Presence of quantum correlations results in a nonvanishing ergotropic gap

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The paradigm of extracting work from an isolated quantum system through a cyclic Hamiltonian process is a topic of immense research interest. The optimal work extracted under such a process is termed *ergotropy* [Europhys. Lett. **67**, 565 (2004)]. Here, in a multiparty scenario, we consider only a class of such cyclic processes that can be implemented locally, giving rise to the concept of *local ergotropy*. Eventually, the presence of quantum correlations results in a nonvanishing thermodynamic quantity called an *ergotropic gap*, measured by the difference between global and local ergotropy. However, the converse does not hold in general, i.e., its nonzero value does not necessarily imply the presence of quantum correlations. For arbitrary multiparty states, we quantify this gap. We also evaluate the difference between maximum global and local extractable work for arbitrary states when the system is no longer isolated but put in contact with a bath of the same *local* temperature.

I. INTRODUCTION

The idea of information is deeply connected with physics, especially thermodynamics [1–3]. Considerable efforts have been devoted to ameliorating this connection in the quantum regime [4,5]. As a consequence, resource-theoretic aspects of quantum thermodynamics have flourished [6–9]. Although the importance of quantum correlations in the context of quantum thermodynamics is not yet understood in full, a substantial amount of progress has been made in this direction in the recent past [10–12].

The presence of correlations that have no classical counterpart is one of the striking features of multiparty quantum systems. One much studied way to capture the notion of quantumness present in a correlation is *entanglement* [13]. However, there exist several tasks in which a multiparty quantum state, not being entangled at all, can be more advantageous than classical correlations. In a bipartite scenario, the quantumness present in a correlation has been quantified by a quantity known as *quantum discord* [14–16]. Quantum discord has been found to be a useful resource for various information-theoretic tasks viz. extended-state merging [17,18] and remote-state preparation [19], although in a restricted sense. Thus, there is a practical motivation for the study of quantumness in a more general framework than entanglement. In this work, we investigate whether quantumness in correlations has any implications in quantum thermodynamics. Interestingly, we show that there exists a thermodynamic quantity, namely an *ergotropic gap*. This gap, which represents the difference between the maximum extractable work under global and local cyclic Hamiltonian processes, is nonvanishing whenever the multiparty quantum state is not classically correlated.

Extracting work from a quantum system is one of the important areas of study in quantum thermodynamics [20,21].

The study of optimal work extraction from an isolated quantum system under a cyclic Hamiltonian process was first performed in the mathematical framework of C^* algebra [22], which was later explored in the well-known Hilbert space formalism [23]. The aim is to transform a quantum system from a higher to a lower internal energy state, extracting the difference in internal energy as work. It has been shown that the optimal amount of work is extracted under a cyclic Hamiltonian process whenever the system evolves into a state from which no further work can be extracted. Such a state is called a *passive* state [22–24]. The authors of Ref. [24] coined the term *ergotropy* for optimal extractable work.

In the recent past, the topic of extracting work from a quantum system gained renewed interest [25–30]. In Ref. [30], the authors designed a scenario in which correlations in multiparty quantum systems enable work extraction. Given a noninteracting Hamiltonian of a multiparty system, any cyclic unitary process can be realized by switching on a suitable external interaction field. Here we consider a situation in which subsystems are spatially separated and no global external interacting field can be implemented on the total system. Each subsystem can only be acted upon by a local field. We call the optimal extractable work *local ergotropy*. We find that there exist classically correlated states for which the ergotropic gap, i.e., the difference between global and local ergotropy, can be nonzero. However, this does not lead to the conclusion that classical correlations always possess a nonzero *ergotropic gap* because there exist classically correlated states for which this gap turns out to be zero. Interestingly, we find that whenever the multiparty system is not classically correlated, the optimal amount of extractable work under cyclic local interaction is strictly less than that obtained under global interaction, i.e., the presence of quantum correlations always results in a nonvanishing ergotropic gap. Given a noninteracting Hamiltonian and an arbitrary initial state of a multiparty system, we quantify this gap. We also consider the scenario in which the system is no longer isolated but put in contact with a bath of the same *local* temperature, and we evaluate the difference between maximum global and local extractable work for an arbitrary state.

