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Diffusion with stochastic resetting at power-law times

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What happens when a continuously evolving stochastic process is interrupted with large changes at random intervals τ distributed as a power law $\sim\tau^{-(1+\alpha)}$; $\alpha > 0$? Modeling the stochastic process by diffusion and the large changes as abrupt resets to the initial condition, we obtain *exact* closed-form expressions for both static and dynamic quantities, while accounting for strong correlations implied by a power law. Our results show that the resulting dynamics exhibits a spectrum of rich long-time behavior, from an ever-spreading spatial distribution for $\alpha < 1$, to one that is time independent for $\alpha > 1$. The dynamics has strong consequences on the time to reach a distant target for the first time; we specifically show that there exists an optimal α that minimizes the mean time to reach the target, thereby offering a step towards a viable strategy to locate targets in a crowded environment.

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Introduction. A wide variety of physical phenomena during evolution undergo sudden *large* changes over a time substantially shorter than the typical dynamical time scale, e.g., financial crashes due to a fall in stock prices [1], a sudden reduction in population size due to catastrophes [2], and sudden changes in tectonic plate location in earthquakes. Often the time series of these phenomena exhibits bursts of intense activity separated by intervals distributed as a power law, e.g., in earthquakes [3], material failure under load fatigue [4], coronal mass ejection from the sun [5], fluorescence decay of nanocrystals and biomolecules [6,7], neuron firings [8], successive crashes in stock exchanges [1,9,10], and email sending times [11]. Considering the underlying generic situation of a continuously evolving process interrupted by sudden large changes at random times, a pertinent question of theoretical and practical relevance is then: How do these interruptions affect the observable properties at long times? To get a first answer, one may model the continuously evolving process by the widely relevant example of diffusion, and the large changes as resets to the initial state.

Diffusion with stochastic resetting has been extensively studied in recent times. Starting with a single diffusing particle resetting to its initial position [12,13], subsequent works studied motion in a bounded domain [14], in a potential [15], for many choices of resetting position [16–18], for a continuous-time random walk [19,20], for Lévy [21] and exponential constant-speed flights [22]. Resetting was also studied in interacting particle systems such as fluctuating interfaces [13,23] and reaction-diffusion models [24]. Diffusion combined with stochastic resetting mimics the natural search strategy, whereby an unsuccessful search continues by returning to the starting position [12], and was used to optimize search in combinatorial problems [25–27]. A naturally occurring example of resetting in many-particle systems is during protein production by ribosomes moving on mRNA, when the latter suddenly degrades at random times and the dynamics resets to the initial condition with the production of a new mRNA [28–30].

While the above works considered resetting at exponentially distributed times (or, a generalized exponential [31]), we consider here a power-law distribution. Even with random walks, changing the waiting time distribution for jumps from an exponential to a power law leads to significant consequences, e.g., rendering normal diffusion anomalous [32–34]; we may then already anticipate our model with a power law instead of an exponential for resetting times to result in dramatic changes. Diffusion involves spreading out of a dynamical observable from a region of high to low concentration, which in the absence of boundaries continues for all times. In the presence of resetting, the opposing tendencies of diffusive spreading and confinement around the initial state due to the abrupt resets lead to surprisingly rich behaviors. As the exponent of the power law varies, the change in the relative dominance of diffusion *vis-à-vis* resetting results in significantly different behaviors. Strong correlations implied by a power law pose a challenge for analytic tractability, yet, remarkably, we are able to characterize these multiple behaviors by *exact* closed-form expressions for both static and dynamic quantities.

