Critical behavior of su(1|1) supersymmetric spin chains with long-range interactions

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We introduce a general class of su(1|1) supersymmetric spin chains with long-range interactions which includes as particular cases the su(1|1) Inozemtsev (elliptic) and Haldane-Shastry chains, as well as the *XX* model. We show that this class of models can be fermionized with the help of the algebraic properties of the su(1|1) permutation operator and take advantage of this fact to analyze their quantum criticality when a chemical potential term is present in the Hamiltonian. We first study the low-energy excitations and the low-temperature behavior of the free energy, which coincides with that of a $(1+1)$-dimensional conformal field theory (CFT) with central charge $c=1$ when the chemical potential lies in the critical interval $(0,\mathcal{E}(\pi))$, $\mathcal{E}(p)$ being the dispersion relation. We also analyze the von Neumann and Rényi ground state entanglement entropies, showing that they exhibit the logarithmic scaling with the size of the block of spins characteristic of a one-boson $(1+1)$-dimensional CFT. Our results thus show that the models under study are quantum critical when the chemical potential belongs to the critical interval, with central charge $c=1$. From the analysis of the fermion density at zero temperature, we also conclude that there is a quantum phase transition at both ends of the critical interval. This is further confirmed by the behavior of the fermion density at finite temperature, which is studied analytically (at low temperature), as well as numerically for the su(1|1) elliptic chain.

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I. INTRODUCTION

Exactly solvable one-dimensional quantum models are widely used as proving grounds for key ideas in condensed matter physics and the theory of critical phenomena, since their conceptual simplicity often makes it possible to derive exact analytical expressions for the relevant physical quantities. Historically, most of the work in this field has been focused on systems with short-range interactions, like the well-known Heisenberg and (quantum) Ising chains. In the last few years, however, it has become feasible to realize in the laboratory quantum spin chains featuring various types of long-range interactions through different experiments involving, e.g., optical lattices of ultracold Rydberg atoms and trapped ions, or neutral atoms in optical cavities [1–5]. In particular, with the help of hyperfine "clock" states of trapped $^{171}\mathrm{Yb}^{+}$ ions it is now possible to simulate quantum spin chains in which the coupling h_{ij} between the ith and jth sites is inversely proportional to a power $\alpha\in(0,3)$ of their distance [1,4]. An important model of this type is the integrable Haldane-Shastry (HS) chain [6,7], whose sites are the equispaced points $z_k=\mathrm{e}^{2\pi\mathrm{i}k/N}$ $(1\leqslant k\leqslant N)$ on the unit circle with a coupling proportional to $|z_i-z_j|^{-2}$. In fact, this chain is a limiting case of a more general model due to Inozemtsev, in which the coupling h_{ij} is an elliptic function of the difference $i-j$ with real period N [8].

Although the particles in the original HS chain carried spin $1/2$, the model was shortly generalized to su(m) spin without losing its remarkable integrability properties [9]. As a matter of fact, the su($m|n$) supersymmetric version of the HS chain, originally introduced by Haldane [10], has also been studied in the literature [11,12]. Of particular interest is the su($m|1$) HS chain (with $m>1$), since it is essentially equivalent to an su(m) supersymmetric t-J model [13–15] with exchange and transfer energies proportional to $|z_i-z_j|^{-2}$. This chain, first introduced by Kuramoto and Yokoyama in the su(2) case [16], is an exactly solvable model which provides one of the simplest realizations of spin-charge separation.

In this work we introduce a wide class of su(1|1) supersymmetric spin chains with general translation-invariant couplings $h_{ij}>0$ and a chemical potential term. For zero chemical potential, these models include in particular the supersymmetric elliptic chain studied in Ref. [17] and its two limiting cases, the su(1|1) HS chain and the *XX* model. The class of models under study are technically simpler than their su($m|1$) counterparts, essentially due to the fact that they can be transformed into a system of free spinless fermions in a straightforward way. However, they still exhibit a sufficiently rich structure which makes it possible to examine a number of key properties in the theory of quantum critical systems in an analytic fashion.

More precisely, our main objective is to study whether the models under consideration are quantum critical for suitable values of the chemical potential, and to determine their corresponding central charge. As is well known, a characteristic feature of $(1+1)$-dimensional CFTs is the fact that at low temperature T their free energy per unit length is approximately given (in appropriate units) by $f_0-\pi cT^2/(6v)$, where f_0 is a constant and v is the effective speed of "sound" [18,19]. Since the low-temperature behavior of f is determined by the low-lying states of the theory, this should also be the case for any one-dimensional quantum system whose low energy spectrum is described by a $(1+1)$-dimensional CFT. In particular, the determination of the low-temperature behavior of the free energy of a one-dimensional critical model provides an efficient way of determining the central charge of its

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underlying CFT. In this way we have been able to show that if the dispersion relation $\mathcal{E}(p)$ is monotonic in the range $[0,\pi]$ the models under study are critical when the chemical potential λ belongs to the open interval $(0,\mathcal{E}(\pi))$, with central charge $c=1$. As further confirmation of this result, we have studied the ground state entanglement entropy, i.e., the entropy of the reduced density matrix of a block of L consecutive spins when the whole chain is in its ground state. Indeed, it is well known that in a $(1+1)$-dimensional CFT the Rényi and von Neumann entanglement entropies scale as $(c/6)(1+1/q)\log L$ and $(c/3)\log L$, respectively, where c is the central charge and q is the Rényi parameter [20–22]. Thus the entanglement entropy of a quantum critical one-dimensional system should be proportional to $\log L$ for $L \gg 1$, where the proportionality constant fixes the central charge of the underlying CFT. Again, we have verified that when the chemical potential belongs to the open critical interval $(0,\mathcal{E}(\pi))$ the entanglement entropy of the models under consideration scales as that of a $(1+1)$-dimensional CFT with $c=1$. We have also examined the behavior of the entanglement entropy and the zero-temperature fermion density as λ approaches the endpoints of the critical interval, showing that it is consistent with a quantum phase transition at both ends. For the su(1|1) chain with elliptic interactions we have studied numerically the fermion density at finite temperature, finding that its behavior is far more complex when the chemical potential lies in the critical interval. More precisely, for suitable values of λ inside this interval the fermion density is not a monotonic function of the temperature, but can rather present up to two extrema.

The paper is organized as follows. In Sec. II we introduce the class of supersymmetric spin chains with which this work is concerned and recall how these models can be fermionized using the algebraic properties of the su(1|1) permutation operator. In Sec. III we analyze the thermodynamics of the general su(1|1) chain (2) when the dispersion relation is monotonic in the range $0 \leqslant p \leqslant \pi$, showing that at low temperature it behaves as a $(1+1)$-dimensional CFT with $c=1$. In Sec. IV we outline the computation of the von Neumann and Rényi ground state entanglement entropies of the latter models in terms of the eigenvalues of the ground state correlation matrix. Section V is devoted to deriving asymptotic formulas for these entropies, both when the size of the block of spins tends to infinity and when the chemical potential approaches the endpoints of the critical interval. In Sec. VI we perform a numerical study of the fermion density of the elliptic su(1|1) chain at finite temperature and determine analytically its low-temperature behavior for arbitrary interactions. Finally, in Sec. VII we summarize our conclusions and discuss some future developments suggested by the present work.

II. THE MODELS

Consider a translation-invariant (closed) spin chain whose N sites are occupied by either a boson or a (spinless) fermion. If we denote by $b_i^\dagger$ and $f_i^\dagger$ the operators that respectively create a boson or a fermion at the ith site, the Hilbert space of the model is the 2^N-dimensional subspace of the infinite-dimensional Fock space determined by the constraints

$$b_i^\dagger b_i + f_i^\dagger f_i = 1, \quad 1 \leqslant i \leqslant N\,. \tag{1}$$

We shall take as the model's Hamiltonian the operator [23]

$$H = \sum_{i<j} h_N(j-i)(1-S_{ij}) - \lambda N_{\mathrm{f}}, \tag{2}$$

where $\lambda \in \mathbb{R}$, $N_{\mathrm{f}} = \sum_i f_i^\dagger f_i$ is the total fermion number operator, h_N is a nonnegative smooth function and S_{ij} is the su(1|1) spin permutation operator [10] defined by

$$S_{ij} = b_i^\dagger b_j^\dagger b_i b_j + f_i^\dagger f_j^\dagger f_i f_j + f_j^\dagger b_i^\dagger f_i b_j + b_j^\dagger f_i^\dagger b_i f_j\,.$$

If we denote by $|0\rangle$ and $|1\rangle$ respectively the states occupied by a boson or a fermion, the action of the operator S_{ij} on the canonical spin basis with elements $|s_1\rangle \otimes \cdots \otimes |s_N\rangle \equiv |s_1,\ldots,s_N\rangle$, with $s_i \in \{0,1\}$, is given by

$$S_{ij}|\ldots,s_i,\ldots,s_j,\ldots\rangle = (-1)^n|\ldots,s_j,\ldots,s_i,\ldots\rangle, \tag{3}$$

where $n = s_i = s_j$ if $s_i = s_j$ while for $s_i \neq s_j$ n equals the number of fermions at the sites $i+1,\ldots,j-1$. Note that S_{ij} is invariant under the supersymmetry transformation $b_i \leftrightarrow f_i$, so that the term $\sum_{i<j} h_N(i-j)(1-S_{ij})$ in H is su(1|1) supersymmetric, while the last term λN_{f}, i.e., the chemical potential of the fermions, transforms into $\lambda(N-N_{\mathrm{f}})$ due to the constraints (1). Furthermore, we shall exclusively be concerned in this paper with *closed* (i.e., periodic) chains, for which $h_N(x) = h_N(N-x)$. It is customary to extend the function h_N to the whole real line as an N-periodic function, so that

$$h_N(x) = h_N(-x) = h_N(x+N) \geqslant 0, \quad \forall x \in \mathbb{R}\,. \tag{4}$$

