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Onsager's-principle-consistent 13-moment transport equations

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A new set of generalized transport equations is derived for higher-order moments which are generated in evolution equation for stress tensor and heat flux vector in 13-moment equations. The closure we employ satisfies Onsager's symmetry principle. In the derivation, we do not employ a phase density function based on Hermite polynomial series in terms of higher-order moments, unlike Grad's approach. The distribution function is rather chosen to satisfy collision invariance, and H-theorem and capture relatively strong deviations from equilibrium. The phase density function satisfies the linearized Boltzmann equation and provides the correct value of the Prandtl number for monatomic gas. The derived equations are compared with Grad's 13-moments equations for gas modeled as Maxwellian molecule. The merits of the proposed equations against Grad's and R13 equations are discussed. In particular, it is noted that the proposed equations contain higher-order terms compared to these equations but require a fewer number of boundary conditions as compared to the R13 equations. The Knudsen number envelope which can be covered to describe flows with these equations is therefore expected to be larger as compared to the earlier equations.

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I. INTRODUCTION

Macroscopic transport equations are of significant interest to researchers for the complete description of physics of rarefied gas flows. In the transition regime, where the Knudsen number (Kn, the ratio of the mean free path to the characteristic length scale) is greater than 0.1, solutions of the Navier-Stokes-Fourier equations deviate significantly when compared against the computationally expensive direct simulation Monte Carlo (DSMC) technique [1]. Consequently, a number of transport equations have been derived starting from the Boltzmann equation for the description of rarefied gas flows and heat transfer. The transport models derived so far can be grouped into two categories: Burnett-type and 13- (or 26-) moment models. Burnett-type models intend to provide higher-order (in terms of Kn) corrections to the Navier-Stokes-Fourier constitutive relationships, while 13- (or 26-) moment models involve higher-order moments of the Boltzmann equation and then seek for an appropriate closure for the unclosed terms (higher moments generated in the derivation procedure). Burnett-type models mainly include original Burnett [2], Burnett [3,4], super-Burnett [5], BGK-Burnett [3,6], augmented Burnett [7], thermomechanically consistent Burnett [8], simplified Burnett [9], and reduced Burnett [10]. These set of equations have several merits and issues which have been outlined in great detail in Refs. [5–7,10–18]. Without reproducing any of the pros [10–13,19] and cons [3,5,8,11,18,20,21] here, we note that these models have suffered from various issues because of not complying with the principles of nonequilibrium thermodynamics. Garcia-Colin *et al.* [11] have recently reiterated the importance of deriving transport equations consistent with the principles of nonequilibrium thermodynamics.

The second set of equations are higher-order moment equations: 13-moment [22,23], regularized 13-moment [24,25], and 26-moment [26] equations. Grad [23] derived the first set of higher-order moment equations, popularly known as Grad's 13-moment equations. He represented the phase density function in terms of a series of Hermite polynomials and closed the higher-order moments generated in the derivation with the constructed function. Grad's equations have, however, not been applied to many boundary value problems [27]. Weiss [28] noted that the hyperbolic character of Grad's equations results in spurious subshocks in shock structure calculation. Moreover, the choice of higher-order moments is also arbitrary in Grad's approach as it does not relate to the order of Knudsen number. Despite these issues, it is well established that appropriate closure in these higher-order moment models can predict many important rarefied phenomena [29,30]. The most recent and rigorously tested 13-moment equations are by Struchtrup [24], which are third-order accurate in Knudsen number, thereby containing terms of super Burnett order [18,31–33]. Many important rarefied phenomena have been successfully investigated with R13 and R26 equations [24,26,32,34–37]. It is important to note that R13 and R26 equations have shown success in predicting the rarefaction effect for $\mathrm{Kn} \leqslant 0.5$ and $\mathrm{Kn} < 1.0$, respectively [38]. These successful predictions are sufficient enough for one to look for even more appropriate closures which can further extend the Knudsen number envelope and may provide solutions up to the middle or end of the transition regime. It important to note that the R13 equations have a purely mathematical basis [24] and are derived based on an order-of-magnitude analysis. Here, our aim is to take advantage of the higher-order moment approach and construct equations which are not just mathematically sound but also physically consistent. We recognize that transport models are required to capture effects which are outcome of strong deviations from equilibrium. Hence, it becomes necessary to employ principles of nonequilibrium thermodynamics in arriving at better transport models. The earlier models have ignored Onsager's symmetry principle in their derivation [39–41]. This motivated us to seek transport models for rarefied gas flows whose foundations lie in these advanced thermodynamic principles [40,42].

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Transport equations which comply with the principles of nonequilibrium thermodynamics have also been derived by Dadzie [8] and Bobylev and Cercignani [43]. In order to take into account the local number of molecules and their spatial distribution, Dadzie [8] introduced a moment of an additional variable. Bobylev and Cercignani [43] suggested employing a low-entropy-producing family of distribution functions.

Transport equations can model strong deviations from equilibrium if these equations can approximate the solution of the full Boltzmann equation. These equations should also have closures which satisfy the principles of thermodynamics. With Onsager's symmetry at the center of derivation, we provide a phase density function which satisfies the collision invariance property. This phase density function is then employed to close the higher-order moments generated in the evolution equations for heat fluxes and stresses. The single-particle distribution function provides the correct value of Prandtl number. The equations are presented for Maxwellian molecules for which the production terms are known. These equations [Eqs. (35) and (36)] have the desirable features of the moments-based approach which does not require the small-Knudsen-number assumption and also closes the set with distribution function whose core lies in satisfying the required physics.

II. GENERALIZED 13-MOMENT EQUATIONS

The probability of finding a molecule in the phase-space element $d\mathbf{x}d\mathbf{c}$ is characterized by the single-particle distribution function (f) and can be obtained from the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{x}} \cdot \mathbf{c} f = J(f,f), \tag{1}$$

where $\mathbf{c}$ and $\mathbf{x}$ are molecule velocity and space vectors, respectively, and $J(f,f)$ is a binary particle collision operator. The Maxwellian distribution which corresponds to equilibrium distribution for monatomic gas is

$$f_0 = \frac{\rho}{m}\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}} \exp[-\beta(|\mathbf{c}-\mathbf{u}|)^2], \tag{2}$$

where $\beta = 1/(2RT)$, R is a specific gas constant, T is the absolute temperature, $\mathbf{u}$ is the bulk velocity, ρ is the density, and m is the molecular mass.

