Probability characteristics of nonlinear dynamical systems driven by δ-pulse noise

Alexander A. Dubkov,[1,*] Oleg V. Rudenko,[2,3,4,1,5,†] and Sergey N. Gurbatov[1,‡]

[1]*Radiophysics Department, Lobachevsky State University, Gagarin Ave. 23, 603950 Nizhni Novgorod, Russia*
[2]*Faculty of Physics, Lomonosov Moscow State University, Leninskie Gory, 119991 Moscow, Russia*
[3]*Prokhorov General Physics Institute, Russian Academy of Sciences, ul. Vavilova 38, 119991 Moscow, Russia*
[4]*Schmidt Institute of the Earth, Russian Academy of Sciences, ul. B. Gruzinskaya 10, 123242 Moscow, Russia*
[5]*Blekinge Institute of Technology, SE-371 Karlskrona, Sweden*

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For a nonlinear dynamical system described by the first-order differential equation with Poisson white noise having exponentially distributed amplitudes of δ pulses, some exact results for the stationary probability density function are derived from the Kolmogorov-Feller equation using the inverse differential operator. Specifically, we examine the "effect of normalization" of non-Gaussian noise by a linear system and the steady-state probability density function of particle velocity in the medium with Coulomb friction. Next, the general formulas for the probability distribution of the system perturbed by a non-Poisson δ-pulse train are derived using an analysis of system trajectories between stimuli. As an example, overdamped particle motion in the bistable quadratic-cubic potential under the action of the periodic δ-pulse train is analyzed in detail. The probability density function and the mean value of the particle position together with average characteristics of the first switching time from one stable state to another are found in the framework of the fast relaxation approximation.

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I. INTRODUCTION

As is well known, an adequate description for a broad class of continuous random processes that are frequently met in physical, biological, and chemical systems can be provided by excitation in the form of Gaussian white noise. At the same time, in electronics, optics, neurodynamics, and acoustics, there are other stochastic processes that exhibit instantaneous discrete jumps and therefore must be modeled differently in terms of δ-pulse noise. In contrast to Gaussian white noise, the statistical characteristics of these disturbances may be different and are determined by the statistics of pulse amplitudes and intervals between stimuli.

The well-known experimentally observed pulse noise is so-called shot noise. Shot noise, first discovered in vacuum tubes [1], is a basic manifestation of the discreteness of charge carriers, and it occurs in many solid-state devices such as tunnel junctions [2], Shottky barrier diodes, *p-n* junctions [3], quantum point contacts [4], mesoscopic conductors [5], and even thin-film metallic resistors [6]. It also occurs in photon counting in optical devices, such as photomultiplier tubes and avalanche photodiode detectors [7], where shot noise is associated with the particle nature of light. This is therefore another consequence of discretization, in this case of the energy in the electromagnetic field in terms of photons.

To describe the shot noise, there is an appropriate mathematical model of a δ-pulse train, which occurs at times forming a Poisson point process (Poisson white noise). The probability characteristics of nonlinear dynamical systems perturbed by Poisson white noise can be analyzed in the framework of Markovian theory, and they are described by the Kolmogorov-Feller equation [8], which is a generalization of the Fokker-Planck equation for Gaussian white noise. Moreover, Gaussian white noise can be obtained as a limiting case of Poisson white noise [9].

Processes driven by Poisson white noise have been studied for various problems, such as parametric and nonlinear oscillators [10–16], transport of Brownian particles in spatially periodic potentials [17–19], thermal ratchets [20–22], active Brownian motion [23,24], thermally activated switching [25], energetics of classical stochastic systems [26], neocortical model neurons [27], population dynamics [28–31], vibro-impact Duffing systems [32], and a quantum absorption refrigerator [33].

The analytical treatment of dynamical systems driven by Poisson white noise poses more difficulties compared to systems driven by Gaussian white noise. As a result, the exact expressions for the probability characteristics of these systems have been derived only for very limited cases, particularly if the amplitudes of pulses are exponentially distributed [9,17,34–36]. Also, analytical studies have been pursued for dynamical properties such as mean first-passage times [37]. It is worth mentioning some approximate methods proposed for investigations of nonlinear systems driven by Poisson white noise, such as the cumulant truncation procedure [12], the exponential-polynomial closure method [13], the generalized cell mapping method [30], the path-integral approach [38], numerical integration schemes [19,39], and adiabatic elimination [40]. The famous Itó-Stratonovich dilemma that arises when one examines a nonlinear dynamical system with multiplicative Poisson white noise was discussed in the literature as well [41].