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II. FRAMEWORK FOR WORK EXTRACTION

Consider a quantum system, composed of N subsystems, prepared in the state $\varrho_{A_1,\dots,A_N} \in \mathcal{D}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N)$, where $\mathcal{H}_i$ denotes the Hilbert space corresponding to the ith subsystem, and $\mathcal{D}(\mathcal{X})$ denotes the set of density operators acting on Hilbert space $\mathcal{X}$. Consider that the local Hamiltonian for the ith party is given by $H_i = \sum_j e_i^j |j_i\rangle\langle j_i|$, where $|j_i\rangle$ denotes the jth energy eigenstate of the ith particle with energy eigenvalue e_i^j. No interactions are considered among the various subsystems. Therefore, the total Hamiltonian of the composite system takes the form

$$H = \sum_{i=1}^{N} H_i \bigotimes_{k \in \bar{i}} \mathbf{I}_k = \sum_{i=1}^{N} \tilde{H}_i, \tag{1}$$

where $H_i \bigotimes_{k \in \bar{i}} \mathbf{I}_k = \mathbf{I}_1 \otimes \cdots \otimes \mathbf{I}_{i-1} \otimes H_i \otimes \mathbf{I}_{i+1} \otimes \cdots \otimes \mathbf{I}_N$, with $\mathbf{I}_l$ denoting an identity operator acting on the Hilbert space $\mathcal{H}_l$.

In the paradigm of work extraction from an isolated system under a cyclic unitary process, the protocol is to transform the state from $\varrho_{A_1,\dots,A_N}$ to some $\sigma_{A_1,\dots,A_n}$ by using some time-dependent unitary operation $U(\tau)$ such that $\sigma_{A_1,\dots,A_n}$ has less internal energy than $\varrho_{A_1,\dots,A_N}$. Note that, since only unitary operations are used, the entropy of the final state is the same as that of the initial state. Any such unitary can be generated by applying a time-dependent interaction $V(t)$ among the N subsystems, such that $V(t)$ is nonvanishing only when $0 \leqslant t \leqslant \tau$. The corresponding evolution can be described by the unitary operator $U(\tau) = \overrightarrow{\exp}(-\iota \int_0^\tau dt[H + V(t)])$, where $\overrightarrow{\exp}$ denotes the time-ordered exponential. In this setup, the optimally extractable work is therefore

$$W_{\text{opt}}^G = \max_{U(\tau)} \text{Tr}\left\{\left[\varrho_{A_1,\dots,A_N} - U(\tau)\varrho_{A_1,\dots,A_N}U^\dagger(\tau)\right]H\right\}$$
$$= \text{Tr}\left[\varrho_{A_1,\dots,A_N}H\right] - \min_{U(\tau)} \text{Tr}\left[U(\tau)\varrho_{A_1,\dots,A_N}U^\dagger(\tau)H\right],$$

where optimization is done over all unitaries. It has been shown that this optimization causes the system to evolve into a state $\varrho_{A_1,\dots,A_N}^{\text{passive}}$, called a passive state [22–24]. Thus the optimal amount of extractable work, namely ergotropy [24], amounts to

$$W_{\text{opt}}^G = \text{Tr}\left[\varrho_{A_1,\dots,A_N}H\right] - \text{Tr}\left[\varrho_{A_1,\dots,A_N}^{\text{passive}}H\right]. \tag{2}$$

Among the passive states there is a special one, called the thermal or Gibbs state. Given many copies of the system, it may be possible that work can be extracted even from passive states. However, no such work extraction is possible from the thermal states, so it is called a *completely* passive state [24,31].