The moments of the single-particle distribution function are assumed to describe the state of the gas completely. Defining Ψ $(= m\{1, c_i, \frac{1}{2}|\mathbf{C}^2|, C_{\langle i}C_{j\rangle}, \frac{1}{2}|\mathbf{C}|^2 C_i\}$, where $\mathbf{C} = \mathbf{c} - \mathbf{u}$ is the peculiar velocity) and taking its inner product with distribution function as $(\langle \Psi, f\rangle \equiv \int \Psi f\, d\mathbf{c})$ generates 13 moments $(\rho, \rho u_i, \rho\epsilon, p_{ij}, q_i)$. (The angular brackets are defined below.)

The moments of the Boltzmann equations result in conservation laws for the macroscopic quantities, i.e., density (ρ), momentum (ρu_i), energy ($\rho\epsilon$), and evolution equations for stress tensor (p_{ij}) and heat flux vector (q_i) [22,23]. The generalized equations for any phase density function obtained after taking its moments with the Boltzmann equation are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} = 0, \tag{3}$$

$$\rho\frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ik}}{\partial x_k} = 0, \tag{4}$$

$$\rho\frac{\partial \epsilon}{\partial t} + \rho u_k \frac{\partial \epsilon}{\partial x_k} + \frac{\partial q_k}{\partial x_k} + p\frac{\partial u_k}{\partial x_k} + \sigma_{ij}\frac{\partial u_i}{\partial x_j} = 0, \tag{5}$$

$$\frac{\partial \sigma_{ij}}{\partial t} + u_k\frac{\partial \sigma_{ij}}{\partial x_k} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + 2\sigma_{k\langle i}\frac{\partial u_{j\rangle}}{\partial x_k} + 2p\frac{\partial u_{\langle i}}{\partial x_{j\rangle}} + \sigma_{ij}\frac{\partial u_k}{\partial x_k} + \frac{\partial}{\partial x_k} m\int C_{\langle i}C_j C_{k\rangle}\, f\, d\mathbf{c} = m\int C_{\langle i}C_{j\rangle}\, J(f,f_0)\, d\mathbf{c}, \tag{6}$$

$$\begin{aligned}\frac{\partial q_i}{\partial t} &+ u_k\frac{\partial q_i}{\partial x_k} + \frac{5}{2}\left(\frac{p}{\rho}\frac{\partial p}{\partial x_i} - \frac{p^2}{\rho^2}\frac{\partial \rho}{\partial x_i}\right) + \frac{\partial}{\partial x_k}\frac{m}{2}\int |\mathbf{C}|^2 C_{\langle i}C_{k\rangle}\, f\, d\mathbf{c} - \frac{5}{2}\frac{p}{\rho}\frac{\partial \sigma_{ik}}{\partial x_k} - \frac{\sigma_{ik}}{\rho}\frac{\partial p}{\partial x_k} + \frac{1}{6}\frac{\partial}{\partial x_i} m\int |\mathbf{C}|^4 (f - f_0)\, f\, d\mathbf{c} \\ &- \frac{\sigma_{ij}}{\rho}\frac{\partial \sigma_{jk}}{\partial x_k} + \frac{7}{5}q_k\frac{\partial u_i}{\partial x_k} + \frac{7}{5}q_i\frac{\partial u_k}{\partial x_k} + \frac{2}{5}q_k\frac{\partial u_k}{\partial x_i} + \frac{\partial u_j}{\partial x_k} m\int C_{\langle i}C_j C_{k\rangle}\, f\, d\mathbf{c} = \frac{m}{2}\int C_i|\mathbf{C}|^2\, J(f,f_0)\, d\mathbf{c},\end{aligned} \tag{7}$$

where p is pressure, $\epsilon = (3/2)RT$, $p_{ij} = p\delta_{ij} + \sigma_{ij}$ with δ_{ij} being the δ function, and

$$\begin{aligned} A_{\langle ij\rangle} &= \frac{1}{2}(A_{ij} + A_{ji}) - \frac{\delta_{ij}}{3}A_{kk} \\ A_{\langle ijk\rangle} &= A_{(ijk)} - \frac{1}{5}(A_{(ill)}\delta_{jk} + A_{(jll)}\delta_{ik} + A_{(kll)}\delta_{ij}) \\ A_{(ijk)} &= \frac{1}{6}(A_{ijk} + A_{ikj} + A_{jik} + A_{jki} + A_{kij} + A_{kji}). \end{aligned} \tag{8}$$

Note that Eqs. (3)–(7) do not form a closed set of equations. The challenge of the moments-based approach is in identifying a suitable closure. The production terms on the right-hand side of Eqs. (6) and (7) for molecules being assumed as Maxwellian are as follows [14]:

$$m\int C_{\langle i}C_{j\rangle}\, J(f,f_0)\, d\mathbf{c} = -\frac{p}{\mu}\sigma_{ij}, \tag{9}$$

$$\frac{m}{2}\int C_i|\mathbf{C}|^2\, J(f,f_0)\, d\mathbf{c} = -\frac{2}{3}\frac{p}{\mu}q_i, \tag{10}$$

where μ is viscosity. Grad [22,23] expressed the distribution function required in the evaluation of integrals on the left-hand side of Eqs. (6) and (7) in terms of orthogonal Hermite polynomials. Hermite polynomials have several desirable properties; however, the truncation of the infinite series results in loss of their properties. This truncation is required because it is possible to evaluate only a finite number of coefficients involved in the series. The distribution function after truncation and evaluation of the coefficients becomes

$$f_{G13} = f_0\left[1 + \frac{m^2}{2\rho k^2 T^2}\sigma_{jk}C_{\langle j}C_{k\rangle} - \frac{m^2}{\rho k^2 T^2}q_k C_k\left(1 - \frac{1}{5}\frac{m}{kT}|\mathbf{C}|^2\right)\right], \tag{11}$$

where k is Boltzmann constant. The equations obtained by substituting Eq. (11) in Eqs. (6) and (7) are known as Grad's moments equations.

III. DISTRIBUTION FUNCTION

In this work we do not follow the earlier approach of expressing the distribution function in terms of Hermite polynomials. We rather employ a distribution function consistent with Onsager's symmetry principle as discussed in Sec. III A. The properties of the distribution function are checked explicitly in Sec. III B. This distribution function is then employed to obtain the unknown integrals on the left-hand side of Eqs. (6) and (7) in Sec. IV.

As is well known, a thermodynamic force X_i ($X_i = \frac{\partial\sigma}{\partial\alpha_i}, i = 1,2,3,\ldots$) disturbs the state of equilibrium and generates its conjugate flux J_i [40], where $\boldsymbol{\sigma}$ is entropy and α_i is a thermodynamic variable. The entropy produced in the process can be expressed in terms of the thermodynamic forces and fluxes as $\boldsymbol{\sigma} = \sum_{i=1}^{n} J_i X_i$. As per Onsager [44,45], the fluxes are related to the thermodynamic forces and governed by the phenomenological linear law: $J_i = \sum_{k=1}^{n} L_{ik}X_k$. Further, the phenomenological coefficients L_{ik}, L_{ki} are related as ($L_{ik} = L_{ki}$ with $i,k = 1,2,\ldots,n$).

