

## Percolation model with an additional source of disorder

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The ranges of transmission of the mobiles in a mobile *ad hoc* network are not uniform in reality. They are affected by the temperature fluctuation in air, obstruction due to the solid objects, even the humidity difference in the environment, etc. How the varying range of transmission of the individual active elements affects the global connectivity in the network may be an important practical question to ask. Here a model of percolation phenomena, with an additional source of disorder, is introduced for a theoretical understanding of this problem. As in ordinary percolation, sites of a square lattice are occupied randomly with probability  $p$ . Each occupied site is then assigned a circular disk of random value  $R$  for its radius. A bond is defined to be occupied if and only if the radii  $R_1$  and  $R_2$  of the disks centered at the ends satisfy a certain predefined condition. In a very general formulation, one divides the  $R_1$ - $R_2$  plane into two regions by an arbitrary closed curve. One defines a point within one region as representing an occupied bond; otherwise it is a vacant bond. The study of three different rules under this general formulation indicates that the percolation threshold always varies continuously. This threshold has two limiting values, one is  $p_c(\text{sq})$ , the percolation threshold for the ordinary site percolation on the square lattice, and the other is unity. The approach of the percolation threshold to its limiting values are characterized by two exponents. In a special case, all lattice sites are occupied by disks of random radii  $R \in \{0, R_0\}$  and a percolation transition is observed with  $R_0$  as the control variable, similar to the site occupation probability.

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A simple way to describe the phenomenon of percolation is to consider a rectangular slab of porous material placed horizontally and ask, if some liquid is poured on the top surface, will it appear at the bottom surface? The answer is yes (no), depending on if the fraction  $p$  of the porous volume is larger (smaller) than a threshold value  $p_c$  of the porosity [1–3]. It was Broadbent and Hammersley who introduced the percolation model by occupying (pore space) randomly the sites of a regular lattice with probability  $p$  and keeping them vacant (rock matrix) with probability  $1 - p$  while trying to understand better the mechanism of gas masks [4]. The percolation model can also be described by randomly occupying the bonds of the lattice. At present, the percolation model is considered one of the simplest models for introducing students to the study of order-disorder phenomena [5].

Consider a pair of connected neighboring occupied sites. A group of such occupied sites forms a cluster if they are reachable by the nearest neighbor connections. Any two occupied sites, separated at a certain distance, are connected if both belong to the same cluster of occupied sites. The correlation between them decreases as their distance of separation increases and its functional form is exponential when the distance is large. The length scale that characterizes such a form is known as the correlation length  $\xi(p)$ , which diverges as  $p$  approaches a critical value  $p_c$ , known as the percolation threshold, which marks the transition point between the ordered and disordered phases. The best value of  $p_c(\text{sq})$  for site percolation on the square lattice is 0.592 746 050 792 10(2) [6] and  $1/2$  for the bond percolation [7]. In both cases, the nature of transition is continuous and they belong to the same universality class.

Over the years a number of variants of the percolation model have been studied [8]. In the continuum percolation [9,10], one finds the minimal density of equal-size overlapping lilies, floating at random positions on the water surface of a pond, such that an ant will be able to cross the pond walking on the

lilies [2]. In a mobile *ad hoc* network (MANET) each node represents a mobile phone with a fixed range of transmission that is capable of receiving as well as transmitting signals [11]. Depending on the value of the range, there exists a critical density of lilies or phones where the long-range correlation appears [1].

Recently, it was suggested that a discontinuous transition may be possible in a model of percolation and termed it explosive percolation [12–15]. Subsequently, it has been shown that, though such models show very sharp changes in their order parameters for finite-size systems and therefore appear like discontinuous transitions, they indeed exhibit a continuous transition in the asymptotic limit of very large system sizes [16–19].

Here we introduce a very general formulation of the percolation model. Sites of a square lattice of size  $L \times L$  are occupied randomly using circular disks of random radii values  $R$ . The transmission range of a mobile phone in a MANET may be compared to the radius  $R$  of a disk. This range is affected by the temperature fluctuation in air, obstruction due to the solid objects, humidity difference in the environment, etc., and therefore, assuming random values for the radii of the disks is a better description than using the identical disks. In this prescription, a bond is defined to be occupied if and only if the radii  $R_1$  and  $R_2$  of disks centered at the ends satisfy a certain predefined rule; otherwise it is vacant. Most generally, the  $R_1$ - $R_2$  plane is divided into two different regions by an arbitrary closed curve. Any point within one region represents an occupied bond; otherwise it is a vacant one. The percolation thresholds vary continuously between  $p_c(\text{sq})$  and unity.

