

# Defocusing complex short-pulse equation and its multi-dark-soliton solution

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In this paper, we propose a complex short-pulse equation of both focusing and defocusing types, which governs the propagation of ultrashort pulses in nonlinear optical fibers. It can be viewed as an analog of the nonlinear Schrödinger (NLS) equation in the ultrashort-pulse regime. Furthermore, we construct the multi-dark-soliton solution for the defocusing complex short-pulse equation through the Darboux transformation and reciprocal (hodograph) transformation. One- and two-dark-soliton solutions are given explicitly, whose properties and dynamics are analyzed and illustrated.

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## I. INTRODUCTION

It is well known that the nonlinear Schrödinger (NLS) equation, which describes the evolution of slowly varying wave packets in weakly nonlinear dispersive media under quasimonochromatic assumption, has been very successful in many applications such as nonlinear optics and water waves [1–4]. However, as the width of optical pulses is of the order of femtoseconds ( $10^{-15}$  s), the spectrum of these ultrashort pulses is approximately of the order  $10^{15}$  s<sup>-1</sup>, and the monochromatic assumption to derive the NLS equation is not valid anymore [5]. Description of ultrashort processes requires a modification of standard slowly varying envelope models. This is the motivation for the study of the short-pulse equation, the complex short-pulse equation, and their coupled models.

In 2004, Schäfer and Wayne derived a short-pulse (SP) equation [6]

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \quad (1)$$

to describe the propagation of ultrashort optical pulses in nonlinear media [7]. Here  $u = u(x, t)$  is a real-valued function, representing the magnitude of the electric field; the subscripts  $t$  and  $x$  denote partial differentiation. The SP equation has been shown to be completely integrable [8–10], whose periodic and soliton solutions of the SP equation were found in [11–15].

Similar to the NLS equation, it is known that the complex-valued function has advantages in describing optical waves which have both the amplitude and phase information [1]. Following this spirit, one of the authors recently proposed a complex short-pulse (CSP) equation [16,17],

$$q_{xt} + q + \frac{1}{2}(|q|^2 q_x)_x = 0. \quad (2)$$

In contrast with no physical interpretation of the one-soliton solution to the SP equation (1), the one-soliton solution of the CSP equation (2) is an envelope soliton with a few optical cycles.

The CSP equation can be viewed as an analog of the NLS equation in the ultrashort-pulse regime when the width of optical pulse is of the order  $10^{-15}$  s. The NLS equation has the focusing and defocusing cases, which admits the bright- and dark-type soliton solutions, respectively. As a matter of fact, the dark soliton in optical fibers was predicted in 1973 [18], and was observed experimentally in 1988 [19,20], a decade earlier than the observation of the bright soliton [21,22]. Therefore it is natural that the CSP equation can also have the focusing and defocusing type, which may be proposed as

$$q_{xt} + q + \frac{1}{2}\sigma(|q|^2 q_x)_x = 0, \quad (3)$$

where  $\sigma = 1$  represents the focusing case, and  $\sigma = -1$  stands for the defocusing case. It turns out that this is indeed the case as shown in the subsequent section. The same as the focusing CSP equation discussed in [16,17], the defocusing CSP equation can also occur in nonlinear optics when ultrashort pulses propagate in a nonlinear media of defocusing type.

The remainder of the present paper is organized as follows. In Sec. II, the CSP equation of both the focusing and defocusing types is derived from the context of nonlinear optics based on Maxwell's equations. Then, based on the reciprocal link between the defocusing CSP equation and the complex coupled dispersionless (CCD) equation, the Darboux transformation to the CCD equation is derived to give a general solitonic formula to the defocusing CSP equation in Sec. III. We continue to derive explicit formulas for one- and multi-dark-soliton solutions to the defocusing CSP equation by a limiting process in Sec. IV. The one- and two-dark-soliton solution is analyzed in details, which can be classified into smoothed, cusponed, and looped ones depending on the parameters. The paper is concluded by some comments and remarks in Sec. V.

## II. DERIVATION OF THE FOCUSING AND DEFOCUSING COMPLEX SHORT-PULSE EQUATION

The starting point to derive the CSP equation is the same as the one for the NLS equation [2–4], which is the Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \quad (4)$$

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where  $\mathbf{E}$  and  $\mathbf{H}$  are electric and magnetic field vectors, and  $\mathbf{D}$  and  $\mathbf{B}$  are corresponding electric and magnetic flux densities. The relations between  $\mathbf{D}$ ,  $\mathbf{B}$  and  $\mathbf{E}$ ,  $\mathbf{H}$  are called the constitutive relations given by

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad (5)$$

where  $\epsilon$  is the permittivity and  $\mu$  is the permeability. In a vacuum,  $c^2 = 1/(\epsilon_0 \mu_0)$  with  $c$  the velocity of light in vacuum. In the frequency-dependent media,

$$\mathbf{D} = \epsilon \star \mathbf{E}, \quad \mathbf{B} = \mu \star \mathbf{H}, \quad \mathbf{D} = \mathbf{E} + \mathbf{P}, \quad (6)$$

where  $\star$  means the convolution, and  $\mathbf{P}$  is the electric induced polarization. By eliminating  $\mathbf{B}$  and  $\mathbf{D}$ , the resulting wave equation follows:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \mathbf{E}_{tt} = \mu_0 \mathbf{P}_{tt}, \quad (7)$$

which describes light propagation in optical fibers. If we assume the local medium response and only the third-order nonlinear effects governed by  $\chi^{(3)}$ , the induced polarization consists of linear and nonlinear parts,  $\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_L(\mathbf{r}, t) + \mathbf{P}_{NL}(\mathbf{r}, t)$ , with the linear part

$$\mathbf{P}_L(\mathbf{r}, t) = \epsilon_0 \int_{-\infty}^{\infty} \chi^{(1)}(t - t') \cdot \mathbf{E}(\mathbf{r}, t') dt', \quad (8)$$

and the nonlinear part

$$\begin{aligned} \mathbf{P}_{NL}(\mathbf{r}, t) = & \epsilon_0 \int_{-\infty}^{\infty} \chi^{(3)}(t - t_1, t - t_2, t - t_3) \\ & \times \mathbf{E}(\mathbf{r}, t_1) \mathbf{E}(\mathbf{r}, t_2) \mathbf{E}(\mathbf{r}, t_3) dt_1 dt_2 dt_3. \end{aligned} \quad (9)$$