It was shown in Ref. [17] that any chain of the form (2) can be recast into a model of spinless hopping fermions by identifying the boson state $|0\rangle$ with the fermion vacuum. More precisely, we define a new set of fermion creation operators $a_i^\dagger = f_i^\dagger b_i$, $1 \leqslant i \leqslant N$, which indeed satisfy the canonical anticommutation relations (CAR) on account of (1). For instance, we have

$$\begin{aligned} a_i^\dagger a_i + a_i a_i^\dagger &= f_i^\dagger f_i b_i b_i^\dagger + f_i f_i^\dagger b_i^\dagger b_i = \{f_i^\dagger, f_i\} b_i^\dagger b_i + f_i^\dagger f_i \\ &= b_i^\dagger b_i + f_i^\dagger f_i = 1. \end{aligned}$$

The chain sites can now be either empty (i.e., in the state $|0\rangle$) or occupied by a fermion (in the state $|1\rangle$), and thus the Hilbert space is the whole 2^N-dimensional Fock space built acting on the vacuum $|0,\ldots,0\rangle$ with the operators $a_i^\dagger$. As first shown by Haldane [10], from Eqs. (3) and the constraints (1) it follows that the su(1|1) exchange operator S_{ij} admits the following simple expression in terms of the new fermion operators $a_i, a_i^\dagger$:

$$S_{ij} = 1 - a_i^\dagger a_i - a_j^\dagger a_j + a_i^\dagger a_j + a_j^\dagger a_i\,.$$

Likewise,

$$f_i^\dagger f_i = f_i^\dagger f_i (b_i^\dagger b_i + f_i^\dagger f_i) = f_i^\dagger f_i (b_i b_i^\dagger + f_i^\dagger f_i - 1) = a_i^\dagger a_i$$

(since $f_i^\dagger f_i$ is idempotent), so that $\lambda N_{\mathrm{f}} = \lambda \sum_i a_i^\dagger a_i$ is simply the chemical potential for the new fermions. Taking into account the latter identities, the Hamiltonian (2) can be rewritten as

$$H = -\sum_{i,j} h_N(i-j) a_i^\dagger a_j - \lambda \sum_i a_i^\dagger a_i, \tag{5}$$

where we have set $h_N(0) = -\sum_{j=1}^{N-1} h_N(j)$ (see Ref. [17] for more details). This Hamiltonian describes a system of N hopping (spinless) free fermions on a circle, with hopping amplitude between the ith and jth sites given by $h_N(i-j)$ and chemical potential λ. The translation invariance of this model (encoded in the periodicity of the function h) suggests introducing the Fourier-transformed operators

$$c_l = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \mathrm{e}^{-2\pi \mathrm{i} k l/N} a_k, \quad 0 \leqslant l \leqslant N-1\,. \tag{6}$$

It can be readily shown that these operators satisfy the CAR and can therefore be considered as a new set of fermionic operators; in fact, as we shall see below, $c_l^\dagger$ creates a fermion with momentum $p = 2\pi l/N \pmod{2\pi}$. It is shown in Ref. [17] that H is diagonal when written in terms of the new operators c_l and their adjoints. In fact, we have

$$H = \sum_{l=0}^{N-1} [\varepsilon_N(l) - \lambda] c_l^\dagger c_l, \tag{7}$$

where

$$\varepsilon_N(l) = \sum_{j=1}^{N-1} [1 - \cos(2\pi j l/N)] h_N(j)\,. \tag{8}$$

Likewise, the system's total momentum operator P is given by

$$P = \sum_{l=0}^{N-1} \frac{2\pi l}{N} c_l^\dagger c_l,$$

which shows that the operator $c_l^\dagger$ creates a fermion with momentum $2\pi l/N \pmod{2\pi}$. In this work we shall be concerned with systems for which $\varepsilon_N(l)$ depends on l and N only through the corresponding momentum $2\pi l/N$, i.e.,

$$\varepsilon_N(l) = \mathcal{E}(2\pi l/N), \quad 0 \leqslant l \leqslant N-1,$$

where the *dispersion relation* $\mathcal{E}$ is a smooth function defined in the interval $[0,2\pi]$. It easily follows from Eq. (8) that if such a function $\mathcal{E}$ exists it is necessarily unique and that $\mathcal{E}(p) = \mathcal{E}(2\pi - p)$. An important type of interaction $h_N(x)$ satisfying the above requirement is given by the elliptic function

$$h_N(x) = \left(\frac{\alpha}{\pi}\right)^2 \sinh^2\!\left(\frac{\pi}{\alpha}\right) \left[\wp_N(x) - \frac{2\hat{\eta}_1}{\alpha^2}\right], \tag{9}$$

where $\alpha > 0$ is a real parameter, $\wp_N(x) \equiv \wp(x; N/2, \mathrm{i}\alpha/2)$, and $\hat{\eta}_1 = \zeta(1/2; 1/2, \mathrm{i}N/(2\alpha))$, $\wp(x;\omega_1,\omega_3)$ and $\zeta(x;\omega_1,\omega_3)$ denoting, respectively, the Weierstrass elliptic and zeta functions with half-periods ω_1 and ω_3 [24,25]. It can be shown [17] that the function (9) satisfies the three conditions in Eq. (4). Moreover, since

$$\lim_{\alpha\to 0+} h_N(x) = \delta_{1,x} + \delta_{N-1,x}, \qquad \lim_{\alpha\to\infty} h_N(x) = \frac{(\pi/N)^2}{\sin^2\left(\frac{\pi x}{N}\right)},$$

the model (2) with interaction strength (9) smoothly interpolates between the Heisenberg (for $\alpha = 0$) and Haldane-Shastry (for $\alpha = \infty$) su(1|1) chains (with a chemical potential term added). In fact, the former of these models can be transformed into the spin 1/2 (closed) XX Heisenberg Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right) + \left(1 - \frac{\lambda}{2}\right) \sum_{i=1}^{N} \left(1 + \sigma_i^z\right),$$

where σ_k^a is the ath Pauli matrix acting on the kth site and $\sigma_{N+1}^a \equiv \sigma_1^a$, with the help of the standard Wigner-Jordan transformation [26]

$$a_k = \sigma_1^z \cdots \sigma_{k-1}^z \cdot \tfrac{1}{2}\left(\sigma_k^x - \mathrm{i}\sigma_k^y\right), \quad 1 \leqslant k \leqslant N\,.$$

The dispersion relation $\mathcal{E}(p)$ for the elliptic interaction (9) was computed in closed form in Ref. [17]. More precisely, from Eq. (2.21b) in the latter reference and the homogeneity properties of the Weierstrass functions we have

$$\mathcal{E}(p) = 2\sinh^2(\pi/\alpha)\left\{\wp(p) - \left[\zeta(p) - \frac{\eta_1 p}{\pi}\right]^2 - \frac{2\eta_1}{\pi}\right\}, \tag{10}$$

where

$$\wp(p) \equiv \wp(p;\pi,\mathrm{i}\pi/\alpha), \quad \zeta(p) \equiv \zeta(p;\pi,\mathrm{i}\pi/\alpha), \quad \eta_1 = \zeta(\pi)\,.$$

In particular, we see that in this case the dispersion relation is a pure 2π-periodic [27] function, independent of the number of particles N. Taking the $\alpha \to 0+$ and $\alpha \to \infty$ limits in the above equation for $\mathcal{E}(p)$ one recovers the well-known dispersion relations of the XX model [28] and the su(1|1) Haldane-Shastry chain, namely,

$$\mathcal{E}_{XX}(p) = 2(1 - \cos p), \quad \mathcal{E}_{\mathrm{HS}}(p) = \tfrac{1}{2}\, p(2\pi - p)\,. \tag{11}$$

III. CRITICALITY AND THERMODYNAMICS

In this section we shall exploit the equivalence of the su(1|1) supersymmetric chain (2) to the free fermion model (5) to analyze the critical behavior of this chain as a function of the chemical potential λ. To this end, we first need to determine the ground state of the model (5), which is straightforward from Eq. (7). Indeed, it is obvious from the latter equation that the modes excited in the ground state are precisely those whose momenta $p = 2\pi l/N$ satisfy the condition $\lambda > \mathcal{E}(p)$, so that the ground state is nondegenerate. Strictly speaking, this is true only if we assume that $\mathcal{E}(2\pi l/N) \neq \lambda$ for $l = 0, \ldots, N-1$. Indeed, if $\mathcal{E}(2\pi l/N) = \lambda$ the mode with momentum $2\pi l/N$ [and $2\pi(N-l)/N$, if $l > 0$ and $l \neq N/2$] can be either present or absent in the ground state, which is therefore degenerate. Since we shall be exclusively concerned with the thermodynamic limit $N \to \infty$, from now on we shall implicitly assume without loss of generality that $\mathcal{E}(2\pi l/N) \neq \lambda$ for $0 \leqslant l \leqslant N-1$.

We shall also assume in what follows that the dispersion relation has a positive derivative in the interval $(0,\pi)$, so that it is monotonically increasing in the latter interval and reaches its maximum at $p = \pi$. This is "generically" true, and it certainly holds for the dispersion relation (10) of the elliptic interaction (9) and, in particular, for the XX model and the su(1|1) Haldane-Shastry chains. If this is the case, it is straightforward to show that *the model is gapless for* $\lambda \in [0, \mathcal{E}(\pi)]$.