The entropy production and thermodynamic fluxes obtained from the kinetic theory satisfy Onsager's symmetry principle only for the first approximation of the phase density function in the Chapman and Enskog series [40]. De Groot and Mazur [40] further showed through detailed derivation that the expressions for entropy production and thermodynamic fluxes obtained from Gibbs' relation are identical with that from the kinetic theory for this first approximation. However, the second approximation of Chapman and Enskog expansion which yields higher-order continuum equations formulates entropy production and fluxes that differ from Gibbs' relation and do not satisfy the symmetry principle [39–41,46]. This suggests that, whereas Chapman and Enskog expansion is appropriate for small deviations from equilibrium, it should not be employed for large deviations. This could be one of the problems with the Chapman and Enskog expansion. Since the source of the problem is an incorrect distribution function generated via the Chapman-Enskog expansion, here we employ an alternate distribution function.

A. Distribution function in terms of thermodynamic force and flux

Since the first-order distribution function obtained from the Chapman and Enskog expansion is consistent with the entropy principle, we can write [40] (p. 184):

$$f^{(1)} = f_0 - (\Upsilon_\tau : X_\tau + \Upsilon_q \cdot X_q) \tag{12}$$

with $\Upsilon_j \odot X_j$ given as [47] (p. 713):

$$\Upsilon_j \odot X_j = t_{r(j)}\left(\frac{\partial f_0}{\partial t} + \nabla_{\mathbf{x}}\cdot(\mathbf{c}f_0)\right)_{X_j=0\,\forall j\neq i}, \tag{13}$$

where $\odot$ denotes full tensor contraction of the tensors of same tensorial order, Υ_j is the microscopic flux, and X_j is its conjugate force. Note that the macroscopic thermodynamic flux J_i can be obtained microscopically as $J_i = \langle\bar{\Upsilon}_i, f\rangle$. The challenge is to now obtain the correct form of the thermodynamic forces X_j and fluxes Υ_j. De Groot and Mazur [40] (p. 184) suggested that the thermodynamic forces and fluxes can be cast around the equilibrium Maxwellian distribution as

$$\Upsilon_j = -f_0\, t_{r(j)}\, \bar{\Upsilon}_j, \tag{14}$$

$$\bar{\Upsilon}_\tau = -\left\{\mathbf{C}\otimes\mathbf{C} - \frac{1}{2}[|\mathbf{C}|^2(\gamma-1)]\mathbf{I}\right\}, \tag{15}$$

$$\bar{\Upsilon}_q = -\left(\frac{5}{2\beta} - |\mathbf{C}|^2\right)\mathbf{C}, \tag{16}$$

$$X_\tau = \beta[\nabla\otimes\mathbf{u} + (\nabla\otimes\mathbf{u})^T];\quad X_q = \nabla\beta. \tag{17}$$

In the above equations, $\otimes$ denotes outer product, subscripts τ and q have been used for fluxes and forces associated with stress and heat flux, respectively, $t_{r(\tau)}$ is relaxation time for momentum transport ($=\mu/p$), $t_{r(q)}$ is relaxation time for energy transport [$=\kappa(\gamma-1)/(R\gamma p)$], κ is thermal conductivity, and γ is the ratio of the specific heat. The above form of the distribution function is rather standard and yields the correct constitutive relationships satisfying Onsager's symmetry principle as shown in De Groot and Mazur [40] (p. 178).

Note that the standard BGK collision model with two different time scales corresponding to momentum and thermal diffusions is assumed in the above formulation. That is, the thermodynamic forces (X_τ and X_q) relax the nonequilibrium state to the equilibrium state at different rates, corresponding to these characteristic time scales. The variation of the momentum diffusion and thermal diffusion time scales with temperature is taken into account through the following relations: $\mu = \mu_0(T/T_0)^\varphi$ and $\kappa = \kappa_0(T/T_0)^\varphi$, where μ_0 and κ_0 are, respectively, the viscosity and thermal conductivity at reference temperature T_0. Further, φ (≈ 0.75) depends on interaction between molecules. Employing two different relaxation times for momentum transport and energy transport resolves the issue of Prandtl number being nonunity for most gases.

The second-order correction to the distribution function can be further constructed to keep the functional form of distribution function in terms of forces and fluxes. As per Mahendra [48], the distribution function with second- (and higher-) order correction can be expanded in terms of $\Upsilon_j \odot X_j$

as:

$$f = f_0 - \sum_j \Upsilon_j \odot X_j + \sum_{k,j} (\Upsilon_{kj} \odot X_k) \odot X_j + \cdots . \tag{18}$$

The last term in the above equation can be evaluated in a similar manner as in Eq. (13) [47] (p. 713), i.e.,

$$\Upsilon_{kj} \odot X_k = t_{r(j)} \left[\frac{\partial \Upsilon_j}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{c} \Upsilon_j) \right]_{X_j = 0 \forall j \neq i} . \tag{19}$$

As already argued, the first-order correction satisfies the Onsager's symmetry principle due to the form in which the distribution function is constructed. In a similar manner, the continuum equations from the above-suggested second-order correction to the Maxwellian distribution also satisfies the symmetry principle, as shown in the Appendix.

The second-order corrections in Eq. (19) can be evaluated in the form consistent with first-order correction as [48]:

$$\Upsilon_{jj} = -f_0 \, t_{r(j)}^2 \, \bar{\Upsilon}_{jj} . \tag{20}$$

Mahendra [48] termed the distribution function suggested above as the Onsager-BGK kinetic model because of its consistency with the Onsager's reciprocity principle. Employing the above distribution function, he successfully derived the Euler and Navier-Stokes equations (along with the energy equation). Further, the numerical results obtained while employing the above distribution function for a variety of problems were successfully compared against DSMC data. These measures enhance confidence in the derived distribution function. He notes that the agreement is particularly good because the employed distribution function ensures correct amount of entropy generation associated with different thermodynamic forces.