Here  $\epsilon_0$  is the vacuum permittivity and  $\chi^{(j)}$  is the  $j$ th-order susceptibility. As discussed in [23], the nonlinear response is due to the induced atomic dipole with a response time of the order  $1/\Delta$ , where  $\Delta = |\omega_{ik} - \omega_0|$ .  $\omega_{ik}$  represents the transition frequency from the initial (usually ground) quantum state  $i$  into some excited state  $k$ , and  $\omega_0$  is the central carrier frequency. Since the typical transition frequency from the atomic ground state to the lowest excited state significantly exceeds the usual carrier frequency,  $1/\Delta$  is typically less than 1 fs. Therefore, we can assume an instantaneous nonlinear response in the femtosecond regime. Moreover, the nonlinear effects are relatively small in silica fibers;  $\mathbf{P}_{NL}$  can be treated as a small perturbation. Therefore, we first consider Eq. (7) with  $\mathbf{P}_{NL} = 0$ . Furthermore, we restrict ourselves to the case that the optical pulse maintains its polarization along the optical fiber, and the transverse diffraction term can be neglected. In this case, the electric field can be considered to be one dimensional and expressed as

$$\mathbf{E} = \frac{1}{2} \mathbf{e}_1 [E(z, t) + \text{c.c.}], \quad (10)$$

where  $\mathbf{e}_1$  is a unit vector in the direction of the polarization,  $E(z, t)$  is the complex-valued function, and c.c. stands for the complex conjugate. Under this case, it is useful to transform Eq. (7) into the frequency domain, which reads

$$\tilde{E}_{zz}(z, \omega) + \epsilon(\omega) \frac{\omega^2}{c^2} \tilde{E}(z, \omega) = 0, \quad (11)$$

where  $\tilde{E}(z, \omega)$  is the Fourier transform of  $E(z, t)$  defined as

$$\tilde{E}(z, \omega) = \int_{-\infty}^{\infty} E(z, t) e^{i\omega t} dt. \quad (12)$$

The frequency-dependent dielectric constant occurring in Eq. (12) is defined as

$$\epsilon(\omega) = 1 + \tilde{\chi}^{(1)}(\omega), \quad (13)$$

where  $\tilde{\chi}^{(1)}(\omega)$  is the Fourier transform of  $\chi^{(1)}(t)$ . Up to now, the consideration is exactly the same as the one for deriving the NLS equation. To derive the NLS equation, the optical field is assumed to be quasimonochromatic, i.e., the pulse spectrum, centered as  $\omega_0$ , is assumed to have a spectral width  $\Delta\omega$  such that  $\Delta\omega/\omega_0 \ll 1$ . Under this assumption, the NLS equation can be derived to govern the slowly varying envelope of the optical wave packet in weakly nonlinear dispersive media. However, when the width of the optical pulse is of the order of femtoseconds ( $10^{-15}$  s), the monochromatic assumption to derive the NLS equation is not valid anymore. We need to construct a suitable fit to the frequency-dependent dielectric constant  $\epsilon(\omega)$  in the desired spectral range. More specifically, for the frequency-dependent dielectric constant  $\epsilon(\omega) = 1 + \tilde{\chi}^{(1)}(\omega)$ , we assume  $\tilde{\chi}^{(1)}$  can be approximated by

$$\tilde{\chi}^{(1)} = \tilde{\chi}_0^{(1)} \mp \tilde{\chi}_2^{(1)} \lambda^2, \quad \tilde{\chi}_2^{(1)} > 0. \quad (14)$$

As discussed subsequently, the negative sign represents the focusing media with anomalous group velocity dispersion (GVD), and the positive sign stands for the defocusing media with normal GVD.

Next we proceed to the consideration of the nonlinear effect. Assuming the nonlinear response is instantaneous so that  $P_{NL}$  is given by  $P_{NL}(z, t) = \epsilon_{0NL} E(z, t)$  [3], the nonlinear contribution to the dielectric constant is defined as

$$\epsilon_{NL} = \frac{3}{4} \chi_{xxxx}^{(3)} |E(z, t)|^2. \quad (15)$$

Therefore, the Helmholtz equation can be modified as

$$\tilde{E}_{zz}(z, \omega) + \tilde{\epsilon}(\omega) \frac{\omega^2}{c^2} \tilde{E}(z, \omega) = 0, \quad (16)$$

where

$$\tilde{\epsilon}(\omega) = 1 + \tilde{\chi}_0^{(1)} \mp \tilde{\chi}_2^{(1)} \lambda^2 + \epsilon_{NL}. \quad (17)$$

In summary, Eq. (16) with Kerr cubic nonlinearity reads

$$\tilde{E}_{zz} + \frac{1 + \tilde{\chi}_0^{(1)}}{c^2} \omega^2 \tilde{E} \mp (2\pi)^2 \tilde{\chi}_2^{(1)} \tilde{E} + \epsilon_{NL} \frac{\omega^2}{c^2} \tilde{E} = 0. \quad (18)$$

By applying the inverse Fourier transform to Eq. (18), the nonlinear wave equation in physical domain is

$$E_{zz} - \frac{1}{c_1^2} E_{tt} = \pm \frac{1}{c_2^2} E + \frac{3}{4} \chi_{xxxx}^{(3)} (|E|^2 E)_{tt}, \quad (19)$$

where

$$c_1 = \frac{c}{\sqrt{1 + \tilde{\chi}_0^{(1)}}}, \quad c_2 = \frac{1}{2\pi \sqrt{\tilde{\chi}_2^{(1)}}}. \quad (20)$$