Indeed, first of all, it is clear that the system is gapped for $\lambda < 0$ or $\lambda > \mathcal{E}(\pi)$. For instance, for $\lambda < 0$ the gap

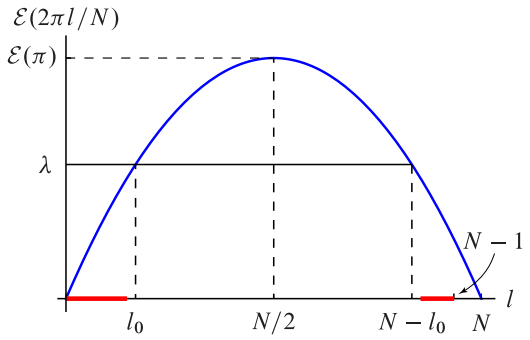

FIG. 1. Dispersion relation $\mathcal{E}(2\pi l/N)$ as a function of the mode number $l = 0,\ldots,N-1$ (the range of modes excited in the ground state for the given λ has been represented by a thick red line).

between the first excited state $c_0^\dagger|0,\ldots,0\rangle$ and the ground state is $\Delta E = |\lambda| > 0$, which remains positive as $N \to \infty$. Similarly, when $\lambda > \mathcal{E}(\pi)$ the gap is approximately equal to $\Delta E = \lambda - \mathcal{E}(\pi) > 0$ independently of N. Suppose, on the other hand, that $0 \leqslant \lambda \leqslant \mathcal{E}(\pi)$, and let l_0 be the root of the equation $\mathcal{E}(2\pi l_0/N) = \lambda$ in the interval $[0,N/2]$, which exists and is unique on account of the monotonicity of $\mathcal{E}$ in the interval $[0,\pi]$. The modes excited in the ground state are now those with $0 \leqslant l \leqslant \lfloor l_0 \rfloor$ and $N - \lfloor l_0 \rfloor \leqslant l \leqslant N-1$, where $\lfloor l_0 \rfloor$ denotes the integer part of l_0 (see Fig. 1). Thus if $0 \leqslant \lambda \leqslant \mathcal{E}(\pi)$ the gap between the first excited state and the ground state, given by

$$\Delta E = \min\bigl(\lambda - \mathcal{E}\{2\pi\lfloor l_0\rfloor/N\}, \mathcal{E}\{2\pi(\lfloor l_0\rfloor+1)/N\} - \lambda\bigr),$$

is $O(1/N)$, since $\lambda = \mathcal{E}(2\pi l_0/N)$. Thus ΔE tends to zero as $N \to \infty$ and the system is gapless, as claimed. (In fact, when l_0 is an integer the modes with $l = l_0$ or $l = N - l_0$ may or may not be present in the ground state, but this does not affect the ground state energy and therefore the foregoing argument.)

We shall next show that when the chemical potential λ belongs to the open interval $(0,\mathcal{E}(\pi))$ the su(1|1) chain (2) is indeed critical, or, more precisely, that at low energies its spectrum is that of a $(1+1)$-dimensional CFT with one free boson. To begin with, we note that when $0 < \lambda < \mathcal{E}(\pi)$ the low-energy excitations of the chain (2) are linear in the excitation momentum. Indeed, let

$$p_0 = 2\pi l_0/N \equiv \mathcal{E}^{-1}(\lambda) \in (0,\pi) \tag{12}$$

denote the Fermi momentum, where $\mathcal{E}^{-1}$ is the inverse function of the restriction of the dispersion relation to the interval $[0,\pi]$. Adding a fermion with momentum $p_0 + \Delta p$ (or, equivalently, $2\pi - p_0 - \Delta p$), with $0 < \Delta p \ll 1$, to the ground state increases the energy by $\mathcal{E}(p_0+\Delta p) - \lambda = \mathcal{E}(p_0+\Delta p) - \mathcal{E}(p_0) \simeq \mathcal{E}'(p_0)\Delta p$. The same excess energy is approximately obtained when removing from the ground state a fermion with momentum $p_0 - \Delta p$ (or $2\pi - p_0 + \Delta p$). Thus for low excitation momenta we have $\Delta E \simeq \mathcal{E}'(p_0)\Delta p$, as in a $(1+1)$-dimensional CFT with speed of "sound" $v = \mathcal{E}'(p_0)$.

The simple argument outlined above, based on linearizing the dispersion relation near the Fermi momentum p_0 (or $2\pi - p_0$), the only region in momentum space relevant at low excitation energies, does not provide any information on the central charge of the underlying CFT. A more precise way of establishing the equivalence at low energies of the su(1|1) spin chain (2) with a $(1+1)$-dimensional CFT, and in particular of determining its central charge, is based on the analysis of the chain's free energy. Indeed, as mentioned in the Introduction, at low temperatures the free energy per unit length of a $(1+1)$-dimensional CFT is given (in natural units $\hbar = k_B = 1$) by

$$f(T) \simeq f_0 - \frac{\pi c T^2}{6v}, \tag{13}$$

where $f_0 = f(0)$ is a constant, c is the central charge, and v is the effective speed of sound. On the other hand, by Eq. (7) the free energy of the spin chain (2) is simply given by

$$F(T) = -T\log Z = -T\sum_{l=0}^{N-1}\log Z_l,$$

where $Z_l = 1 + \mathrm{e}^{-\beta(\mathcal{E}(2\pi l/N)-\lambda)}$ (with $\beta \equiv 1/T$) is the partition function of the lth normal mode. Substituting in the previous equation and using the relation $\mathcal{E}(p) = \mathcal{E}(2\pi - p)$ we obtain the closed formula

$$f(T) = \lim_{N\to\infty}\frac{F(T)}{N} = -\frac{T}{\pi}\int_0^\pi \log[1+\mathrm{e}^{-\beta(\mathcal{E}(p)-\lambda)}]\,dp. \tag{14}$$

In order to determine the low-temperature behavior of $f(T)$, we note that $\mathcal{E}(p) - \lambda$ is negative for $0 < p < p_0$ and positive for $p_0 < p < \pi$, so that $f(T) = f_0 + f_1(T) + f_2(T)$, where

$$f_0 = \frac{1}{\pi}\int_0^{p_0}[\mathcal{E}(p) - \lambda]\,dp = f(0) \tag{15}$$

is constant and

$$f_1(T) = -\frac{T}{\pi}\int_0^{p_0}\log[1+\mathrm{e}^{-\beta[\lambda-\mathcal{E}(p)]}]\,dp, \tag{16}$$

$$f_2(T) = -\frac{T}{\pi}\int_{p_0}^{\pi}\log[1+\mathrm{e}^{-\beta[\mathcal{E}(p)-\lambda]}]\,dp \tag{17}$$

vanish at $T = 0$. The low-temperature behavior of $f_1(T)$ can be determined by performing the change of variable $x = [\lambda - \mathcal{E}(p)]/T$, which yields

$$f_1(T) = -\frac{T^2}{\pi}\int_0^{\lambda\beta}\log(1+\mathrm{e}^{-x})\frac{dx}{\mathcal{E}'(p)}. \tag{18}$$

The condition $\mathcal{E}'(p_0) \neq 0$ implies that $p - p_0 = O(Tx)$ and hence $\mathcal{E}'(p) = v + O(Tx)$, where $v = \mathcal{E}'(p_0)$ is the Fermi velocity. We thus have

$$f_1(T) = -\frac{T^2}{\pi v}\int_0^{\lambda\beta}\log(1+\mathrm{e}^{-x})\,dx + O(T^3),$$

so that for $T \ll 1$ we obtain [29]

$$f_1(T) = -\frac{T^2}{\pi v}\int_0^{\infty}\log(1+\mathrm{e}^{-x})\,dx + O(T^3)$$
$$= -\frac{\pi T^2}{12v} + O(T^3).$$

The last term $f_2(T)$ can be similarly dealt with through the change of variable $x = \beta(\mathcal{E}(p) - \lambda)$, with the same result.

Hence at low temperatures we have

$$f(T) = f_0 - \frac{\pi T^2}{6v} + O(T^3),$$

which coincides with Eq. (13) with $c = 1$. This shows that the spin chain (2) is indeed critical for $0 < \lambda < \mathcal{E}(\pi)$, with central charge $c = 1$.

The critical behavior of the su(1|1) chain at the endpoints $\lambda = 0, \mathcal{E}(\pi)$ can be similarly investigated. Indeed, suppose to begin with that $\lambda = 0$. In this case $f(T) = f_2(T)$, where f_2 is as in Eq. (17) with $p_0 = 0$, so that performing the change of variable $x = \beta\mathcal{E}(p)$ we obtain

$$f(T) = -\frac{T^2}{\pi}\int_0^{\beta\mathcal{E}(\pi)} \log(1 + \mathrm{e}^{-x})\frac{dx}{\mathcal{E}'(p)}\,.$$

The dispersion relation can be expanded around $p = 0$ as $\mathcal{E}(p) = (p/a)^\kappa + O(p^{\kappa+1})$, where $\kappa \geqslant 1$ denotes the order of the lowest nonvanishing derivative of $\mathcal{E}$ at the origin (generically, therefore, $\kappa = 1$) and

$$a \equiv \left[\frac{\kappa!}{\mathcal{E}^{(\kappa)}(0)}\right]^{1/\kappa}. \tag{19}$$

From the latter expansion we have $p/a = (Tx)^{1/\kappa} + O[(Tx)^{2/\kappa}]$, and therefore

$$\mathcal{E}'(p) = \frac{\kappa}{a}\left(\frac{p}{a}\right)^{\kappa-1} + O(p^\kappa) = \frac{\kappa}{a}\,(Tx)^{1-\frac{1}{\kappa}} + O(Tx)\,. \tag{20}$$

Substituting into the previous equation for $f(T)$ we thus obtain

$$f(T) = -\frac{aI_\kappa}{\kappa\pi}\,T^{1+\frac{1}{\kappa}} + O(T^{1+\frac{2}{\kappa}}), \quad T \ll 1, \tag{21}$$

with

$$I_\kappa \equiv \int_0^\infty x^{\frac{1}{\kappa}-1}\,\log(1 + \mathrm{e}^{-x})\,dx\,.$$