Consider a situation in which each subsystem of the joint system $\varrho_{A_1,\dots,A_N}$ is spatially separated, and implementation of any global interaction field is not allowed. Each party can only apply a time-dependent local field on its respective subsystem. Hence the interaction on the composite system reads

$$V(t) = \sum_{i=1}^{N} V_i(t) \bigotimes_{k \in \bar{i}} \mathbf{I}_k = \sum_{i=1}^{N} \tilde{V}_i(t). \tag{3}$$

The class of unitaries generated from such interactions is therefore

$$U(\tau) = \overrightarrow{\exp}\left(-\iota \int_0^\tau dt \sum_{i=1}^{N} (\tilde{H}_i + \tilde{V}_i(t))\right)$$

$$= \prod_{i=1}^{N} \overrightarrow{\exp}\left(-\iota \int_0^\tau dt\, \{H_i + V_i(t)\} \bigotimes_{k \in \bar{i}} \mathbf{I}_k\right)$$

$$= \prod_{i=1}^{N} \left\{\overrightarrow{\exp}\left(-\iota \int_0^\tau dt\, \{H_i + V_i(t)\}\right) \bigotimes_{k \in \bar{i}} \mathbf{I}_k\right\}$$

$$= \bigotimes_{i=1}^{N} \overrightarrow{\exp}\left(-\iota \int_0^\tau dt(H_i + V_i(t))\right)$$

$$= \bigotimes_{i=1}^{N} U_i(\tau),$$

where $U_i(\tau)$ is the unitary on the ith particle. Here we ignore the global constant factor, which is not relevant. Let us denote this class of unitaries as

$$\mathcal{LU} := \left\{U(\tau) | U(\tau) = \bigotimes_{i=1}^{N} U_i(\tau)\right\}. \tag{4}$$

The optimal work that can be extracted under such local interactions is thus

$$W_{\text{opt}}^L := \max_{U \in \mathcal{LU}} \text{Tr}\left[\left(\varrho_{A_1,\dots,A_n} - U\varrho_{A_1,\dots,A_n}U^\dagger\right)H\right]$$
$$= \text{Tr}\left[\varrho_{A_1,\dots,A_n}H\right] - \min_{U \in \mathcal{LU}} \text{Tr}\left[U\varrho_{A_1,\dots,A_n}U^\dagger H\right]. \tag{5}$$

In this scenario, since work is extracted by applying local unitaries, we call the optimal extractable work the *local ergotropy* of the state ϱ given the Hamiltonian H. In the above notation, superscript L is introduced to distinguish this quantity from the one defined in Eq. (2), where the superscript G has been used to indicate that global unitaries are allowed.

At this point, we define a quantity that is the difference of the global and local ergotropy, called the ergotropic gap (EG),

$$W_{\text{EG}} := W_{\text{opt}}^G - W_{\text{opt}}^L. \tag{6}$$

Replacing W_{opt}^G and W_{opt}^L from Eqs. (2) and (5), respectively, we have

$$W_{\text{EG}} = \min_{U \in \mathcal{LU}} \text{Tr}\left[U\varrho_{A_1,\dots,A_n}U^\dagger H\right] - \text{Tr}\left[\rho_{A_1,\dots,A_n}^{\text{passive}}H\right]. \tag{7}$$

It is easy to see that W_{EG} cannot be negative. This is because local operations are restricted to extract energy from subsystems only, whereas the global unitary has the power to extract energy from subsystems as well as from correlations. In the following, we study this quantity in the presence of correlations between the subsystems of multiparty systems.

III. CORRELATIONS AND THE ERGOTROPIC GAP

In physics, the study of correlations is quite important as it is the most significant feature to characterize multiparticle systems. However, its characterization and quantification become notoriously difficult when one shifts from the classical

realm to the quantum realm. The core focus of quantum-information theory is to study these correlations, which are also important from a foundational perspective. Depending on the situation, correlations can be characterized in different ways, e.g., *nonlocal* [32], *steerable* [33], *entanglement* [13], *quantum correlation (discord)* [16], etc., and there are a number of practical applications [17–19,34–45]. They also play an important role in quantum thermodynamics [30]. Here we are interested in the role of quantum correlations in *ergotropy*.

