Hydrogen atom in a quantum plasma environment under the influence of Aharonov-Bohm flux and electric and magnetic fields

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This study presents the confinement influences of Aharonov-Bohm (AB) flux and electric and magnetic fields directed along the z axis and encircled by quantum plasmas on the hydrogen atom. The all-inclusive effects result in a strongly attractive system while the localizations of quantum levels change and the eigenvalues decrease. We find that the combined effect of the fields is stronger than a solitary effect and consequently there is a substantial shift in the bound state energy of the system. We also find that to perpetuate a low-energy medium for the hydrogen atom in quantum plasmas, a strong electric field and weak magnetic field are required, whereas the AB flux field can be used as a regulator. The application of the perturbation technique utilized in this paper is not restricted to plasma physics; it can also be applied in molecular physics.

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I. INTRODUCTION

Plasma is one of the four fundamental states of matter. It is a gas, but not a usual kind of gas. This is because in a normal gas, the atoms or molecules are electrically neutral, but in a plasma at least some of these particles have either lost or gained an electron, so a plasma consists of free electrons and positively or negatively charged atoms and molecules known as ions. Thus, we can describe a plasma as an ionized gas, a gas in which sufficient energy is provided to free electrons from atoms or molecules and to allow both species, i.e., ions and electrons, to coexist. Moreover, to transform a normal gas into plasma, a very high temperature is required.

The Debye-Hückel theory, as originally proposed [1], gave a theoretical explanation for departures from ideality in solutions of electrolytes and plasmas. The Debye-Hückel model provides a modern treatment of nonideality in plasma via the screening effect. This model is used to simulate plasma screening effect of weakly coupled plasmas and it is given by [2] $V(r) = -(Ze^2/r)\exp(-r/\lambda_D)$, where λ_D represents the Debye length or Debye screening parameter and determines the interaction between electrons in Debye plasma. This model generally accounts for pair correlations. It can be observed from the model that the effect of plasmas on a test charge is just a replacement of the Coulomb potential by an effective screened potential. It was shown in Ref. [3] that the effective screened potential of a test charge of mass m in a dense quantum plasma can be modeled using a modified Debye-Hückel potential also known as the exponential-cosine-screened Coulomb potential $V(r) = -(Ze^2/r)\exp(-r/\lambda_D)\cos(r/\lambda_D)$. It was shown in Ref. [2] that $V(r) = -(Ze^2/r)\exp(-r/\lambda_D)\cos(g\ r/\lambda_D)$ can be used to model weakly coupled plasmas with $g = 0$ and dense quantum plasmas with $g = 1$.

There has been ceaseless interest (see [4–7] and references therein) in studying atomic and molecular processes in the plasma environment due to their applications in distinguishing various plasmas and also providing passable knowledge of collision dynamics. The ionization processes and atomic excitation play a crucial role in the conceptual understanding of various phenomena related to hot plasma physics, astrophysics, and experiments performed with charged ions. It has been discerned that a long-range Coulomb field plays a decidedly salient role in the electron-ion scattering problem [4]. For instance, at small scattering angles, the total cross section for elastic scattering of a charged particle in a Coulomb field diverges and the impact excitation cross sections for electron-positive ion collisions have finite values at the reaction threshold.

Strictly speaking, there are many studies focusing on studying the effects of several fields on the hydrogen atom embedded in plasmas. For instance, Bahar and Soylu [8] studied the confinement effect of a magnetic field on the two-dimensional hydrogen atom in plasmas. The plasma screening effect of dense quantum plasmas on the photodetachment cross section of a hydrogen negative ion within the framework of the dipole approximation was presented in [2]. It was found in Ref. [9] that an anomalous resistance in plasma occurs when current flows through a plasma in a strong magnetic field. Lumb *et al.* [10] reported the effects of the shape of a laser pulse, confinement radius, Debye screening length, and different laser parameters on the dynamics of a spherically confined hydrogen atom embedded in an exponential-cosine-screened Coulomb potential using the Bernstein-polynomial method. The screening and weak external electric field effects on the hydrogen atom in plasmas were also reported in Ref. [11].

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The effect of plasma screening on various properties such as transition energy and polarizability of hydrogenlike ions was studied recently by Das [12].

Based on the information we have gathered, we have adequate materials to proceed to study the hydrogen atom in dense and weakly coupled quantum plasmas under the influence of the Aharonov-Bohm (AB) flux field, an electric field, and a uniform magnetic field directed along the z axis. The hydrogen atom has a notable importance in quantum mechanics and quantum field theory as a simple two-body problem that has yielded an analytical solution in a closed form [13]. Comprehension of its simple structure is very important when investigating quantum effects in more complex structures. The influences of an external electric field and a magnetic field on the hydrogen atom have been studied in numerous papers [2,8–13]. Besides using electric and magnetic fields to manipulate the energy levels or localization of the quantum state of a hydrogen atom in quantum plasmas, we suggest that the AB flux field could be used as well. In fact, as we show in this paper, the dominance of the AB flux field on other external fields justifies its superiority.

The AB effect is a quantum mechanical phenomenon in which an electrically charged particle is affected by an electromagnetic field despite being confined to a region in which both the magnetic field and electric field are zero. Experimental confirmation of its existence was presented in Ref. [14]. In this paper the influences of these three external fields on the hydrogen atom within a dense and weakly coupled quantum plasma are studied. Consequently, we feel this work will be of interest in the areas of atomic structure and collisions in plasmas.

II. THEORY AND CALCULATIONS

The model equation for the hydrogen atom under the influences of AB flux and electric and uniform magnetic fields directed along the z axis and surrounded by a quantum plasma environment can be written in cylindrical coordinates as

$$\left[\frac{1}{2\mu}\left(-i\hbar\vec{\nabla}+\frac{e}{c}\vec{A}\right)^2-\frac{Ze^2}{r}\exp\left(-\frac{r}{\lambda_D}\right)\cos\left(g\frac{r}{\lambda_D}\right)\right.$$
$$\left.-Fr\cos(\theta)\right]\psi(r,\theta)=E_{nm}\psi(r,\theta), \tag{1}$$

where E denotes the energy levels, μ is the effective mass of the electron, the vector potential $\vec{A}$ can be written as a sum of two terms $\vec{A}=\vec{A}_1+\vec{A}_2$ having the azimuthal components [15] $\vec{A}_1=\frac{Br}{2}\hat{\phi}$ and $\vec{A}_2=\frac{\phi_{AB}}{2\pi r}\hat{\phi}$, $\vec{B}=B\hat{z}$ is the applied external magnetic field with $\vec{\nabla}\times\vec{A}_1=\vec{B}$, $\vec{A}_2$ represents the additional magnetic flux ϕ_{AB} created by a solenoid inserted inside the antidot with $\vec{\nabla}\cdot\vec{A}_2=0$, and Z denotes the atomic number that is found useful in describing energy levels of light to heavy neutral atoms. In the study of atomic structure, the motion of the electron in a potential created by $+Ze$ charged nuclei has been found to be a very important problem. The results obtained from such studies can be applied to the hydrogen atom (with $Z=1$), He^{+} (with $Z=2$), and Li^{2+} (with $Z=3$) [11]. Moreover, the characteristic properties of plasmas can be represented by the coupling parameter $\Gamma=(Ze)^2/\alpha k_\beta T$ (where α is the average distance between the particles). The ranges of electron density n_e and temperature T are known to be 10^{18}–10^{23} cm^{-3} and 10^2–10^5 K, respectively, in quantum plasmas with $\Gamma>1$. Furthermore, F represents an electric field strength with an angle θ between F and r. With $\theta=0$, $Fr\cos(\theta)$ becomes Fr [11]. The variation of the effective potential energy as a function of various model parameters is displayed in Fig. 1.