B. Second-order correction to the distribution function

The collision invariance and Onsager symmetry properties of the distribution function have been explicitly checked for in this section. Explicit expression for various terms in Eq. (19) has also been evaluated by the present authors, as

$$\begin{aligned} \bar{\Upsilon}_{\tau\tau} \odot X_\tau = {} & -\overbrace{C_i \left[\mathbf{C} \otimes \frac{\partial \mathbf{u}}{\partial x_i} + \left(\mathbf{C} \otimes \frac{\partial \mathbf{u}}{\partial x_i} \right)^T \right]}^{\omega_1} + \overbrace{\frac{1}{2\beta} [\mathbf{C} \otimes \nabla g + (\mathbf{C} \otimes \nabla g)^T]}^{\omega_2} - \left[\overbrace{\frac{1}{2\beta} (\gamma - 1) \mathbf{C} \cdot \nabla g}^{\omega_3} - \overbrace{\frac{1}{2\beta} (\gamma - 1) (\mathbf{C} \otimes \mathbf{C}) : X_\tau}^{\omega_4} \right] \\ & \times \mathbf{I} - \overbrace{\bar{\Upsilon}_\tau \left(\frac{1}{t_{r(\tau)}} \sum_j \Upsilon_j \odot X_j \right)}^{\omega_5} + \overbrace{\bar{\Upsilon}_\tau \left\{ [\varphi(\gamma - 1) - \gamma] \nabla \cdot \mathbf{u} + \frac{\varphi}{\beta} \mathbf{C} \cdot \nabla \beta + \mathbf{C} \cdot \nabla g \right\}}^{\omega_6} , \end{aligned} \tag{21}$$

$$\begin{aligned} \bar{\Upsilon}_{qq} \odot X_q = {} & -\overbrace{\bar{\Upsilon}_q \left(\frac{1}{t_{r(q)}} \sum_j \Upsilon_j \odot X_j \right)}^{\xi_1} - \left\{ \overbrace{\frac{1}{\beta} [\mathbf{C} \cdot \nabla g]}^{\xi_2} - \overbrace{\frac{1}{\beta} (\mathbf{C} \otimes \mathbf{C}) : X_\tau}^{\xi_3} + \overbrace{\left(\frac{5}{2\beta} \right) (\gamma - 1) \nabla \cdot \mathbf{u}}^{\xi_4} + \overbrace{\left(\frac{5}{2\beta^2} \right) (\mathbf{C} \cdot \nabla \beta)}^{\xi_5} \right\} \\ & \times \mathbf{C} - \overbrace{\left[\frac{5}{2\beta} - |\mathbf{C}|^2 \right] \left[\frac{1}{2\beta} \nabla g \right]}^{\xi_6} - \overbrace{\left[\frac{5}{2\beta} - |\mathbf{C}|^2 \right] [\mathbf{C} \otimes \nabla \mathbf{u}]}^{\xi_7} + \overbrace{\bar{\Upsilon}_q \left\{ [\varphi(\gamma - 1) - \gamma] \nabla \cdot \mathbf{u} + \frac{\varphi}{\beta} \mathbf{C} \cdot \nabla \beta + \mathbf{C} \cdot \nabla g \right\}}^{\xi_8} , \end{aligned} \tag{22}$$

where $g = \log(\rho/\beta)$. Note that terms $\bar{\Upsilon}_{\tau q}$ and $\bar{\Upsilon}_{q\tau}$ do not appear in Eq. (19) because shear stress and heat flux are of different tensorial orders and therefore do not interact as per Curie's principle [40]. The terms ω_6 and ξ_8 in the above equations are obtained from the variation of time scales and deserve special attention.

The distribution function with known contractions of forces and fluxes can be expressed as:

$$\begin{aligned} f = {} & f_0 - [\Upsilon_\tau : X_\tau + \Upsilon_q \cdot X_q - (\Upsilon_{\tau\tau} \odot X_\tau) : X_\tau \\ & - (\Upsilon_{qq} \odot X_q) \cdot X_q] . \end{aligned} \tag{23}$$

The terms capturing deviations from equilibrium should also satisfy an additional constraint of the additive invariants property of kinetic theory:

$$\langle \Psi, (f - f_0) \rangle = 0. \tag{24}$$

That is, no mass, momentum, or energy should be generated (or destroyed) during inter-molecular collisions. The authors have explicitly checked for these constraints and found that the distribution function in Eq. (23) does not satisfy Eq. (24) in the present form. The obtained second-order correction to distribution function is therefore modified such that it satisfies the additive invariance property without breaking Onsager's symmetry.

The correction procedure has been adopted from Refs. [6,49]; the earlier authors used this approach for deriving the BGK Burnett equations. If we use the distribution function [Eq. (23)] in Eq. (24), then integrals of most of the terms in the distribution function add up to zero; however, terms ω_1, ω_5, ω_6, ξ_1, and ξ_5 in Eqs. (21) and (22) provide nonvanishing contribution. So terms with similar forms with different coefficients are added to the function. Next, the added terms are contracted with thermodynamic forces of the same tensorial order. We therefore obtain three equations of "nonvanishing or diminishing moments" of mass, momentum, and energy with the collision integral. These equations are solved for the three unknowns coefficients. Note that one may always add more unknowns and still satisfy the collision invariance property; however, this may break Onsager's symmetry.

Mathematically for f in Eq. (23),

$$\langle m,(f-f_0)\rangle = 0. \tag{25}$$

$$\langle c_i,(f-f_0)\rangle = -\frac{\rho}{\beta^2}\left(\varphi t_{r(\tau)^2} - t_{r(q)}^2 + 2t_{r(\tau)}t_{r(q)}\right) \times \left[\frac{\partial\beta}{\partial x_j}\frac{\partial u_{\langle j}}{\partial x_{i\rangle}} - \left(\frac{17}{12}-\frac{5\gamma}{4}\right)\frac{\partial\beta}{\partial x_i}\frac{\partial u_j}{\partial x_j}\right] \neq 0. \tag{26}$$

Similarly, we find that moment of $\frac{1}{2}|\mathbf{C}^2|,(f-f_0)$ is nonzero (but has a complicated form and therefore is not presented here), i.e,

$$\left\langle \tfrac{1}{2}|\mathbf{C}^2|,(f-f_0)\right\rangle \neq 0. \tag{27}$$

We then identify terms which are contributing to the moment, say, to momentum (we refer to this moment as a nondiminishing moment above). For instance,

$$\langle c_i,\omega_5\rangle = \frac{\rho}{\beta^2}(t_{r(\tau)}t_{r(q)})\left[\frac{\partial\beta}{\partial x_j}\frac{\partial u_{\langle j}}{\partial x_{i\rangle}} - \left(\frac{17}{12}-\frac{5\gamma}{4}\right)\frac{\partial\beta}{\partial x_i}\frac{\partial u_j}{\partial x_j}\right] + \frac{5\rho u_i}{4\beta^3}t_{r(q)}^2\left(\frac{\partial\beta}{\partial x_i}\frac{\partial\beta}{\partial x_i}\right). \tag{28}$$