An N-particle quantum state, with d levels for each particle ($\varrho_{A_1,...,A_n}$), has d^N eigenvalues (possibly degenerate) forming a normalized probability vector $(\lambda)_1^{d^N}$, represented in a row. Rearranging the eigenvalues lets us form a vector $\vec{\lambda} = (\lambda_\alpha)_{\alpha=1}^{d^N}$, where $\lambda_\alpha \geqslant \lambda_{\alpha+1} \forall\ \alpha$. The d^N energy eigenstates of the Hamiltonian of Eq. (1) are denoted as $\{|\xi_\alpha\rangle\}_{\alpha=1}^{d^N}$, with energy eigenvalues $\xi_\alpha \leqslant \xi_{\alpha+1} \forall\ \alpha$ (there may be degeneracy). In this notation, the passive state $\varrho_{A_1,...,A_n}^{\text{passive}}$ reads [22–24]

$$\varrho_{A_1,...,A_n}^{\text{passive}} := \sum_\alpha \lambda_\alpha |\xi_\alpha\rangle\langle\xi_\alpha|. \qquad (8)$$

$\varrho_{A_1,...,A_n}^{\text{passive}}$ commutes with the Hamiltonian that is diagonalizable in the orthonormal product basis (ONPB) $\{\bigotimes_i |j_i\rangle\}_{j,i}$, where $\{|j_i\rangle\}_j$ forms an orthonormal basis (ONB) (energy eigenbasis) of the ith party Hilbert space $\mathcal{H}_i$.

If the multiparticle system is in a pure product state, then the EG is always zero. Consider an arbitrary pure product state,

$$\rho_{A_1,...,A_n}^{\text{product}} = |\psi\rangle_{A_1}\langle\psi| \otimes |\psi\rangle_{A_2}\langle\psi| \cdots \otimes |\psi\rangle_{A_n}\langle\psi|, \qquad (9)$$

where $|\psi\rangle_{A_i} \in \mathbb{C}^{d_i}, \forall\ i$. Let the ground energy state of the ith particle be $|0\rangle_{A_i}$. Applying local unitaries, the state of each subsystem can be transformed from $|\psi\rangle_{A_1}$ to $|0\rangle_{A_i}$, changing the global state into its passive form $\varrho_{A_1,...,A_n}^{\text{passive}} = \bigotimes_i |0\rangle_{A_i}\langle 0|$. It readily follows that the EGs for pure product states are vanishing. Now we ask whether the EGs of correlated states are vanishing. We first start with CC states.

An N-particle state is called classically correlated (CC) if it can be written as [46]

$$\varrho_{\text{CC}} = \sum_{\{\beta_i\}\in\text{ONB}[\mathcal{H}_i]} p_{\beta_1,...,\beta_N} \bigotimes_{i=1}^N |\beta_i\rangle\langle\beta_i|, \qquad (10)$$

where $\{|\beta_i\rangle\}_\beta$ is an ONB for the ith particle Hilbert space $\mathcal{H}_i$, and $(p_{\beta_1,...,\beta_N})_1^{d^N}$ is a probability vector. Clearly the state ϱ_{CC} is diagonalized in the ONPB $\{\bigotimes_i |\beta_i\rangle\}_{\beta,i}$. Consider such a two-qubit CC state of the following form:

$$\varrho_{A_1A_2} = \lambda_1 |0\rangle_{A_1}\langle 0| \otimes |0\rangle_{A_2}\langle 0| + \lambda_2 |1\rangle_{A_1}\langle 1| \otimes |1\rangle_{A_2}\langle 1|, \qquad (11)$$

where $0 < \lambda_2 < \lambda_1 < 1$, $\lambda_1 + \lambda_2 = 1$, and $|0\rangle_{A_i}(|1\rangle_{A_i})$ represents the ground (excited) energy eigenstate of the ith particle Hamiltonian $H_i = e^0|0\rangle_{A_i}\langle 0| + e^1|1\rangle_{A_i}\langle 1|$, with e^0 and e^1 denoting ground and excited energy eigenvalues, respectively. Here the Hamiltonian for the composite system is $H = H_1 \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes H_2$. The corresponding passive state reads

$$\varrho_{A_1A_2}^{\text{passive}} = \lambda_1 |0\rangle_{A_1}\langle 0| \otimes |0\rangle_{A_2}\langle 0| + \lambda_2 |0\rangle_{A_1}\langle 0| \otimes |1\rangle_{A_2}\langle 1|. \qquad (12)$$