The integral I_κ can actually be evaluated using the technique of Ref. [17], namely:

$$\begin{aligned} I_\kappa &= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}\int_0^\infty x^{\frac{1}{\kappa}-1}\,\mathrm{e}^{-nx}\,dx \\ &= \Gamma(\kappa^{-1})\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^{1+\frac{1}{\kappa}}} \equiv \Gamma(\kappa^{-1})\eta(1+\kappa^{-1}) \\ &= (1 - 2^{-1/\kappa})\Gamma(\kappa^{-1})\zeta_{\mathrm{R}}(1+\kappa^{-1}), \end{aligned}$$

where $\zeta_{\mathrm{R}}(z)$ is Riemann's zeta function, $\eta(z)$ is Dirichlet's eta function, and we have used the identity $\eta(z) = (1 - 2^{1-z})\zeta_{\mathrm{R}}(z)$. Substituting into Eq. (21) we finally obtain

$$f(T) = -\gamma\,T^{1+\frac{1}{\kappa}} + O\big(T^{1+\frac{2}{\kappa}}\big), \tag{22}$$

with

$$\gamma = \frac{a}{\pi}\,(1 - 2^{-1/\kappa})\Gamma(1+\kappa^{-1})\zeta_{\mathrm{R}}(1+\kappa^{-1})\,. \tag{23}$$

We thus see that for $\lambda = 0$ the chain (2) cannot be critical unless $\kappa = 1$, i.e., $\mathcal{E}'(0) \neq 0$. Moreover, for $\kappa = 1$ we have $a = 1/v$, and therefore

$$f(T) = -\frac{\pi T^2}{12v} + O(T^3)\,.$$

This shows that when $\lambda = 0$ and $\mathcal{E}'(0) \neq 0$ the chain (2) is still critical but has central charge $c = 1/2$, and its low-energy behavior is therefore described by a CFT with one free *fermion*. For instance, for the elliptic interaction (9) $\kappa = 2$ for $0 \leqslant \alpha < \infty$, while $\kappa = 1$ for $\alpha = \infty$. In particular, for $0 \leqslant \alpha < \infty$ Eqs. (22)–(23) with $\kappa = 2$ reproduce the result in Ref. [17]. On the other hand, it is well known that the su(1|1) Haldane-Shastry chain (i.e., the $\alpha = \infty$ case) can indeed be described at low energies by a $(1 + 1)$-dimensional CFT with one free fermion [12].

The analysis is totally analogous at the other endpoint $\lambda = \mathcal{E}(\pi)$. Indeed, since now $\mathcal{E}(p) - \mathcal{E}(\pi) < 0$ for $0 \leqslant p < \pi$ we have $f(T) = f_0 + f_1(T)$, where f_0 and f_1 are, respectively, given by Eqs. (15) and (16) with $\lambda = \mathcal{E}(\pi)$. Performing the usual change of variable $x = \beta[\mathcal{E}(\pi) - \mathcal{E}(p)]$ we thus arrive at Eq. (18). Near $p = \pi$ we have $\mathcal{E}(\pi) - \mathcal{E}(p) = [(\pi - p)/b]^\nu + O[(\pi - p)^{\nu+1}]$, where ν denotes the lowest nonvanishing derivative of the dispersion relation at $p = \pi$ and

$$b \equiv \left[-\frac{\nu!}{\mathcal{E}^{(\nu)}(\pi)}\right]^{1/\nu}. \tag{24}$$

Note that, by the symmetry $\mathcal{E}(p) = \mathcal{E}(2\pi - p)$, ν is necessarily *even* and $\mathcal{E}^{(\nu)}(\pi) < 0$. Proceeding as before we readily obtain Eqs. (22)–(23) with a and κ respectively replaced by b and ν. In particular, since in this case $\nu \geqslant 2$ we see that at the endpoint $\lambda = \mathcal{E}(\pi)$ the model (2) is not critical. In summary, our analysis indicates that the latter model is critical for $0 < \lambda < \mathcal{E}(\pi)$, and for $\lambda = 0$ when $\mathcal{E}'(0) \neq 0$.

IV. GROUND STATE ENTANGLEMENT ENTROPY

We shall study in this section the von Neumann entanglement entropy S of the ground state of the su(1|1) supersymmetric model (2), defined as the von Neumann entropy of the reduced density matrix ρ_L of a block of L consecutive sites when the system is in its ground state. In other words, if we denote by $|\psi\rangle$ the ground state of the chain (2), then $\rho_L = \mathrm{tr}_{N-L}\,|\psi\rangle\langle\psi|$, where tr_{N-L} denotes the trace over the Hilbert space of the remaining $N - L$ sites, and the von Neumann entanglement entropy is given by

$$S = -\,\mathrm{tr}(\rho_L \log \rho_L)\,.$$

More generally, we shall also consider the Rényi entropy

$$S_q = \frac{\log \mathrm{tr}\,(\rho_L^q)}{1 - q},$$

where $q > 0$ is a real parameter, which reduces to that of von Neumann in the $q \to 1$ limit. As pointed out in the Introduction, the von Neumann and Rényi ground-state entanglement entropies of a $(1 + 1)$-dimensional CFT scale as $r_q \log L$ when $L \to \infty$, where the coefficient r_q is related to the holomorphic and antiholomorphic central charges c and $\bar{c}$ by $r_q = (1 + q^{-1})(c + \bar{c})/12$ (with $q = 1$ for the von Neumann entropy). Since the su(1|1) supersymmetric chain (2) is critical for $0 < \lambda < \mathcal{E}(\pi)$, with central charge $c = \bar{c} = 1$, it is to be expected that for this model

$$S_q \simeq \frac{1}{6}\,(1 + q^{-1})\log L$$

in the limit $L \to \infty$. In fact, we shall rigorously establish this asymptotic formula in the next section [cf. Eq. (32)].

Before addressing the actual computation of the entanglement entropy of the su(1|1) chain (2), we note that the result is the same for its "antiferromagnetic" version $-H$. This is most easily proved by considering the equivalent Hamiltonian (5), whose ground state entanglement entropy is obviously unchanged if we reverse the roles of the occupied and empty sites. In other words, the entanglement entropy is the same for the Hamiltonian (5) as for its image under the replacement $a_i \leftrightarrow a_i^\dagger$. Using the CAR and the even character of the interaction h, it is immediate to show that the latter transformation maps H into $-H - N[\lambda + h_N(0)]$, which establishes our claim.

First of all, it is clear that the ground state is not entangled for λ outside the interval $[0,\mathcal{E}(\pi)]$. Indeed, if (for instance) $\lambda < 0$ the ground state is obviously the vacuum $|0,\dots,0\rangle$ (i.e., the state with all sites occupied by bosons for the original Hamiltonian (2)), since in this case all the modes have positive energy $\mathcal{E}(2\pi l/N) - \lambda$. In particular, the ground state is a product state ($|0\rangle^{\otimes N}$) and is therefore not entangled. The situation is completely analogous for $\lambda > \mathcal{E}(\pi)$, since in this case $\mathcal{E}(2\pi l/N) - \lambda < 0$ for all l, and therefore all the modes are excited in the ground state. Thus $c_l^\dagger|\psi\rangle = 0$ for all $l = 0,\dots,N-1$, and therefore

$$a_k^\dagger|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \mathrm{e}^{-2\pi \mathrm{i} kl/N} c_l^\dagger|\psi\rangle = 0, \quad 1 \leqslant k \leqslant N\,.$$

Hence $|\psi\rangle = |1,\dots,1\rangle = |1\rangle^{\otimes N}$ (i.e., the state with all sites occupied by fermions), which is again a product state and therefore not entangled. (This is also true when $\lambda = \mathcal{E}(\pi)$ if N is *odd*.) From the previous considerations it follows that the ground state entanglement entropy of the model (2) vanishes for λ outside the interval $[0,\mathcal{E}(\pi)]$ (and when $\lambda = \mathcal{E}(\pi)$, if N is odd), since in these cases the ground state is a product state. For this reason, in the rest of this section we shall suppose that λ belongs to the open critical interval $(0,\mathcal{E}(\pi))$.

We shall next find a closed form expression for the entanglement entropy of the su(1|1) chain (2) by applying the method of Ref. [30] to the equivalent fermionic Hamiltonian (5). The first step in our computation is the evaluation of the ground-state correlation matrix A of the latter model, with matrix elements

$$A_{mn} = \langle\psi|a_m^\dagger a_n|\psi\rangle \equiv \langle a_m^\dagger a_n\rangle, \quad 1 \leqslant m,n \leqslant N\,.$$

This matrix can be easily determined (in the thermodynamic limit) from the relations

$$\langle c_j^\dagger c_k\rangle = \begin{cases} 0, & \lfloor l_0\rfloor + 1 \leqslant j \leqslant N - \lfloor l_0\rfloor - 1 \\ \delta_{jk}, & \text{otherwise,} \end{cases}$$

which in turn are a straightforward consequence of the CAR and the conditions

$$\begin{cases} c_j|\psi\rangle = 0, & \lfloor l_0\rfloor + 1 \leqslant j \leqslant N - \lfloor l_0\rfloor - 1 \\ c_j^\dagger|\psi\rangle = 0, & \text{otherwise} \end{cases}$$

characterizing the ground state. Indeed, from the inverse Fourier transform formula

$$a_k = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} kl/N} c_l$$

and the previous relations it immediately follows that [31]

$$\begin{aligned} A_{mn} &= \frac{1}{N}\left(\sum_{l=0}^{\lfloor l_0\rfloor} + \sum_{l=N-\lfloor l_0\rfloor}^{N-1}\right) \mathrm{e}^{-2\pi \mathrm{i}(m-n)l/N} \\ &= \frac{1}{N} + \frac{2}{N}\sum_{l=1}^{\lfloor l_0\rfloor} \cos[2\pi(m-n)l/N] \\ &\overset{N\gg 1}{\simeq} \frac{1}{\pi}\int_0^{p_0} \cos[p(m-n)]\,dp = \frac{\sin[p_0(m-n)]}{\pi(m-n)}\,. \end{aligned} \tag{25}$$