In this study, the case of external excitation by a sequence of stimuli with non-Poisson statistics is extremely difficult, because the resulting process is non-Markovian. Here, the nonlinear dynamical system perturbed by the δ-pulse train with arbitrary probability distributions of amplitudes and time intervals between pulses is considered. In addition to obtaining some exact results for the steady-state probability

*dubkov@rf.unn.ru
†rudenko@acs366.phys.msu.ru
‡gurb@rf.unn.ru

density function (PDF) in the case of a Poisson δ-pulse train, we propose a general method to find the probability characteristics. The paper is organized as follows. In Sec. II, using the Markovian approach and the Kolmogorov-Feller equation, we find the PDF in the case of exponential probability distribution of spike amplitudes, and we consider a number of examples. In Sec. III, the technique based on an analysis of system behavior in the intervals between neighboring δ-pulses and averaging is presented. As an example, a particle motion in the bistable quadratic-cubic potential under the action of the periodic δ-pulse train is investigated in detail in Sec. IV. The probability density function and the mean value of the particle displacement in conjunction with the first two moments of the switching time from one stable state of a bistable system to another are obtained in the framework of the fast relaxation approximation, including the limit of large observation times. A brief discussion of the main results is presented in the Conclusions.

II. MARKOVIAN APPROACH FOR WHITE SHOT NOISE EXCITATION

A. Probability characteristics of a nonlinear system subject to white Poisson noise

We investigate the nonlinear dynamical system, which is governed by the following stochastic first-order differential equation:

$$\dot{x} = f(x) + \xi(t), \tag{1}$$

where $\xi(t)$ is an external noise in the form of a δ-pulse train,

$$\xi(t) = \sum_k a_k\, \delta(t - t_k). \tag{2}$$

Next, we assume that the intervals between neighboring pulses $\tau_k = t_k - t_{k-1}$ $(t_0 = 0)$ of the additive noise (2) are statistically independent and identically distributed with the PDF $w(\tau)$ (i.e., the moments of pulse appearance represent the point renewal process [42]), and the amplitudes a_k are also statistically independent and have the same probability distribution $W_a(x)$. It should be noted that the problem of mean first-passage times for such types of noise was analyzed in Ref. [43], the closed set of integral equations for the characteristic functional of the noise (2) was first obtained in Ref. [44] (see the Appendix), and its spectral power density was found in Ref. [45]. At the same time, the noise (2) is the first derivative of the well-known continuous-time random-walk (CTRW) model (see, for example, the review [46]).

In the case of exponentially distributed time intervals τ_k,

$$w(\tau) = \nu e^{-\nu\tau}, \quad \tau \geqslant 0, \tag{3}$$

the external perturbation $\xi(t)$ transforms to the white shot noise with Poisson statistics of spikes. As a result, the random process $x(t)$ in Eq. (1) becomes Markovian, and we can obtain the following Kolmogorov-Feller integrodifferential equation for its PDF $P(x,t)$ corresponding to stochastic Eq. (1) (see [8]):

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[f(x)P] + \nu\int_{-\infty}^{\infty} W_a(z)[P(x-z,t) - P(x,t)]dz. \tag{4}$$

It should be noted that Eq. (4) is a particular case of the generalized Kolmogorov equation obtained in Ref. [47] for arbitrary non-Gaussian white noise $\xi(t)$. Taking into account the following formula for the shift operator:

$$P(x-z,t) = e^{-z\frac{\partial}{\partial x}} P(x,t),$$

we can write Eq. (4) in the differential form

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[f(x)P] + \nu\left[\theta_a\left(i\frac{\partial}{\partial x}\right) - 1\right]P, \tag{5}$$

where $\theta_a(u) = \langle e^{iua_k}\rangle$ is the characteristic function of random amplitudes a_k.

