

Nonergodic complexity management

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Linear response theory, the backbone of nonequilibrium statistical physics, has recently been extended to explain how and why nonergodic renewal processes are insensitive to simple perturbations, such as in habituation. It was established that a permanent correlation results between an external stimulus and the response of a complex system generating nonergodic renewal processes, when the stimulus is a similar nonergodic process. This is the principle of complexity management, whose proof relies on ensemble distribution functions. Herein we extend the proof to the nonergodic case using time averages and a single time series, hence making it usable in real life situations where ensemble averages cannot be performed because of the very nature of the complex systems being studied.

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I. INTRODUCTION

The mathematician Norbert Wiener, in the middle of the last century [1], speculated that a system high in energy can be controlled by one that is low in energy. The necessary force is produced by the low-energy system being high in information content and the high-energy system being low in information content. Consequently, there is an information gradient that produces the force by which the low-energy system controls the high-energy system, through a flow of information against the traditional energy gradient. Quantifying the transfer of information from a complex system high in information to one low in information is the first articulation of a universal principle of network science and we refer to this speculation as Wiener's rule (WR).

In a modern context WR can be understood as an entropic force, used to explain such diverse phenomena as the elasticity of freely jointed polymer molecules [2], oceanic forces [3] and the conscious states in the human brain, through neuroimaging [4]. Over the past decade the nascent field of network science has been applied to determining the conditions under which WR is facilitated or suppressed. After half a century WR has been shown to be correct and has been superseded by the more detailed principle of complexity management (PCM) [5,6].

One result of the many analyses of information transfer, which is being continually rediscovered, is that complex networks in living systems exist at, or on the edge of, phase transitions, which optimizes both intra- and internetwork information transmission [7]. Moreover, the statistical distributions of a diverse collection of complex systems are inverse power laws, whether modeling the connectivity of the internet or social groups, the frequency or magnitude of earthquakes, the number of solar flares, the time intervals in conversational turn taking, and many other phenomena (see, for example, Ref. [8]). The power-law index is the measure of complexity in each system.

Traditional methods of nonequilibrium statistical physics have not been successful in addressing the question of information transfer between complex networks. For example, in studying the response of complex systems to harmonic

perturbations it was determined by many authors [9–12] that *linear response theory* (LRT), a cornerstone of physics, was “dead.” An assessment of this premature death, made by Aquino *et al.* [5,6], resulted in a generalization of LRT (GLRT) that was successfully applied to the question of information transfer between complex dynamic networks. These latter authors focused on the intimate connection between neural organization and information theory, as well as the production of $1/f$ noise. Their research supported the observation that $1/f$ signals are encoded and transmitted by sensory neurons with greater efficiency than are white noise signals [13]. Psychologists interpret the generation of $1/f$ noise as a manifestation of cognition [14,15], although no psychologically well-founded model for the origin of $1/f$ noise yet exists [16]. However, experimental observation of brain dynamics either monitoring EEG activity [17] or through actigraphy [18] confirm that the awake condition of the brain is a source of $1/f$ noise [19].

Despite its successes, GLRT, like its predecessor LRT, has a fundamental limitation that hinders its application to many real world systems. In this article we review the current results obtained using GLRT, demonstrate how to overcome its limitations by using theoretical arguments, and verify the theory using numerical simulations.

II. RENEWAL EVENTS

It is useful to introduce the notion of a *renewal event*, which is an event associated with a reorganization of the system under study. It is customary to call the time between two renewal events a *laminar region*; the lengths of two consecutive laminar regions are independent. We study complex systems that exhibit inverse-power-law dynamical behavior. A good approximation for the waiting time distribution (WTD) between two renewal events in these systems is

$$\psi(t) = \frac{(\mu - 1)T^{\mu-1}}{(T + t)^\mu}, \quad (1)$$

where T and μ are parameters characterizing the complex system under study. Normalizability of ψ requires that the power-law index μ , the index of complexity, must be larger

than 1. When $2 < \mu < 3$, the second moment is infinite, so these systems obey the generalized central limit theorem (GCLT) [20] and are in the Lévy basin of attraction. When $1 < \mu < 2$, the mean time also becomes infinite; in this case the GCLT does not apply. These systems are *nonergodic*.