To evolve the state $\varrho_{A_1A_2}$ into $\varrho_{A_1A_2}^{\text{passive}}$, one needs to apply the following unitary:

$$|0\rangle_{A_1} \otimes |0\rangle_{A_2} \longmapsto |0\rangle_{A_1} \otimes |0\rangle_{A_2},$$
$$|1\rangle_{A_1} \otimes |1\rangle_{A_2} \longmapsto |0\rangle_{A_1} \otimes |1\rangle_{A_2}, \qquad (13)$$

which is the inverse of the CNOT operation and hence an entangling unitary and cannot be realized by unitaries of the form $U_{A_1} \bigotimes U_{A_2}$.

Naturally, the question arises whether all CC states possess a nonvanishing EG. However, the following example shows that this is not the case in general. Consider the class of CC states of the form

$$\rho_{AB} = p_0|0\rangle_{A_1}\langle 0| \otimes |0\rangle_{A_2}\langle 0| + p_1|0\rangle_{A_1}\langle 0| \otimes |1\rangle_{A_2}\langle 1|$$
$$+ p_2|1\rangle_{A_1}\langle 1| \otimes |0\rangle_{A_2}\langle 0|$$
$$+ p_3|1\rangle_{A_1}\langle 1| \otimes |1\rangle_{A_2}\langle 1|, \qquad (14)$$

where $p_0 < p_1 < p_2 < p_3$, $0 \leqslant p_k \leqslant 1, \forall\ k$, and $\sum_{k=0}^3 p_k = 1$. The corresponding passive state reads

$$\rho_{AB}^{\text{passive}} = p_3|0\rangle_{A_1}\langle 0| \otimes |0\rangle_{A_2}\langle 0| + p_2|0\rangle_{A_1}\langle 0| \otimes |1\rangle_{A_2}\langle 1|$$
$$+ p_1|1\rangle_{A_1}\langle 1| \otimes |0\rangle_{A_2}\langle 0|$$
$$+ p_0|1\rangle_{A_1}\langle 1| \otimes |1\rangle_{A_2}\langle 1|. \qquad (15)$$

Given such CC states, they can be transformed into passive form by applying the σ_x operation on each site locally, which thus implies a vanishing EG. Thus for CC states, the EG can be zero as well as nonzero. Here we ask whether there exists any correlation that always possesses a nonzero EG. In the following proposition, we answer this question.

Proposition 1. The ergotropic gap is always nonvanishing in the presence of quantum correlations.

Proof. A quantum state is said to contain quantumness in the correlation if it is not CC, i.e., there is no ONPB that diagonalized the state, and such states are called *quantum correlated*. In the bipartite case, quantumness is quantified by a quantity called *discord* [16], which has attracted a great deal of research interest recently.

It is clear that a quantum-correlated state must contain entangled state(s) in its spectrum. However, the passive state corresponding to such a state is diagonal in the product basis (ONPB of the Hamiltonian). The fact that it is impossible to arrive at some product basis starting from a basis containing entangled state(s) by implementing only local unitaries implies that $W_{\text{opt}}^G > W_{\text{opt}}^L$, i.e., not all the ergotropy of the system is locally accessible for quantum-correlated states. ∎

The converse of the above proposition does not hold, i.e., a nonzero ergotropic gap does not imply the presence of quantum correlations, which we have already shown in the previous example.