Next, we demonstrate that for some probability distributions of pulse amplitudes a_k, the stationary solution of Eq. (4) can be found exactly. So, for the one-sided exponential distribution

$$W_a(x) = \lambda\, e^{-\lambda x}, \quad x > 0, \tag{6}$$

where λ is a positive parameter, Eq. (4) takes the following form with the inverse differential operator:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[f(x)P + \frac{\nu}{\lambda + \partial/\partial x} P\right]. \tag{7}$$

Assuming the existence of a steady-state probability distribution $P_{\rm st}(x)$ and applying the conditions of zero probability flux at $x = \pm\infty$, we arrive from Eq. (7) at the first-order differential equation

$$\frac{d}{dx}[f(x)P_{\rm st}] + [\lambda f(x) + \nu]P_{\rm st} = 0,$$

the solution of which reads [48]

$$P_{\rm st}(x) = \frac{e^{-\lambda x}}{f(x)} \exp\left\{-\nu\int\frac{dx}{f(x)}\right\}. \tag{8}$$

The exact result (8) corresponds to the excitation (2) with positive pulses.

In a more interesting (from a physical point of view) situation of spikes of both polarities, the random amplitudes of which are distributed according to the Laplace law

$$W_a(x) = \frac{\beta}{2}\, e^{-\beta|x|}, \tag{9}$$

from Eq. (5) we get

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[f(x)P + \frac{\nu}{\partial^2/\partial x^2 - \beta^2}\frac{\partial P}{\partial x}\right]. \tag{10}$$

Corresponding to Eq. (10), the equation for the steady-state PDF takes the form

$$\frac{d^2}{dx^2}[f(x)P_{\rm st}] + \nu\frac{dP_{\rm st}}{dx} - \beta^2 f(x)P_{\rm st} = 0. \tag{11}$$

Unfortunately, the second-order differential equation (11) cannot be solved analytically in the general case.

B. Transformation of pulse noise by a linear system

Let us analyze the case of linear system (1). Substituting in Eq. (11) $f(x) = -kx$, we arrive at

$$\frac{d^2P_{\rm st}}{dx^2} + \frac{1}{x}\left(2 - \frac{\nu}{k}\right)\frac{dP_{\rm st}}{dx} - \beta^2 P_{\rm st} = 0. \tag{12}$$

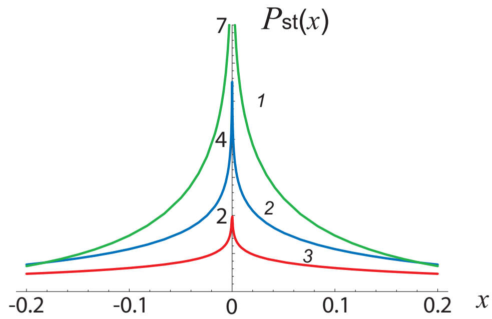

FIG. 1. The steady-state PDF of the random process (15) at the output of the linear system for different values of parameter β: curve 1, $\beta = 7$; curve 2, $\beta = 2$; curve 3, $\beta = 0.7$.

Equation (12) can be solved exactly in the case $k = \nu$, when it takes the form of the modified Bessel differential equation after replacing the independent variable $y = \beta x$. As a result, the solution of Eq. (12) can be written as

$$P_{st}(x) = c_1\, I_0(\beta x) + c_2\, K_0(\beta|x|), \tag{13}$$

where $I_0(x)$ is the modified Bessel function of the first kind of zero order, and $K_0(x)$ $(x > 0)$ is the modified Bessel function of the second kind of zero order (MacDonald function). Taking into account the behavior of functions $I_0(x)$ and $K_0(x)$ for large arguments $x \to +\infty$,

$$I_0(x) \simeq \frac{e^x}{\sqrt{2\pi x}}, \quad K_0(x) \simeq \sqrt{\frac{\pi}{2x}}e^{-x},$$

one needs to set $c_1 = 0$ and to find the constant c_2 from the normalization condition. Finally, we obtain

$$P_{st}(x) = \frac{\beta}{\pi}\, K_0(\beta|x|). \tag{14}$$

The result (14) demonstrates how a linear system can provide the normalization of input noise. As follows from Eq. (1), the random process $x(t)$ in this case represents the integral sum of statistically independent variables,

$$x(t) = x(0) + \int_0^t e^{-k(t-\tau)}\xi(\tau)d\tau, \tag{15}$$

and, according to the central limit theorem, its PDF should be close to the Gaussian distribution in the limit of large times. Indeed, the wide probability distribution of white shot noise $\xi(t)$ having the infinite variance transforms to something like the Gaussian probability density function but with singularity at zero point: $P_{st}(x) \sim -\ln|x|$ when $x \to 0$ (see Fig. 1).