Let us now consider the analogous correlation matrix A_L for a block of L consecutive sites, which by translation invariance we can take as the first L ones. By the defining property of the reduced density matrix ρ_L [32], for $1 \leqslant m,n \leqslant L$ we have

$$\begin{aligned} (A_L)_{mn} &= \langle a_m^\dagger a_n\rangle_L \equiv \mathrm{tr}_L(a_m^\dagger a_n \rho_L) = \mathrm{tr}(a_m^\dagger a_n|\psi\rangle\langle\psi|) \\ &= \langle\psi|a_m^\dagger a_n|\psi\rangle \equiv A_{mn}, \end{aligned}$$

where tr_L denotes the trace over the Hilbert space of the first L sites. Thus A_L is just the submatrix of A consisting of its first L rows and columns. Following Ref. [30], we now consider an alternative basis of fermionic operators whose correlation matrix is diagonal. More precisely, let $U = (u_{mn})_{1\leqslant m,n\leqslant L}$ be a unitary matrix diagonalizing the Hermitian matrix A_L, i.e., satisfying

$$U A_L U^\dagger = \mathrm{diag}(\mu_1,\dots,\mu_L) \tag{26}$$

where $\mu_1,\dots,\mu_L \in [0,1]$ are the eigenvalues of A_L. We then define the operators g_k ($1 \leqslant k \leqslant L$) by $g_k = \sum_{m=1}^L u_{km}^* a_m$; note that g_k, though certainly nonlocal, acts on the Hilbert space of the first L sites. The operators g_k and their adjoints satisfy the CAR by the unitarity of the matrix U, and their correlation matrix is given by

$$\langle g_k^\dagger g_l\rangle_L = \mu_k \delta_{kl}$$

on account of Eq. (26). As shown in Ref. [33], the latter equation and Wick's theorem for Gaussian states imply that the correlation matrix factorizes as $\rho_L = \otimes_{k=1}^L \varrho_k$, with

$$\varrho_k = \mu_k g_k^\dagger g_k + (1-\mu_k) g_k g_k^\dagger\,.$$

The Hilbert space of the system is the tensor product of the two-dimensional spaces spanned by the vectors $|v\rangle_k, g_k^\dagger|v\rangle_k$ ($1 \leqslant k \leqslant N$), where $g_k|v\rangle_k = 0$. Moreover, from the CAR it easily follows that ϱ_k is diagonal in the basis $\{|v\rangle_k, g_k^\dagger|v\rangle_k\}$, with respective eigenvalues $1-\mu_k$ and μ_k. Thus the von Neumann and Rényi entropies of ϱ_k are respectively equal to $s(\mu_k)$ and $s_q(\mu_k)$, where

$$\begin{cases} s(x) = -x\log x - (1-x)\log(1-x), \\ s_q(x) = (1-q)^{-1}\log[x^q + (1-x)^q]. \end{cases} \tag{27}$$

By the additivity property of both of these entropies we then have

$$S = \sum_{k=1}^{L} s(\mu_k), \quad S_q = \sum_{k=1}^{L} s_q(\mu_k). \tag{28}$$

Equations (27)–(28), which are *exact* for any L, make it possible to evaluate numerically the ground state entanglement entropy of *any* supersymmetric su(1|1) chain of the form (2) in *polynomial* time, since they are based on the diagonalization of the $L \times L$ matrix with elements (25). From the latter equations it also follows that the entropy of all of these models is a *universal* function of the Fermi momentum p_0, the difference between two models being manifested only in the dependence of p_0 on the parameter λ through Eq. (12).

V. ASYMPTOTIC FORMULAS FOR THE ENTANGLEMENT ENTROPY

Equations (27)–(28) can be used to obtain approximate expressions for the entanglement entropy of the general su(1|1) supersymmetric chain (2) in several interesting regimes. To begin with, we shall investigate the behavior of the entropy as λ approaches its extreme critical values 0 and $\mathcal{E}(\pi)$. Suppose, in the first place, that λ tends to zero for fixed L, so that the Fermi momentum p_0 is much smaller than $1/L$. In this case all the matrix elements of the correlation matrix A_L in Eq. (25) are approximately equal to p_0/π, so that $A_L = p_0 B_L/\pi$, where B_L is the $L \times L$ matrix with all matrix elements equal to 1. Since the eigenvalues of B_L are 0 (with multiplicity $L-1$) and L, when $Lp_0 \ll 1$ the Rényi entanglement entropy is approximately given by

$$S_q \simeq s_q(Lp_0/\pi) \simeq \begin{cases} \frac{(Lp_0/\pi)^q}{1-q}, & 0 < q < 1; \\ \frac{q}{q-1}\frac{Lp_0}{\pi}, & q > 1. \end{cases}$$

For the same reason, when $Lp_0 \ll 1$ the von Neumann entropy can be approximated by

$$S \simeq s(Lp_0/\pi) \simeq -\frac{Lp_0}{\pi} \log\left(\frac{Lp_0}{\pi}\right).$$

In particular, we see that both S_q and S are continuous [34] at $\lambda = 0$. Similarly, suppose now that p_0 is close to its upper critical value $\mathcal{E}(\pi)$, so that $p_0 = \pi - \varepsilon$ with $\varepsilon \ll 1/L$. In this case we have

$$A_{mn} \simeq -(-1)^{m-n}\frac{\varepsilon}{\pi}, \quad m \neq n,$$

while $A_{nn} = (\pi - \varepsilon)/\pi$. Thus $A_L = \mathbb{1} - (\varepsilon C_L)/\pi$, where C_L is the $L \times L$ matrix with matrix elements $C_{mn} = (-1)^{m-n}$. It is easy to check that the eigenvalues of C_L are again 0 (with multiplicity $L-1$) and L, so that the previous asymptotic expressions for S_q and S still hold with p_0 replaced by $\pi - p_0$. In particular, this shows that the von Neumann and Rényi entanglement entropies are both continuous [35] also at the upper critical value $\lambda = \mathcal{E}(\pi)$. On the other hand, it is clear that these entropies have a discontinuous first derivative (with respect to the chemical potential λ) at both endpoints $\lambda = 0$ and $\lambda = \mathcal{E}(\pi)$. For instance, for $0 < \lambda \ll 1$ we have

$$p_0 \simeq a\,\lambda^{1/\kappa}, \tag{29}$$

where κ is the order of the first nonvanishing derivative of $\mathcal{E}$ at $p = 0$ and a is defined in Eq. (19). Thus $dS/d\lambda$ diverges as $\lambda^{1/\kappa - 1}|\log\lambda|$ when $\lambda \to 0^+$. Similarly, for $0 < q < 1$ the derivative of the Rényi entropy diverges as $\lambda^{q/\kappa - 1}$ in this limit, while for $q > 1$ $dS_q/d\lambda$ diverges as $\lambda^{1/\kappa - 1}$ for $\kappa > 1$ and tends to a nonzero finite limit when $\kappa = 1$. The situation is similar at the other endpoint $\lambda = \mathcal{E}(\pi)$, i.e.,

$$\pi - p_0 \simeq b[\mathcal{E}(\pi) - \lambda]^{1/\nu}, \tag{30}$$

with b defined by Eq. (24), except that now ν (the order of the lowest nonvanishing derivative of $\mathcal{E}$ at $p = \pi$) is necessarily even and thus greater than or equal to 2. Hence in all cases the derivatives of S and S_q diverge as $\lambda \to \mathcal{E}(\pi)^-$. The above analysis strongly suggests that there is a quantum phase transition at $\lambda = 0$ and $\lambda = \mathcal{E}(\pi)$ between an ordered (nonentangled) and a disordered (entangled) ground state, with the entanglement entropy as the order parameter. This conclusion is confirmed by the behavior of the zero-temperature fermion density n_f, which by translation invariance is simply given by

$$n_f = \langle a_i^\dagger a_i \rangle \equiv A_{ii} = \frac{p_0}{\pi} \tag{31}$$

in the critical interval $0 < \lambda < \mathcal{E}(\pi)$. Indeed, by Eqs. (29)–(30), near the two critical points $\lambda = 0, \mathcal{E}(\pi)$ the fermion density respectively behaves as $(a/\pi)\lambda^{1/\kappa}$ and $1 - (b/\pi)(\mathcal{E}(\pi) - \lambda)^{1/\nu}$. Since $n_f = 0$ for $\lambda < 0$ and $n_f = 1$ for $\lambda > \mathcal{E}(\pi)$, this behavior is typical of a quantum phase transition with exact exponents $1/\kappa$ and $1/\nu$ at the critical points $\lambda = 0$ and $\lambda = \mathcal{E}(\pi)$. For instance, for the elliptic interaction (9) it is known [17] that $\nu = 2$ and $\kappa = 2$ for $0 \leqslant \alpha < \infty$, while $\kappa = 1$ for $\alpha = \infty$ (i.e., for the su(1|1) HS chain). The parameters a and b can also be exactly computed in this case, namely,

$$a = \frac{\pi}{\sinh(\pi/\alpha)}\left(\frac{\pi^2}{6}g_2 - 2\eta_1^2\right)^{-1/2},$$

$$b = \frac{\pi}{\sinh(\pi/\alpha)}\left[\pi^2\left(\frac{g_2}{2} - 4e_1^2\right) + 2\eta_1(\eta_1 + 2\pi e_1)\right]^{-1/2},$$

where $e_1 = \wp(\pi)$ and g_2 is the second invariant of the Weierstrass function with half-periods $(\pi, \mathrm{i}\,\pi/\alpha)$ [25].