In this work, we consider a particle with diffusion constant D diffusing in one dimension x, and being interrupted at random times by a reset to its initial location x_0. The time τ between successive resets is distributed as a power law:

$$\rho(\tau) = \frac{\alpha}{\tau_0(\tau/\tau_0)^{1+\alpha}};\ \tau \in [\tau_0,\infty),\ \alpha > 0, \tag{1}$$

with τ_0 a microscopic cutoff. Figures 1(a) and 1(b) show typical space-time trajectories for representative α's. Note that for $\alpha < 1$, all moments of $\rho(\tau)$ are infinite. For $\alpha > 1$, the first moment is finite: $\langle\tau\rangle = \tau_0\alpha/(\alpha-1)$; while for $\alpha > 2$, the second moment also becomes finite: $\langle\tau^2\rangle = \tau_0^2\alpha/(\alpha-2)$. By contrast, the previously studied exponential $\rho(\tau)$ always has finite mean and variance. Also, an exponential $\rho(\tau)$ implies a resetting at any time to occur with a constant probability. By contrast, a power-law distribution implies, depending on α, the corresponding probability to depend explicitly on time.

Our exact results for the long-time properties of the system show that the spatial probability distribution exhibits on tuning α a rich behavior with multiple crossovers. For $0 < \alpha < 1$, the average gap $\langle\tau\rangle$ between successive resets being infinite, a typical space-time trajectory in a given time has a small number of reset events, and in between diffuses further away from the initial location, Fig. 1(a); this leads to a spatial distribution with a width that continually increases in

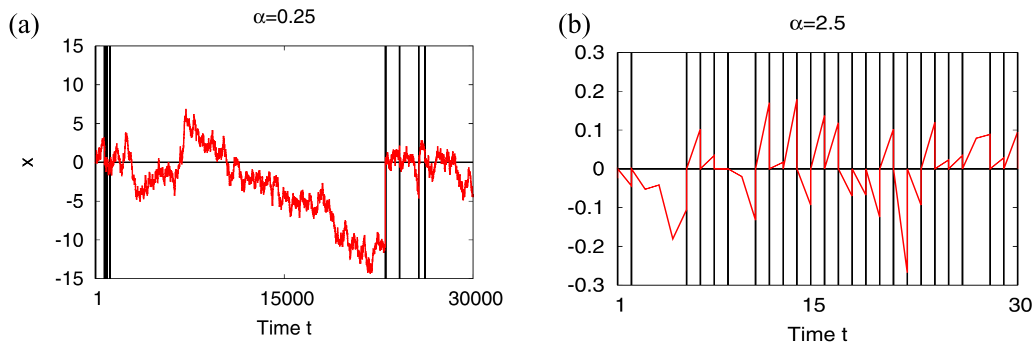

FIG. 1. Typical space-time trajectories (red lines), with black lines marking resetting events: Resetting location $x_0 = 0$, diffusion constant $D = 0.5$, $\tau_0 = 1.0$.

time as $\sqrt{t}$, similar to diffusive spreading. The behavior for $\alpha < 1$ is captured in the scaling plots in Figs. 2(a) and 2(b). By contrast, for $\alpha > 1$, a finite $\langle\tau\rangle$ implies frequent resets in a given time, so that the particle does not diffuse too far from its initial location, Fig. 1(b). Hence, one has at long times a spatial probability distribution that no longer spreads in time, but is time independent with power-law tails [Fig. 2(c)]; nevertheless, fluctuations as characterized by the mean-squared displacement (MSD) diverge with time for $1 < \alpha < 2$, while a time-independent behavior emerges only for $\alpha > 2$. Previous studies for an exponential $\rho(\tau)$ have shown that diffusion with resetting always leads to a time-independent spatial distribution with a finite MSD. Our work highlights that such a scenario does not necessarily hold for a power law $\rho(\tau)$.