Equation (1) can be used to calculate the probability of having a laminar region that is at least as long as t (survival probability):

$$\Psi(t) = 1 - \int_0^t \psi(t) dt = \left(\frac{T}{T+t} \right)^{\mu-1}. \quad (2)$$

Another useful quantity that plays a key role in this article is the rate $R(t)$ at which new events are generated, given that an event occurred at $t = 0$. When $2 < \mu < 3$, we have [21]

$$R(t) \sim \frac{1}{\bar{t}} \left[1 + \left(\frac{T}{t} \right)^{\mu-2} \frac{1}{3-\mu} \right], \quad (3)$$

where \bar{t} is the first moment. In this case the system is Poissonian only in the infinite time limit. In the nonergodic regime ($1 < \mu < 2$), where the first moment diverges, Feller [22] demonstrated that the rate at which new events are generated is

$$R(t) \propto \frac{1}{t^{2-\mu}}, \quad 1 < \mu < 2. \quad (4)$$

The main implication of this result is that a nonergodic system is in a perennial nonequilibrium state, as the rate at which events are generated keeps decreasing forever (notice the difference with the usual Poissonian case, where this rate is constant). A direct consequence of Eq. (4) is that performing ensemble averages of statistical properties, related to renewal events for systems that have an event at $t = 0$, is different from making time averages of the same properties on a single system that was prepared at $t = 0$, since the latter averages change with time. This change of statistical properties with time is a consequence of the fact that they are linked to the rate of event generation. In other words, by definition, these latter systems are nonergodic, as we anticipated while discussing the properties of the moments of ψ .

In order to create a time series $\xi(t)$ for a complex system characterized by the above statistical properties, a value 1 or -1 is associated with each laminar region. At each renewal event a fair coin is tossed to decide whether to switch from one value to the other. The time series $\xi(t)$ allows us to define the autocorrelation function

$$\Phi(t, t') \equiv \langle \xi(t) \xi(t') \rangle, \quad (5)$$

which is needed when the LRT and GLRT are introduced.

As an aside, we notice that the power spectrum of $\xi(t)$ also depends on μ . In the Gauss basin of attraction [20], $\mu > 3$, the spectrum $S(f)$ for $f \ll 1$ is very flat as $\mu \rightarrow \infty$. For $2 < \mu (= 3 - \beta) < 3$, in the asymptotic region $t \gg T$, we have

$$S(f) \propto \frac{1}{f^\beta} \quad (6)$$

[20], which is $1/f$ noise with $\beta < 1$. When $\mu < 2$, we have [23]

$$S(f) \propto \frac{1}{L^{2-\mu} f^\beta}, \quad (7)$$

where L is the length of the time series. We also notice that, under the conditions $t \ll T/(\mu - 2)$ and $\mu > 2$, we have

$$S(f) \propto \frac{1}{f^2}, \quad (8)$$

the same result as that obtained for flicker noise.

III. GENERALIZED LINEAR RESPONSE THEORY: ENSEMBLE AVERAGE

Aquino *et al.* [5,6] applied GLRT to the case of one complex system perturbing another. In the following, the former is denoted by P (perturbing system), while the latter is denoted by S (responding system). Thus, the S system is characterized by the global variable $\xi_S(t)$ and is perturbed by the global variable $\xi_P(t)$. Conventional LRT [24] is given by

$$\langle \xi_S(t) \rangle = \epsilon \int_0^t \chi(t, t') \xi_P(t') dt', \quad (9)$$

where the symbol $\langle \xi_S(t) \rangle$ denotes the Gibbs ensemble average over infinitely many realizations of the response of $\xi_S(t)$ to $\xi_P(t)$. Without loss of generality, in the absence of perturbation this average is assumed to vanish. $\epsilon \ll 1$ is the stimulus strength. LRT predicts the response of S on the basis of the unperturbed autocorrelation function $\Phi_S(t, t')$ of $\xi_S(t)$. In fact, the function $\chi(t, t')$, called the linear response function (LRF), is related to the derivative of the autocorrelation function, normalized so that its quadratic mean value is 1. In LRT the autocorrelation function is assumed to depend only on the difference between t' and t (hence it is stationary, by definition), consequently the derivative with respect to t or t' can be taken, differing only by a change of sign [24].