The radii  $R$  of the disks are drawn from a uniform rectangular distribution  $P(R)$  of half width  $W$  and the center at  $R = 1/2 + S$ , where  $S$  denotes the shift parameter. For the simulation, a random number  $r \in \{0, 1\}$  from a uniform distribution is assigned at each lattice site to calculate  $R = 1/2 + S + (2r - 1)W$ .

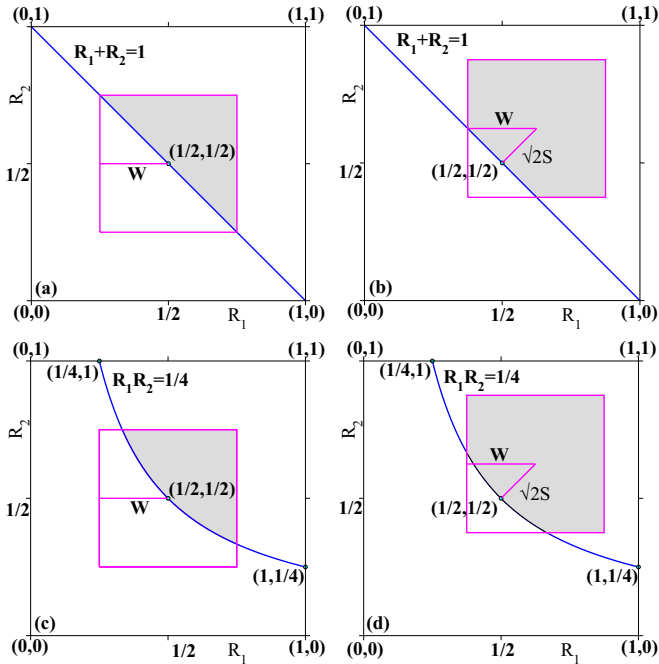


FIG. 1. On the  $R_1$ - $R_2$  plane, for a specific set of values of  $W = 1/4$  and  $S$ , the regions corresponding to the occupied bonds (gray) and unoccupied bonds (white) are indicated. For the sum rule (a)  $S = 0$  and (b)  $S = 1/8$  and for the product rule (c)  $S = 0$  and (d)  $S = 1/8$ .

*The sum rule.* A bond is occupied if and only if

$$R_1 + R_2 \geq 1. \quad (1)$$

For a given pair of  $S$  and  $W$ , the points in the  $R_1$ - $R_2$  plane, representing the occupied and vacant bonds, lie within a square box (Fig. 1). In Figs. 1(a) and 1(b) we exhibit two specific cases with  $S = 0$  and  $1/8$ , respectively, where  $W = 1/4$ . A typical picture of a percolating configuration for the sum rule is shown in Fig. 2.

To generate a single percolation configuration with the occupation probability  $p$ , we start from an empty square lattice of size  $L$  and then drop  $pL^2$  disks, one by one, onto the lattice sites. At every step, an arbitrary site  $i$  is randomly selected and if it is vacant, a disk with a randomly selected radius  $R_i$  is placed at this site. Once  $pL^2$  sites are occupied, all four neighboring bonds of every occupied site are then tested for possible occupation. The number of occupied bonds an occupied site may have varies from 0 to 4 even if all neighboring sites are occupied. In this way, all bonds are assigned their occupied or vacant status. A cluster is a set of occupied sites interlinked by occupied bonds. A random configuration  $\alpha$  has a number of clusters of different shapes and sizes. The size  $s$  of a cluster is the number of sites in the cluster and the size of the largest cluster is denoted by  $s_{\max}^\alpha(p, L)$ . The order parameter  $\Omega(p, L)$  is defined by the configuration averaged fractional size of the largest cluster, i.e.,  $\Omega(p, L) = \langle s_{\max}^\alpha(p, L) \rangle / L^2$ .