Besides the crossovers at $\alpha = 1,2$, there is another one at $\alpha = 1/2$, where the time-dependent spatial distribution near the resetting location changes over from a cusp for $0 < \alpha < 1/2$ [Fig. 2(a)] to a divergence for $1/2 < \alpha < 1$ [Fig. 2(b)]. This feature may be contrasted with exponential resetting, where the spatial distribution at long times always exhibits a cusp singularity [12]. As we will show, this difference in behavior is linked to resetting events occurring with a probability that is time independent for an exponential $\rho(\tau)$, but which has an essential time dependence for a power law $\rho(\tau)$ for $0 < \alpha < 1$. We also study the mean first passage time (MFPT) for the diffusing-resetting particle to reach a distant target fixed in space. The MFPT is an important quantifier of practical relevance, e.g., for a diffusing reactant on a polymer that has to react with an external reactive site fixed in space [35,36]. A surprise emerging from our results is that for $\alpha > 1$, the MFPT exhibits a nonmonotonic dependence on α, implying an optimal α that minimizes the MFPT to reach a given target. The derivation and understanding of these results constitute the rest of this Rapid Communication.

We begin with deriving $P^{\rm r}(x,t|x_0,0)$, the probability density for the particle to be at x at time t, given $x = x_0$ at $t = 0$. This probability depends solely on trajectories originating at the last reset prior to t, when the motion starts afresh (gets "renewed") at x_0. Then, $P^{\rm r}(x,t|x_0,0)$ is given by the propagator $P(x,t|x_0,t-\tau) \equiv \exp[-(x-x_0)^2/(4D\tau)]/\sqrt{4\pi D\tau}$ of free diffusion for time τ ($\tau \in [0,t]$) elapsed since the last reset, weighted by the probability density $f_\alpha(t,t-\tau)$ at time t for the last reset to occur at time $t-\tau$, as [37]

$$P^{\rm r}(x,t|x_0,0) = \int_0^t d\tau\ f_\alpha(t,t-\tau)P(x,t|x_0,t-\tau). \tag{2}$$

To proceed, we require $f_\alpha(t,t-\tau)$, which is given by the probability density $G(t-\tau)$ for a reset at time $t-\tau$ and the probability $\rho_0(\tau)$ for no reset in the interval $[t-\tau,t]$, as $f_\alpha(t,t-\tau) = \rho_0(\tau)G(t-\tau)$, where $\rho_0(\tau) \equiv \int_\tau^\infty d\tau'\,\rho(\tau') = (\tau/\tau_0)^{-\alpha}$; $\tau \geqslant \tau_0$, using Eq. (1). Let $g_n(t)$; $n \geqslant 0$, be the probability density for the nth reset at time t, with $\int_0^\infty dt\ g_n(t) = 1\ \forall n$. Here, $g_0(t) = \delta(t)$ accounts for the initial condition $x = x_0$ at $t = 0$, which itself is a reset. One has [38] $g_n(t) = \int_0^t d\tau\ \rho(t-\tau)g_{n-1}(\tau)$; $n \geqslant 1$, since the probability for the nth reset at time t is given by the probability for the $(n-1)$th reset at an earlier time τ and the probability that the next reset happens after an interval $t-\tau$. By definition, we have $G(t) = \delta(t) + \sum_{n=1}^\infty g_n(t)$, and a straightforward calculation using Laplace transform (LT) to compute $g_n(t)$ yields for large t that $G(t) = 1/\langle\tau\rangle$ for $\alpha > 1$, and $G(t) = t^{\alpha-1}$ for $0 < \alpha < 1$. For an exponential $\rho(\tau) = r\ \exp(-r\tau)$, $G(t) = r$ for all $t > 0$.

