the primitive w.r.t. m_1 and m_2:

$$\int_{m_1,m_2}\mathcal{L}^{-1}_{s_2\to t_2,s_1\to t_1}[e^{-m_1\sqrt{s_1}-m_2\sqrt{s_2}}(e^{-\alpha} - e^{-\alpha(\sqrt{s_1}+\sqrt{s_2})})] = \frac{e^{-\frac{m_2^2}{4t_2}-\frac{m_1^2}{4t_1}}}{\pi\sqrt{t_1t_2}}\frac{e^{-\alpha} - e^{-\alpha^2(\frac{1}{4t_1}+\frac{1}{4t_2})-\alpha(\frac{m_1}{2t_1}+\frac{m_2}{2t_2})}}{\alpha}. \tag{G14}$$

We still have to deal with the integration over α, which is now an integral of the form

$$\int_{\alpha>0}\frac{e^{-\alpha} - e^{-\alpha^2A-\alpha B}}{\alpha}. \tag{G15}$$

We can compute this integral by deriving w.r.t. A, integrating over α, and then integrating over A; alternatively, we can use the same strategy with B. The two results are

$$\int_{\alpha>0} \frac{e^{-\alpha} - e^{-\alpha^2 A - \alpha B}}{\alpha} = \frac{1}{2}\left(\pi \operatorname{erfi}\left(\frac{B}{2\sqrt{A}}\right) + \ln(A) - 2\ln(B) - \gamma_{\mathrm{E}}\right) - \frac{B^2\, {}_2F_2\left(1,1;\frac{3}{2},2;\frac{B^2}{4A}\right)}{4A} + C_B, \tag{G16}$$

$$\int_{\alpha>0} \frac{e^{-\alpha} - e^{-\alpha^2 A - \alpha B}}{\alpha} = \frac{\pi}{2}\operatorname{erfi}\left(\frac{B}{2\sqrt{A}}\right) - \frac{B^2\, {}_2F_2\left(1,1;\frac{3}{2},2;\frac{B^2}{4A}\right)}{4A} + C_A. \tag{G17}$$

Thus

$$C_A - C_B = \tfrac{1}{2}[\ln(A) - 2\ln(B) - \gamma_{\mathrm{E}}], \tag{G18}$$

and the case $A = 0$, $B = 1$ allows us to conclude that $C_A = \frac{1}{2}\ln(A) - \frac{\gamma_{\mathrm{E}}}{2}$ and $C_B = \ln(B)$. The final result for the integral is

$$\begin{aligned}\int_{\alpha>0} \frac{e^{-\alpha} - e^{-\alpha^2 A - \alpha B}}{\alpha} &= \frac{\pi}{2}\operatorname{erfi}\left(\frac{B}{2\sqrt{A}}\right) - \frac{B^2\, {}_2F_2\left(1,1;\frac{3}{2},2;\frac{B^2}{4A}\right)}{4A} + \frac{1}{2}\ln(A) - \frac{\gamma_{\mathrm{E}}}{2}\\ &= \mathcal{J}\left(\frac{B^2}{4A}\right) + \frac{1}{2}\ln(A) - \frac{\gamma_{\mathrm{E}}}{2}.\end{aligned} \tag{G19}$$

We checked this result numerically with a very good precision.

Applying this formula to the integral over α and specifying $A = \frac{1}{4t_1} + \frac{1}{4t_2}$ and $B = \frac{m_1}{2t_1} + \frac{m_2}{2t_2}$, we obtain Eq. (G6). The same computation, with $A = \frac{1}{4t_1}$ and $B = \frac{m_1}{2t_1}$, gives Eq. (G7).

To derive Eq. (G10) (with $m = 1$, for simplicity), we start with the integral representation of the exponential integral function:

$$e^{\sqrt{s}}\mathrm{Ei}(-2\sqrt{s}) = -\int_0^\infty \frac{e^{-\sqrt{s}-x}}{\sqrt{s}(2\sqrt{s}+x)}\,\mathrm{d}x = -\int_0^\infty \frac{e^{-\sqrt{s}(2y+1)}}{y+1}\mathrm{d}y. \tag{G20}$$

Doing the inverse Laplace transform inside the integral leads to

$$\begin{aligned}\mathcal{L}^{-1}_{s\to t}[e^{\sqrt{s}}\mathrm{Ei}(-2\sqrt{s})] &= -\int_0^\infty \frac{(2y+1)e^{-\frac{(2y+1)^2}{4t}}}{2\sqrt{\pi}t^{3/2}(y+1)}\mathrm{d}y = -\frac{e^{-\frac{1}{4t}}}{\sqrt{\pi}t^{3/2}}\int_0^\infty \frac{te^{-u}}{\sqrt{4tu+1}+1}\mathrm{d}u\\ &= \frac{e^{-\frac{1}{4t}}\left[6t\left(\pi \operatorname{erfi}\left(\frac{1}{2\sqrt{t}}\right) + \ln(t) - \gamma + 2\right) - {}_2F_2\left(1,1;2,\frac{5}{2};\frac{1}{4t}\right)\right]}{24\sqrt{\pi}t^{5/2}} - \frac{1}{2t}.\end{aligned} \tag{G21}$$