C. Inertial particle in the medium with Coulomb friction under the action of a δ-pulse train

Let us consider, as in Ref. [49], the random motion of a particle with unit mass in a medium with Coulomb friction caused by white Poisson noise (2). The corresponding equation for the particle velocity $v(t)$ reads

$$\dot{v} = -\gamma \operatorname{sgn} v + \xi(t), \tag{16}$$

where sgn v is the sign function. Substituting, in accordance with Eq. (16), $f(v) = -\gamma \operatorname{sgn} v$ in Eq. (11), we arrive at

$$\frac{d^2}{dv^2}(P_{st}\operatorname{sgn} v) - \frac{\nu}{\gamma}\frac{dP_{st}}{dv} - \beta^2 P_{st}\operatorname{sgn} v = 0. \tag{17}$$

In view of the obvious symmetry of the stationary PDF $P_{st}(v)$ of the particle velocity, we can solve Eq. (17) only for positive arguments. As a result, we have the following homogeneous linear second-order differential equation with constant coefficients:

$$P''_{st} - \frac{\nu}{\gamma}P'_{st} - \beta^2 P_{st} = 0. \tag{18}$$

The general solution of Eq. (18) takes the form

$$P_{st}(v) = C_1 e^{\lambda v} + C_2 e^{-\mu v}, \tag{19}$$

where the positive parameters λ and μ are

$$\lambda = \frac{\nu}{2\gamma}\left(\sqrt{1+\frac{4\gamma^2\beta^2}{\nu^2}}+1\right), \quad \mu = \frac{\nu}{2\gamma}\left(\sqrt{1+\frac{4\gamma^2\beta^2}{\nu^2}}-1\right). \tag{20}$$

From the condition of the limited solution (19) and Eqs. (20), we need to set the constant C_1 equal to zero. Finally, taking into account the normalization condition, we get the following exact formula for the stationary PDF of the particle velocity:

$$P_{st}(v) = \frac{\mu}{2}\, e^{-\mu|v|}. \tag{21}$$

It is interesting to note that the probability distribution (21) follows the distribution of pulse amplitudes (9) and differs from the Maxwell distribution. Finally, in the limiting transition from the white Poisson noise to the white Gaussian noise with intensity D [9],

$$\nu \to \infty, \quad \beta \to \infty, \quad \frac{\nu}{\beta^2} = D,$$

Eq. (21) transforms to the well-known result (see [49])

$$P_{st}(v) = \frac{\gamma}{2D}\, e^{-\gamma|v|/D}. \tag{22}$$

III. GENERAL METHOD TO CALCULATE THE PROBABILITY DENSITY FUNCTION

To find the probability characteristics of the nonlinear dynamical system (1) with non-Poisson pulse excitation, we apply another approach, namely, we rely on the fact that in the interval between the kth and $(k+1)$th pulses, the behavior of the system (1) can be described by the following equation:

$$\dot{x} = f(x) \tag{23}$$

with the initial condition $x(t_k + 0) = x_k + a_k$ $[x_k = x(t_k)]$, taking into account the jump a_k at the time t_k. After integration, the solution of Eq. (23) in the time interval $t_k < t \leqslant t_{k+1}$ can be written in the form

$$x(t) = h^{(-1)}[h(x_k + a_k) + t - t_k] \equiv g(x_k + a_k, t - t_k), \tag{24}$$

where $h'(x) = 1/f(x)$, and $h^{(-1)}(x)$ denotes the inverse function. Setting in Eq. (24) $t = t_{k+1}$, we find the value of random process $x(t)$ at the time of the next pulse,

$$x_{k+1} = x(t_{k+1}) = g(x_k + a_k, \tau_{k+1}). \tag{25}$$

Equation (25) is a recursive formula for the values of the random process $x(t)$ at the moments of the neighboring pulses, which includes pulse amplitudes and intervals between them.