Now let us take a wave function in cylindrical coordinates as $\psi(r,\phi)=\frac{1}{\sqrt{2r\pi}}e^{im\phi}\mathcal{H}_{nm}(r)$, where $m=0,\pm1,\pm2,\ldots$ denotes the magnetic quantum number. Inserting this wave function into Eq. (1), we find a second-order differential equation[1] $d^2\mathcal{H}_{nm}(r)/dr^2+2\mu/\hbar^2[E_{nm}-U_{\rm eff}]\mathcal{H}_{nm}(r)=0$, where the effective potential $U_{\rm eff}$ is

$$U_{\rm eff}=-\frac{Ze^2}{r}\exp\left(-\frac{r}{\lambda_D}\right)\cos\left(g\frac{r}{\lambda_D}\right)-Fr+\frac{\omega_c\hbar}{2}(m+\xi)$$
$$+\left(\frac{\mu\omega_c^2}{8}\right)r^2+\frac{\hbar^2}{2\mu}\left[\frac{(m+\xi)^2-\frac{1}{4}}{r^2}\right], \tag{2}$$

where $\xi=\phi_{AB}/\phi_0$ is an integer with the flux quantum $\phi_0=hc/e$ and $\omega_c=eB/\mu c$ denotes the cyclotron frequency. In order to achieve our goal in this study, we need to solve the radial Schrödinger equation with the effective model (2). However, the equation does not admit an exact solution with this model. Thus, we are constrained to utilize two methods: numerical procedure or perturbation technique. In this paper we employ a perturbative formalism [16] to solve the problem. In this perturbation technique, it is mandatory to split the effective model into two submodels. The main part should correspond to a shape invariant potential in which the superpotential is known analytically and the second part will be considered as the perturbation. This approach has been employed by Ikhdair and Sever to obtain the bound state energy for the exponential-cosine-screened Coulomb potential [17]. Now, in view of the above information, let us rewrite the wave function $\mathcal{H}_{nm}(r)$ to reflect the known normalized eigenfunction of the unperturbed system $\mathcal{P}_{nm}(r)$ and moderating function $\mathcal{Q}_{nm}(r)$ corresponding to the perturbation potential in the form $\mathcal{H}_{nm}(r)=\mathcal{P}_{nm}(r)\mathcal{Q}_{nm}(r)$. Substitution of this expression into the radial Schrödinger equation gives

$$\frac{\hbar^2}{2\mu}\left[\frac{1}{\mathcal{P}_{nm}(r)}\frac{d^2\mathcal{P}_{nm}(r)}{dr^2}+\frac{1}{\mathcal{Q}_{nm}(r)}\frac{d^2\mathcal{Q}_{nm}(r)}{dr^2}\right.$$
$$\left.+\frac{2}{\mathcal{P}_{nm}(r)\mathcal{Q}_{nm}(r)}\frac{d\mathcal{P}_{nm}(r)}{dr}\frac{d\mathcal{Q}_{nm}(r)}{dr}\right]=U_{\rm eff}(r)-E_{nm}. \tag{3}$$

The logarithmic derivatives of the perturbed and unperturbed wave functions can be written as

$$\Delta W_{nm}=-\frac{\hbar}{\sqrt{2\mu}}\frac{1}{\mathcal{Q}_{nm}(r)}\frac{d\mathcal{Q}_{nm}(r)}{dr},$$
$$W_{nm}=-\frac{\hbar}{\sqrt{2\mu}}\frac{1}{\mathcal{P}_{nm}(r)}\frac{d\mathcal{P}_{nm}(r)}{dr}, \tag{4}$$

[1]The detailed derivation of this for a general potential model is given in the Appendix.