For the general elliptic su(1|1) model with interactions (9) (with $0 < \alpha < \infty$) and dispersion relation (10), it is of course unfeasible to explicitly invert $\mathcal{E}$ to obtain a closed-form expression for the Fermi momentum $p_0 = \mathcal{E}^{-1}(\lambda)$. Note, however, that the graph of the fermion density n_f admits the simple parametrization $(\mathcal{E}(p), p/\pi)$, with $0 < p < \pi$. In this way we have generated the plot in Fig. 2, where n_f is represented as a function of the normalized parameter $\lambda/\mathcal{E}(\pi)$, where $\mathcal{E}(\pi) = 2\sinh^2(\pi/\alpha)[e_1 - (2\eta_1/\pi)]$, for several values of α in the range [0,50] and for $\alpha = \infty$. The fermion density can be easily computed in closed form for the limiting cases $\alpha = 0$ and $\alpha = \infty$, i.e., for the XX model and the su(1|1) Haldane-Shastry chain, due to the simple form of their dispersion relations. Indeed, from Eq. (11) we immediately obtain

$$n_{f,XX} = \frac{2}{\pi}\arcsin(\sqrt{\lambda}/2), \quad n_{f,\mathrm{HS}} = 1 - \sqrt{1 - \frac{2\lambda}{\pi^2}},$$

respectively, for $0 < \lambda < 4$ and $0 < \lambda < \pi^2/2$. As expected, the first of these formulas agrees with the result in Ref. [33], taking into account that our parameter λ is related to the parameter h in the latter reference by $h = 2 - \lambda$. On the other

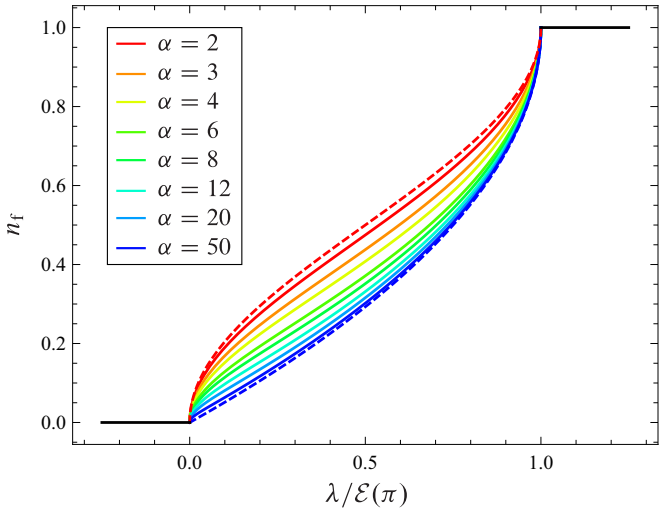

FIG. 2. Zero temperature density of fermions of the su(1|1) chain (2) with elliptic interactions (9) for several values of the parameter α in the range [2,50]. The red (top) and blue (bottom) dashed curves correspond respectively, to the *XX* model ($\alpha = 0$) and the su(1|1) Haldane-Shastry chain ($\alpha = \infty$).

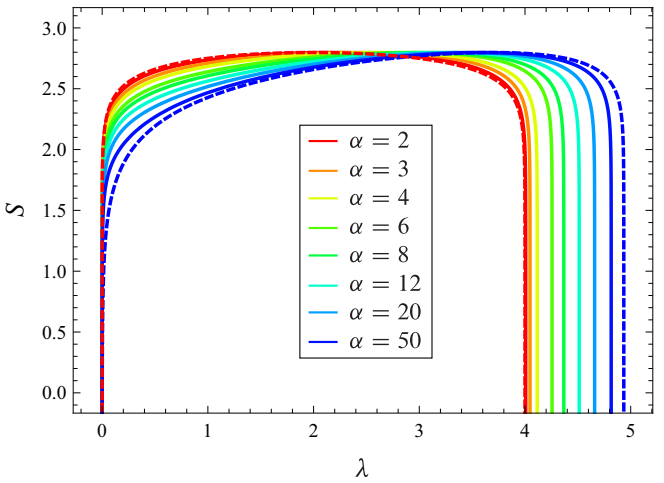

FIG. 3. Approximation (32) to the von Neumann entanglement entropy ($q = 1$) of the elliptic su(1|1) chain (2)–(9) for $L = 1000$ and several values of the parameter α between 2 and 50. The red and blue dashed curves correspond respectively to the *XX* Heisenberg model ($\alpha = 0$) and the su(1|1) Haldane-Shastry chain ($\alpha = \infty$).

hand, the formula for the su(1|1) HS chain is, to the best of our knowledge, new.

It is also of interest to determine the asymptotic behavior of the von Neumann and Rényi entropies for $0 < \lambda < \mathcal{E}(\pi)$ fixed and $L \gg 1$. To this end, we note that Eq. (25) implies that A_{mn} is a function of $m - n$ only, and hence the correlation matrix A_L is a Toeplitz matrix. This fact can be exploited to find a simple asymptotic formula for the von Neumann and Rényi entanglement entropies in the $L \to \infty$ limit, as shown in Ref. [33] for the *XX* model. The formula in the latter reference, which is based on a particular case of the general Fisher-Hartwig conjecture [36] proved by Basor [37], is also valid for a general model of the form (2) provided only that we express the result in terms of the Fermi momentum p_0. Indeed, this formula relies only on Eq. (25) for the correlation matrix, which, as we have just seen, holds for the model (2) with $p_0 = \mathcal{E}^{-1}(\lambda)$. In this way one obtains the following asymptotic formula for the Rényi entropy in the limit $L \sin p_0 \gg 1$:

$$S_q = \frac{q+1}{6q} \log(L \sin p_0) + \gamma_1^{(q)} + o(1), \qquad (32)$$

while the corresponding formula for the von Neumann entropy is obtained from the above by setting $q = 1$. Here $o(1)$ denotes a function of L and p_0 which tends to 0 as $L \sin p_0 \to \infty$, and $\gamma_1^{(q)}$ is a constant (independent of L and p_0) whose precise value, which can be found in Ref. [33], will not be needed in what follows.

Equation (32) can be easily applied in the case of the *XX* and su(1|1) Haldane-Shastry chains. Indeed, for the former of these models we have $\sin p_0 = \sqrt{\lambda(1 - \frac{\lambda}{4})}$, so that (32) agrees with the result in Ref. [33]. On the other hand, for the su(1|1) HS chain $\sin p_0 = \sin(\sqrt{\pi^2 - 2\lambda}\,)$, and hence Eq. (32) yields the following asymptotic formulas for the von Neumann and Rényi ground state entanglement entropies:

$$S_q = \frac{q+1}{6q} \log[L \sin(\sqrt{\pi^2 - 2\lambda})] + \gamma_1^{(q)} + o(1)\,. \qquad (33)$$

These formulas are valid for λ belonging to the critical interval $(0,\pi^2/2)$, in the asymptotic regime $L \sin \sqrt{\pi^2 - 2\lambda} \gg 1$. For the general elliptic su(1|1) model (9) with $0 < \alpha < \infty$ no such closed formulas are available. However, as for the fermion density, the graph of S_q admits the simple parametrization

$$\left(\mathcal{E}(p), \frac{q+1}{6q} \log[L \sin p] + \gamma_1^{(q)}\right), \quad 0 < p < \mathcal{E}(\pi),$$

where for simplicity we have dropped the $o(1)$ terms.

For instance, in Fig. 3 we present a plot of the approximation (32) to the von Neumann entropy of the elliptic su(1|1) chain (2)–(9) for $L = 1000$ and several values of the parameter α, including the limiting cases $\alpha = 0$ (the Heisenberg *XX* model) and $\alpha = \infty$ (the su(1|1) Haldane-Shastry chain). It is apparent that all of these plots are qualitatively similar, although only in the case of the *XX* model ($\alpha = 0$) is the graph of S symmetric about the midpoint $\lambda = \mathcal{E}(\pi)/2$. More precisely, the maximum of S at $\lambda = \mathcal{E}(\pi/2)$ is increasingly displaced towards the right as α tends to infinity, with $\mathcal{E}(\pi/2)/\mathcal{E}(\pi)$ varying continuously from 1/2 to 3/4 as α ranges from 0 to ∞.

VI. FERMION DENSITY AT FINITE TEMPERATURE

In the previous sections we have seen that the su(1|1) chain (2) is critical for $0 < \lambda < \mathcal{E}(\pi)$, with central charge $c = 1$. This is confirmed by the asymptotic behavior of the ground state entanglement entropy when the size of the block of spins considered tends to infinity. In this section we shall show that the fermion density at finite temperature, given by

$$n_f = \lim_{N\to\infty} \frac{1}{N} \sum_{l=0}^{N-1} (1 + e^{\beta[\mathcal{E}(2\pi l/N) - \lambda]})^{-1}$$
$$= \frac{1}{\pi} \int_0^\pi \frac{dp}{1 + e^{\beta[\mathcal{E}(p) - \lambda]}}, \qquad (34)$$

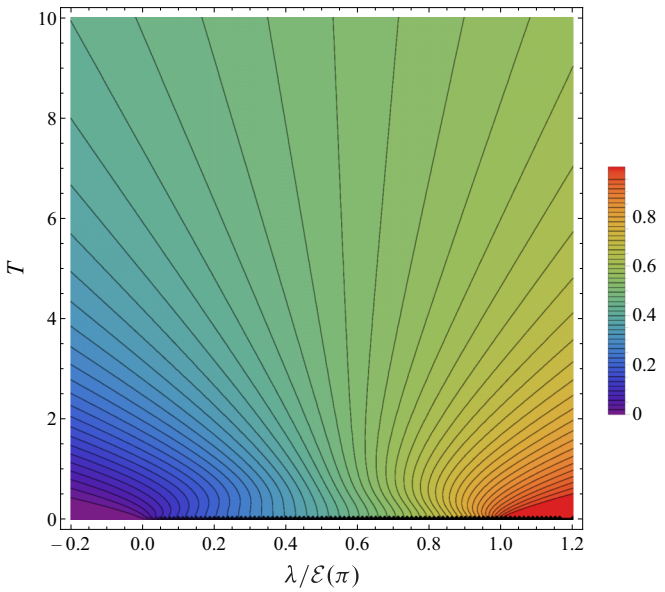

FIG. 4. Contour plot of the fermion density of the elliptic su(1|1) chain (2)–(9) with $\alpha = 5$ for $0 \leqslant T \leqslant 10$ and $-0.2 \leqslant \lambda/\mathcal{E}(\pi) \leqslant 1.2$.