From Eq. (25) we can derive the recurrence relation for the probability distributions of random variables x_k and x_{k+1}. Since all the variables x_k, a_k, and τ_{k+1} in Eq. (25) are statistically independent, the PDF $P_{k+1}(x)$ of the random variable x_{k+1} satisfies the following equation:

$$P_{k+1}(x) = \int_0^\infty w(\tau)d\tau \times \int_{-\infty}^{\infty} \delta(x - g(y,\tau))[P_k(y) * W_a(y)]dy, \quad (26)$$

where $*$ denotes the convolution. Solving Eq. (26), we can find the probability distribution of random process $x(t)$ at any time t $(t_k < t \leqslant t_{k+1})$ using the equation

$$P(x,t) = \int_0^\infty [\underbrace{w(\tau) * w(\tau) * \cdots * w(\tau)}_{k}]\, d\tau \times \int_{-\infty}^{\infty} \delta(x - g(y,t-\tau))[P_k(y) * W_a(y)]dy. \quad (27)$$

Next, we analyze the case of a periodic pulse train when $w(\tau) = \delta(\tau - T)$. In this situation, from Eq. (26) we arrive at

$$P_{k+1}(x) = \int_{-\infty}^{\infty} \delta(x - g(y,T))[P_k(y) * W_a(y)]dy. \quad (28)$$

Integrating into Eq. (28), we get

$$P_{k+1}(x) = \sum_j \left| \frac{\partial g_j^{(-1)}(x,T)}{\partial x} \right| \times \left[P_k\left(g_j^{(-1)}(x,T)\right) * W_a\left(g_j^{(-1)}(x,T)\right)\right], \quad (29)$$

where $g_j^{(-1)}(x,T)$ is the jth branch of the inverse function with respect to $x = g(y,T)$. Taking into account Eq. (24), we arrive at

$$P_{k+1}(x) = \sum_j \left| \frac{f\left(h_j^{(-1)}[h(x) - T]\right)}{f(x)} \right| \times \left[P_k\left(h_j^{(-1)}[h(x) - T]\right) * W_a\left(h_j^{(-1)}[h(x) - T]\right)\right]. \quad (30)$$

Calculating from the recurrence relation (30) the probability distribution $P_k(x)$, we find the PDF of the random process $x(t)$ of the nonlinear dynamical system (1) as [see Eq. (27)]

$$P(x,t) = \sum_j \left| \frac{f\left(h_j^{(-1)}[h(x) - t + kT]\right)}{f(x)} \right| \times \left[P_k\left(h_j^{(-1)}(z)\right) * W_a\left(h_j^{(-1)}(z)\right)\right]_{z=h(x)-t+kT}, \quad (31)$$

where $kT < t \leqslant (k+1)T$.

IV. OVERDAMPED DIFFUSION OF A PARTICLE IN BISTABLE QUADRATIC-CUBIC POTENTIAL

As an example of the approach proposed, we analyze the overdamped diffusion of a particle in the following bistable potential:

$$U(x) = \gamma\left(\frac{x^2|x|}{3} - \frac{bx^2}{2}\right) \quad (b,\gamma > 0). \quad (32)$$

This motion is governed by the following Langevin equation for the particle position $x(t)$:

$$\dot{x} = \gamma x(b - |x|) + \xi(t) \quad (33)$$

with additive noise $\xi(t)$ in the form of a δ-pulse train with the period T and random amplitudes a_k. It should be emphasized that the "modulus" approximation as in Eq. (32) has been applied in Ref. [50] to find the exact solutions of nonlinear systems described by both ordinary and partial differential equations without noise arising from the analysis of some physical problems, while in Ref. [51] the probability and spectral characteristics of nonlinear oscillator with "modulus" potential perturbed by white Gaussian noise have been found.

As seen from Fig. 2, the nonlinear dynamical system (33) has two stable states $x = \pm b$. Substituting $f(x) = \gamma x(b - |x|)$ in Eq. (23) and solving the equation, we get

$$x(t) = \frac{b(x_k + a_k)}{b\, e^{-\gamma b(t-t_k)} + |x_k + a_k|[1 - e^{-\gamma b(t-t_k)}]}. \quad (34)$$

Setting in Eq. (34) $t = (k+1)T$, $t_k = kT$, we arrive at

$$x_{k+1} = \frac{b(x_k + a_k)}{be^{-\gamma bT} + |x_k + a_k|(1 - e^{-\gamma bT})}. \quad (35)$$