When the statistics are nonstationary $\Phi_S(t, t')$ does not depend only on the difference between t' and t , so deriving by t' is in general different from deriving by t . If the generalized (G)LRF $\chi(t, t')$ is derived by differentiating the nonstationary correlation function with respect to $t' < t$, then we have [5]

$$\chi(t, t') = \frac{d\Phi_S(t, t')}{dt'} = R_S(t') \Psi_S(t - t'), \quad (10)$$

where the subscript indicates that the rate of generation of new events $R_S(t)$, the autocorrelation function $\Phi_S(t)$, and the survival probability $\Psi_S(t)$ are those of the responding system. In Ref. [25] this choice was referred to as phenomenological GLRT.

In Ref. [25] it is shown that deriving the GLRF $\chi(t, t')$ by differentiating the nonstationary correlation function $\Phi_S(t, t')$ with respect to t corresponds to the case in which the stimulus has the effect of weakly perturbing the time of occurrence of an event of S . The choice

$$\chi(t, t') = -\frac{d\Phi_S(t, t')}{dt} \quad (11)$$

was referred to by Allegrini *et al.* as dynamical GLRT and has proven to lead to a very accurate agreement with their experiment on the perturbation of a liquid crystal [26]. Although the adoption of the ensemble average done in Refs. [5,6] leads to correlation plottings between S and P that are very similar in both the phenomenological and dynamical cases, it is important to stress that the nonergodic complexity

management of this article is realized with the dynamical procedure.

The PCM is obtained by studying the cross correlation between ξ_S and ξ_P , normalized to ϵ , as a function of μ_S and μ_P , as $t \rightarrow \infty$:

$$\Phi_\infty = \lim_{t \rightarrow \infty} \frac{\langle \xi_S(t) \xi_P(t) \rangle}{\epsilon}. \quad (12)$$

Numerical investigations [5,6] have indicated a number of remarkable properties. For example, if the S system is ergodic and the P system is nonergodic, the cross correlation is maximum: this means that there is a flux of information from the P system to the S system (Wiener's rule). When the P system is ergodic and the S system is nonergodic, the asymptotic cross correlation vanishes; thus, there is no residual response of the S system to the P stimulus. Note that this was the domain that earlier investigators prematurely interpreted as the death of LRT. In the case in which both systems are ergodic, there is a partial positive correlation between S and P that changes with μ_S and μ_P ; as is the case when both systems are nonergodic.

The extraordinary results obtained using the asymptotic cross-correlation function have a fundamental limitation: the predictions of this form of PCM rely on ensemble averages. Thus, the predictions based on the cross correlation are not necessarily valid when we have only a single nonergodic time series for each system, that is, when we cannot apply the equivalence between ensemble averages and time averages. This is a common situation, since many interesting systems cannot be replicated. Consider the response of a single molecule to its environment [27] or a single brain to a unique stimulus, in both the response time series is unique, that is, one of a kind.

IV. ESTABLISHING COMPLEXITY MANAGEMENT WITH TIME RATHER THAN ENSEMBLE AVERAGES

We begin to address the limitation of a single time series by describing how the S system is stimulated. Recalling Eq. (1),

we note that there are two parameters that can be perturbed: μ and T . Since μ quantifies the complexity of the system, it is reasonable to expect that it can be forced to change only in response to very strong stimulation. A noninvasive perturbation, therefore, is expected to only change T . This restriction is in keeping with the dynamical approach to LRT [25] used in Ref. [26] to design the GLRT [5,6] that led to such remarkably good agreement with experimental observation.

The P system exerts its influence on the S system as follows: if S has an event at time t and if its next laminar region is assigned a value with the same sign as $\xi_P(t)$, then S is perturbed so that its next laminar region tends to be longer, by assigning to its parameter T in Eq. (1) the value $T_+ = T(1 + \epsilon)$. On the contrary, if the next laminar region of S has a value with the sign opposite to that of $\xi_P(t)$, then the value $T_- = T(1 - \epsilon)$ is used, thus tending to make the next laminar region shorter.