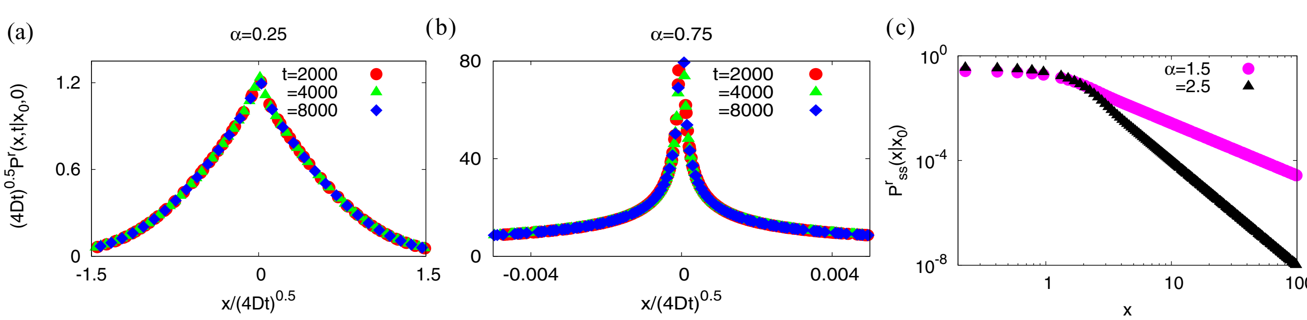

FIG. 2. (a), (b) Data collapse of exact spatial distribution for $\alpha < 1$ for different times, following Eq. (5). (c) Time-independent distribution for $\alpha > 1$, Eq. (8). Resetting location $x_0 = 0$, diffusion constant $D = 0.5$, $\tau_0 = 1.0$.

By contrast, for the power law for $0 < \alpha < 1$, $G(t)$ is time dependent, which we show later to significantly affect the observable properties. We get for $t \gg \tau_0$ [39,40],

$$f_{\alpha<1}(t,t-\tau) = \frac{\sin(\pi\alpha)}{\pi}\tau^{-\alpha}(t-\tau)^{\alpha-1}, \quad (3)$$

$$f_{\alpha>1,\tau\geqslant\tau_0}(t,t-\tau) = \frac{1}{\tau_0}\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{\tau}{\tau_0}\right)^{-\alpha}, \quad (4)$$

and $\int_0^{\tau_0} d\tau\, f_{\alpha>1,\tau<\tau_0}(t,t-\tau) = 1 - \int_{\tau_0}^{t} d\tau\, f_{\alpha>1,\tau\geqslant\tau_0}(t,t-\tau)$. Knowing f_α, Eq. (2) allows one to derive $P^{\mathrm{r}}(x,t|x_0,0)$.

Spatial distribution, $\alpha < 1$. For large $t \gg \tau_0$, we have [40]

$$P^{\mathrm{r}}(x,t|x_0,0) = \frac{\Gamma(\alpha)\sin(\pi\alpha)e^{-z/t}}{\pi\sqrt{4\pi Dt}}U\left(\alpha,\alpha+\frac{1}{2},\frac{z}{t}\right), \quad (5)$$

where $z \equiv (x-x_0)^2/(4D)$, and $U(a,b,x)$ is the confluent hypergeometric function [41]. In the limit $t \to \infty$, the right-hand side does not approach a time-independent form. Since the average time $\langle\tau\rangle$ between successive resets is infinite for $\alpha < 1$, a typical space-time trajectory shows bursts of resets separated by very long time intervals during which the particle diffuses further and further away from its initial position [see Fig. 1(a)] leading to the spatial distribution (5) that continually broadens in time. While $\langle x - x_0\rangle = 0$ due to the mirror symmetry about x_0 of the dynamics, the MSD grows linearly with time as in pure diffusion. The time dependence in Eq. (5) is captured by the data collapse in Figs. 2(a) and 2(b).

The limiting behavior of $P^{\mathrm{r}}(x,t|x_0,0)$ for small and large x reveals rich and hitherto unexpected features. Using large and small x behavior of $U(a,b,x)$ [42] yields

$$P^{\mathrm{r}}(x,t|x_0,0) \sim \begin{cases} \frac{\Gamma(\alpha-1/2)}{(4Dt)^{1-\alpha}}\frac{\sin(\pi\alpha)}{\pi^{3/2}|x-x_0|^{2\alpha-1}}; & |x-x_0|\to 0, \quad \frac{1}{2}<\alpha<1, \\ \frac{\Gamma(1/2-\alpha)\Gamma(\alpha)}{\sqrt{4\pi Dt}}\frac{\sin(\pi\alpha)}{\pi^{3/2}}; & |x-x_0|\to 0, \quad \alpha<\frac{1}{2}, \\ e^{-(x-x_0)^2/(4Dt)}; & |x-x_0|\to\infty. \end{cases} \quad (6)$$