In the approximation of a fast relaxation of the system between pulses, $\gamma bT \gg 1$, Eq. (35) gives

$$x_{k+1} = b\frac{x_k + a_k}{|x_k + a_k|} = b\,\mathrm{sgn}(x_k + a_k), \quad (36)$$

i.e., the nonlinear system (33) is in one of the stable states before the appearance of the next pulse. As a result, the PDF of the random process $x(t)$ at time $t = (k+1)T$ is a superposition of two δ functions,

$$P_{k+1}(x) = p_{k+1}\delta(x - b) + q_{k+1}\delta(x + b), \quad (37)$$

where, in accordance with Eq. (36), the probabilities p_k and q_k can be found as

$$p_{k+1} = \mathrm{Prob}\{x_k + a_k > 0\}, \quad p_{k+1} + q_{k+1} = 1, \quad (38)$$

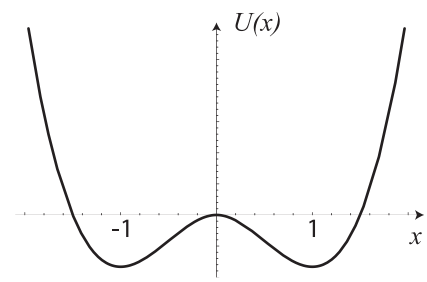

FIG. 2. The bistable quadratic-cubic potential (32). The parameters are $\gamma = 1$, $b = 1$.

where the symbol Prob$\{A\}$ denotes the probability of the event A. From Eqs. (37) and (38), we obtain the mean value of the particle position at time $t = (k+1)T$,

$$\langle x_{k+1}\rangle = b(2p_{k+1} - 1). \tag{39}$$

To calculate the probability in Eq. (38), we have to find the PDF of the sum $x_k + a_k$, which is the convolution of corresponding probability distributions because of the statistical independence of random variables,

$$P_k(x) * W_a(x) = \int_{-\infty}^{\infty} P_k(y) W_a(x-y)dy,$$

or, in accordance with Eq. (37),

$$P_k(x) * W_a(x) = p_k W_a(x-b) + q_k W_a(x+b). \tag{40}$$

As a result, from Eqs. (38) and (40) we have

$$\begin{aligned} p_{k+1} &= p_k \int_{-b}^{\infty} W_a(x)dx + q_k \int_{b}^{\infty} W_a(x)dx \\ &= p_k \int_{-b}^{b} W_a(x)dx + \int_{b}^{\infty} W_a(x)dx. \end{aligned} \tag{41}$$

Thus, Eq. (41) gives the following simple recurrence relation for the probabilities p_k:

$$p_{k+1} = \mathrm{Prob}\{|a| < b\} p_k + \mathrm{Prob}\{a > b\}. \tag{42}$$

It is not difficult to solve the recurrence relation (42). Indeed,

$$\begin{aligned} p_{k+1} &= \mathrm{Prob}\{|a| < b\} p_k + \mathrm{Prob}\{a > b\} \\ &= \mathrm{Prob}\{|a| < b\}(\mathrm{Prob}\{|a| < b\} p_{k-1} \\ &\quad + \mathrm{Prob}\{a > b\}) + \mathrm{Prob}\{a > b\} \\ &= \mathrm{Prob}^2\{|a| < b\} p_{k-1} + \mathrm{Prob}\{a > b\}(1 \\ &\quad + \mathrm{Prob}\{|a| < b\}) = \cdots = \mathrm{Prob}^k\{|a| < b\} p_1 \\ &\quad + \mathrm{Prob}\{a > b\}(1 + \mathrm{Prob}\{|a| < b\} + \cdots \\ &\quad + \mathrm{Prob}^{k-1}\{|a| < b\}). \end{aligned} \tag{43}$$

Applying in Eq. (43) the formula for the sum of geometric progression and taking into account that $p_1 = \mathrm{Prob}\{x(0) > 0\}$, we arrive finally at

$$\begin{aligned} p_{k+1} &= \mathrm{Prob}\{x(0) > 0\}\mathrm{Prob}^k\{|a| < b\} \\ &\quad + \mathrm{Prob}\{a > b\}\frac{1 - \mathrm{Prob}^k\{|a| < b\}}{1 - \mathrm{Prob}\{|a| < b\}}. \end{aligned} \tag{44}$$

It is interesting to note that in the limit $k \to \infty$, the nonlinear system (33) forgets about the initial conditions, and Eq. (44) becomes

$$p_\infty = \frac{\mathrm{Prob}\{a > b\}}{1 - \mathrm{Prob}\{|a| < b\}} = \frac{\mathrm{Prob}\{a > b\}}{\mathrm{Prob}\{|a| > b\}}. \tag{45}$$