In order to assess the influence of P on S for a single time series, using this perturbation procedure, it is natural to consider a time window of size T_W and analyze the time-averaged cross-correlation function:

$$C(t_0, T_W) \equiv \frac{1}{T_W} \int_{t_0}^{t_0+T_W} \xi_S(t') \xi_P(t') dt'. \quad (13)$$

By moving the starting point t_0 of the window and evaluating C , a density plot for the time-averaged cross correlation can be created as a function of the power-law indexes. A measure of the influence of the P system on the S system is the center of gravity (COG) of this density plot. In the domain $1 < \mu_S, \mu_P < 2$, the COG of the density plot is erratic; in sharp contrast with the smooth behavior found in the calculations of the cross-correlation function in this region obtained using ensemble distribution functions by Aquino *et al.* [5,6]. This is clearly shown in Fig. 1(a), where C/ϵ is plotted as a function of μ_S and μ_P . It is worth noting that different realizations of the figure lead to different landscapes in the nonergodic quadrant. The reasons behind this behavior will become clear shortly.

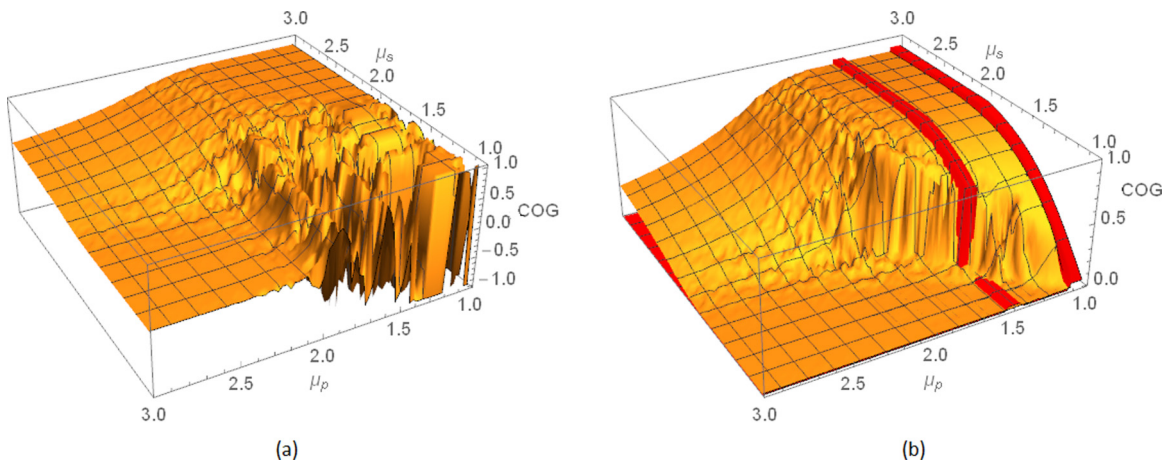


FIG. 1. COG as a function of the inverse-power-law indexes μ_S and μ_P . (a) The time sequence of length L is divided into L/T_W intervals of length T_W . The cross-correlation function of Eq. (13) is evaluated for each interval and the landscape is obtained by plotting the mean of the resulting distribution of values. (b) The times t_0 of Eq. (13) are the times of event occurrence and the landscape is obtained by plotting the mean of the resulting distribution of values. The analytical prediction of Eqs. (19) and (20) is shown by the two red (dark gray) lines.

The main contribution of this article has two parts. The first part is a data processing prescription that enables one to eliminate the erratic behavior observed in Fig. 1(a) and produce the smooth behavior of Fig. 1(b). In the second part we provide a theoretical justification for this prescription and calculate the asymptotic cross-correlation function analytically.

The prescription is to locate the beginning t_0 of the window at which each C is evaluated on an event of either the perturbing or the perturbed system.

We now present the theoretical foundations that led to the data processing prescription given above. We start by considering the following random quantity:

$$\bar{\xi}_S = \frac{1}{t} \int_0^t \xi_S(t') dt', \quad (14)$$

based on different realizations of the unperturbed ξ_S that was prepared so as to have an event at $t_0 = 0$. Notice that the beginning of the window is always located at $t = 0$, in contrast to Eq. (13). In the case of $\mu_S < 2$, it was shown by performing ensemble averages [20] that $\bar{\xi}_S$ is characterized by the Lamperti probability density function [28]:

$$\Pi(\bar{\xi}_S) = \frac{2}{\pi} \frac{(1 - \bar{\xi}_S^2)^{\alpha-1} \sin \pi \alpha}{(1 - \bar{\xi}_S^2)^{2\alpha} + (1 + \bar{\xi}_S^2)^{2\alpha} + (1 - \bar{\xi}_S^2)^{\alpha} \cos \pi \alpha}, \quad (15)$$

where α is $\mu_S - 1$, whose graph is depicted as the symmetric curve in Fig. 2. We notice that this distribution is clearly nonergodic as a single realization is most probably located around 1 or -1 , while the ensemble average is 0.