Thus, as $|x-x_0| \to 0$, the behavior crosses over from being with a cusp for $\alpha < 1/2$ [Fig. 2(a)] to being divergent for $1/2 < \alpha < 1$ [Fig. 2(b)]. This crossover behavior stems from the form of $f_{\alpha<1}(t,t-\tau)$, which is peaked at $\tau = 0,t$, implying that most resets are close to either the present or the initial time. However, as α crosses $1/2$, the relative weight of these peaks changes, with the peak at $\tau = 0$ becoming more dominant for $\alpha > 1/2$; this leads to a significant increase in reset events at small intervals prior to the time of observation, thereby increasing the probability for the particle to be close to the resetting location, and effecting the mentioned crossover from a cusp to a divergence around x_0 across $\alpha = 1/2$. The behavior of $P^{\mathrm{r}}(x,t|x_0,0)$ for $|x-x_0| \gg 1$ is dominated by the propagator of the free diffusing particle, due to many trajectories having last resets close to the initial time and free diffusion without reset at subsequent times.

Spatial distribution, $\alpha > 1$. We get for $t \gg \tau_0$ [40],

$$P^{\mathrm{r}}(x,t|x_0,0) = \left\{1-\frac{1}{\alpha}\left[1-\left(\frac{t}{\tau_0}\right)^{1-\alpha}\right]\right\} \times\frac{\exp(-z/\tau_0)}{\sqrt{4\pi D\tau_0}} + \frac{(\alpha-1)\tau_0^{\alpha-1}}{\alpha\sqrt{4\pi D}} \times\left[\frac{\gamma(\beta,z/\tau_0)}{z^{-\beta}} - \frac{e^{-z/t}}{t^\beta}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha-1/2)(z/t)^k}{\Gamma(\alpha+k+1/2)}\right], \quad (7)$$

where $\beta \equiv \alpha - 1/2$ and $\gamma(a,x)$ is the lower incomplete gamma function. As before, $\langle x - x_0\rangle = 0$ by symmetry, while the MSD for $\alpha > 2$ converges at long times to $2D\tau_0(\alpha-1)^2/[\alpha(\alpha-2)]$, and diverges with time for $1 < \alpha < 2$ as $t^{2-\alpha}$, thus exhibiting a crossover at $\alpha = 2$.

Unlike for $\alpha < 1$, here $P^{\mathrm{r}}(x,t|x_0,0)$ is independent of time as $t \to \infty$ to yield a nontrivial steady state [43]

$$P^{\mathrm{r}}_{\mathrm{ss}}(x|x_0) = \left(\frac{\alpha-1}{\alpha\sqrt{4\pi D\tau_0}}\right)\mathcal{G}\left(\frac{|x-x_0|}{\sqrt{4D\tau_0}}\right); \quad (8)$$

$\mathcal{G}(y) = y^{1-2\alpha}\gamma(\alpha-1/2,y^2) + e^{-y^2}$. Using $\gamma(a,x)/x^a \to 1/a$ as $x \to 0$, $\gamma(a,x) \to \Gamma(a)$ as $x \to \infty$ gives

$$P^{\mathrm{r}}_{\mathrm{ss}}(x|x_0) \sim \begin{cases} \frac{(\alpha-1)(2\alpha+1)}{\alpha(2\alpha-1)\sqrt{4\pi D\tau_0}}; & |x-x_0|\to 0, \\ \frac{(\alpha-1)\Gamma\left(\alpha-\frac{1}{2}\right)}{\alpha\sqrt{4\pi D\tau_0}}\left[\frac{4D\tau_0}{(x-x_0)^2}\right]^{\alpha-1/2}; & |x-x_0|\to\infty. \end{cases} \quad (9)$$