Furthermore, by using the normalized independent variables  $z \rightarrow c_2 z$ ,  $t \rightarrow c_2/c_1$ , and normalized field  $E \rightarrow (3c_1^2 \chi_{xxxx}^{(3)}/4)^{-1/2} E$ , we obtain the normalized wave equation

$$E_{zz} - E_{tt} = \pm E + (|E|^2 E)_{tt}. \quad (21)$$

Next, we focus on only a right-moving wave packet and assume a multiple scales ansatz

$$E(z, t) = \epsilon E_0(\tau, z_1, z_2, \dots) + \epsilon^2 E_1(\tau, z_1, z_2, \dots) + \dots, \quad (22)$$

where  $\epsilon$  is a small parameter,  $\tau$  and  $z_n$  are the scaled variables defined by

$$\tau = \frac{t - x}{\epsilon}, \quad z_n = \epsilon^n z. \quad (23)$$

Substituting (22) with (23) into (19), we obtain the following partial differential equation for  $E_0$  at the order  $O(\epsilon)$ :

$$-2 \frac{\partial^2 E_0}{\partial \tau \partial z_1} = \pm E_0 + 2 \frac{\partial}{\partial \tau} \left( |E_0|^2 \frac{\partial E_0}{\partial \tau} \right). \quad (24)$$

Here the term  $E_0^2 E_{0,\tau}$  is ignored but it is validated subsequently. Finally a general complex short-pulse equation can be obtained,

$$q_{xt} \pm q + \frac{1}{2}(|q|^2 q_x)_x = 0, \quad (25)$$

by the scale transformations

$$x = \frac{1}{\sqrt{2}} \tau, \quad t = \frac{1}{\sqrt{2}} z_1, \quad q = \sqrt{2} E_0. \quad (26)$$

It is obvious that Eq. (25) with positive sign is the same as Eq. (3), while Eq. (25) with negative sign is equivalent to Eq. (3) by a conversion of time  $t \rightarrow -t$ . Consequently, we derive the CSP equation of both the focusing and defocusing types. We should point out that there are typos in the scaling transformations in [16].

To validate the approximation, we compare the solutions to Maxwell equations with the ones to the CSP equation. As a matter of fact, solitary wave solutions with a few cycles derived directly from the Maxwell equations under the assumption of the Kramers-Kronig relation holds have been investigated

in the literature [24–27]. Here we mainly refer to the results in [25] and consider the normalized equation (21) with positive sign. We assume an envelope solitary wave solution is of the form

$$E(z, t) = A(\xi) e^{i\phi(z, t)}, \quad (27)$$

with  $\xi = z - vt$ ,  $\phi(z, t) = \omega(t - vz) + F(\xi)$ . Inserting this ansatz into Eq. (21), one obtains the set of equations

$$A_{zz} - A_{tt} - A(\phi_z^2 - \phi_t^2) - A - (6AA_t^2 + 3A^2A_{tt} - A^3\phi_t^2) = 0, \quad (28)$$

$$2A_z\phi_z + A\phi_{zz} - (2A_t\phi_t + A\phi_{tt}) - (6A^2A_t\phi_t + A^3\phi_{tt}) = 0, \quad (29)$$

$$[1 - v^2 A^2 / (1 - v^2)] A F_{\xi\xi} + 2[1 - 3v^2 A^2 / (1 - v^2)] A_\xi F_\xi + 6v\omega A^2 A_\xi / (1 - v^2) = 0. \quad (30)$$

By introducing normalized amplitude  $a = vA/\sqrt{1 - v^2}$ , we obtain

$$F(\xi) = -\frac{\omega}{2v} \int_{-\infty}^{\xi} \frac{a^2(3 - 2a^2)}{(1 - a^2)} d\xi' \quad (31)$$

by integrating Eq. (30) once. Further, inserting  $\phi_t = \omega - vF_\xi$  and  $\phi_z = -\omega v + F_\xi$  into Eq. (28), one obtains a second-order differential equation

$$a_{\xi\xi} - \frac{6aa_\xi^2}{1 - 3a^2} - \frac{\omega^2 a}{1 - 3a^2} \left( \frac{\delta^2}{v^2} - \frac{a^2(4(1 - a^2)^2 - a^2)}{4v^2(1 - a^2)^3} \right) = 0, \quad (32)$$

where  $\delta^2 = v^2/[\omega^2(1 - v^2)] - v^2$ . Integrating once and requiring  $a, a_\xi \rightarrow 0$  at  $\xi \rightarrow \pm\infty$ , one arrives at

$$a_\xi = \pm \frac{\omega\sqrt{1 + \delta^2}a}{v(1 - 3a^2)(1 - a^2)} \sqrt{\left(1 - \frac{3}{2}a^2\right) \left( \frac{1 + 4\delta^2 + \sqrt{1 - 8\delta^2}}{4(1 + \delta^2)} - a^2 \right) \left( \frac{1 + 4\delta^2 - \sqrt{1 - 8\delta^2}}{4(1 + \delta^2)} - a^2 \right)}. \quad (33)$$

From (33), one can easily show that a localized solution exists with amplitude

$$a_{\max} = \frac{1}{2} \sqrt{\frac{1 + 4\delta^2 + \sqrt{1 - 8\delta^2}}{1 + \delta^2}} \quad (34)$$

provided  $\delta^2 \leq 1/8$ .