We now consider the time-averaged quantity

$$\bar{\xi} \equiv \frac{1}{T_W} \int_0^{T_W} \xi_S(t) \xi_P(t) dt, \quad (16)$$

which is obtained from Eq. (13) with $t_0 = 0$. We employ the same procedure followed in the calculation of $\bar{\xi}_S$. Bologna *et al.* [29], as well as Akimoto [30], demonstrated that the resulting distribution is a skewed Lamperti distribution

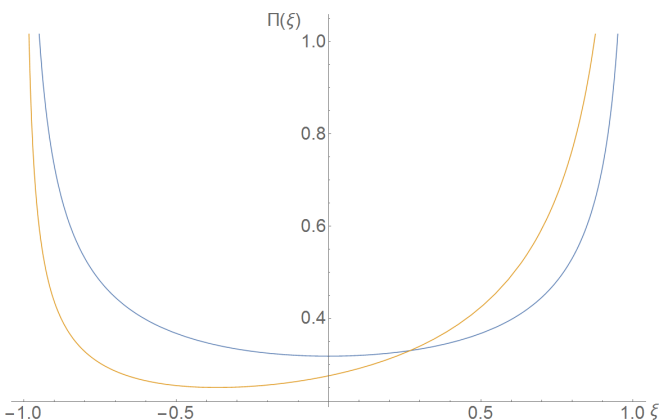


FIG. 2. Unperturbed (blue, symmetric, $\epsilon = 0$) and perturbed (red, asymmetric, $\epsilon = 0.5$) Lamperti distributions with $\mu = 1.5$.

given by

$$\Pi(\bar{\xi}) = \frac{2}{\pi} \frac{(1 - \bar{\xi}^2)^{\alpha-1} \sin \pi \alpha}{(1 - \bar{\xi})^{2\alpha} \eta + (1 + \bar{\xi})^{2\alpha} \frac{1}{\eta} + (1 - \bar{\xi}^2)^{\alpha} \cos \pi \alpha}. \quad (17)$$

The parameter η is responsible for the asymmetry of the curve in Fig. 2 and is related to the intensity of the perturbation ϵ by

$$\eta \equiv \left[\frac{1 + \epsilon}{1 - \epsilon} \right]^{\mu_S - 1}. \quad (18)$$

The COG of the density plot is given by

$$B(\mu_P, \mu_S) = \epsilon \frac{1 - \eta}{1 + \eta}. \quad (19)$$

These results are exact for $t_0 = 0$, but, as we discussed, in many real applications the distribution of C is necessarily determined from a moving time window.

In order to understand how these results can be useful in the latter case, we make some intuitive observations, followed by additional theory. We already noted that, when $\mu < 2$, the mean length of the laminar regions of a system diverges to infinity. This explains why we obtain the erratic plot in Fig. 1(a): for most of the duration of the time series there are no events, thus the cross correlation is either 1 or -1 . This fact can be exploited to obtain the regular behavior of Fig. 1(b): when one system has an event, it is most probably embedded in a long laminar region of the other system. If the P system has an event, then it is most likely embedded in a long laminar region of the S system. In this case the resulting value of C follows the statistics of the unperturbed Lamperti distribution given by Eq. (15), as the S system has no influence on the P system and the latter is nonergodic. If the S system has an event, then it is most likely embedded in a long laminar region of the P system, which is equivalent to saying that S is subject to constant stimulation. In this case the computed value of C follows the statistics of the perturbed Lamperti distribution given by Eq. (17).

The theory behind the above intuitive observations follows. The probability W_S of having an event in S at time t is given by

$$W_S(t) = \frac{R_S(t)}{R_S(t) + R_P(t)}, \quad (20)$$

with $R(t)$ given by Eq. (4) with the μ of the corresponding system. The probability W_P can be obtained from Eq. (20) by exchanging the roles of S and P . When $t \rightarrow \infty$, if $\mu_S > \mu_P$, we have $W_S = 1$ and $W_P = 0$; if $\mu_S < \mu_P$, then $W_S = 0$ and $W_P = 1$. As a side note we observe that this argument implies that the perturbed system does not respond asymptotically to simple perturbations, which corresponds to the phenomenon of habituation.