The steady state distribution has power-law tails and a cusp around x_0, Fig. 2(c). Equation (7) implies a late-time relaxation to the steady state as $\sim t^{1/2-\alpha}$. As for $\alpha < 1$, $f_{\alpha>1}(t,t-\tau)$ explains the above behavior: Eq. (4) implies a large number of resets in the small interval $[t,t-\tau_0]$, while those outside this interval occur with a probability decaying as a power law. Hence, the probability of finding the particle very far from the resetting position is relatively small, explaining the power-law tails in Eq. (9). That the MSD is infinite for $1 < \alpha < 2$ is explained by the fact that in this range, $\langle\tau^2\rangle$ is infinite, so that although trajectories on an *average* are reset after a time $\langle\tau\rangle$, there are huge fluctuations around the average in the *actual* time between resets. This feature leads at a given time t to a finite probability for the particle to be at a position $|x| \gg |x_0|$, owing to trajectories that were last reset in a time of duration substantially longer than $\langle\tau\rangle$. Such events contribute a fat-enough tail to $P^{\mathrm{r}}_{\mathrm{ss}}(x|x_0)$ that the MSD does not have a finite value even at long times. Invoking a similar argument implies a finite MSD at long times for $\alpha > 2$ when $\langle\tau^2\rangle$ is finite.

First-passage time. Let $f^{\mathrm{r}}(x_0,T)$ be the first-passage time distribution (FPTD), i.e., $f^{\mathrm{r}}(x_0,T)dT$ is the probability that the motion starting at $x = 0$ crosses x_0 for the first time between times T and $T + dT$. We have $f^{\mathrm{r}}(x_0,T) = -\partial q(x_0,T)/\partial T$, with $q(x_0,T)$ the probability that the motion has not crossed x_0 up to time T. The MFPT is $\langle T\rangle \equiv \int_0^\infty dT\, T f^{\mathrm{r}}(x_0,T) = \widetilde{q}(x_0,0)$, where $\widetilde{q}(x_0,s)$ is the LT of $q(x_0,T)$, and we have used $q(x_0,\infty) = 0$. A renewal theory argument akin to that used for

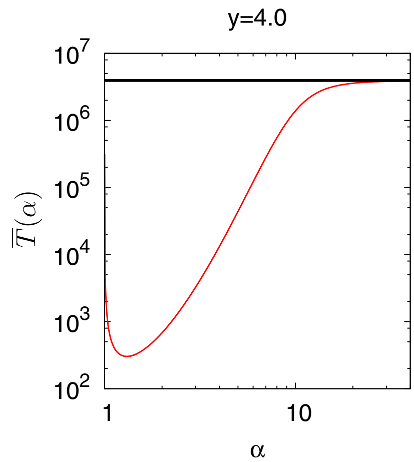

FIG. 3. $\overline{T}(\alpha)$ versus α, showing the existence of a minimum.

$P^{\mathrm{r}}(x,t|x_0,0)$ gives

$$f^{\mathrm{r}}(x_0,T)=\int_0^T d\tau\, q(x_0,T-\tau)f_\alpha(T,T-\tau)f(x_0,\tau), \quad (10)$$

since a trajectory reaching x_0 from $x=0$ for the first time at time T is last reset at an earlier instant $T-\tau;\ \tau\in[0,T]$, and has not passed through x_0 before that.