As mentioned in [25], in the case of  $a_{\max} \ll 1$  where the slowly evolving wave field approximation (SEWA) is valid, the solution to Eq. (33) can be written as

$$\frac{9}{2}\delta\sqrt{\delta^2 - \frac{a^2}{2}} - \cosh^{-1}\left(\frac{\sqrt{2}\delta}{a}\right) = \pm \frac{\omega}{v}\delta\xi. \quad (35)$$

Furthermore, when  $\delta^2 \ll 1/8$  and the first term in Eq. (35) can be neglected, we obtain the one-soliton solution to the NLS

equation,

$$a_{\text{NLS}} = \sqrt{2}\delta \text{sech}(\omega\delta\xi/v). \quad (36)$$

Multiplying Eq. (35) by 2 and taking cosh function, we arrive at a localized solution to the higher order nonlinear Schrödinger (HONLS) equation by taking into account dispersions beyond group velocity dispersion (GVD),

$$a_{\text{HONLS}} = \frac{2\sqrt{2}\delta}{9\delta^2 + \sqrt{81\delta^2 + 4\cosh(2\omega\delta\xi/v)}}, \quad (37)$$

$$F = -\frac{3\omega}{2v} \int_{-\infty}^{\xi} a^2 d\xi'.$$

In Fig. 1, we compare the solutions for Eqs. (21) and (25) with positive sign [16,17,28], and the solution to the NLS equation and the higher order NLS equations (37) and (35) for

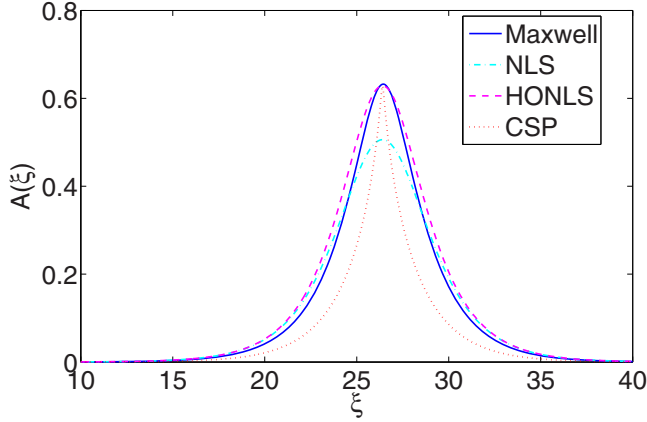


FIG. 1. Comparison of solitary solutions among Maxwell equations (solid blue line), focusing CSP equation (red dotted line), NLS (magenta dashed line), and the higher order NLS equations (cyan dash-dot line).

the parameters  $v = 1/2.25$ ,  $\omega = 1.0$ . Here, a classical Runge-Kutta method is used to integrate Eq. (33). It can be observed that solution of the Maxwell equations lies in between the ones of the CSP equation and the higher order NLS equation.

For the defocusing case, through a similar procedure as the focusing case, we can obtain the following equations:

$$\begin{aligned}
 & [a_\xi(3a^2 - 1)]_\xi \\
 & + \frac{\omega^2 a^4 [-a^4 + 4(1 - a^2)^2] - 4C_0 v^3 (a^4 + v^3 C_0)}{4 v^2 (a^2 - 1)^3 a^3} \\
 & - \frac{(v^2 - 1)^2 \omega^2 - v^2}{v^2 (v^2 - 1)} a = 0, \\
 & F(\xi) = \int_{-\infty}^{\xi} \left[ \frac{\omega(2a^6 - 3a^4 - 2C_0 v^3)}{2v(a^2 - 1)^2 a^2} - C_2 \right] d\xi + C_2 \xi,
 \end{aligned} \tag{38}$$

where  $C_0$  is an integration constant. Integrating (38) once, we arrive at

$$a_\xi = \pm \frac{\sqrt{G(a, v, \omega, C_0, C_1)}}{2v\sqrt{(1 - v^2)(3a^2 - 1)(a^2 - 1)a}},$$

where  $G(a, v, \omega, C_0, C_1)$  is a tenth order polynomial with respect to  $a$  (so we omit the explicit formula), and  $C_1$  is an integration constant. In a special case, we can obtain the dark soliton solution by choosing the parameters  $v = 0.44$ ,  $\omega = 0.4799915339$ ,  $C_0 = -17.65400508$ . Similarly, the dark soliton can be obtained by numerically solving (38) via the classical Runge-Kutta method. The result is compared with the one for the defocusing CSP equation in Fig. 2. As is seen, a good agreement is achieved.

Notice that the CSP equation (3) can be rewritten by

$$(\sqrt{1 + \sigma|q_x|^2})_t + \frac{1}{2}\sigma|q|^2\sqrt{1 + \sigma|q_x|^2}_x = 0,$$

so that we can define a reciprocal (hodograph) transformation

$$dy = \rho^{-1}dx - \frac{1}{2}\sigma\rho^{-1}|q|^2dt, \quad ds = -dt, \tag{39}$$

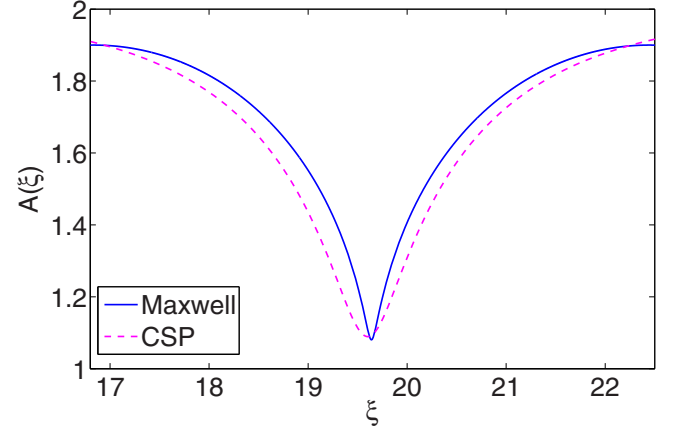


FIG. 2. Comparison of dark solitons between Maxwell equations (blue solid line) and the defocusing CSP equation (magenta dashed line).

where  $\rho^{-1} = \sqrt{1 + \sigma|q_x|^2}$ . By doing so, the CSP equation (3) is converted into the following coupled equation:

$$q_{ys} = \rho q, \tag{40}$$

$$\rho_s + \frac{1}{2}\sigma(|q|^2)_y = 0. \tag{41}$$

We remark here that Eqs. (40) and (41) with  $\sigma = 1$  is the complex coupled dispersionless (CCD) equation studied in [29], while the case of  $\sigma = -1$  is the case which, for some reason, has not been studied in the literature.