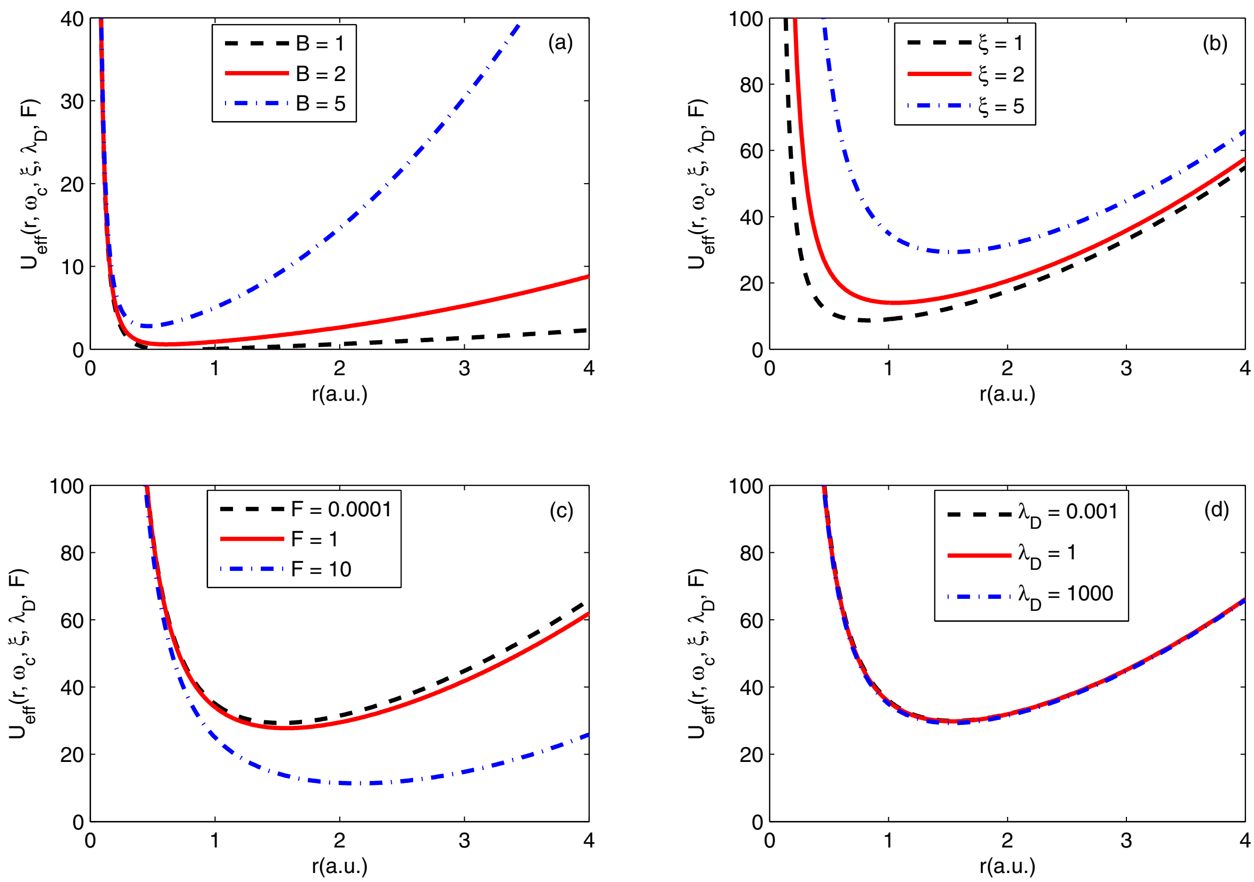

FIG. 1. Effective potential energy to simulate dense quantum plasmas environment with rotational ($m = 1$) levels for (a) various values of B with $\xi = 5$, $F = 0.0001$, and $\lambda_D = 40$. Increasing the intensity of the magnetic field and keeping other fields constant leads to a corresponding increment in the effective potential function. Thus, the potential energy becomes more repulsive. (b) Various values of ξ with $B = 5$, $F = 0.0001$, and $\lambda_D = 40$. (c) Various values of F with $B = 5$ and $\xi = 5$. Increasing the strength of the electric field increases attractiveness of the effective potential. (d) Various values of λ_D with $B = 5$, $\xi = 5$, and $F = 0.0001$. Setting effects of all fields to be constant and then varying the screening parameter up to, say, a factor of 1000 has little or no effect on the effective model, however it has a noticeable effect on its series expansion, as will be shown in Fig. 2(a). Moreover, suppose we neglect the effects of the AB flux field and external magnetic field as shown in Fig. 2(b). Then a significant effect of the screening parameter can be observed. This is an indication of how dominant the effects of these external fields are on the screening parameter. All our computations are in atomic units.

respectively, which result in

$$\frac{\hbar^2}{2\mu\mathcal{P}_{nm}(r)}\frac{d^2\mathcal{P}_{nm}(r)}{dr^2} = W_{nm}^2 - \frac{\hbar}{\sqrt{2\mu}}\frac{dW_{nm}}{dr} = \left(V_0(r) + \frac{\hbar^2}{2\mu}\left[\frac{\left(\sigma_{0m}-\frac{1}{2}\right)^2-\frac{1}{4}}{r^2}\right]\right) - \epsilon_{nm} \tag{5}$$

and

$$\frac{\hbar^2}{2\mu}\left[\frac{1}{\mathcal{Q}_{nm}(r)}\frac{d^2\mathcal{Q}_{nm}(r)}{dr^2} + \frac{2}{\mathcal{P}_{nm}(r)\mathcal{Q}_{nm}(r)}\frac{d\mathcal{P}_{nm}(r)}{dr}\frac{d\mathcal{Q}_{nm}(r)}{dr}\right] = \Delta W_{nm}^2 - \frac{\hbar}{\sqrt{2\mu}}\frac{d\Delta W_{nm}}{dr} + 2W_{nm}\Delta W_{nm} = \Delta U_{\text{eff}}(r) - \Delta\epsilon_{nm}, \tag{6}$$

where $V_0(r)$ denotes the unperturbed potential, ϵ_{nm} represents the eigenvalues of the unperturbed system, and $\Delta\epsilon_{nm} = E_{nm}^{(1)} + E_{nm}^{(2)} + E_{nm}^{(3)} + \cdots$ are the energy eigenvalues of the perturbed system, which provide correction term to the energy such that the total eigenvalues become $E_{nm} = \epsilon_{nm} + \Delta\epsilon_{nm}$. By comparing supersymmetric perturbation theory with the logarithmic perturbation theory [18], Eq. (6) seems to be in a closed analytical form. This makes the approach utilized in this study more advantageous than those available in the literature [18–20].

As we have mentioned earlier, it is necessary to split the effective potential into two parts. Within this context, the zeroth-order term corresponds to the Coulomb potential while the higher-order terms constitute the perturbation expressions. However, the perturbation equation (5) cannot be exactly solved in its present form. It is therefore required to expand the related functions to the perturbation in terms of the perturbation parameter η (which we eventually set as unity):

$$\Delta U_{\text{eff}}(r;\eta) = \sum_{i=1}^{\infty}\eta_i U_{\text{eff}}(r)^{(i)}, \quad \Delta W_{nm}(r;\eta) = \sum_{i=1}^{\infty}\eta_i W_{nm}^{(i)},$$