This shows that if we augment our distribution function such that we add terms *similar* to ω_5 with an unknown coefficient, then we can, in principle, get a vanishing moment by choosing the value of coefficient appropriately. Therefore, we add ω_7 and ξ_9. This is the general framework in which we identify terms and then augment our distribution function. Care needs to be taken as the added terms may generate non-vanishing moment for mass ($\langle m,(f-f_0)\rangle = 0$), which was, however, satisfied earlier. We can add at most three terms to ensure that three constraints imposed by collision invariant property are satisfied. Second, we must not add any terms which may break Onsager's symmetry principle. We have checked this point explicitly, and the proof is included at the end in the Appendix. The modified second-order correction to the distribution function therefore becomes (where prime denotes corrected terms):

$$\begin{aligned}\bar{\Upsilon}'_{\tau\tau}\odot X_\tau &= \bar{\Upsilon}_{\tau\tau}\odot X_\tau + \overbrace{\frac{\Delta}{t_{r(\tau)}}(\Upsilon_q\cdot X_q)\bar{\Upsilon}_\tau}^{\omega_7} + \left(\frac{5}{2\beta}-|\mathbf{C}|^2\right)\left(\overbrace{\Lambda(\omega_6+\omega_1)}^{\omega_8}\right)\\ \bar{\Upsilon}'_{qq}\odot X_q &= \bar{\Upsilon}_{qq}\odot X_q + \overbrace{\frac{\Delta}{t_{r(q)}}(\Upsilon_\tau : X_\tau)\bar{\Upsilon}_q}^{\xi_9} + \left(\frac{5}{2\beta}-|\mathbf{C}|^2\right)\left(\overbrace{\Omega\mathbf{C}\cdot\nabla\beta\mathbf{C}}^{\xi_{10}}\right).\end{aligned} \tag{29}$$

Note that the choice of the added terms is not arbitrary. As mentioned in the modification above, terms associated with Ω, Λ, and Δ are identified based on their contribution to nondiminishing moments in the invariance property. These coefficients are obtained by following the constraints of Eq. (24) as

$$\Omega = \varphi + 2\,; \quad \Lambda = -\frac{2}{5}; \quad \Delta = \frac{1}{2}\left[-\varphi\frac{t_{r(\tau)}}{t_{r(q)}} - \frac{t_{r(q)}}{t_{r(\tau)}} + 2\right]. \tag{30}$$

These modifications do not change the functional form of fluxes and forces and hence preserve Onsager's symmetry. With these modifications, the distribution function changes to:

$$f' = f_0 - (\Upsilon_\tau : X_\tau + \Upsilon_q\cdot X_q - (\Upsilon'_{\tau\tau}\odot X_\tau) : X_\tau - (\Upsilon'_{qq}\odot X_q)\cdot X_q), \tag{31}$$

where $\Upsilon'_{jj} = -f_0\,t_{r(j)}^2\,\bar{\Upsilon}'_{jj}$. The distribution function which satisfies the Boltzmann equation and additive invariance property also maximizes entropy production [50].

IV. PROPOSED MOMENT EQUATIONS

The aforementioned distribution function being consistent with the principles of nonequilibrium thermodynamics has the promise of yielding more accurate higher-order continuum transport equations. These higher-order continuum equations are now derived as part of this work. We have closed the 13-moment equations [Eqs. (6) and (7)] using the distribution function given by Eq. (31) which yields:

$$m\int C_{\langle i}C_jC_{k\rangle}\, f\, d\mathbf{c} = \frac{3}{2}\frac{\rho}{\beta^3}\left\{\left[\varphi\left(\frac{\kappa(\gamma-1)}{R\gamma p}\right)^2 - \left(\frac{\mu}{p}\right)^2\right]\left[\frac{\partial\beta}{\partial x_{\langle i}}\frac{\partial u_j}{\partial x_{k\rangle}}\right] + \beta\left(\frac{\mu}{p}\right)^2\left[\frac{\partial g}{\partial x_{\langle i}}\frac{\partial u_j}{\partial x_{k\rangle}}\right]\right\}, \tag{32}$$

$$\begin{aligned}\frac{m}{2}\int |\mathbf{C}|^2 C_{\langle i}C_{k\rangle}\, f\, d\mathbf{c} = 7\mu\frac{p}{\rho}\frac{\partial u_{\langle i}}{\partial x_{k\rangle}} + \frac{1}{20}\frac{\rho}{\beta^2}\left(\frac{\mu}{p}\right)^2\bigg[&[342+35\varphi-7\gamma(31+5\varphi)]\frac{\partial u_{\langle i}}{\partial x_{k\rangle}}\frac{\partial u_l}{\partial x_l} + 27\frac{\partial u_{\langle i}}{\partial x_l}\frac{\partial u_l}{\partial x_{k\rangle}}\\ &+ 45\frac{\partial u_{\langle i}}{\partial x_l}\frac{\partial u_{k\rangle}}{\partial x_l} - 18\frac{\partial u_l}{\partial x_{\langle i}}\frac{\partial u_l}{\partial x_{k\rangle}}\bigg] + \frac{7}{8}\frac{\rho}{\beta^5}\left(\frac{\kappa(\gamma-1)}{R\gamma p}\right)^2\left[2\frac{\partial\beta}{\partial x_{\langle i}}\frac{\partial\beta}{\partial x_{k\rangle}} + \beta\frac{\partial\beta}{\partial x_{\langle i}}\frac{\partial g}{\partial x_{k\rangle}}\right],\end{aligned} \tag{33}$$

$$\begin{aligned} m\int |\mathbf{C}|^4 (f-f_0)\, f\, d\mathbf{c} &= (50-30\gamma)\mu \frac{p}{\rho}\frac{\partial u_i}{\partial x_i} + \frac{5}{2}\frac{\rho}{\beta^5}\left(\frac{\kappa(\gamma-1)}{R\gamma p}\right)^2 \left[\frac{7}{2}\frac{\partial \beta}{\partial x_i}\frac{\partial \beta}{\partial x_i} + \beta^2 \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_i}\right] \\ &\quad + \frac{\rho}{\beta^2}\left(\frac{\mu}{p}\right)^2 \frac{7}{16}(-5+3\gamma)(-63-10\varphi+5\gamma(7+2\varphi))\left[\left(\frac{\partial u_i}{\partial x_i}\right)^2\right]. \end{aligned} \tag{34}$$