### III. DARBOUX TRANSFORMATION AND MULTI-DARK-SOLITON SOLUTION TO THE DEFOCUSING CSP EQUATION

In the present section, we aim at finding the multi-dark-soliton solution of the defocusing CSP equation

$$q_{xt} + q - \frac{1}{2}(|q|^2 q_x)_x = 0 \tag{42}$$

via the Darboux transformation method. First, it is noted that the CSP equation is invariant under the following scaling transformations:  $q \rightarrow cq$ ,  $\partial_x \rightarrow \frac{1}{c}\partial_x$ , and  $\partial_t \rightarrow c\partial_t$ . Thus, without loss of generality, we can fix either the amplitude or the wave number of  $q$ . Second, due to the fact that the CSP equation belongs to the Wadati-Konno-Ichikawa (WKI) hierarchy, it is not feasible to construct the Darboux transformation (DT) from the spectral problem of the CSP equation directly. Instead, we can develop the DT for the CCD equation which is linked to the CSP equation by the hodograph transformation (39).

In what follows, we present the Lax pair and the corresponding DT of the CCD equations (40) and (41) with  $\sigma = -1$ . It can be easily shown that the compatibility condition  $\Psi_{ys} = \Psi_{sy}$  of the following linear problems:

$$\Psi_y = U(\rho, q; \lambda)\Psi, \tag{43}$$

$$\Psi_s = V(q; \lambda)\Psi, \tag{44}$$

where

$$U(q, \rho; \lambda) = \lambda^{-1} \begin{bmatrix} -i\rho & \bar{q}_y \\ q_y & i\rho \end{bmatrix},$$

$$V(q; \lambda) = \frac{i}{4} \lambda \sigma_3 + \frac{i}{2} Q, \quad Q = \begin{bmatrix} 0 & -\bar{q} \\ q & 0 \end{bmatrix},$$

with the overbar representing the complex conjugate and  $\sigma_3$  being the third Pauli matrix, yields the defocusing CCD equation

$$q_{ys} = \rho q, \quad (45)$$

$$\rho_s - \frac{1}{2}(|q|^2)_y = 0. \quad (46)$$

Through the hodograph transformation

$$dx = \rho dy + \frac{1}{2}|q|^2 ds, \quad dt = -ds,$$

one can obtain the defocusing CSP equation (42). To obtain the soliton equation, we give the following Darboux matrix for the defocusing CSP equation (42) (we omitted the proof here; the interested reader can refer to [28,30–32] for details):

$$T = I + \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1} P_1, \quad P_1 = \frac{|y_1\rangle\langle y_1|\sigma_3}{\langle y_1|\sigma_3|y_1\rangle}, \quad (47)$$

$$\langle y_1| = |y_1\rangle^\dagger, \quad |y_1\rangle = \begin{bmatrix} \psi_1(y, s; \lambda_1) \\ \phi_1(y, s; \lambda_1) \end{bmatrix},$$

where  $|y_1\rangle$  denotes the special solution for the system (43) and (44) with  $\lambda = \lambda_1$ , and we can convert system (43) and (44) into a new system:

$$\Psi[1]_y = U(q[1], \rho[1]; \lambda) \Psi[1], \quad (48)$$

$$\Psi[1]_s = V(q[1]; \lambda) \Psi[1]. \quad (49)$$

The Bäcklund transformations between  $(q[1], \rho[1])$  and  $(q, \rho)$  are given through

$$q[1] = q + \frac{(\bar{\lambda}_1 - \lambda_1)\bar{\psi}_1\phi_1}{\langle y_1|\sigma_3|y_1\rangle}, \quad (50)$$

$$\rho[1] = \rho - 2 \ln_{ys} \left( \frac{\langle y_1|\sigma_3|y_1\rangle}{\bar{\lambda}_1 - \lambda_1} \right), \quad (51)$$

$$|q[1]|^2 = |q|^2 - 4 \ln_{ss} \left( \frac{\langle y_1|\sigma_3|y_1\rangle}{\bar{\lambda}_1 - \lambda_1} \right). \quad (52)$$

Furthermore, we have the following  $N$ -fold Darboux matrix: The  $N$ -fold Darboux matrix can be represented as

$$T_N = I + Y M^{-1} D^{-1} Y^\dagger \sigma_3, \quad (53)$$

where

$$Y = [|y_1\rangle, |y_2\rangle, \dots, |y_N\rangle],$$

$$M = \left( \frac{\langle y_i|\sigma_3|y_j\rangle}{\bar{\lambda}_i - \lambda_j} \right)_{1 \leq i, j \leq N},$$

$$D = \text{diag}(\lambda - \bar{\lambda}_1, \lambda - \bar{\lambda}_2, \dots, \lambda - \bar{\lambda}_N),$$

the vector  $|y_i\rangle$  represents the special solution for system (43) and (44) with  $\lambda = \lambda_i$ , and the Bäcklund transformations for

$q[N]$  and  $\rho[N]$  are

$$q[N] = q + \frac{\det(\hat{M})}{\det(M)}, \quad (54)$$

$$\rho[N] = \rho - 2 \ln_{ys} [\det(M)], \quad (55)$$

$$|q[N]|^2 = |q|^2 - 4 \ln_{ss} [\det(M)], \quad (56)$$

where

$$\hat{M} = \begin{bmatrix} M & Y_1^\dagger \\ -Y_2 & 0 \end{bmatrix},$$

and  $Y_k$  represents the  $k$ th row of matrix  $Y$ . The proof can be given similar to the one in [32], which is omitted here. Instead, we merely comment that the following identities associated with the matrix and determinant are used:

$$\phi M^{-1} \psi^\dagger = \begin{vmatrix} M & \psi^\dagger \\ -\phi & 0 \end{vmatrix} / |M|,$$

$$1 + \phi M^{-1} \psi^\dagger = \begin{vmatrix} M & \psi^\dagger \\ -\phi & 1 \end{vmatrix} / |M| = \frac{\det(M + \psi^\dagger \phi)}{\det(M)}.$$

Here  $M$  is a  $N \times N$  matrix;  $\phi, \psi$  are the  $N \times 1$  matrix. Based on the above  $N$ -fold Darboux transformation for the CCD system (40) and (41), we have the solitonic solution formula for the defocusing CSP equation (42) in parametric form,

$$q[N] = q + \frac{\det(\hat{M})}{\det(M)}, \quad (57)$$

$$x = \int \rho dy + \frac{1}{2} \int |q|^2 ds - 2 \ln_s [\det(M)], \quad t = -s. \quad (58)$$

#### IV. ONE- AND MULTI-DARK SOLUTIONS TO THE DEFOCUSING CSP EQUATION

In this section, we derive an explicit expression for the one- and multi-dark-soliton solution to the defocusing CSP equation through formulas (54)–(56) by a limit technique.