$$\Delta E_{nm}^{(i)}(\eta) = \sum_{i=1}^{\infty}\eta_i E_{nm}^{(i)}, \tag{7}$$

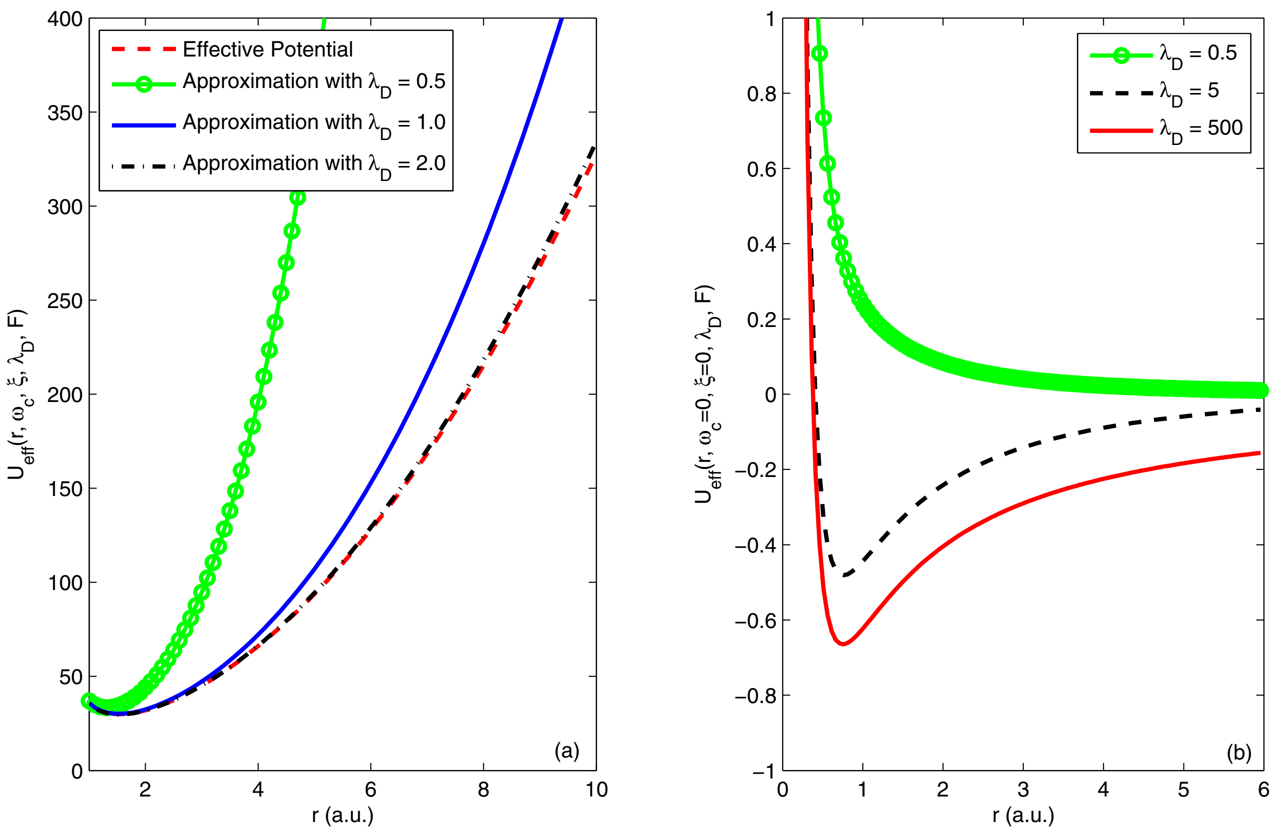

FIG. 2. (a) Variation of the effective potential and its approximation with r. We have used the fitting parameters $g = 1$, $\xi = 5$, and $F = 0.001$. The approximation is only valid for small λ_D^{-1}. For this reason, we choose $\lambda_D \geqslant 2$ where necessary in our computation. However, one may wonder why the series have been truncated at order 3. With respect to this, convergence is not an important property for series approximations in physical problems. A slowly convergent approximation that requires many terms to achieve reasonable accuracy is much less valuable than the divergent series, which gives accurate answers in a few terms [17]. This is the reason why we truncate the series expansion in Eq. (9) at a lower-order term. (b) Effective potential energy to simulate the dense quantum plasma environment with rotational ($m = 1$) levels for various values of λ_D with $F = 0.0001$. All our computations are in a.u.

where i represents the order of perturbation. We substitute Eq. (7) into Eq. (6) and then equate terms with the same power of η on both sides to obtain the following expressions:

$$2W_{nm}(r)W_{nm}^{(1)}(r) - \frac{\hbar}{\sqrt{2\mu}}\frac{dW_{nm}^{(1)}(r)}{dr} = V_1(r) - E_{nm}^{(1)}, \tag{8a}$$

$$W_{nm}^{(1)2}(r) + 2W_{nm}(r)W_{nm}^{(2)}(r) - \frac{\hbar}{\sqrt{2\mu}}\frac{dW_{nm}^{(2)}(r)}{dr} = V_2(r) - E_{nm}^{(2)}, \tag{8b}$$

$$2\big[W_{nm}(r)W_{nm}^{(3)}(r) + W_{nm}^{(1)}(r)W_{nm}^{(2)}(r)\big] - \frac{\hbar}{\sqrt{2\mu}}\frac{dW_{nm}^{(3)}(r)}{dr} = V_3(r) - E_{nm}^{(3)}, \tag{8c}$$

$$2\big[W_{nm}(r)W_{nm}^{(4)}(r) + W_{nm}^{(1)}(r)W_{nm}^{(3)}(r)\big] + W_{nm}^{(2)}(r)W_{nm}^{(2)}(r) - \frac{\hbar}{\sqrt{2\mu}}\frac{dW_{nm}^{(4)}(r)}{dr} = V_4(r) - E_{nm}^{(4)}. \tag{8d}$$

We now apply this background information to our problem. Thus, the effective potential in Eq. (2) can be expanded in a power series of the Debye screening parameter λ_D as

$$U_{\rm eff}(r) = -\frac{A}{r} + \frac{\hbar^2}{2\mu}\left[\frac{(m+\xi)^2 - \frac{1}{4}}{r^2}\right] + \left[\frac{A}{\lambda_D} + \frac{\omega_c\hbar}{2}(m+\xi)\right] - \left[F + \frac{A}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right]r$$
$$+ \left[\frac{A}{\lambda_D^3}\left(\frac{1}{6} - \frac{g^2}{2}\right) + \frac{\mu\omega_c^2}{8}\right]r^2 - \left[\frac{A}{\lambda_D^4}\left(\frac{1}{24} - \frac{g^2}{4} + \frac{g^4}{24}\right)\right]r^3 + O(r^4), \tag{9}$$

where $A = Ze^2$. The accuracy of this approximation is shown in Fig. 2(a). It is only valid for large values of the Debye screening parameter λ_D. The first term in the series expansion (9) is the unperturbed term, which is the Coulomb potential with a well known solution. Alternatively, it can be easily obtained using the recently proposed formula method [21]. The second term is the centrifugal term and the remaining terms constitute the perturbation expression. Within this context, the unperturbed energy and the corresponding normalized wave function can be written as

$$E_{nm}^{(0)} = -\frac{\mu}{2\hbar^2}\frac{A^2}{\sigma_{nm}^2}, \quad \mathcal{P}_{nm}^{(0)}(r) = N_{nm}^{(0)} r^{\sigma_{0m}} e^{-\varrho r} L_n^{2\sigma_{0m}-1}[2\varrho r], \tag{10}$$

respectively, with $n = 0,1,2,\ldots$. The ground state superpotential and the normalization factor can be written as

$$W_{n=0,m}^{(0)}(r) = -\frac{\hbar}{\sqrt{2\mu}}\frac{\sigma_{0m}}{r} + \frac{1}{\hbar}\sqrt{\frac{\mu}{2}}\frac{A}{\sigma_{0m}}, \quad N_{nm}^{(0)} = \frac{(2\varrho)^{\sigma_{0m}}}{\sigma_{nm}}\left[\frac{\hbar^2(2\sigma_{nm}-1-n)!}{\mu n! A}\right]^{-1/2}, \tag{11}$$

where $\varrho = A\mu/\hbar^2\sigma_{nm}$. Now let us consider the expressions leading to the first-, second-, and third-order perturbations given by Eqs. (8a)–(8d). Using superpotentials given in Eq. (4) and multiplying each term in Eqs. (8a)–(8d) by $P_{nm}^{(0)^2}$, we obtain the first-order correction to the energy and its superpotential as follows:

$$E_{nm}^{(1)} = \int_{-\infty}^{\infty} \mathcal{P}_{nm}^{(0)^2}(r)\left\{-\left[F + \frac{A}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right]r\right\}dr, \tag{12a}$$

$$W_{nm}^{(1)} = \sqrt{\frac{2\mu}{\hbar}}\frac{1}{\mathcal{P}_{nm}^{(0)^2}(r)}\int^r \mathcal{P}_{nm}^{(0)^2}(y)\left\{E_{nm}^{(1)} + \left[F + \frac{A}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right]y\right\}dy. \tag{12b}$$

Also, the second-order correction to the energy and its superpotential can be written as follows:

$$E_{nm}^{(2)} = \int_{-\infty}^{\infty} \mathcal{P}_{nm}^{(0)^2}(r)\left\{\left[\frac{A}{\lambda_D^3}\left(\frac{1}{6} - \frac{g^2}{2}\right) + \frac{\mu\omega_c^2}{8}\right]r^2 - W_{nm}^{(1)^2}\right\}dr, \tag{13a}$$

$$W_{nm}^{(2)} = \sqrt{\frac{2\mu}{\hbar}}\frac{1}{\mathcal{P}_{nm}^{(0)^2}(r)}\int^r \mathcal{P}_{nm}^{(0)^2}(y)\left\{E_{nm}^{(2)} - \left[\frac{A}{\lambda_D^3}\left(\frac{1}{6} - \frac{g^2}{2}\right) + \frac{\mu\omega_c^2}{8}\right]y^2 + W_{nm}^{(1)^2}\right\}dy. \tag{13b}$$