By definition, as  $p$  is gradually increased, the largest cluster grows monotonically. Around the transition point, it makes several jumps in size when it merges with other clusters. For an arbitrary configuration, the largest cluster executes the

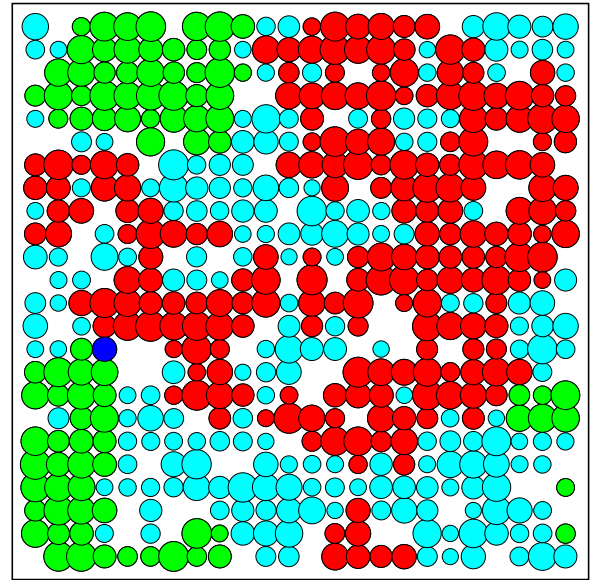


FIG. 2. Percolating configuration of 501 circular disks drawn using the sum rule (with periodic boundary conditions) for  $L = 24$ ,  $W = 0.15$ ,  $S = 0$ , and  $p \approx 0.87$ . The largest and the second largest clusters are of sizes 208 (red) and 90 (green), respectively. Because of the blue disk, these two clusters merge and the maximal jump in the order parameter takes place. Disks at all other occupied sites are in cyan.

maximal jump  $\Delta_m s_{\max}^\alpha(p, L)$  at  $p = p_c^\alpha$ , when it merges with the maximal of the second largest cluster [20]. An average over many such configurations is considered as the percolation threshold  $p_c(L) = \langle p_c^\alpha \rangle$  for the system of size  $L$ .

For the percolation model, it is well known that the correlation length diverges like  $\xi(p) \propto |p_c - p|^{-\nu}$  as  $p \rightarrow p_c$  for the infinite system, where  $\nu$  is the correlation length exponent and its value is  $4/3$  in two dimensions [1,21]. However, for a finite-size system  $\xi$  may be at most  $L$  and that is attained at  $p = p_c(L)$ . Therefore, one gets  $p_c(L) = p_c - AL^{-1/\nu}$  and the asymptotic value of  $p_c$  is obtained by extrapolating  $p_c(L)$  against  $L^{-1/\nu}$ . It is also known that right at the percolation threshold the largest cluster is a fractal object and its size grows as  $\langle s_{\max}^\alpha(p_c, L) \rangle \sim L^{d_f}$ , where  $d_f$  is its fractal dimension in two dimensions [22]. Similarly, the maximal of the second largest cluster is also a fractal with the same fractal dimension  $d_f$ . As a consequence, the amount of the maximal jump in the order parameter decreases with increasing  $L$  as  $\langle \Delta_m s_{\max}^\alpha(p_c, L) \rangle / L^2 \sim L^{d_f-2}$ .

For  $S = 0$  and  $W = 0$ , the bond between any pair of neighboring occupied sites is occupied. Therefore,  $p_c(S = 0, W = 0) = p_c(\text{sq})$ . When  $W > 0$ , though only half of the disks have radii larger than  $1/2$ , a global connectivity can still be achieved. The small size disks certainly contribute to the density of occupied sites but may or may not take part in the bond density. Consequently, it takes the higher density of occupied sites to attain the global connectivity. The growth of the largest cluster is therefore retarded, i.e.,  $p_c(S = 0, W > 0) > p_c(\text{sq})$ . Again, because of the small disks, in the limit of  $p \rightarrow 1$ , the size  $s_{\max}^\alpha(p, L) / L^2$  converges to a value that is well below unity and depends on the parameters  $S$  and  $W$ .