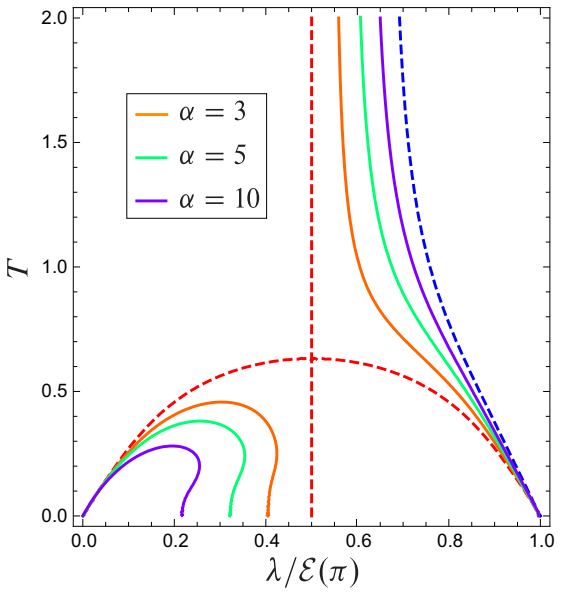

FIG. 5. Plot of the implicit curve $\partial n_f/\partial T = 0$ for the elliptic chain (2)–(9) and several values of the parameter α. The red and blue dashed curves correspond respectively to the XX model ($\alpha = 0$) and the su(1|1) Haldane-Shastry chain ($\alpha = \infty$).

also exhibits a qualitatively richer behavior when λ lies in the critical interval $(0,\mathcal{E}(\pi))$.

As a concrete example, we shall first focus on the su(1|1) elliptic chain (2)–(9). In Fig. 4 we present a contour plot of n_f for this model with $\alpha = 5$ for $T \in [0,10]$ and $\lambda/\mathcal{E}(\pi) \in [-0.2,1.2]$, obtained by numerically evaluating the integral in Eq. (34). For λ outside the critical interval $(0,\mathcal{E}(\pi))$, it is clear that n_f is a monotonic function of T [increasing for $\lambda \leqslant 0$, decreasing for $\lambda \geqslant \mathcal{E}(\pi)$], since

$$\frac{\partial n_f}{\partial T} = \frac{\beta^2}{4\pi}\int_0^\pi \frac{\mathcal{E}(p)-\lambda}{\cosh^2\{\beta[\mathcal{E}(p)-\lambda]/2\}}\,dp\,.$$

On the other hand, it is apparent from Fig. 4 that there is a range of values of λ in the interval $(0,\mathcal{E}(\pi))$ for which the fermion density is not a monotonic function of the temperature. Remarkably, the su(1|1) elliptic chain exhibits this interesting behavior for all values of the parameter α, including the limiting cases $\alpha = 0$ and $\alpha = \infty$. More precisely, for each α there are three critical values λ_i $(i = 1,2,3)$ of the chemical potential λ such that (i) for $0 < \lambda \leqslant \lambda_1$, the fermion density reaches an absolute minimum at some positive temperature and then increases monotonically towards its limiting value $1/2$; (ii) for $\lambda_1 < \lambda < \lambda_2$, n_f first reaches a maximum at some $T > 0$ and then a minimum, after which it tends monotonically to $1/2$; (iii) for $\lambda_2 \leqslant \lambda \leqslant \lambda_3$, n_f is monotonically increasing, and (iv) for $\lambda_3 < \lambda < \mathcal{E}(\pi)$, the fermion density attains an absolute maximum at some $T > 0$ and then decreases monotonically towards $1/2$. This is also true for the limiting values $\alpha = 0$ (XX model) and $\alpha = \infty$ (su(1|1) HS chain), for which $\lambda_1 = \lambda_2 = \lambda_3 = \mathcal{E}(\pi)/2 = 2$ and $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 2\mathcal{E}(\pi)/3 = \pi^2/3$, respectively. This behavior is qualitatively apparent from Fig. 5, where we have represented the implicit curve $\partial n_f/\partial T = 0$ versus λ and T for $\alpha = 0,3,5,10,\infty$, and is also confirmed by the plots of n_f versus T for these values of α and $\lambda = \mathcal{E}(\pi)/3,3\mathcal{E}(\pi)/4$ presented in Fig. 6.

Although in general the integral in Eq. (34) cannot be computed in closed form, its low-temperature behavior can be readily determined, as we shall now explain. To begin with, when $\lambda < 0$ the exponent $\beta(\mathcal{E}(p)-\lambda)$ is positive throughout the whole integration range, so that

$$n_f \simeq \frac{1}{\pi}\int_0^\pi \mathrm{e}^{-\beta(\mathcal{E}(p)-\lambda)}\,dp = \frac{T\mathrm{e}^{-|\lambda|\beta}}{\pi}\int_0^{\beta\mathcal{E}(\pi)} \frac{\mathrm{e}^{-x}}{\mathcal{E}'(p)}\,dx,$$

where $x = \beta\mathcal{E}(p)$. Using Eq. (20) and extending the integration range to $+\infty$ we obtain

$$\begin{aligned} n_f &\simeq \frac{a}{\kappa\pi}\,T^{1/\kappa}\mathrm{e}^{-|\lambda|\beta}\int_0^\infty x^{\frac{1}{\kappa}-1}\mathrm{e}^{-x}\,dx \\ &= \frac{a}{\pi}\,\Gamma(1+\kappa^{-1})\,T^{1/\kappa}\mathrm{e}^{-|\lambda|\beta}, \quad \lambda < 0\,. \end{aligned}$$

Proceeding in a similar way we obtain an analogous formula when $\lambda > \mathcal{E}(\pi)$:

$$n_f \simeq 1 - \frac{b}{\pi}\,\Gamma(1+\nu^{-1})\,T^{1/\nu}\mathrm{e}^{-\beta[\lambda-\mathcal{E}(\pi)]}, \quad \lambda > \mathcal{E}(\pi)\,.$$

We thus see that for $\lambda \notin [0,\mathcal{E}(\pi)]$ the fermion density at low temperature is monotonic, approaching exponentially its zero temperature values 0 (for $\lambda < 0$) and 1 (for $\lambda > \mathcal{E}(\pi)$). For $\lambda = 0$, the change of variable $x = \beta\mathcal{E}(p)$ and Eq. (20) easily

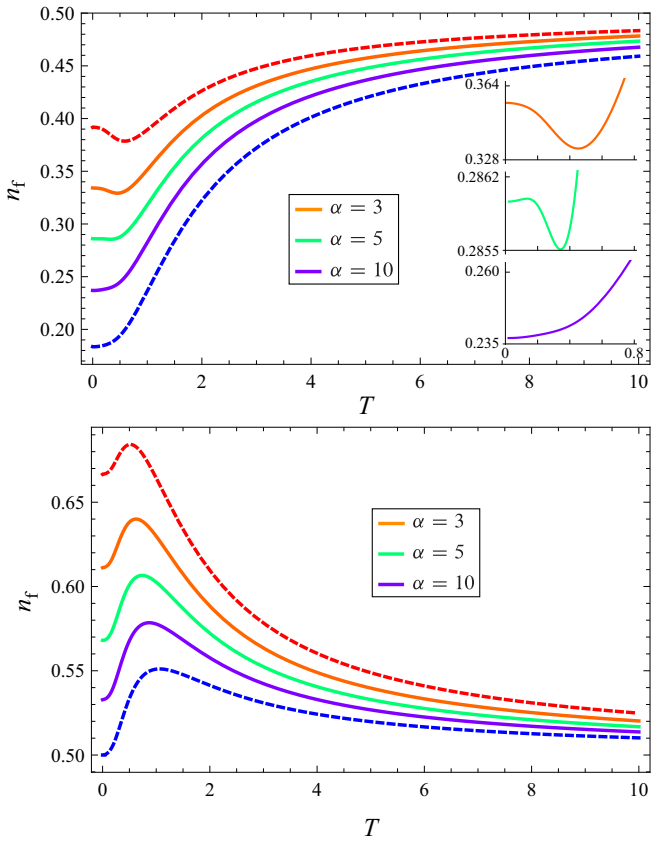

FIG. 6. Top: Plot of the fermion density of the chain (2)–(9) as a function of the temperature T for several values of the parameter α and $\lambda = \mathcal{E}(\pi)/3$ (inset: blowup of the range $0 \leqslant T \leqslant 0.8$). Bottom: Analogous plot for $\lambda = 3\mathcal{E}(\pi)/4$. In both plots, the red and blue dashed lines correspond to the limiting cases $\alpha = 0$ and $\alpha = \infty$.

yield

$$n_f \simeq \frac{aT^{1/\kappa}}{\kappa\pi}\int_0^\infty \frac{x^{\frac{1}{\kappa}-1}\,dx}{1+e^x} = \frac{aT^{1/\kappa}}{\kappa\pi}\int_0^\infty \frac{x^{\frac{1}{\kappa}-1}e^{-x}}{1+e^{-x}}\,dx$$
$$= \frac{aT^{1/\kappa}}{\kappa\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\infty x^{\frac{1}{\kappa}-1}e^{-nx}\,dx$$
$$= \frac{a}{\pi}\,\Gamma(1+\kappa^{-1})\eta(\kappa^{-1})\,T^{1/\kappa},$$

and therefore

$$n_f = \begin{cases} \frac{\log 2}{\pi v}\,T, & \kappa = 1, \\ \frac{a}{\pi}\,(1-2^{1-1/\kappa})\Gamma(1+\kappa^{-1})\zeta_R(\kappa^{-1})\,T^{1/\kappa}, & \kappa > 1\,. \end{cases}$$

Likewise, at the other endpoint $\lambda = \mathcal{E}(\pi)$ we have

$$n_f = 1 - \frac{b}{\pi}\,(1-2^{1-1/\nu})\Gamma(1+\nu^{-1})\zeta_R(\nu^{-1})\,T^{1/\nu},$$

since now ν is even and hence greater than 1.