The third-order correction to the energy and its superpotential becomes

$$E_{nm}^{(3)} = \int_{-\infty}^{\infty} \mathcal{P}_{nm}^{(0)^2}(r)\left\{\left[\frac{A}{\lambda_D^4}\left(\frac{1}{24} - \frac{g^2}{4} + \frac{g^4}{24}\right)\right]r^3 + W_{nm}^{(1)}(r)W_{nm}^{(2)}(r)\right\}dr, \tag{14a}$$

$$W_{nm}^{(2)}(r) = \sqrt{\frac{2\mu}{\hbar^2}}\frac{1}{\mathcal{P}_{nm}^{(0)^2}(r)}\int^r \mathcal{P}_{nm}^{(0)^2}(y)\left\{E_{nm}^{(3)} + W_{nm}^{(1)}(r)W_{nm}^{(2)}(r) + \left[\frac{A}{\lambda_D^4}\left(\frac{1}{24} - \frac{g^2}{4} + \frac{g^4}{24}\right)\right]y^3\right\}dy. \tag{14b}$$

Using the expressions for $E_{nm}^{(1)}$, $E_{nm}^{(2)}$, and $E_{nm}^{(3)}$, one can calculate superpotentials $W_{nm}^{(1)}$, $W_{nm}^{(2)}$, and $W_{nm}^{(3)}$ explicitly. Consequently, the superpotentials can be used to calculate the moderating wave function $\mathcal{Q}_{nm}(r) \approx \exp\{-\sqrt{2\mu}/\hbar\int^r [W_{nm}^{(1)}(r) + W_{nm}^{(2)}(r)]\}$. Since we now have all necessary formulas needed for our calculation, let us now focus our attention on how to utilize them to deduce ground and exited state energies together with their moderating superpotentials. We start with the ground state such that from Eqs. (12a)–(14b) we have

$$E_{0m}^{(1)} = -\frac{\hbar^2}{2\mu}\left\{3\sigma_{0m}^2 - \left(\sigma_{0m} - \frac{1}{2}\right)^2 + \frac{1}{4}\right\}\left\{\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right\},$$

$$W_{0m}^{(1)}(r) = -\frac{\hbar}{\sqrt{2\mu}}\left\{\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right\}\sigma_{0m} r; \tag{15a}$$

$$E_{0m}^{(2)} = \left\{\left[\frac{A}{\lambda_D^3}\left(\frac{1}{6} - \frac{g^2}{2}\right) + \frac{\mu\omega_c^2}{8}\right] - \frac{\hbar^2\sigma_{0m}^2}{2\mu}\left[\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right]^2\right\}\frac{\hbar^4\sigma_{0m}^2}{2\mu^2A^2}\left\{5\sigma_{0m}^2 - 3\left(\sigma_{0m} - \frac{1}{2}\right)^2 + \frac{7}{4}\right\}; \tag{15b}$$

$$W_{0m}^{(2)}(r) = -\left\{\frac{\hbar^2\sigma_{0m}^2}{2\mu}\left[\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right]^2 - \left[\frac{A}{\lambda_D^3}\left(\frac{1}{6} - \frac{g^2}{2}\right) + \frac{\mu\omega_c^2}{8}\right]\right\}\frac{\hbar r\sigma_{0m}}{\mu A^2\sqrt{2\mu}}[\sigma_{0m}\sigma_1\hbar^2 + \mu A r]. \tag{15c}$$

Consequently, the approximate expressions for the ground state energy and radial wave function of the hydrogen atom in the AB flux and electric and uniform magnetic fields directed along the z axis and surrounded by quantum plasmas can be written as

$$E_{0m} \approx E_{0m}^{(0)} + \left[\frac{A}{\lambda_D} + \frac{\omega_c \hbar}{2}(m+\xi)\right] + E_{0m}^{(1)} + E_{0m}^{(2)}, \tag{16a}$$

$$\psi(r,\phi) \approx \frac{1}{\sqrt{2r\pi}} e^{im\phi} \mathcal{P}_{nm}(r) \exp\left(-\frac{\sqrt{2\mu}}{\hbar}\int^r \left(W_{0m}^{(1)} + W_{0m}^{(2)}\right)\right). \tag{16b}$$

It is worth mentioning that there is a corresponding relationship between two and three dimensions that can be obtained by making a replacement $m + \xi = \ell + 1/2$. Therefore, the bound state energy levels for $1s$ in the absence of an external magnetic and AB flux field in three dimensions can also be deduced from the above equations.

Let us now proceed to excited state calculations. We calculate the energy shift and superpotentials of first and second order as follows:

$$E_{1m}^{(1)} = -\frac{\hbar^2}{2\mu}\left\{3\sigma_{1m}^2 - \left(\sigma_{0m} - \frac{1}{2}\right)^2 + \frac{1}{4}\right\}\left\{\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right\},$$

$$W_{1m}^{(1)}(r) = -\frac{\hbar}{\sqrt{2\mu}}\left\{\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right\}\sigma_{1m} r; \tag{17a}$$

$$E_{1m}^{(2)} = \left\{\left[\frac{A}{\lambda_D^3}\left(\frac{1}{6} - \frac{g^2}{2}\right) + \frac{\mu\omega_c^2}{8}\right] - \frac{\hbar^2\sigma_{1m}^2}{2\mu}\left[\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right]^2\right\}\frac{\hbar^4\sigma_{1m}^2}{2\mu^2 A^2}\left\{5\sigma_{1m}^2 - 3\left(\sigma_{0m} - \frac{1}{2}\right)^2 + \frac{7}{4}\right\}; \tag{17b}$$

$$W_{1m}^{(2)}(r) = -\left\{\frac{\hbar^2\sigma_{1m}^2}{2\mu}\left[\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right]^2 - \left[\frac{A}{\lambda_D^3}\left(\frac{1}{6} - \frac{g^2}{2}\right) + \frac{\mu\omega_c^2}{8}\right]\right\}\frac{\hbar r \sigma_{1m}}{\mu A^2\sqrt{2\mu}}[\sigma_{1m}\sigma_{2m}\hbar^2 + \mu A r]. \tag{17c}$$

Therefore, the approximate energy eigenvalues of the hydrogen atom in the AB flux and electric and uniform magnetic fields directed along the z axis and surrounded by quantum plasmas, corresponding to the first excited state ($n = 1$), are

$$E_{1m} \approx E_{1m}^{(0)} + \left[\frac{A}{\lambda_D} + \frac{\omega_c \hbar}{2}(m+\xi)\right] + E_{1m}^{(1)} + E_{1m}^{(2)}. \tag{18}$$