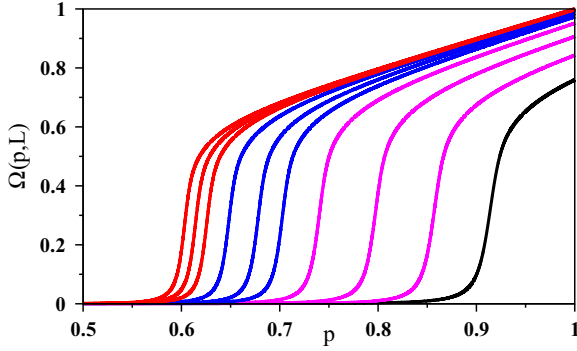


FIG. 3. For the sum rule, the order parameter  $\Omega(p, L)$  is plotted against the probability  $p$  for  $L = 512$  with red for  $S = 0.03$  and  $W = 0.04, 0.045, 0.05$ ; blue for  $S = 0.02$  and  $W = 0.04, 0.05, 0.06$ ; magenta for  $S = 0.01$  and  $W = 0.04, 0.2/3, 0.15$ ; and black for  $S = 0$  and  $W = 0.04$ ; the curves are arranged from left to right.

The  $p_c(L)$  values are extrapolated against  $L^{-1/\nu}$  with different trial values of  $\nu$ . The best fit corresponds to  $\nu = 1/0.7502 \approx 1.333(5)$  and  $p_c(S = 0, W > 0) \approx 0.9191(2)$ . This is independent of  $W$  since the bond occupation probability is  $1/2$  for all values of  $W > 0$ . In addition, the average fractional size of the largest cluster has been found to decay like  $L^{-0.105}$  and gives an estimate of  $d_f = 1.895(5)$  compared to the exact value of  $d_f = 91/48$  [1]. The average value of the maximal jump in the largest cluster varies as  $L^{-0.104}$  and equating the power to  $d_f - 2$  one gets  $d_f = 1.896(5)$ .

Figure 3 exhibits the variation of the order parameter  $\Omega(p, L)$  against the site occupation probability  $p$ . For  $S = 0$ , the curve is independent of  $W$ . Further, for a fixed value of  $S > 0$ , the curve shifts to higher values of  $p$  as  $W$  increases, whereas for a fixed value of  $W$  the curve shifts towards the smaller values of  $p$  as  $S$  increases. Numerically it appears that  $p_c(L)$  depends only on ratio of  $S$  and  $W$ .

For  $S > 0$ , in the limit of  $L \rightarrow \infty$ , first the extrapolated values  $p_c(S, W)$  are calculated. Then a scaling analysis is done where we plot  $p_c(S, W) - p_c(S)$  against  $W/S - 1$  in Fig. 4 and obtain a good data collapse. Tuning the values of  $p_c(S)$ , the curves for different  $S$  fit to a straight line as  $W/S - 1 \rightarrow 0$ , indicating a scaling form

$$p_c(S, W) - p_c(S) \sim (W/S - 1)^{\zeta_S}, \quad (2)$$

where we estimated  $\zeta_S = 1.95(5)$ . The best tuned values of  $p_c(S)$  are consistent with  $p_c(\text{sq})$ .

On the other hand, when  $S$  is negative, the vacant area in Fig. 1(a) increases, the occupied area decreases, and therefore the percolation threshold increases. For a specific threshold value of  $S = S_c = -0.0201(5)$  the  $p_c(S_c) = 1$  for  $W = 1/4$ . It has been observed that  $p_c(S_c, W) - p_c(S, W) \sim (S - S_c)^{\eta_S}$  with  $\eta_S \approx 1.003(5)$ . For other  $W$  values  $S_c(W)$  varies, but  $S_c(W)/W$  remains constant.

*The product rule.* Here the condition for occupation of a bond is

$$R_1 R_2 \geq 1/4. \quad (3)$$

Figures 1(c) and 1(d) represent occupied and vacant bonds determined by the product rule for  $S = 0$  and  $1/8$ , respectively, with  $W = 1/4$ .