Suppose next that λ lies in the critical interval $(0,\mathcal{E}(\pi))$. We start by writing the fermion density as the sum

$$n_f = \frac{p_0}{\pi} - \frac{1}{\pi}\int_0^{p_0}\frac{dp}{1+e^{-\beta[\mathcal{E}(p)-\lambda]}} + \frac{1}{\pi}\int_{p_0}^{\pi}\frac{dp}{1+e^{\beta[\mathcal{E}(p)-\lambda]}}$$
$$\equiv \frac{p_0}{\pi} - n_{f,1} + n_{f,2},$$

where the first term is the value of n_f at $T = 0$ [cf. Eq. (31)]. In this case the leading ($O(T)$) contributions to the two integrals $n_{f,i}$ cancel each other, so that we need to evaluate the $O(T^2)$ corrections. For the first integral, after performing the change of variable $x = \beta[\lambda - \mathcal{E}(p)]$ we have

$$n_{f,1} = \frac{T}{\pi}\int_0^{\lambda\beta}\frac{1}{\mathcal{E}'(p)}\,\frac{dx}{1+e^x}\,.$$

Expanding $\mathcal{E}'(p)$ to first order in $p - p_0$ we obtain

$$\mathcal{E}'(p) = \mathcal{E}'(p_0) + \mathcal{E}''(p_0)(p-p_0) + O[(p-p_0)^2]$$
$$= v - \frac{\mathcal{E}''(p_0)}{v}\,Tx + O[(Tx)^2],$$

where we have used the expansion $Tx = \mathcal{E}(p_0) - \mathcal{E}(p) = -v(p-p_0) + O[(p-p_0)^2]$. We thus have

$$\mathcal{E}'(p)^{-1} = \frac{1}{v}\left\{1 + \frac{\mathcal{E}''(p_0)}{v^2}\,Tx + O[(Tx)^2]\right\}.$$

Substituting in the definition of $n_{f,1}$ and using the estimate

$$\int_{\lambda\beta}^\infty x^j(1+e^x)^{-1}\,dx \leqslant \int_{\lambda\beta}^\infty x^j e^{-x}\,dx = O(\beta^j e^{-\lambda\beta})$$

(with $j = 0,1,\ldots$) we obtain

$$n_{f,1} = \frac{T}{\pi v}\int_0^\infty\frac{dx}{1+e^x} + \frac{\mathcal{E}''(p_0)T^2}{\pi v^3}\int_0^\infty\frac{x\,dx}{1+e^x} + O(T^3)$$
$$= \frac{\log 2}{\pi v}\,T + \frac{\pi\mathcal{E}''(p_0)}{12v^3}\,T^2 + O(T^3)\,. \qquad (35)$$

The term $n_{f,2}$ can be similarly dealt with through the analogous change of variable $x = \beta[\mathcal{E}(p) - \lambda]$, so that $Tx = \mathcal{E}(p) - \mathcal{E}(p_0) = v(p-p_0) + O[(p-p_0)^2]$ and hence

$$\mathcal{E}'(p)^{-1} = \frac{1}{v}\left\{1 - \frac{\mathcal{E}''(p_0)}{v^2}\,Tx + O[(Tx)^2]\right\}.$$

From the definition of $n_{f,2}$ we immediately obtain

$$n_{f,2} = \frac{T}{\pi v}\int_0^\infty\frac{dx}{1+e^x} - \frac{\mathcal{E}''(p_0)T^2}{\pi v^3}\int_0^\infty\frac{x\,dx}{1+e^x} + O(T^3)$$
$$= \frac{\log 2}{\pi v}\,T - \frac{\pi\mathcal{E}''(p_0)}{12v^3}\,T^2 + O(T^3),$$

and combining this result with Eq. (35) we finally have

$$n_f = \frac{p_0}{\pi} - \frac{\pi\mathcal{E}''(p_0)}{6v^3}\,T^2 + O(T^3)\,. \qquad (36)$$

In particular, for the XX and su(1|1) HS chains the low-temperature expansion (36) reads

$$n_{f,XX} = \frac{2}{\pi}\arcsin(\sqrt{\lambda}/2) - \frac{\pi(2-\lambda)}{6\lambda^{3/2}}\,T^2 + O(T^3),$$
$$n_{f,HS} = 1 - \frac{1}{\pi}\sqrt{\pi^2 - 2\lambda} + \frac{\pi T^2}{6(\pi^2-2\lambda)^{3/2}} + O(T^3)\,.$$

The absence of a term linear in T in Eq. (36) is in agreement with the low-temperature behavior of n_f apparent from Fig. 6. It is also interesting to observe that the sign of the leading correction to the $T = 0$ value of n_f is opposite to that of the second derivative of $\mathcal{E}$ at the Fermi momentum p_0. This behavior can be understood by noting that the energy difference between adding a fermion with momentum $p_0 + \Delta p$

(or $2\pi - p_0 - \Delta p$) and removing a fermion with momentum $p_0 - \Delta p$ (or $2\pi - p_0 + \Delta p$), with $0 < \Delta p \ll 1$, is given by $\mathcal{E}(p_0 + \Delta p) + \mathcal{E}(p_0 - \Delta p) - 2\mathcal{E}(p_0) \simeq \mathcal{E}''(p_0)\Delta p^2$. Thus when $\mathcal{E}''(p_0) < 0$ the addition of a fermion is energetically more favorable than its removal for momenta close to the Fermi momentum p_0 (or to $2\pi - p_0$), and consequently the fermion density should increase at sufficiently low temperatures. For instance, for the elliptic interaction (9) with $\alpha \geqslant 0$ finite $\mathcal{E}''(p)$ is positive for p less than a critical momentum (which depends on α) and negative for larger momenta, while for the su(1|1) HS chain (i.e., for $\alpha = \infty$) $\mathcal{E}''(p) = -1$ is always negative. Again, these facts are consistent with the behavior of n_f observed in Fig. 6.

VII. SUMMARY AND OUTLOOK

In this paper we introduce a general class of su(1|1) supersymmetric spin chains with long-range interactions generalizing the su(1|1) Haldane-Shastry and Inozemtsev (elliptic) chains, which can be fermionized using the algebraic properties of the su(1|1) permutation operator. We exploit this fact to study the critical behavior of this class of models (with nonzero chemical potential λ) in terms of their dispersion relation $\mathcal{E}(p)$. More precisely, we show that they are gapless when the chemical potential lies on the interval $[0,\mathcal{E}(\pi)]$, and that their ground state is a product state unless λ belongs to this interval. We prove that the models under study are actually critical when $0 < \lambda < \mathcal{E}(\pi)$ by verifying that their low-energy excitations are linear in the excitation momentum, and that their free energy at low temperature exhibits the characteristic quadratic behavior found in a $(1 + 1)$-dimensional CFT with $c = 1$ [18,19]. As further confirmation of this critical behavior, we find an exact asymptotic formula for the von Neumann and Rényi entanglement entropies for the ground state, showing that when λ belongs to the open interval $(0,\mathcal{E}(\pi))$ they both scale as $\log L$ when the size L of the block of spins considered tends to infinity. Moreover, in both cases the constant multiplying $\log L$ is the same as for a $(1 + 1)$-dimensional CFT with central charge $c = 1$ [20–22]. Likewise, we show that the asymptotic behaviors of the entanglement entropy and the zero-temperature fermion density as λ approaches the endpoints of the critical interval $(0,\mathcal{E}(\pi))$ are consistent with a quantum (continuous) phase transition. We also analyze the fermion density at finite temperature for a particular class of models with elliptic interactions, finding that its behavior is nontrivial (for instance, it is not always a monotonic function of the temperature, and it can in fact present up to two extrema at finite temperature) when λ belongs to the critical interval.

The results of this paper suggest several lines for future research. For one thing, they might prove relevant for the su(2) analogs of the models discussed in this paper, and most notably the spin 1/2 Inozemtsev and HS chains in the presence of a magnetic field. Indeed, it has been analytically shown that the su(1|1) HS chain with zero chemical potential λ is equivalent in the thermodynamic limit to its su(2) counterpart with zero magnetic field [17]. More recently, a numerical computation of the free energy of the spin 1/2 elliptic chain with no magnetic field suggests that this model and its su(1|1) version with $\lambda = 0$ studied in this paper are also equivalent in the thermodynamic limit [38]. If this equivalence could be extended to the case of nonzero chemical potential (or magnetic field strength, for the su(2) models), the results of this work could be used, for instance, to evaluate the ground state entanglement entropy of the spin 1/2 elliptic chain and its asymptotic limit when L tends to infinity.

Another line of research suggested by the present work is the study of the entanglement entropy of the low-lying states of the su(1|1) supersymmetric model (2) when the chemical potential lies in the critical interval $(0,\mathcal{E}(\pi))$. Indeed, it has been recently shown [39,40] that in a $(1 + 1)$-dimensional CFT the quotient between the entanglement entropy of an excited state created by acting on the vacuum with a primary field and that of the ground state is a universal quantity, essentially determined by the conformal weights of the field. Thus the computation of the entanglement entropy of the lowest excited states of the model (2) when $\lambda \in (0,\mathcal{E}(\pi))$, which can be constructed from the equivalent fermionic model (5), could shed some light on its underlying CFT.

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