##### A. One-dark-soliton solution

We start with the seed solution

$$\rho[0] = -\frac{\gamma}{2}, \quad q[0] = \frac{\beta}{2} e^{i\theta}, \quad \theta = y + \frac{\gamma}{2}s, \quad \gamma > 0. \quad (59)$$

Introducing a gauge transformation with  $\Psi = K \hat{\Psi}$  with

$$K = \text{diag}(e^{(i/2)\theta}, e^{-(i/2)\theta}),$$

we can solve the Lax pair equations (43) and (44) at  $\lambda = \lambda_1$ , finding the fundamental matrix solution as follows:

$$\Psi = K L_1 M_1,$$

where

$$L_1 = \begin{bmatrix} 1 & 1 \\ \frac{\beta}{\chi_1^\pm + \gamma} & \frac{\beta}{\chi_1^\pm + \gamma} \end{bmatrix}, \quad M_1 = \text{diag}(e^{\omega_1^\pm}, e^{\omega_1^\mp}),$$

with

$$\omega_1^\pm = \frac{i}{4} (\chi_1^\pm - \lambda_1) \left( s + \frac{2}{\lambda_1} y \right) \pm a_1,$$

$$\chi_1^\pm = \lambda_1 \pm \sqrt{(\lambda_1 + \gamma)^2 - \beta^2}.$$



However, the soliton solution obtained above is usually singular. In order to derive the one-dark-soliton solution through the DT method, a limit process  $\lambda_1 \rightarrow \bar{\lambda}_1$  is needed. To this end, we first pick up one special solution,

$$|y_1\rangle = KL_1M_1 \begin{bmatrix} 1 \\ \alpha_1(\bar{\lambda}_1 - \lambda_1) \end{bmatrix};$$

further, for the sake of convenience, we set

$$\lambda_1 = \beta \cosh(\epsilon + i\varphi_1) - \gamma, \quad \chi_1^\pm = \beta e^{\pm(\epsilon + i\varphi_1)} - \gamma, \\ \alpha_1 = -\frac{e^{-i\varphi_1}}{4\beta \sin^2 \varphi_1},$$

where  $\varphi_1 \in (0, \pi)$ . By taking a limit  $\epsilon \rightarrow 0$ , we can obtain

$$\frac{\langle y_1 | \sigma_3 | y_1 \rangle}{2(\bar{\lambda}_1 - \lambda_1)} = \frac{e^{2\omega_1} + 1}{\beta(e^{-i\varphi_1} - e^{i\varphi_1})}, \quad (60)$$

where

$$\omega_1 = -\frac{\beta \sin \varphi_1}{4} \left( s + \frac{2y}{\beta \cos \varphi_1 - \gamma} \right) + a_1.$$

Thus, the single dark soliton can be written as

$$q[1] = \frac{\beta}{2} \left[ \frac{1 + e^{2(\omega_1 - i\varphi_1)}}{1 + e^{2\omega_1}} \right] e^{i\theta} \\ = \frac{\beta}{4} [(1 + e^{-2i\varphi_1}) + (e^{-2i\varphi_1} - 1) \tanh \omega_1] e^{i\theta}, \quad (61)$$

$$x = -\frac{\gamma}{2}y + \frac{\beta^2}{8}s + \frac{\beta \sin \varphi_1 e^{2\omega_1}}{1 + e^{2\omega_1}}, \quad t = -s, \quad (62)$$

The nonsingularity condition for the single dark soliton is  $\rho[1] \neq 0$  for all  $(x, t) \in \mathbb{R}^2$ . To analyze the property for the one-soliton solution, we calculate out

$$\rho[1] = \frac{\partial x}{\partial y} = -\frac{\gamma e^{4\omega_1} + (2 + \frac{2\beta^2 \sin^2 \varphi_1}{\beta \cos \varphi_1 - \gamma})e^{2\omega_1} + \gamma}{2(1 + e^{2\omega_1})^2}, \quad (63)$$

thus we can classify this one-dark-soliton solution as follows:

(i) Smooth soliton: when  $\gamma - \beta \cos \varphi_1 < 0$ , or  $\gamma - \beta \cos(\varphi_1) > 0$  and  $\Delta_1 > 0$ , where  $\Delta_1 = 2\gamma^2 - 2\gamma\beta \cos \varphi_1 - \beta^2 \sin^2 \varphi_1$ , the single-dark-soliton solution is always smooth. An example is illustrated in Fig. 3(a).

(ii) Cuspon soliton: when  $\gamma - \beta \cos \varphi_1 > 0$  and  $\Delta_1 = 0$  then  $\rho[1]$  attains zero at only one point, which leads to a cusponed dark soliton as displayed in Fig. 3(b).

(iii) Loop soliton: when  $\gamma - \beta \cos \varphi_1 > 0$ ,  $\Delta_1 < 0$ , then  $\rho[1]$  attains two zeros, which leads to a looped dark soliton.