Substituting the above higher-order moments [Eqs. (32)–(34)] results in the following final evolution equations for heat flux vector and stress, respectively:

$$\begin{aligned} &\frac{\partial \sigma_{ij}}{\partial t} + u_k \frac{\partial \sigma_{ij}}{\partial x_k} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + 2\sigma_{k\langle i}\frac{\partial u_{j\rangle}}{\partial x_k} + 2p\frac{\partial u_{\langle i}}{\partial x_{j\rangle}} + \sigma_{ij}\frac{\partial u_k}{\partial x_k} + \frac{3}{2}\frac{\partial}{\partial x_k}\Bigg\{\beta\left(\frac{\mu}{p}\right)^2\left[\frac{\partial g}{\partial x_{\langle i}}\frac{\partial u_j}{\partial x_{k\rangle}}\right] \\ &\quad + \frac{\rho}{\beta^3}\left[\varphi\left(\frac{\kappa(\gamma-1)}{R\gamma p}\right)^2 - \left(\frac{\mu}{p}\right)^2\right]\left[\frac{\partial \beta}{\partial x_{\langle i}}\frac{\partial u_j}{\partial x_{k\rangle}}\right]\Bigg\} = -\frac{p}{\mu}\sigma_{ij}, \end{aligned} \tag{35}$$

$$\begin{aligned} &\frac{\partial q_i}{\partial t} + u_k\frac{\partial q_i}{\partial x_k} + \frac{5}{2}\left(\frac{p}{\rho}\frac{\partial p}{\partial x_i} - \frac{p^2}{\rho^2}\frac{\partial \rho}{\partial x_i}\right) - \frac{5}{2}\frac{p}{\rho}\frac{\partial \sigma_{ik}}{\partial x_k} - \frac{\sigma_{ik}}{\rho}\frac{\partial p}{\partial x_k} + 7\frac{\partial}{\partial x_k}\left(\mu\frac{p}{\rho}\frac{\partial u_{\langle i}}{\partial x_{k\rangle}}\right) + \frac{7}{5}q_k\frac{\partial u_i}{\partial x_k} \\ &\quad + \frac{\partial}{\partial x_k}\Bigg\{\frac{1}{20}\frac{\rho}{\beta^2}\left(\frac{\mu}{p}\right)^2\Bigg[[342+35\varphi-7\gamma(31+5\varphi)]\frac{\partial u_{\langle i}}{\partial x_{k\rangle}}\frac{\partial u_l}{\partial x_l} + 27\frac{\partial u_{\langle i}}{\partial x_l}\frac{\partial u_l}{\partial x_{k\rangle}} + 45\frac{\partial u_{\langle i}}{\partial x_l}\frac{\partial u_{k\rangle}}{\partial x_l} \\ &\quad -18\frac{\partial u_l}{\partial x_{\langle i}}\frac{\partial u_l}{\partial x_{k\rangle}}\Bigg] + \frac{7}{8}\frac{\rho}{\beta^5}\left(\frac{\kappa(\gamma-1)}{R\gamma p}\right)^2\left[2\frac{\partial \beta}{\partial x_{\langle i}}\frac{\partial \beta}{\partial x_{k\rangle}} + \beta\frac{\partial \beta}{\partial x_{\langle i}}\frac{\partial g}{\partial x_{k\rangle}}\right]\Bigg\} + \frac{7}{5}q_i\frac{\partial u_k}{\partial x_k} + \frac{2}{5}q_k\frac{\partial u_k}{\partial x_i} \\ &\quad + \frac{1}{6}\frac{\partial}{\partial x_i}\Bigg\{(50-30\gamma)\mu\frac{p}{\rho}\frac{\partial u_j}{\partial x_j} + \frac{\rho}{\beta^2}\left(\frac{\mu}{p}\right)^2\frac{7}{16}(-5+3\gamma)(-63-10\varphi+5\gamma(7+2\varphi))\left[\left(\frac{\partial u_j}{\partial x_j}\right)^2\right] \\ &\quad + \frac{5}{2}\frac{\rho}{\beta^5}\left(\frac{\kappa(\gamma-1)}{R\gamma p}\right)^2\left[\frac{7}{2}\frac{\partial \beta}{\partial x_j}\frac{\partial \beta}{\partial x_j} + \beta^2\frac{\partial g}{\partial x_j}\frac{\partial g}{\partial x_j}\right]\Bigg\} - \frac{\sigma_{ij}}{\rho}\frac{\partial \sigma_{jk}}{\partial x_k} + \frac{3}{2}\frac{\rho}{\beta^3}\frac{\partial u_j}{\partial x_k}\Bigg\{\beta\left(\frac{\mu}{p}\right)^2\left[\frac{\partial g}{\partial x_{\langle i}}\frac{\partial u_j}{\partial x_{k\rangle}}\right] \\ &\quad + \left[\varphi\left(\frac{\kappa(\gamma-1)}{R\gamma p}\right)^2 - \left(\frac{\mu}{p}\right)^2\right]\left[\frac{\partial \beta}{\partial x_{\langle i}}\frac{\partial u_j}{\partial x_{k\rangle}}\right]\Bigg\} = -\frac{2}{3}\frac{p}{\mu}q_i. \end{aligned} \tag{36}$$

Equations (3)–(5), (35), and (36) form a closed set of equations for the 13 moments $(\rho, \rho u_i, \rho\epsilon, p_{ij}, q_i)$. Derivation of Eqs. (35) and (36) is the primary contribution of this work.

V. DISCUSSION

A rigorous testing of the equations proposed above for several problems is required to fully establish the validity of the derived equations, which is, however, beyond the scope of the present work. Some comments about the derived equations vis-à-vis the existing equations is nonetheless made in the following subsections.

A. Comparison with derivation procedures of existing transport models

The conventional Burnett equations (derived in late 1930s) do not satisfy the H-theorem. Violation of the H-theorem has been reported as the cause of numerical instabilities in these equations. The augmented Burnett equations derived half-a-century later by Zhong [51], involved terms added on an *ad hoc* basis to get rid of the numerical instability. Again, the H-theorem is not proven for these equations. Nonetheless, the augmented equations remained in extensive use in the study of hypersonic flows, despite the fact that the equations do not capture the flow physics accurately in the transition regime. Recent and probably most used Burnett equations are by Agarwal *et al.* [6], where the distribution function is constructed to satisfy the H-theorem. Therefore, it is not surprising that these equations capture the flow better than any other equations proposed so far. This is because the equations are physically consistent as far as the second law of thermodynamics is concerned.

In the moment approach, the Grad's 13-moment equations, which are the most used and cited, satisfy the H-theorem for small deviations from nonequilibrium. The modified R13 equations, the most popular in the higher-order moments approach, also satisfy the H-theorem but only for weak deviations. This is perhaps not surprising as the derivation of the latter equations builds on the approach of Grad.

In all of the works mentioned above (and numerous variants of these works, not mentioned here), we hardly see any remark related to Onsager's symmetry. It is proven that the Burnett equations do not satisfy the reciprocal relations [41]. The H-theorem and Onsager's symmetry are rigid constraints which have largely been ignored while deriving the transport models. In view of the above, it is clear that satisfying Onsager's symmetry is not trivial. However, here we have been successful in overcoming this issue.