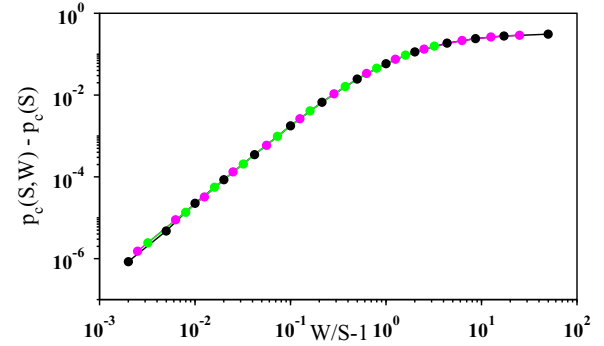


FIG. 4. For the sum rule, the scaling plot of  $p_c(S, W) - p_c(S)$  against  $W/S - 1$  is shown for  $S = 0.1$  (green),  $0.01$  (magenta), and  $0.001$  (black). The values of  $p_c(S)$  required to make the curves straight in the  $W/S \rightarrow 1$  limit are  $0.5927675$ ,  $0.5927684$ , and  $0.5927662$ , respectively, which are very close to  $p_c(\text{sq})$ . The slopes of the linear portions are  $1.96$ ,  $1.93$ , and  $1.94$  respectively, giving  $\zeta_S = 1.95(5)$ .

It can be seen from the Fig. 1(c) that for  $S = 0$ , the probability of an occupied bond (the shaded area) for the product rule decreases with increasing  $W$  and for this reason the order parameter depends explicitly on the value of the width  $W$  and the critical percolation probability increases with  $W$ . On the other hand, for a general value of  $S > 0$ , the  $\Omega(p, L)$  plots are quite similar to those of the sum rule, but  $p_c(L)$  values are slightly larger. First, the asymptotic values of the critical percolation probabilities  $p_c(S, W)$  for  $S = 0$  and  $W \rightarrow 0$  are again found to be  $0.9191(2)$ . For  $S > 0$ , again a scaling plot of  $p_c(S, W) - p_c(S)$  against  $W/S - 1$  gives a very nice data collapse and we find  $\zeta_P \approx 1.93(10)$ . Here also the shift  $S$  may take negative values, so the percolation threshold would increase to unity, i.e.,  $p_c(S_c, W) = 1$  for  $S_c = -0.0117(5)$  for  $W = 1/4$ . The approach to this limit is again characterized by  $\eta_P \approx 1$ .

*The circular rule.* Here a circular region, centered around the point  $(1/2, 1/2)$  of radius  $\Delta$  in the  $R_1$ - $R_2$  plane, is selected. The radii  $R$  of the disks are again distributed by  $P(R)$ , but only  $S = 0$  and  $W = 1/2$  are used. The region inside the circle represents the occupied bonds, whereas the outside region represents the vacant bonds.

Evidently, the critical percolation threshold  $p_c(\Delta, L)$  depends on the value of  $\Delta$ . It has been observed that if the size of the circular region is too small, the size of the largest cluster becomes minuscule even when the occupation probability  $p = 1$ . Consequently, one defines a threshold value  $\Delta_c$  such that a global percolation transition can occur only when  $\Delta > \Delta_c$ . Clearly, the critical percolation probability at  $\Delta_c$  is denoted by  $p_c(\Delta_c) = 1$ . As before,  $p_c(\Delta_c) - p_c(\Delta)$  varies as  $(\Delta - \Delta_c)^{\eta_C}$ . The best-fit value of  $\Delta_c$  is found to be  $0.3488(5)$  with  $\eta_C \approx 0.96(5)$ . Also, the other limit corresponds to  $\Delta_L = 1/\sqrt{2}$  when all points in the  $R_1$ - $R_2$  plane correspond to the occupied bonds. In this case  $p_c(\Delta) - p_c(\text{sq})$  varies as  $(\Delta_L - \Delta)^{\zeta_C}$  and we estimate  $\zeta_C \approx 1.95(5)$ .

Our model is distinctly different from the random site-bond percolation [23,24]. In this model, sites and bonds of the same lattice are occupied independently. A connecting path is therefore a sequence of alternate occupied sites and bonds and

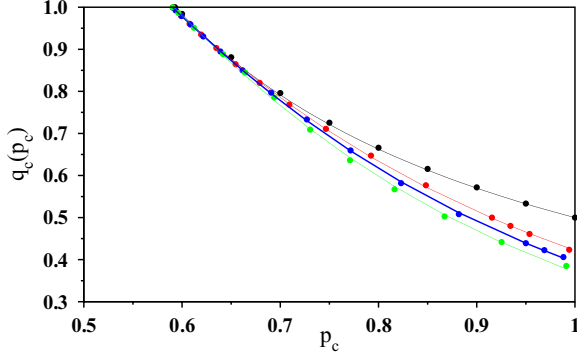


FIG. 5. Critical values of the site  $p_c$  and the bond  $q_c$  occupation probabilities are plotted for the site-bond percolation [24] (black), sum rule (red), product rule (blue), and circular rule (green) (in the sequence from top to bottom). The solid lines are the best-fit forms given in Eq. (4).

the global connectivity is determined by the appearance of such paths across the system. In comparison, in our model when two neighboring sites are occupied, the occupied or vacant status of the bond between them is immediately determined, subject to the fulfillment of certain condition.