The velocity can be solved with the following relation:

$$x - v_{sp,1}t - c_1 = \frac{\gamma(\beta \cos \varphi_1 - \gamma)}{\beta \sin \varphi_1} \omega_1 + \frac{\beta}{2} \sin \varphi_1 \tanh(\omega_1).$$

The velocity of the dark soliton is

$$v_{sp,1} = -\frac{2\gamma(\beta \cos \varphi_1 - \gamma) + \beta^2}{8},$$

and the initial center is

$$c_1 = \frac{\gamma(\gamma - \beta \cos \varphi_1)}{\beta \sin(\varphi_1)} a_1 + \frac{\beta}{2} \sin \varphi_1.$$

The trough of the dark soliton  $|q|$  is along the line

$$x - v_{sp,1}t - c_1 = 0,$$

and the depth of the trough is  $\frac{1}{2}|\beta(1 - \cos \varphi_1)|$ .

## B. Multi-dark-soliton solution

Similar to the process of obtaining the single-dark-soliton solution, starting with the same seed solution, and solving the Lax pair equations (43) and (44) with  $(q = q[0], \rho = \rho[0])$  at  $\lambda = \lambda_i$ , we have

$$|y_i\rangle = KL_iM_i \begin{bmatrix} 1 \\ \alpha_i(\bar{\lambda}_i - \lambda_i) \end{bmatrix} \equiv K \begin{bmatrix} \hat{\phi}_i \\ \beta \hat{\psi}_i \end{bmatrix},$$

where

$$L_i = \begin{bmatrix} 1 & 1 \\ \frac{\beta}{\chi_i^+ + \gamma} & \frac{\beta}{\chi_i^- + \gamma} \end{bmatrix}, \quad M_i = \text{diag}(e^{\omega_i^+}, e^{\omega_i^-}),$$

with

$$\omega_i^\pm = \frac{i}{4}(\chi_i^\pm - \lambda_i) \left( s + \frac{2}{\lambda_i} y \right) \pm a_i, \\ \chi_i^\pm = \lambda_i \pm \sqrt{(\lambda_i + \gamma)^2 - \beta^2},$$

$\alpha_i$ s are appropriate complex parameters and  $a_i$ s are real parameters. Based on the  $N$ -soliton solution (57) and (58) to the defocusing CSP equation, it then follows that

$$q[N] = \frac{\beta}{2} [1 + \hat{Y}_2 M^{-1} \hat{Y}_1^\dagger] e^{i\theta} = \frac{\beta}{2} \left[ \frac{\det(H)}{\det(M)} \right] e^{i\theta}, \quad (64)$$

$$x = -\frac{\gamma}{2}y + \frac{\beta^2}{8}s - 2 \ln_s[\det(M)], \quad t = -s,$$

where

$$M = \left( \frac{\langle y_i | \sigma_3 | y_j \rangle}{2(\bar{\lambda}_i - \lambda_j)} \right)_{1 \leq i, j \leq N}, \quad H = M + Y_1^\dagger Y_2, \\ \hat{Y}_1 = [\hat{\phi}_1, \hat{\phi}_2, \dots, \hat{\phi}_N], \quad \hat{Y}_2 = [\hat{\psi}_1, \hat{\psi}_2, \dots, \hat{\psi}_N].$$

In general, the above  $N$ -soliton solution (64) is singular. In order to derive the  $N$ -dark soliton solution through the DT method, we need to take a limit process  $\lambda_i \rightarrow \bar{\lambda}_i$  ( $i = 1, 2, \dots, N$ ) similar to the one in [32]. By a tedious procedure which is omitted here, we finally have the  $N$ -dark-soliton solution to the defocusing CSP equation (42) as follows:

$$q = \frac{\beta \det(H)}{2 \det(M)} e^{i[y + (\gamma/2)s]}, \quad (65)$$

$$x = -\frac{\gamma}{2}y + \frac{\beta^2}{8}s - 2 \ln_s[\det(M)], \quad t = -s, \quad (66)$$

where the entries of the matrices  $M$  and  $H$  are

$$m_{i,j} = \frac{e^{\omega_i + \omega_j} + \delta_{i,j}}{\beta(e^{-i\varphi_i} - e^{i\varphi_j})}, \quad h_{i,j} = \frac{e^{(\omega_i - i\varphi_i) + (\omega_j - i\varphi_j)} + \delta_{i,j}}{\beta(e^{-i\varphi_i} - e^{i\varphi_j})}, \\ 1 \leq i, j \leq N, \quad (67)$$

$\delta_{i,j}$  is a Kronecker's  $\delta$  and

$$\omega_i = -\frac{\beta}{4} \sin \varphi_i \left( s + \frac{2y}{\beta \cos \varphi_i - \gamma} \right) + a_i, \quad \varphi_i \in \mathbf{R}. \quad (68)$$

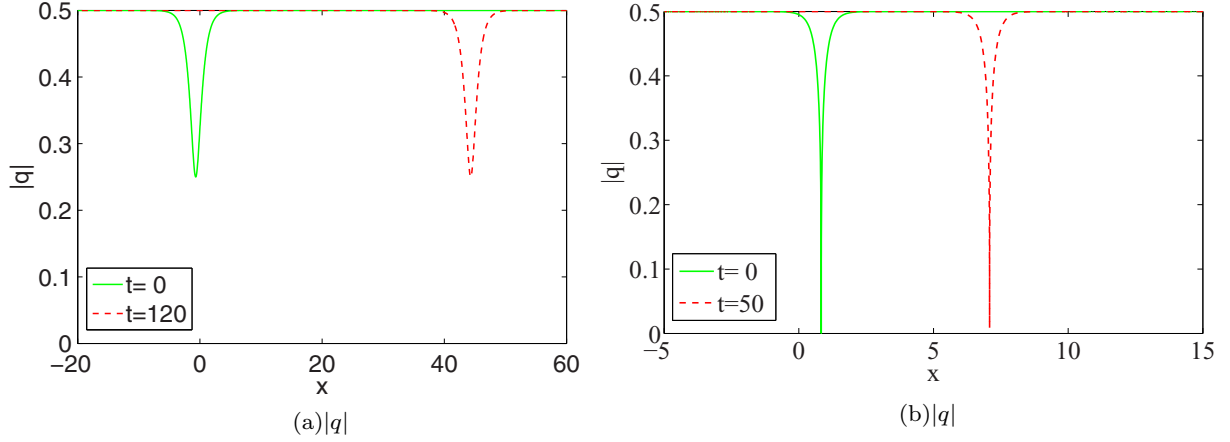


FIG. 3. (a) Smooth dark soliton ( $t = 0$ : blue solid line;  $t = 120$ : red dashed line). Parameters  $\beta = 1$ ,  $\gamma = 1$ ,  $\varphi_1 = 2\pi/3$ ,  $a_1 = 0$ . (b) Cuspon dark soliton ( $t = 0$ : blue solid line;  $t = 50$ : red dashed line). Parameters  $\beta = 1$ ,  $\gamma = \sqrt{2}/2$ ,  $\varphi_1 = \pi/2$ ,  $a_1 = 0$ .