To express this result in terms of our special function $\mathcal{J}$, we can use the following relation between hypergeometric functions:

$${}_2F_2\left(1,1;2,\frac{5}{2};a\right) = 3\,{}_2F_2\left(1,1;\frac{3}{2},2;a\right) - \frac{3\left[e^a\sqrt{\frac{\pi}{4a}}\operatorname{erf}(\sqrt{a}) - 1\right]}{a}. \tag{G22}$$

This can be checked by Taylor expansion. With that, and the definition of $\mathcal{J}$ in Eq. (G2), we obtain the announced result, (G11). Equation (G10) is obtained from there by taking one derivative.

APPENDIX H: CHECK OF THE COVARIANCE FUNCTION

As a check of the action, we computed the two-point correlation function (i.e., the covariance function). The needed path integral is

$$\langle X_{t_1}X_{t_2}\rangle = \int_x \int_{X_0=0}^{X_T=x} \mathcal{D}[X] X_{t_1}X_{t_2}e^{-S[X]}. \tag{H1}$$

At first order in ε, we can expand this path integral using Eq. (14) :

$$\langle X_{t_1}X_{t_2}\rangle = \langle X_{t_1}X_{t_2}\rangle_0 + \frac{\varepsilon}{2}\int_0^{t-\tau}\mathrm{d}\tau_1\int_{\tau_1+\tau}^{t}\mathrm{d}\tau_2\frac{\langle X_{t_1}X_{t_2}\dot{X}_{\tau_1}\dot{X}_{\tau_2}\rangle_0}{\tau_2-\tau_1} + O(\varepsilon^2). \tag{H2}$$

Here, averages $\langle\cdot\rangle_0$ are performed with the action $S_0[X]$ given in Eq. (20), i.e., the action of standard Brownian motion with diffusive constant $D_{\varepsilon,\tau} = 1 + 2\varepsilon[1 + \ln(\tau)] + O(\varepsilon^2)$. This action is quadratic, and using Wick contractions allows us to write

$$\langle X_{t_1}X_{t_2}\dot{X}_{\tau_1}\dot{X}_{\tau_2}\rangle_0 = 4(\min(t_1,t_2)\delta(\tau_1-\tau_2) + \theta(t_1-\tau_1)\theta(t_2-\tau_2) + \theta(t_1-\tau_2)\theta(t_2-\tau_1)) + O(\varepsilon). \tag{H3}$$

In this equation, we used only the zeroth order for the diffusive constant [$D_{\varepsilon,\tau} = 1 + O(\varepsilon)$]; the first term does not contribute since τ_1 and τ_2 do not coincide due to the time regularization.

The last two terms require us to compute the integrals:

$$\int_0^{\min(t_1,t_2-\tau)} d\tau_1 \int_{\tau_1+\tau}^{t_2} d\tau_2 \frac{1}{\tau_2-\tau_1} + \int_0^{\min(t_2,t_1-\tau)} d\tau_1 \int_{\tau_1+\tau}^{t_1} d\tau_2 \frac{1}{\tau_2-\tau_1}$$
$$= t_1\ln(t_1) + t_2\ln(t_2) - |t_1-t_2|\ln|t_1-t_2| - 2\min(t_1,t_2)(\ln(\tau)+1). \tag{H4}$$

We now sum all contributions to order ε, the Brownian result with the rescaled diffusive constant being $\langle X_{t_1}X_{t_2}\rangle_0 = 2D_{\varepsilon,\tau}\min(t_1,t_2)$. This gives

$$\begin{aligned}\langle X_{t_1}X_{t_2}\rangle &= 2D_{\varepsilon,\tau}\min(t_1,t_2) + 2\varepsilon\,(t_1\ln(t_1)+t_2\ln(t_2)-|t_1-t_2|\ln|t_1-t_2|) - 4\varepsilon\min(t_1,t_2)(\ln(\tau)+1) + O(\varepsilon^2)\\ &= 2\min(t_1,t_2) + 2\varepsilon\,(t_1\ln(t_1)+t_2\ln(t_2)-|t_1-t_2|\ln|t_1-t_2|) + O(\varepsilon^2)\\ &= t_1^{1+2\varepsilon} + t_2^{1+2\varepsilon} - |t_1-t_2|^{1+2\varepsilon} + O(\varepsilon^2).\end{aligned} \tag{H5}$$

The τ dependence of the diffusive constant and of the first correction to the action cancel, and we recover the fBm correlation function at first order in ε. We also see that the correction to the diffusive constant is equivalent to setting $\ln(\tau) = -1$.

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