IV. ERGOTROPIC GAP FOR ARBITRARY STATES

Given the Hamiltonian H of the form of Eq. (1) and an arbitrary state, it is possible to quantify W_{EG} in terms of the parameters of the Hamiltonian and the state. First we consider two two-level systems and discuss a few special subclasses of states of this system, and then we consider multiparty-multilevel systems.
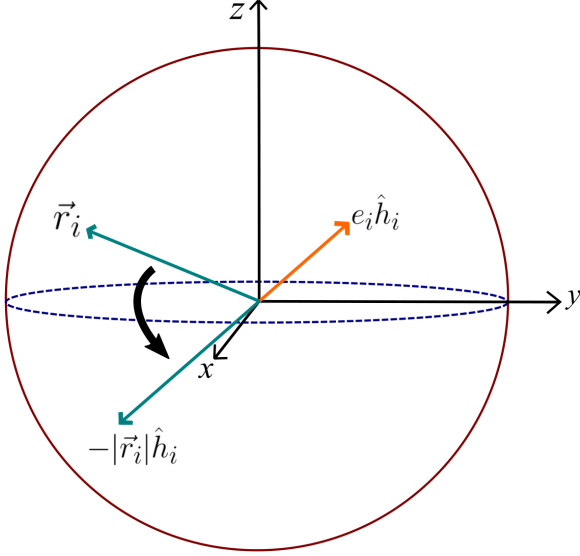
FIG. 1. Bloch sphere for a qubit system. $e_i \hat{h}_i$ represents the Bloch vector of the Hamiltonian of Eq. (16). To optimize the second term in Eq. (5), each party applies local unitary U_i, which rotates the reduced Bloch vector $\vec{r}_i$ along $-\hat{h}_i$.

A. Two-particle–two-level system

Consider an arbitrary two-particle–two-level system with the Hamiltonian

$$
\begin{aligned}
H_i &= e_i^0 |0_i\rangle\langle 0_i| + e_i^1 |1_i\rangle\langle 1_i| \\
&= \tfrac{1}{2}(e_i^+ \mathbf{I} + e_i^- \hat{h}_i \cdot \vec{\sigma}), \quad i \in 1,2,
\end{aligned} \tag{16}
$$

where $e_i^\pm = e_i^1 \pm e_i^0$, $\hat{h}_i$ is a vector in the Bloch sphere (see Fig. 1) and $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$, with $\sigma_x, \sigma_y, \sigma_z$ denoting the Pauli matrices. The total Hamiltonian of the composite system is thus

$$
H = H_1 \otimes \mathbf{I} + \mathbf{I} \otimes H_2. \tag{17}
$$

An arbitrary two-qubit state can be expressed as the following canonical form [47]:

$$
\varrho_{A_1 A_2} = \frac{1}{4}\left[\mathbf{I} \otimes \mathbf{I} + \vec{r}_1 \cdot \vec{\sigma} \otimes \mathbf{I} + \mathbf{I} \otimes \vec{r}_2 \cdot \vec{\sigma} + \sum_{m,n} t_{mn} \sigma_m \otimes \sigma_n \right],
$$

where the reduced state of the ith party is

$$
\varrho_{A_i} = \mathrm{Tr}_{A_{\bar{i}}}(\varrho_{A_1 A_2}) = \tfrac{1}{2}[\mathbf{I} + \vec{r}_i \cdot \vec{\sigma}],
$$

$\vec{r}_i$ being the vectors in $\mathbb{R}^3$ with $|\vec{r}_i| \leqslant 1$.

Let spectral values of $\varrho_{A_1 A_2}$ be $\lambda_{00}, \lambda_{01}, \lambda_{10}$, and λ_{11}, where $\lambda_{00} \geqslant \lambda_{01} \geqslant \lambda_{10} \geqslant \lambda_{11}$. Given the Hamiltonian of Eq. (17), the passive state reads

$$
\varrho_{A_1 A_2}^{\mathrm{passive}} = \sum_{x,y} \lambda_{xy} |x_1\rangle\langle x_1| \otimes |y_2\rangle\langle y_2|,
$$

with $x,y \in \{0,1\}$. We have

$$
\mathrm{Tr}(\varrho_{A_1 A_2}^{\mathrm{passive}} H) = \sum_{x,y=0}^{1} \lambda_{xy} e_{xy}.
$$