From Eqs. (37), (39), and (45), we find the PDF of the particle position and the mean value in asymptotics,

$$\begin{aligned} P_\infty(x) &= \frac{\mathrm{Prob}\{a > b\}}{\mathrm{Prob}\{|a| > b\}}\,\delta(x-b) \\ &\quad + \frac{\mathrm{Prob}\{a < -b\}}{\mathrm{Prob}\{|a| > b\}}\,\delta(x+b), \end{aligned} \tag{46}$$

$$\langle x_\infty\rangle = b\,\frac{\mathrm{Prob}\{a > b\} - \mathrm{Prob}\{a < -b\}}{\mathrm{Prob}\{|a| > b\}}. \tag{47}$$

The mean switching time from one stable state to another caused by an external pulse perturbation is one of the important characteristic of multistable nonlinear systems. The above-mentioned approximation of the fast relaxation of bistable system (33) allows us to calculate the mean crossing time of the potential barrier by a particle, which coincides in such a situation with the mean switching time of the system between two stable states.

To be specific, we assume that the initial value of the particle position is positive: $x(0) > 0$. As a result, before the first δ-pulse, a particle would occupy the stable state $x = b$. At the same time, by virtue of the δ-pulse train excitation, the first switching of the bistable system from the stable state $x = b$ to the stable state $x = -b$ can only occur at the times of external stimuli. Thus, the first switching time τ is a discrete random variable that takes the values $\tau = nT$, $n = 1,2,3,\ldots$ with probabilities $P_n = p\,q^{n-1}$, where $p = \mathrm{Prob}\{a < -b\}$ is the probability of switching at the first stimulus and $p + q = 1$. In the probability theory, this distribution is known as geometric and is associated with the number of independent (Bernoulli) trials needed to achieve success with probability p of success on each trial (see, for example, [52]).

To find the mean value and the variance of the first switching time, we calculate its characteristic function. In accordance with the definition, we have

$$\theta_\tau(u) = \langle e^{iu\tau}\rangle = \sum_{n=1}^{\infty} pq^{n-1}e^{iunT} = \frac{p}{e^{-iuT} - q}, \tag{48}$$

and, consequently,

$$\ln\theta_\tau(u) = \ln p - \ln(e^{-iuT} - q). \tag{49}$$

By definition, the nth-order cumulant of random variable τ can be found from Eq. (49) as

$$\kappa_n = \frac{1}{n!}\left.\frac{d^n \ln\theta_\tau(u)}{d(iu)^n}\right|_{u=0}. \tag{50}$$

For the mean first switching time ($n = 1$), Eqs. (49) and (50) give

$$\langle\tau\rangle = \frac{T}{p}. \tag{51}$$

As seen from Eq. (51), the mean first switching time increases with decreasing probability of switching at first stimulus. For the variance ($n = 2$) from Eqs. (49) and (50), we obtain

$$\sigma_\tau^2 = \frac{q\,T^2}{2\,p^2}. \tag{52}$$

It should be emphasized that all results obtained in this section in the framework of the approximation of fast relaxation $\gamma bT \gg 1$ depend only on one of the parameters of the nonlinear system (33). Upon increasing this parameter b, i.e., both the distance between the potential barrier and wells and the height of the barrier, the probability p decreases. As a result, the mean value of the first switching time (51) and its variance (52) increase.

V. CONCLUSIONS

Based on the Kolmogorov-Feller equation, some exact results for the steady-state probability distributions of a nonlinear dynamical system perturbed by white shot noise with exponentially distributed amplitudes have been obtained. Using the reverse differential operator, we have found the stationary probability distribution of white Poisson noise after its transformation by a linear system and the steady-state PDF of particle velocity in the medium with Coulomb friction. This method can be effectively applied to the analysis of stationary spectral-correlation characteristics of these systems as well. For the case of a non-Poisson δ-pulse train, a general approach to probability analysis of this system, which is based on the study of system dynamics in the intervals between neighboring δ-pulses, has been offered. The proposed method has been demonstrated in detail on an example of overdamped motion of a particle in the bistable quadratic-cubic potential under the action of the periodic δ-pulse train. The probability distribution and the mean value of the particle position have been obtained in the framework of the fast relaxation approximation. The mean value and the variance of the first switching time from one stable state of a bistable system to another have been found as well.

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