The red stripes superimposed on the numerical calculations in Fig. 1(b) are determined using Eqs. (19) and (20) and show excellent agreement with the numerical simulations. The above derivation is valid also in the case in which one of the systems is ergodic and the other is not ergodic: in the long-time limit, only the former has events. This fact and the considerations above

imply that, in complete agreement with the PCM, the response of an ergodic system to a nonergodic system is maximal. On the other hand, the response of a nonergodic system to an ergodic system vanishes. In the case in which both systems are ergodic, the above theory is not applicable, but, given the equivalence (by definition) of ensemble averages and time averages, in this case we again recover the results of the PCM, as expected.

V. PRACTICAL APPLICATIONS

Let us consider the diffusing variable $x(t)$ defined by

$$v(t) = \dot{x} = \xi_S(t)\xi_P(t). \quad (21)$$

Let us assume that the time series $v(t)$ goes from $t = 0$ to L , with the symbol L denoting the length of the time series. We are interested in the case where we have only one realization of $v(t)$. As a consequence to establish the influence that $\xi_P(t)$ exerts on $\xi_S(t)$ the prescription of Eq. (13) seems to be the most natural procedure to adopt, with t_0 moving from 0 to $L - T_w$. If L is sufficiently large, the adoption of Eq. (13) corresponds to generating a number, $L/T_w \gg 1$, of Gibbs realizations sufficient to establish a statistically significant influence of the stimulus ξ_P on the time series $\xi_S(t)$.

As we said, in the case where both S and P are the generators of nonergodic fluctuations, the adoption of the prescription Eq. (13) results in the erratic behavior shown by Fig. 1(a) for $1 < \mu_S, \mu_P < 2$. Our main result is that the beginning of the moving window, t_0 , must be chosen to coincide with the occurrence of a renewal event of either S or P . At first sight this choice is in apparent conflict with the fact that the time average is done only on the renewal events, which are a very small fraction of the total number of events. Naively, one would expect this prescription to make the statistical average less accurate. Instead the theoretical approach developed herein demonstrates that this choice of initial value leads to a smooth function of μ_S and μ_P , in very good agreement with numerical simulation.

To describe the practical applications of the results of this article, following Ref. [31], we establish a connection between two emerging theoretical perspectives that physicists are adopting in their attempts to address fundamental biological issues beyond the limits of reductionism. The former theoretical perspective is illustrated by the debate regarding the effect of finite size on criticality in natural swarms [32]. In fact, the recent experiment of Ref. [32] is attracting [33] the attention of researchers to the key role of criticality in biology, thereby leading them to look, for instance, at the very interesting proposals of Refs. [34,35]. In the conclusion of their article [35], Mora and Bialek emphasize the phase-transition-related property of *critical slowing down*, namely, the infinitely slow regression to equilibrium of processes at criticality. They point out, however, the existence of a possible conflict with the resilience of complex biological systems that are expected to promptly adapt themselves to the changes of their environment. Flocks of birds [32] are an outstanding example of biological resilience.

An interesting experiment concerning the cognition of living beings is given by the work done at Duke University by the group of Nicoliles [36]. In this experiment, information was transmitted from rat A moving in a box to rat B moving in

a different box through a cable connecting the neural network of the brain of rat A to the neural network of the brain of rat B . This experiment is the *in vivo* counterpart of an *in vitro* experiment done in 1999 at the University of North Texas by the group of Gross [37], interpreted by them as a form of chaos synchronization. Thus, we see rapid progress from the 1999 *in vitro* experiment [36] that culminates in the 2014 experiment [38], concerning the same kind of information transfer from the brain of one human subject to that of another.

This form of synchronization seems to be a natural property of the dialogue between two individuals [39], though the statistical roots of these synchronization processes has remained unknown. The latter remarkable example of biological complexity is illustrated in a recent review paper [40], reflecting the growing interest regarding anomalous diffusion in biological cells, a paradigm of the special nature of biological processes.