This difference shows up in the following example. In Fig. 1(a), the gray area represents the bond occupation probability  $q = 1/2$ , where the percolation threshold is estimated to be  $p_c \approx 0.9191$ . This is clearly different from the random site-bond percolation on square lattice, which gives  $p_c = 1$  when  $q_c$  is set at  $1/2$  [24].

In random site percolation, the bond density grows with the site density as  $q(p) = p^2$ . In comparison, in our case, this form is modulated by a function as  $q(p) = \mathcal{H}(S, W)p^2$ , where, for the sum rule,

$$\mathcal{H}(S, W) = 1/2 + S/W - S^2/(2W^2) \quad \text{for } S > 0,$$

$$\mathcal{H}(S, W) = 1/2 - S/W + S^2/(2W^2) \quad \text{for } S < 0.$$

For the product rule, there exists a threshold value  $S_W$  such that, for  $S \leq S_W$ ,

$$4W^2\mathcal{H}(S, W) = (S + W)^2 + (S + W) - \ln(1 + 2S + 2W)/2,$$

$$4W^2\mathcal{H}(S, W) = 4W^2 - (S - W)^2 - (S - W) + \ln(1 + 2S - 2W)/2$$

for  $S \geq S_W$ , where  $S_W = [(1 + 4W^2)^{1/2} - 1]/2$ . Our numerical estimations are very much consistent with these expressions.

In Fig. 5 we show the phase diagram, similar to the site-bond percolation. The phase space in this diagram is divided into two regions, namely, the percolating and the nonpercolating regions. Therefore, every point on the boundary between the two regions signifies a critical point, represented by  $[p_c, q_c(p_c)]$ . The data for the random site-bond percolation have been collected from [24]. Similar phase boundaries for the sum, product, and circular rules have also been shown for comparison. All four phase boundaries are completely distinct in general, but they meet only at the point  $[p_c(\text{sq}), 1]$ . For the random site-bond percolation, the functional form of the

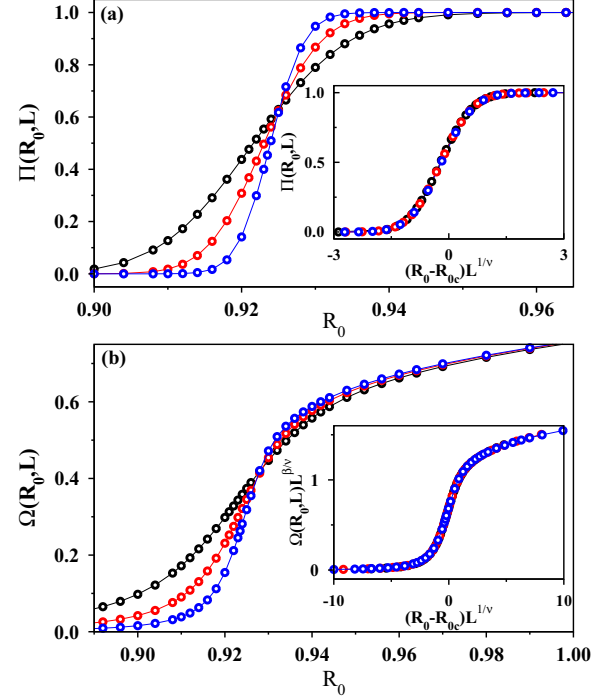


FIG. 6. For the system sizes  $L = 256$  (black),  $512$  (red), and  $1024$  (blue) with  $R_{0c} = 0.925$ ,  $1/\nu = 0.75$ , and  $\beta/\nu = 0.11$ . (a) Plot of the percolation probability  $\Pi(R_0, L)$  against  $R_0$ . The inset shows a scaling by  $(R_0 - R_{0c})L^{1/\nu}$  exhibiting the data collapse. (b) Plot of the order parameter  $\Omega(R_0, L)$  against  $R_0$ . The inset shows a scaling by  $\Omega(R_0, L)L^{\beta/\nu}$  against  $(R_0 - R_{0c})L^{1/\nu}$  exhibiting an excellent data collapse.

critical curve is  $q_c(p_c) = B/(A + p_c)$  [24] and is represented by the black solid line. Here we have tried a modified functional form to fit our data as

$$q_c(p_c) = B/(A + p_c^\theta) \quad (4)$$

and we have observed that  $\theta = 2.41, 2.70$ , and  $2.81$  for the sum, product, and circular rules, respectively. For the sum and product rules  $W = 1/4$  is used.