We can proceed further to obtain expression for states $n = 2,3,4,5,\ldots$. However, we leave the calculations as exercises in elementary integrals. From the supersymmetry, we can write the nth state energy shifts and the corresponding superpotentials as

$$E_{nm}^{(1)} = -\frac{\hbar^2}{2\mu}\left\{3\sigma_{nm}^2 - \left(\sigma_{0m} - \frac{1}{2}\right)^2 + \frac{1}{4}\right\}\left\{\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right\},$$

$$W_{nm}^{(1)}(r) = -\frac{\hbar}{\sqrt{2\mu}}\left\{\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right\}\sigma_{nm} r; \tag{19a}$$

$$E_{nm}^{(2)} = \left\{\left[\frac{A}{\lambda_D^3}\left(\frac{1}{6} - \frac{g^2}{2}\right) + \frac{\mu\omega_c^2}{8}\right] - \frac{\hbar^2\sigma_{nm}^2}{2\mu}\left[\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right]^2\right\}\frac{\hbar^4\sigma_{nm}^2}{2\mu^2 A^2}\left\{5\sigma_{nm}^2 - 3\left(\sigma_{0m} - \frac{1}{2}\right)^2 + \frac{7}{4}\right\}; \tag{19b}$$

$$W_{nm}^{(2)}(r) = -\left\{\frac{\hbar^2\sigma_{nm}^2}{2\mu}\left[\frac{F}{A} + \frac{1}{\lambda_D^2}\left(\frac{1}{2} - \frac{g^2}{2}\right)\right]^2 - \left[\frac{A}{\lambda_D^3}\left(\frac{1}{6} - \frac{g^2}{2}\right) + \frac{\mu\omega_c^2}{8}\right]\right\}\frac{\hbar r \sigma_{nm}}{\mu A^2\sqrt{2\mu}}[\sigma_{nm}\sigma_{n+1,m}\hbar^2 + \mu A r]. \tag{19c}$$

Consequently, we obtain the approximate energy eigenvalues of the hydrogen atom in quantum plasmas environment under the influences of AB flux and electric and uniform magnetic fields, directed along the z axis, corresponding to the nth state as

$$E_{nm} \approx E_{nm}^{(0)} + \left[\frac{A}{\lambda_D} + \frac{\omega_c \hbar}{2}(m+\xi)\right] + E_{nm}^{(1)} + E_{nm}^{(2)}. \tag{20}$$

It is worth mentioning that in all our calculations for energy, we have changed the lower limit of the integration from $-\infty$ to 0 so as to accommodate the fact that r is never negative. Tables I and II display eigenvalues for the hydrogen atom in quantum plasmas under the influence and the absence of external fields (the magnetic field, AB flux field, and electric field) in a.u. and in low vibrational n and rotational m. From the tables, in the absence of external fields (i.e., when $B = \xi = F = 0$), the spacing between the energy levels of the effective potential is narrow and decreases with increasing n. We notice that there exists degeneracy among some states (n,m) [for instance, (1,1) and (3, − 1), and (0,1) and (2, − 1)] and quasidegeneracy of the energy levels among some states [for instance, (2,0) and (1,1), and (2,1) and (3,0)], but application of the magnetic field strength not only increases the energy levels of the effective potential and spacings between states but also transforms the degeneracies to quasidegeneracies. Moreover, the quasidegeneracies among the states are also removed and the energy values shift up.

TABLE I. Energy values for the hydrogen atom in dense quantum plasma under the influence of AB flux and external magnetic and electric fields with various values of magnetic quantum numbers. The following fitting parameters have been employed: $A = 1$, $\lambda_D = 20$, $m = 1$, and $g = 1$. All values are in a.u.

m	n	$F = 0, \xi = 0, B = 0$	$F = 0, \xi = 0, B = 5$	$F = 0, \xi = 5, B = 0$	$F = 5, \xi = 0, B = 0$	$F = 5, \xi = 5, B = 5$
0	0	−1.95001560	−0.7781406	−0.0156852	−5.6218906	−442558.77
	1	−0.17283160	45.530293	−0.0833031	−429.00096	−1530748.8
	2	−0.03429688	322.23133	−0.2026389	−8104.1749	−4095347.7
	3	−0.00689445	1205.8525	−0.3904204	−59179.616	−9364067.9
1	0	−0.17269097	37.483559	−0.0542562	−331.57894	−1164728.2
	1	−0.03390625	295.43484	−0.1639670	−7369.2527	−3445630.2
	2	−0.00612883	1150.9314	−0.3407485	−56363.444	−8291125.3
	3	−0.01687886	3166.5456	−0.6042120	−256439.08	−17514714
−1	0	−1.95000000	−4.4500000	0.0021055	−1.9500000	−139321.25
	1	−1.95000000	−4.4500000	−0.0327008	−1.9500000	−595724.41
	2	−0.17269097	32.483559	−0.1070687	−331.57894	−1830220.7
	3	−0.03390625	290.43484	−0.2342795	−7369.2527	−4626935.2

By subjecting the hydrogen atom in quantum plasmas to only the AB flux field, the energy values are reduced and degeneracies are removed, whereas the quasidegeneracies among the states are not affected. The energy levels become more negative and the system becomes strongly attractive as the quantum number n increases for fixed m. When only an electric field is applied, the degeneracies and the quasidegeneracies are not affected and the attractiveness of the total interaction potential increases. The overall effects indicate that the system is strongly attractive while the localizations of quantum levels change and the eigenvalues decrease. Also, the combined effect of the fields is stronger than the individual effects and consequently, there is a considerably shift in the bound state energy of the system.

In Fig. 3 we show the combined effect of the AB flux and magnetic and electric fields on the energy values of the hydrogen atom in the quantum plasma environment. The confinement effect of the AB flux field on the hydrogen atom in quantum plasmas is stronger than that of the magnetic field. This can be seen in Fig. 3(a) by comparing the energy values when B is 1.7 and when ξ is 1, 2, and 4. For instance, when the AB flux field is small, say, $\xi = 1$, and the intensity of the magnetic field increases, it can be seen that the energy shift is approximately 5. However, within the same range of magnetic field intensity, a little distortion, say, $\xi = 2$, leads to a huge shift in the energy level. Figure 3(b) also shows similar properties. However, in Fig. 3(c), we show the effect of high electric field intensity on the hydrogen atom in quantum plasmas. As can be seen, a low intensity of the AB flux field, say, $\xi = 1$, and a high electric field ($F = 1.2$) cannot affect the hydrogen atom in quantum plasmas, even while increasing the magnetic field intensity gradually. However, if we adjust the intensity of the AB flux field (say, $\xi = 3$), a very low energy can be obtained provided the magnetic field is low. However, as the intensity of the magnetic field increase so does the energy. This indicates that the energy values of the hydrogen atom in the quantum plasma environment or localization of quantum states can be changed or adjusted to a maximum level by applying a strong magnetic field and electric field intensity. A weak magnetic field and strong electric field intensity will reduce the energy of the hydrogen atom to a minimum level. In either way, the AB flux field can act as a catalyst to boost the process.