Notice that the present derivation procedure does not make any assumption or reference to Knudsen number in its entire derivation. The Chapman-Enskog expansion inherently involves expansion in terms of Knudsen number and therefore yields accurate results for small Knudsen numbers only. Therefore, most of the equations derived through this approach break down much before Kn equals unity even for very simple flows. However, as our approach does not assume Kn to be small, we expect our equations to work in the slip and transition flow regimes. The basis of making such a claim is the derivation of R13 equations [24] where higher-order terms as a consequence of order of magnitude of analysis make equations more accurate than Grad's equations in terms of Knudsen number.

B. Comparison of proposed equations with Grad's moment and R13 equations

In this section, we compare the proposed set of equations with Grad's 13-moment, R13, and 26-moment equations. The higher-order moments in Eqs. (6) and (7) obtained after substitution of Grad's distribution function [Eq. (11)] are as follows:

$$m\int C_{\langle i}C_jC_{k\rangle}\, f\, d\mathbf{c}|_{G13} = 0, \tag{37}$$

$$\frac{m}{2}\int |\mathbf{C}|^2 C_{\langle i}C_{j\rangle}\, f\, d\mathbf{c}|_{G13} = 7\frac{p}{\rho}\sigma_{ij}, \tag{38}$$

$$m\int |\mathbf{C}|^4 (f-f_0)\, f\, d\mathbf{c}|_{G13} = 0. \tag{39}$$

Equations (37)–(39) from Grad's approach can be compared against Eqs. (32)–(34) from the present approach. Note that most of the terms in Eqs. (32)–(34) have $(\mu/p)^2$ as their coefficient, indicating that these are higher-order terms. {The term $[\kappa(\gamma-1)/(R\gamma p)]^2$ can be written as $1/\mathrm{Pr}^2(\mu/p)^2$ where Pr is Prandtl number.} This suggests that there are no first-order terms in Eqs. (32) and (34), thereby agreeing with Eqs. (37) and (39), respectively. To first order, Eq. (33) has $7\mu(p/\rho)(\partial u_{\langle i}/\partial x_{k\rangle})$, which is similar to $7(p/\rho)\sigma_{ij}$ in Grad's Eq. (38). This agreement enhances confidence in the derived equations.

Clearly, the proposed equations have additional terms as compared to Grad's 13-moment equations. The additional higher-order terms in our equation involve products of derivatives of velocities, derivatives of temperature, and derivatives of density. Various combinations of these quantities are involved in the equations. Due to the additional terms in the proposed equations, the applicability of the proposed equations is expected to be higher in terms of Knudsen number as compared to the Grad's equations. It is easy to see that these additional terms vanish on linearization of the equations (see the Appendix for linearized equations) and therefore do not contribute to the stability of the equations for simple linear waves. That is, the stability of the proposed equations is the same as that for the Grad's equations. The reason for this similarity (in terms of stability) is the absence of second (or higher) derivatives of velocity and temperature in the proposed equations. Such terms arise in the process of the regularization of the Grad's equations and in the conventional Burnett equations. The presence of these terms in the equations is known to be the cause of instability; this issue is clearly not applicable to the present equations.

As already recognized earlier, solving 13-moment equations requires the appropriate boundary conditions for the higher-order variables introduced [i.e., σ_{ij} and q_i in Eqs. (6) and (7) respectively]. See, for example, Struchtrup [52], on how boundary conditions can be prescribed for these quantities. This requirement is common between the present and Grad's equations. However, the requirement of number of boundary conditions is less than the R13 equations. This is because second-order derivatives of σ_{ij} and q_i are involved in the R13 equations; these higher-order derivative terms get added in the equations in the process of the regularization of the equations. Clearly, it would be more feasible to solve the present equations as compared to the R13 equations, making the present equations attractive.

The proposed equations can also be compared against the 26-moment equations of Gu *et al.* [26,53]. In their approach, the unclosed terms in equations for stress and heat flux in Eqs. (6) and (7) are not evaluated using a distribution function (as done by Grad and followed here); instead, evolution equations for the unknown integrals in Eqs. (6) and (7) (which are still higher-order moments) are written down. Their approach therefore yields a total of 26 moments (or unknowns). The unknown integrals generated are finally closed using a distribution function similar to that suggested by Grad. Gu and Emerson further regularized their equations in a manner suggested by Struchtrup. Clearly, while their approach is interesting, it leads to additional variables and the requirement of more boundary conditions. This limitation is linked to Grad's approach where improvement in the accuracy is inherently coupled with the number of unknowns that needs to be solved. In contrast, in our approach we can improve the performance of the equations either by adding more moments or more terms to the distribution function. That is, the evolution equations of the still-higher moments can be introduced by taking further moments of the Boltzmann equations and evaluating the unknown terms using the same distribution function as employed here. Similarly, the present equations can be improved by accounting for more terms in the series given in Eq. (18).

We recognize that, whereas the regularized 26-moment equations are potentially more accurate than the derived equations, our derived equations have the simplicity of having to deal with fewer number of variables and having to pose fewer boundary conditions (requirement of number of boundary conditions increases with both increase in number of variables and the regularization process) than the R26 equations.

VI. CONCLUSIONS

A new set of transport equations using a higher-order moments approach and nonequilibrium thermodynamics principles is derived. The higher-order moments equations are closed by Onsager's-reciprocity-principle-consistent phase density function. The proposed phase density function satisfies linearized Boltzmann equation and collision invariance property.

The distribution function is then utilized to close the higher-order terms generated in the evolution equation for stress tensor and heat flux vector. This approach differs substantially from the previous approaches of deriving 13-moment equations which rely on Grad's formulation of phase density function in terms of orthogonal Hermite polynomials.

The difference in the derived 13-moment equations from Grad's moment equations are outlined. In particular, our equations contain almost the same terms as Grad's equations to the first order, but products of derivatives of velocity, temperature, and density get added to the second order. The number of boundary conditions required for the solution of these equations is same as the Grad's equations but less than the R13 and R26 equations.

The employed distribution function ensures a correct value of the Prandtl number (=2/3 for monatomic gas) as it involves two different relaxation times for momentum and energy transport. The proposed equations have additional terms as compared to Grad; therefore, the applicability of the proposed equations is expected to be higher in terms of Knudsen number as compared to the Grad's equations. However, solving the proposed equations is expected to be easier as compared to R13 and R26 equations.

Since no arbitrary assumption is employed in our approach and the equations have a sound physical basis, we expect the proposed equations to describe the high-Knudsen-number flow phenomenon better than the existing set of equations. These equations need to be tested rigorously in various cases which we aim to provide in the future.

APPENDIX: ONSAGER'S SYMMETRY PROOF

Here we follow the procedure suggested in Romero and Velasco [41] to show that the derived equations satisfy the Onsager's symmetry.