By taking  $N = 2$  in (67), the determinants corresponding to the two-dark-soliton solution can be calculated as

$$|M| = 1 + e^{2\omega_1} + e^{2\omega_2} + a_{12}e^{2(\omega_1+\omega_2)}, \quad (69)$$

$$|H| = 1 + e^{2(\omega_1 - i\varphi_1)} + e^{2(\omega_2 - i\varphi_2)} + a_{12}e^{2(\omega_1 + \omega_2 - i\varphi_1 - i\varphi_2)}, \quad (70)$$

where

$$a_{12} = \frac{\sin^2\left(\frac{\varphi_2 - \varphi_1}{2}\right)}{\sin^2\left(\frac{\varphi_2 + \varphi_1}{2}\right)}. \quad (71)$$

The collision processes between smooth-smooth dark solitons and smooth-cuspon dark solitons are illustrated in Figs. 4(a) and 4(b), respectively. It is seen that the interactions between dark solitons are elastic. Different from the interaction between

two smooth bright solitons, which could develop singularity, the interaction between two smooth dark solitons never appears as singularity. When a smoothed dark soliton interacts with a cusponed dark soliton, the singularity of the cusponed dark soliton could vanish as observed in Fig. 4(b).

## V. CONCLUSIONS AND DISCUSSIONS

In this paper, we derived the complex short-pulse equation of both focusing and defocusing types from the context of nonlinear optics and found the multi-dark-soliton solution of the defocusing type. Comparing with the classical theory for the SP equation, there are several advantages in using complex representation. First, amplitude and phase are two fundamental characteristics for a wave packet, the information of these two factors is nicely combined into a single complex-valued function. Second, the use of complex representation can allow us to model the propagation of optical pulses in both the focusing and defocusing nonlinear media. Such advantages

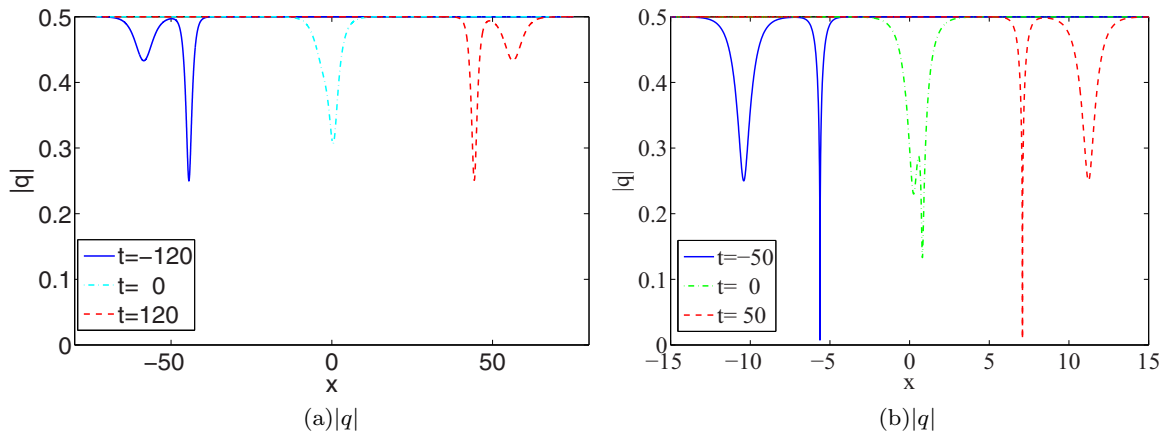


FIG. 4. (a) Smooth-smooth dark soliton ( $t = -120$ : blue solid line;  $t = 0$ : magenta dash-dot line;  $t = 120$ : red dashed line). Parameters  $\beta = 1$ ,  $\gamma = 1$ ,  $\varphi_1 = 2\pi/3$ ,  $\varphi_2 = 5\pi/6$ ,  $a_1 = a_2 = 0$ . (b) Smooth-cuspon dark soliton ( $t = -50$ : blue solid line;  $t = 0$ : green dash-dot line;  $t = 50$ : red dashed line). Parameters  $\beta = 1$ ,  $\gamma = \sqrt{2}/2$ ,  $\varphi_1 = \pi/2$ ,  $\varphi_2 = 2\pi/3$ ,  $a_1 = a_2 = 0$ ; It is seen that the interaction of smooth dark soliton and cuspon dark soliton is elastic.

can be observed in many analytical results related to the NLS equation and the CSP equation. Therefore, by using a complex representation, we have shown that the focusing CSP equation admits the bright soliton solution [16, 17], the breather solution, as well as the rogue wave solution [28], whereas, as shown in the present paper, the defocusing CSP equation has the multi-dark-soliton solution the same as the defocusing NLS equation. It would be a very interesting topic to compare the properties of ultrashort optical pulses experimentally with the theoretical predictions for the CSP equation and the ones for the NLS equations. This, of course, is beyond the scope of the present paper.

The dynamics of the dark soliton has been a hot topic in nonlinear optics. The history for the observation of the dark soliton is even earlier than the bright soliton. In this work, we proposed an integrable equation which admits the multi-dark-

soliton solution. Moreover, we provided the dynamics analysis for the single-dark soliton and two-dark soliton in detail. The results would further enrich our understanding of dark solitons in the ultrashort-pulse model.

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