Figure 4 displays the dominance of the AB flux field on external electric and magnetic fields. This can be seen via a

TABLE II. Energy values for the hydrogen atom in a weakly coupled plasma under the influence of AB flux and external magnetic and electric fields with various values of magnetic quantum numbers. The following fitting parameters have been employed: $A = 1$, $\lambda_D = 20$, $m = 1$, and $g = 0$. All values are in a.u.

m	n	$F = 0, \xi = 0, B = 0$	$F = 0, \xi = 0, B = 5$	$F = 0, \xi = 5, B = 0$	$F = 5, \xi = 0, B = 0$	$F = 5, \xi = 5, B = 5$
0	0	−1.9506173	−0.7787423	−0.0110816	−5.6230782	−442781.81
	1	−0.1763182	45.526807	−0.0610759	−429.21011	−1531518.6
	2	−0.0402301	322.22539	−0.1840939	−8108.2092	−4097404.3
	3	−0.0095952	1205.8498	−0.4474275	−59209.163	−9368766.0
1	0	−0.1757576	37.480492	−0.0457138	−331.74021	−1165313.9
	1	−0.0397546	295.42900	−0.1557682	−7372.9206	−3447360.5
	2	−0.0091772	1150.9283	−0.3980985	−56391.583	−8295285.1
	3	−0.0071157	3166.5554	−0.8757537	−256567.21	−17523495
−1	0	−1.9500000	−4.4500000	−0.0000247	−1.9500000	−139391.73
	1	−1.9500000	−4.4500000	−0.0178498	−1.9500000	−596024.65
	2	−0.1757576	32.480492	−0.0736449	−331.74021	−1831141.1
	3	−0.0397546	290.42900	−0.2072696	−7372.9206	−4629258.8

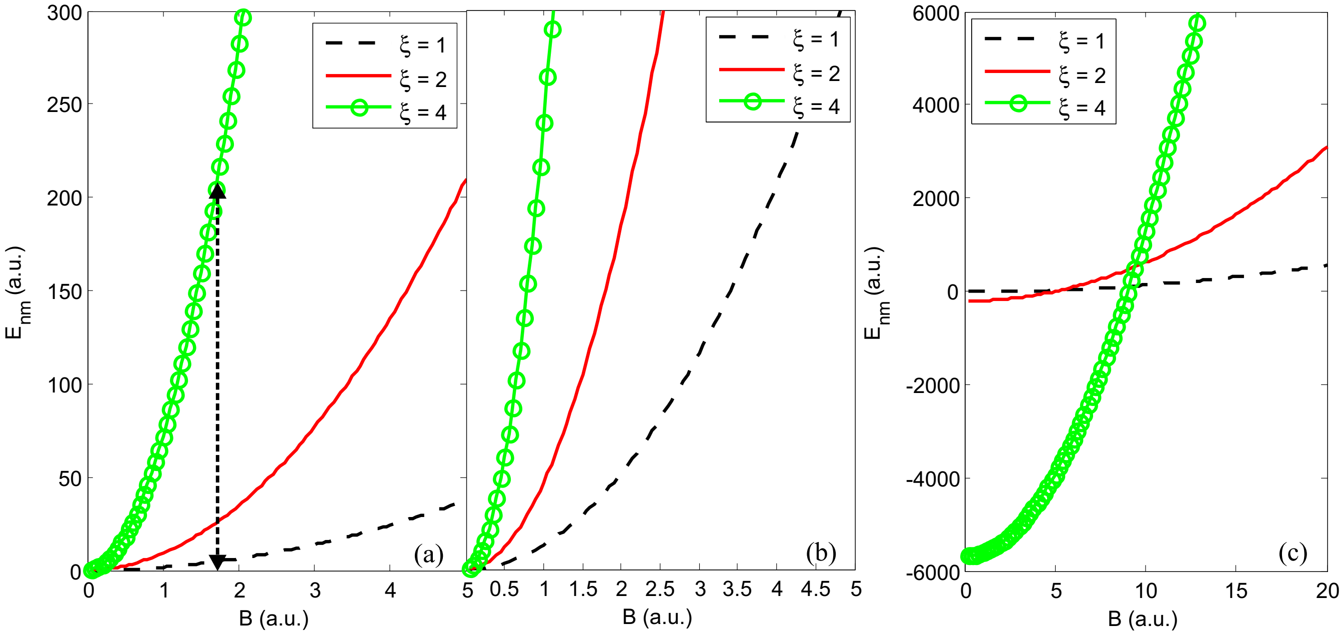

FIG. 3. Variation of energy values for the hydrogen atom in quantum plasmas and under the influence of the magnetic field and the AB flux field and electric field in atomic units using the fitting parameters $m = n = 0$ and $\lambda_D = 20$ (a) as a function of external magnetic field with various ξ and $F = 0.0001$. (b) Same as (a) but with $m = -1$ and $n = 2$. (c) Same as (a) but with $F = 1.2$. All values are expressed in a.u.

comparison of Figs. 4(a)–4(c). From Fig. 4(a) it can be seen that when the hydrogen atom is under a low AB flux field, the gap between energy levels when $F = 0.0001$ and 1.2 is tiny. However, in Fig. 4(b), where we increase the intensity a little (say, $\xi = 2$), it can be observed that the energy values of the process gradually increase from negative to positive as the magnetic field increases. Moreover, the gap between $F = 0.0001$ and 1.2 becomes wide. Furthermore, we double the intensity of the formal AB flux (i.e., $\xi = 4$) as displayed in the inset of Fig. 4(b). We observe that for a very weak electric field and strong magnetic field, there exists a positive energy, whereas for a low magnetic field, the energy of the hydrogen atom in quantum plasma becomes negative. From these figures we observe that the AB flux field seems to be