First, we linearize O13 moment equations around equilibrium state in following manner:

$$\begin{aligned} \rho = \rho_0 + \hat{\rho} \quad T = T_0 + \hat{T} \quad p = p_0 + \hat{p} \\ u_i = \hat{u_i} \quad \sigma_{ij} = \hat{\sigma_{ij}} \quad q_i = \hat{q_i}, \end{aligned} \tag{A1}$$

where quantities with subscript 0, $X_0 = 0$ correspond to equilibrium state and $\hat{X}$ are *small* perturbations away from equilibrium. Putting above equations into Eqs. (3)–(7) leads to following equations linearized equations:

$$\frac{\partial \hat{\rho}}{\partial t} = -\rho_0 \frac{\partial \hat{u}_k}{\partial x_k}, \tag{A2}$$

$$\frac{\partial \rho_0 \hat{u}_i}{\partial t} = -\frac{\partial \hat{p}}{\partial x_i} - \frac{\partial \hat{\sigma}_{ik}}{\partial x_k}, \tag{A3}$$

$$\frac{\partial \rho_0 \hat{\epsilon}}{\partial t} = -\frac{\partial \hat{q}_k}{\partial x_k} - p_0 \frac{\partial \hat{u}_k}{\partial x_k}, \tag{A4}$$

$$\frac{\partial \hat{\sigma}_{ij}}{\partial t} = -\frac{4}{5} \frac{\partial \hat{q}_{\langle i}}{\partial x_{j\rangle}} - 2p_0 \frac{\partial \hat{u}_{\langle i}}{\partial x_{j\rangle}} - \frac{p}{\mu} \hat{\sigma}_{ij}, \tag{A5}$$

$$\frac{\partial \hat{q}_i}{\partial t} = \frac{5}{8} \frac{p_0}{R\beta_0^2} \frac{\partial (2R\delta\beta)}{\partial x_i} - \frac{5}{2} \frac{p_0}{\rho_0} \frac{\partial \hat{\sigma_{ik}}}{\partial x_k} - \frac{2}{3} \frac{p_0}{\mu} \hat{q}_i. \tag{A6}$$

The appropriate thermodynamic forces are as follows:

$$X_\rho = \frac{2\beta_0 R}{\rho_0} \hat{p} - \frac{p_0}{\rho_0} X_\epsilon \quad X_{u,i} = 2R\beta_0 u_i \quad X_\epsilon = -2R\delta\beta$$

$$X_{\sigma_{ij}} = \frac{\hat{\sigma}_{ij} R\beta_0}{p_0} \quad X_{q,i} = \frac{8 q_i \beta_0^2 R}{5 p_0}. \tag{A7}$$

Now expressing Eqs. (A2)–(A6) in terms of thermodynamic forces [Eq. (A7)] yields:

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{\rho_0}{2R\beta_0} \frac{\partial X_{u,k}}{\partial x_k}, \tag{A8}$$

$$\frac{\partial \rho_0 \hat{u}_i}{\partial t} = -\frac{\rho_0}{2\beta_0 R} \frac{\partial X_\rho}{\partial x_i} - \frac{p_0}{2\beta_0 R} \frac{\partial X_\epsilon}{\partial x_i} - \frac{p_0}{R\beta_0} \frac{\partial X_{\sigma_{ik}}}{\partial x_k}, \tag{A9}$$

$$\frac{\partial \rho_0 \hat{\epsilon}}{\partial t} = -\frac{5p_0}{8\beta_0^2 R} \frac{\partial X_{q,k}}{\partial x_k} - \frac{p_0}{2R\beta_0} \frac{\partial X_{u,k}}{\partial x_k}, \tag{A10}$$

$$\frac{\partial \hat{\sigma}_{ij}}{\partial t} = -\frac{5}{2} \frac{p_0}{\beta_0^2 R} \frac{\partial X_{q,\langle i}}{\partial x_{j\rangle}} - \frac{p_0}{R\beta_0} \frac{\partial X_{u,\langle i}}{\partial x_{j\rangle}} - \frac{p_0^2}{R\beta_0 \mu} X_{\sigma_{ij}}, \tag{A11}$$

$$\frac{\partial \hat{q}_i}{\partial t} = -\frac{5}{8} \frac{p_0}{R\beta_0^2} \frac{\partial X_\epsilon}{\partial x_i} - \frac{5}{2} \frac{p_0}{R\beta_0^2} \frac{\partial X_{\sigma_{ik}}}{\partial x_k} - \frac{5}{12} \frac{p_0^2}{\mu R\beta_0^2} X_{q,i} \tag{A12}$$

Phenomenological (L_{ik} mentioned in Sec. III) coefficients in thermodynamic fluxes [left-hand side of Eqs. (A8)–(A12)] are as follows:

$$\begin{aligned} L_a &= -\frac{\rho_0}{2R\beta_0} \frac{\partial}{\partial x_i} & L_b &= -\frac{p_0}{2\beta_0 R} \frac{\partial}{\partial x_i} \\ L_c &= -\frac{p_0}{R\beta_0} \frac{\partial}{\partial x_i} & L_d &= -\frac{5p_0}{8\beta_0^2 R} \frac{\partial}{\partial x_i} \\ L_e &= -\frac{5}{2} \frac{p_0}{\beta_0^2 R} \frac{\partial}{\partial x_i} & L_f &= -\frac{p_0^2}{R\beta_0 \mu} \\ & & L_g &= -\frac{5}{12} \frac{p_0^2}{\mu R\beta_0^2}. \end{aligned} \tag{A13}$$

In order to satisfy Onsager's reciprocal relations, we must satisfy $L^\dagger = TLT$, where the dagger denotes Hermitian conjugate. Note that T is a diagonal matrix such that a diagonal element is $+1$ or -1 for a thermodynamic flux being even or odd under time reversal, respectively. The $L^\dagger$ matrix can be constructed from Eqs. (A13) in following manner (see also Ref. [41]):

$$L = \begin{bmatrix} 0 & L_a & 0 & 0 & 0 \\ L_a & 0 & L_b & L_c & 0 \\ 0 & L_b & 0 & 0 & L_d \\ 0 & L_c & 0 & L_f & L_e \\ 0 & 0 & L_d & L_e & L_g \end{bmatrix}. \tag{A14}$$

Further,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \times L \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -L_a & 0 & 0 & 0 \\ -L_a & 0 & -L_b & -L_c & 0 \\ 0 & -L_b & 0 & 0 & -L_d \\ 0 & -L_c & 0 & L_f & -L_e \\ 0 & 0 & -L_d & -L_e & L_g \end{bmatrix} = L^\dagger. \tag{A15}$$

This completes the required proof.

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