Note that in the absence of resetting, we have the FPTD $f(x_0,T)=|x_0|/\sqrt{4\pi DT^3}\exp[-x_0^2/(4DT)]$, thus $\langle T\rangle=\infty$ [44]. In our case, the existence of a steady state for $\alpha>1$ allows for a finite MFPT, which we now demonstrate. Let us introduce a dimensionless variable $y\equiv|x_0|/\sqrt{4D\tau_0}$, given by the ratio of the distance to the location of desired first passage to the diffusive length scale in the system. The LT of Eq. (10) gives the dimensionless MFPT $\overline{T}(\alpha)\equiv\langle T\rangle/\tau_0$ as a function of $y\gg 1$ [40]:

$$\overline{T}(\alpha)=\sqrt{\pi}\left(\frac{\alpha}{\alpha-1}\right)\left[ye^{-y^2}+\frac{\gamma(\alpha+1/2,y^2)}{y^{2\alpha}}\right]^{-1}. \quad (11)$$

As $\alpha\to\infty$, $\overline{T}(\alpha\to\infty)=(\sqrt{\pi}/y)\exp(y^2)$. The expression for $f_{\alpha>1}$ implies that this limit corresponds to resetting deterministically after every τ_0 time, so that the FPTD is $re^{-rt};\ r\equiv y/(\sqrt{\pi})\exp(-y^2)$, leading to the form of $\overline{T}(\alpha\to\infty)$. Figure 3 shows that the MFPT at a fixed y changes nonmonotonically with α; the value at which $\overline{T}(\alpha)$ shows a minimum as a function of α can be obtained numerically. The existence of a minimum implies a result relevant both physically and in the context of search processes in a crowded environment. Namely, for a given distance $|x_0|$ to a fixed target and a given diffusion constant D, an optimal α minimizes the time to get to the target for the first time.

Equation (11) implies that the MFPT diverges as α approaches unity from above, and in fact, the MFPT is infinite for $\alpha<1$. This is because for $\alpha<1$, the long-time behavior is similar to free diffusion, with the spatial distribution expanding indefinitely in time. Then, the probability of a typical trajectory to achieve a first passage through a given location fixed in space gets smaller with time, and only an atypical one reaches the target, resulting in an infinite MFPT.

Conclusions. We considered the dynamics of a particle diffusing and resetting to its initial position at random times sampled from a power law $\sim\tau^{-(1+\alpha)}$. Our exact calculations demonstrated many interesting effects: on tuning α across 1, the motion at long times crosses over from being unbounded in time to one that is time independent even in the absence of boundaries. This behavior may be contrasted with resetting at exponentially distributed times that always leads to a time-independent state at long times. A surprising behavior emerges in the time-dependent spatial distribution around the resetting location for $\alpha<1$: it shows a crossover from a cusp for $\alpha<1/2$ to a divergence for $1/2>\alpha>1$. Although the motion at long times is time independent for $\alpha>1$, the mean-squared displacement diverges with time for $1<\alpha<2$, but is time independent for $\alpha>2$. For the mean time to reach for the first time a distant target fixed in space, we revealed for $\alpha>1$ that there exists of all possible reset strategies an optimal one corresponding to a particular α that minimizes the mean time.

Our investigations open up many possibilities for future studies. In the context of search problems, it is interesting to study the time to reach targets randomly distributed in space by one or many independent searchers. Such a situation emerges in the context of animal foraging, where a reset corresponds to returning to the nest [45]. One may further study the effects of disorder in space due to geographical obstructions and predators that alter the path of a searcher. To this end, our setup can be generalized to a motion on a lattice with every site having as a waiting time a random variable quenched in space and time. Another interesting followup of our work is to extend it to many-particle interacting systems, and investigate how dynamics at multiple scales interplays with resetting. Our observed crossovers arise from the nontrivial time dependence of the probability of last reset, and should be observable in other systems; our initial results on interfaces confirm this expectation [46].

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[40] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.93.060102 for details on the derivation of Eqs. (3), (4), (5), (7), and (11).
[41] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[42] The large-x behavior is $U(a,b,x) \sim x^{-a}$, and that for small x is $U(a,b,x) \approx [\Gamma(b-1)/\Gamma(a)]x^{1-b}$ for $1 < b < 2$, and $U(a,b,x) \approx \Gamma(1-b)/\Gamma(1+a-b)$ for $0 < b < 1$ [41].
[43] Any steady state reached by resetting manifestly breaks detailed balance, and is thus a generic nonequilibrium steady state [12].
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