A very interesting special case of our model is the situation when all sites of the lattice are occupied ( $p = 1$ ) by disks of uniformly distributed radii  $R \in \{0, R_0\}$ . A related model in continuum percolation considers disks of randomly selected radii [25,26]. The set of occupied bonds is then determined by the sum rule using the periodic boundary condition along the horizontal direction and the open boundary condition along the vertical direction. For any value of  $R_0 < 1/2$ , none of the bonds become occupied. When  $R_0$  is further increased, the size of the largest cluster exhibits a sharp increase, similar to the ordinary percolation, for a critical value  $R_{0c}$ . We defined  $\Pi(R_0, L)$  as the spanning probability from the top to the bottom of the lattice. We also calculated the order parameter  $\Omega(R_0, L) = \langle s_{\max}^\alpha(R_0, L) \rangle / L^2$ .

In Fig. 6(a) we plot  $\Pi(R_0, L)$  against  $R_0$  for three different system sizes that meet at approximately same value of  $R_0 = R_{0c} = 0.925(5)$ . A finite-size scaling of  $\Pi(R_0, L)$  plotted against the scaled variable  $(R_0 - R_{0c})L^{1/\nu}$  with  $1/\nu = 0.75$

works very well [Fig. 6(a) inset], implying

$$\Pi(R_0, L) \sim \mathcal{F}[(R_0 - R_{0c})L^{1/\nu}]. \quad (5)$$

In Fig. 6(b) we plot  $\Omega(R_0, L)$  against  $R_0$  and the scaling form [Fig. 6(b) inset]

$$\Omega(R_0, L)L^{\beta/\nu} \sim \mathcal{G}[(R_0 - R_{0c})L^{1/\nu}] \quad (6)$$

works excellently. Compared with the ordinary percolation, we recognize  $\nu$  as the correlation length exponent and  $\beta$  as the order parameter exponent. Our best collapse of the data corresponds to  $1/\nu = 0.75$  and  $\beta/\nu = 0.110(5)$ . These values are to be compared with the exact values of the two-dimensional percolation exponents  $\nu = 4/3$  and  $\beta = 5/36$ , i.e.,  $\beta/\nu = 5/48 \approx 0.1042$  [27,28]. The entire calculation has been repeated using the product rule and the results are found to be very similar to those of the sum rule except for  $R_{0c} = 0.978(5)$  and  $\beta/\nu \approx 0.104(5)$ .

To summarize, in the statistical physics framework of the percolation phenomena we have attempted to study the global connectivity problem in a mobile *ad hoc* network, where all active elements are not of uniform transmitting capacities. Transmission ranges of different mobile elements may be different. We asked the question of whether the network is still

globally connected. Our theoretical study in this paper answers this question in the affirmative, which is also interesting from the point of view of critical phenomena of disordered systems.

A very general percolation problem has been formulated with two different types of randomness. A bond is occupied if the pair of neighboring disks of randomly distributed radii  $R_1$  and  $R_2$  fulfills a certain condition. Such a condition is most generally described by dividing the  $R_1$ - $R_2$  plane into two regions by a closed curve of arbitrary shape: One region represents the connected bonds, whereas the other region represents the vacant bonds. The percolation threshold varies within  $p_c(\text{sq}) \leq p_c \leq 1$ . The nature of the percolation transition is continuous, but the approach of the percolation threshold to its limiting values is described in terms of the exponents  $\zeta$  and  $\eta$ . Moreover, our analysis even on a fully occupied lattice reveals that a percolation transition can occur where the control parameter is the maximal radius  $R_0$  of the disks. The set of critical exponents exhibits excellent agreement with those of the ordinary percolation, implying that both may belong to the same universality class.

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