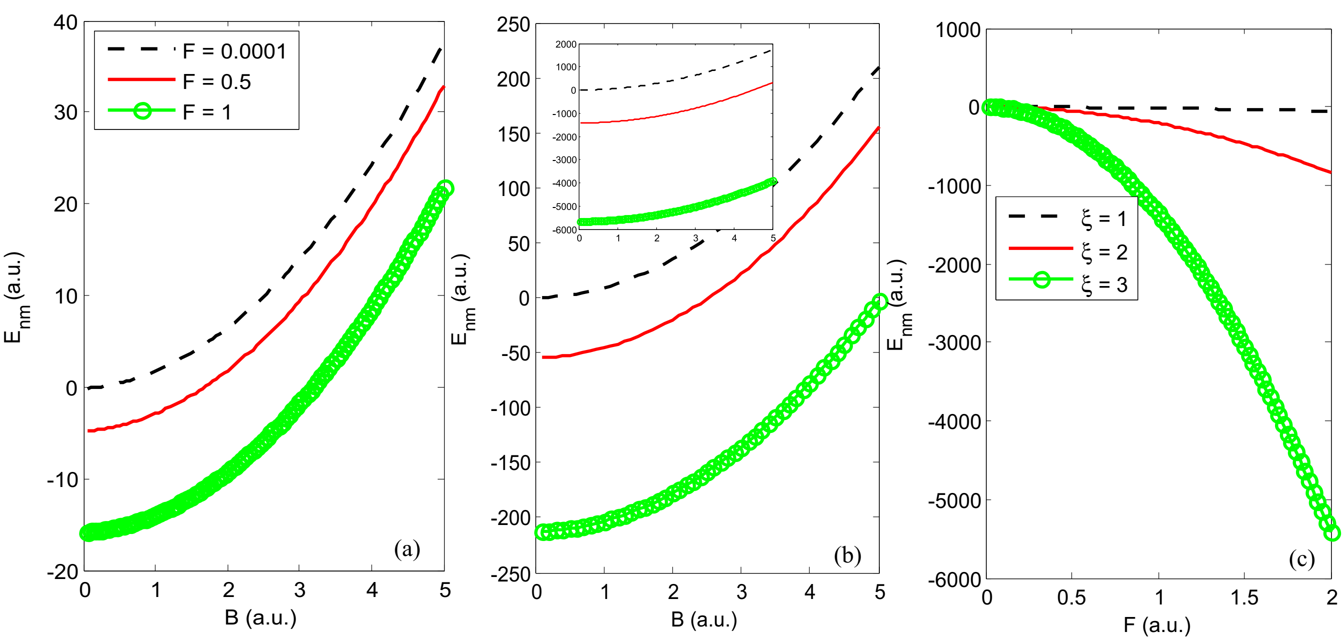

FIG. 4. Variation of energy values for the hydrogen atom in quantum plasmas and under the influence of the magnetic field and the AB flux field and electric field in atomic units using the fitting parameters $m = n = 0$ and $\lambda_D = 20$ (a) as a function of external magnetic field with various F and $\xi = 1$. (b) Same as (a) but with $\xi = 2$; the inset is for $\xi = 4$. (c) Same as (b) but as a function of magnitude of the external electric field with various ξ and $B = 1$.

the key parameter. All these justify the superior effect of the AB flux over magnetic and electric fields on the hydrogen atom in quantum plasmas. This can be understood further if we consider a strong electric field (i.e., $F = 1.2$) and then calculate $\Delta E_{nm} = E_{nm}|_{B=5} - E_{nm}|_{B=0}$ for the three plots we discussed above: in Fig. 4(a), where $\xi = 1$ and $\Delta E_{nm} \approx 38$; in Fig. 4(b), where $\xi = 2$ and $\Delta E_{nm} \approx 160$, and in the inset of Fig. 4(b), where $\xi = 4$ and $\Delta E_{nm} \approx 2500$.

In Fig. 3(c) we find that the external electric field either will have no effect on the energy of the hydrogen atom in quantum plasmas or will decrease the energy values under high intensity. It can be concluded from Figs. 3 and 4 that the confinement effect of the AB flux field on the hydrogen atom dominates on the external electric field and is even more dominant on the magnetic field. Therefore, the AB flux field can be regarded as a key control parameter for energy levels or localization of the quantum state of the hydrogen atom in quantum plasmas. In other words, to maintain a low energy for the hydrogen atom in quantum plasmas, a strong electric field and weak magnetic field are required, whereas the AB flux field can serve as a regulator.

III. CONCLUSION

In this paper we have studied the effects of the electric field, AB flux field, and uniform magnetic field directed along the z axis on the hydrogen atom in quantum plasmas. The overall effects indicate that the system is strongly attractive while the localizations of quantum levels change and the eigenvalues decrease. Also, as we have demonstrated, the combined effect of the fields is stronger than individual effects and consequently there is a considerably shift in the bound state energy of the system. We found that to maintain a low energy for the hydrogen atom in quantum plasmas, a strong electric field and weak magnetic field are required, whereas the AB flux field can be used as a regulator or a booster. The application of the perturbation technique we utilized in this paper is not limited to plasma physics; it can also be applied in molecular physics. Finally, we suggest a possible extension of the present work for inclusion of quantum effects with two and three- particle correlations [22,23].

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APPENDIX: EXACT SOLUTION TO A GENERAL POTENTIAL FORM UNDER THE INFLUENCE OF AB FLUX AND EXTERNAL MAGNETIC FIELDS

In this appendix we show in detail the derivation of Eq. (1) for a general form of potential model $V(r)$, i.e., $(i\hbar\vec{\nabla} - e/c\vec{A})^2\psi = 2\mu[E_{nm} - V(r)]\psi$. For convenience, let us introduce $\mathcal{K} = -e/c$, so that Eq. (1) becomes

$$-\hbar^2\nabla^2\psi + i\hbar\mathcal{K}\vec{\nabla}\cdot(\vec{A}\psi) + i\hbar\mathcal{K}\vec{A}\cdot\vec{\nabla}\psi + \hbar\mathcal{K}^2\vec{A}\cdot\vec{A}\Psi = 2\mu[E_{nm} - V(r)]\psi. \tag{A1}$$

Using the property $\vec{\nabla}\cdot(\vec{A}\psi) = \vec{A}\cdot\vec{\nabla}\psi + \psi\vec{\nabla}\cdot\vec{A}$ and

$$\vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\frac{Br}{2} + \frac{\phi_{AB}}{2\pi r}\right)\hat{\phi} = \frac{1}{r}\frac{\partial}{\partial\phi}\vec{A} = 0, \tag{A2}$$

Eq. (A1) becomes

$$-\hbar^2\nabla^2\psi + 2i\hbar\mathcal{T}\vec{A}\cdot\vec{\nabla}\psi + \hbar\mathcal{K}^2\vec{A}\cdot\vec{A}\Psi = 2\mu[E_{nm} - V(r)]\psi. \tag{A3}$$

Now we obtain an expression for $\nabla^2\psi$ and $\vec{A}\cdot\vec{\nabla}\psi$ as

$$\begin{aligned}\nabla^2\psi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\\ &= r^{-1/2}e^{im\phi}\left(\frac{\mathcal{H}(r)}{4r^2} + \mathcal{H}''(r) - \frac{m^2}{r^2}\mathcal{H}(r)\right),\\ \vec{A}\cdot\vec{\nabla}\psi &= \left(\frac{Br}{2} + \frac{\phi_{AB}}{2\pi r}\right)\hat{\phi}\cdot\left(\frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{z}\right)\\ &= \frac{im}{r}\left(\frac{Br}{2} + \frac{\phi_{AB}}{2\pi r}\right).\end{aligned} \tag{A4}$$

Substituting Eq. (A4) into Eq. (A3), we have

$$\begin{aligned}&-\hbar^2\mathcal{H}''(r) - \frac{\hbar^2\mathcal{H}(r)}{4r^2} + \frac{m^2\mathcal{H}(r)\hbar^2}{r^2}\\ &\quad -\frac{2\hbar m\mathcal{K}}{r}\left(\frac{Br}{2} + \frac{\phi_{AB}}{2\pi r}\right)\mathcal{H}(r) + \mathcal{K}^2\vec{A}\cdot\vec{A}\mathcal{H}(r)\\ &= 2\mu[E_{nm} - V(r)]\mathcal{H}(r).\end{aligned} \tag{A5}$$

If we use $e = c = 1$, then $\mathcal{K} = -1$, so we obtain a more explicit expression

$$\mathcal{H}''(r) + \frac{2\mu}{\hbar^2}\left\{E - \left[V(r) - Fr + \frac{\hbar^2}{2\mu}\left(\frac{(m+\xi)^2 - \frac{1}{4}}{r^2}\right) + \frac{\omega_c\hbar}{2}(m+\xi) + \left(\frac{\mu\omega_c^2}{8}\right)r^2\right]\right\}\mathcal{H}(r) = 0, \tag{A6}$$

where we have introduced $\xi = \phi_{AB}/\phi_0$ with $\phi_0 = hc/e$ and electric field F.

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