It turns out that the connection between these two forms of biological complexity requires a deeper understanding of the origin of ergodicity breakdown. We invite the readers to focus their attention on the recent theoretical remarks in Ref. [41]. Critical slowing down is a property of critical systems involving interactions among an infinitely large number of units. The authors of Ref. [41] emphasized the importance of *temporal complexity*, which must not be confused with critical slowing down, even if in some processes of phase transitions the inverse-power-law index of temporal complexity is the same as that of critical slowing down. At the onset of criticality, a complex system makes a transition from a condition, where the single units are essentially statistically independent of one another, to an organized state of highly correlated behavior. This condition, however, does not last forever, and, from time to time, a system, with a finite number of interacting units, undergoes organizational collapses [42]. We have interpreted these intermittent collapses as *free will states* [43]. These free will states are renewal events and consequently make the dynamics of criticality nonergodic, thereby establishing a connection, while retaining the distinction, between temporal complexity and critical slowing down in complex biological networks.

It is important to stress that, as previously shown [41], ergodicity breaking is confined to the time region $t < T_{\text{eq}}$, where $T_{\text{eq}} \propto \sqrt{N}$ and N denotes the number of interacting units. We believe this to be a general property of criticality and that evaluating the transmission of information from one complex system at criticality to another complex system at criticality in the time scale $t > T_{\text{eq}}$ [43–47] gives the misleading impression that the network entrainment one finds may be a form of chaos synchronization [37]. This is a consequence of the fact that an evaluation of the correlation between the perturbed complex network and its stimulus done in the time scale $t < T_{\text{eq}}$ would generate the erratic behavior shown by Fig. 1(a) for $1 < \mu_S, \mu_P < 2$, giving the false impression of a lack of correlation.

This leads us to another practical application of the present results: the dynamics of the brain. The widely accepted belief that brain activity is not dominated by a characteristic time scale has led the investigators in this field of research to make experimental observations that, in turn, have revealed the crucial role of renewal events. According to analysis of the EEG of individuals performing a task done by Buiatti *et al.* [48], brain activity reveals the action of renewal events

with the waiting time distribution density of Eq. (1) with the power index μ in the region $1 < \mu < 3$. More precisely moving from one patient to another the index μ is found to fluctuate from values of $\mu < 2$ to values of $\mu > 2$. As pointed out in this article, the region $\mu < 2$ is not ergodic. As a consequence, there are aging effects that make the statistical evaluation of the effects of perturbation very challenging. The authors of Ref. [48] had to use the theoretical procedure of Ref. [49] to evaluate the scaling of the time series generated by a single EEG. The aging procedure of Ref. [49] is equivalent to establishing the index μ from the observation of the time series $\xi_S(t)$ of Eq. (21) in the absence of a stimulus. In other words, the theoretical approach of Ref. [49] is equivalent to studying the scaling of the diffusing variable $x(t)$ generated by

$$\dot{x}(t) = \xi_S(t), \quad (22)$$

namely, from Eq. (21) when $\xi_P(t)$ is time independent.

Buiatti *et al.* [48] used this procedure to establish for some subjects the power index $\mu < 2$. It is really remarkable that the research work done by the neurophysiologists of the University of Pisa [50] led to the same conclusion, with the additional important remark that the anomalous scaling properties revealed by their experimental observations are, in fact, a manifestation of temporal complexity [51]. The discovery that, in the case of the dynamics of the brain, the power-law index of ξ_S of Eq. (21) may have the nonergodic value of $\mu_S < 2$ raised the challenging problem of assessing the cross correlation between S and P . This problem, very difficult in the case when both S and P are nonergodic, is

now satisfactorily resolved, both theoretically and numerically, using the methods developed herein.

In conclusion, the theoretical and numerical results of this article afford a solid procedure to analyze the time series $v(t)$ of Eq. (21), thereby opening the road to studying the entrainment between a perturbed complex network and a stimulus that can be derived from another network with the same temporal complexity, without confining the analysis to the long-time regime that may not be available due to the finite size of the experimental time series. On the other hand, the theoretical analysis, which may be done in the long-time limit with surrogate sequences, would prevent the investigators from realizing that the important transfer of information is realized in the short-time limit where both networks must be thought of as nonstationary and nonergodic complex systems. The list of practical applications is not limited to networks of neurophysiological interest. We believe that the emergence of cognition occurs at criticality [52]. Criticality generates temporal complexity [41,50,51] and consequently the analysis of the transfer of information from a sociological (psychological) network to another requires the adoption of the theoretical and numerical prescriptions afforded by this article.

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