where $e_{xy} = e_1^x + e_2^y$. Hence, we have

$$
W_{\mathrm{opt}}^G = \mathrm{Tr}(\varrho_{A_1 A_2} H) - \left[\sum_{x,y=0}^{1} \lambda_{xy} e_{xy} \right]
$$

and

$$
W_{\mathrm{opt}}^L = \mathrm{Tr}(\varrho_{A_1 A_2} H) - \min_{U_1 \otimes U_2} \mathrm{Tr}[U \varrho_{A_1 A_2} U^\dagger H].
$$

Now observe that

$$
\begin{aligned}
&\min_{U_1 \otimes U_2} \mathrm{Tr}[U_1 \otimes U_2 \varrho_{A_1 A_2} U_1^\dagger \otimes U_2^\dagger H] \\
&= \min_{U_1 \otimes U_2} \mathrm{Tr}[\varrho_{A_1 A_2} U_1^\dagger \otimes U_2^\dagger H U_1 \otimes U_2] \\
&= \min_{U_1 \otimes U_2} \mathrm{Tr}[\varrho_{A_1 A_2} U_1^\dagger \otimes U_2^\dagger (H_1 \otimes \mathbf{I} + \mathbf{I} \otimes H_2) U_1 \otimes U_2] \\
&= \sum_{i=1}^{2} \min_{U_i} \mathrm{Tr}[\varrho_{A_i} U_i^\dagger H_i U_i] \\
&= \sum_{i=1}^{2} \min_{U_i} \mathrm{Tr}[U_i \varrho_{A_i} U_i^\dagger H_i].
\end{aligned}
$$

Therefore, we need to independently minimize $\mathrm{Tr}[U_i \varrho_{A_i} U_i^\dagger H_i]$ over all U_i. To obtain the minimum expectation value of the Hamiltonian H_i, each party applies the local unitary that rotates the local state vectors $\vec{r}_i$ along $-\hat{h}_i$ (see Fig. 1), which actually transform the states $\varrho_{A_i} = \tfrac{1}{2}(\mathbf{I} + \vec{r}_i \cdot \vec{\sigma})$ to the state $\tfrac{1}{2}(\mathbf{I} - |\vec{r}_i| \hat{h}_i \cdot \vec{\sigma})$. Thus we have

$$
\begin{aligned}
&\min_{U_i} \mathrm{Tr}[\varrho_{A_i} U_i^\dagger H_i U_i] \\
&= \mathrm{Tr}\left[\tfrac{1}{2}(\mathbf{I} - |\vec{r}_i| \hat{h}_i \cdot \vec{\sigma}) \tfrac{1}{2}(e_i^+ \mathbf{I} + e_i^- \hat{h}_i \cdot \vec{\sigma})\right] \\
&= \tfrac{1}{2}(e_i^+ - e_i^- |\vec{r}_i|).
\end{aligned}
$$

Hence the local extractable work will be

$$
W_{\mathrm{opt}}^L = \mathrm{Tr}(\varrho_{A_1 A_2} H) - \frac{1}{2} \sum_{i=1}^{2} [e_i^+ - e_i^- |\vec{r}_i|],
$$

which further implies

$$
W_{\mathrm{EG}} = \frac{1}{2} \sum_{i=1}^{2} [e_i^+ - e_i^- |\vec{r}_i|] - \left[\sum_{x,y=0}^{1} \lambda_{xy} e_{xy} \right]. \tag{18}
$$

For any pure product state, $|\vec{r}_1| = |\vec{r}_2| = 1$ and the passive state is $|00\rangle$, i.e., $\lambda_{00} = 1$, which, from the above expression, immediately implies $W_{\mathrm{EG}} = 0$, which is compatible with our previous observation that for any pure state the ergotropic gap is zero.

Nonzero EG for a mixed product state: Consider a mixed product state of the particular form $\rho_{AB} = \rho_A \otimes \rho_B = \mathrm{diag}\{\alpha, 1-\alpha\} \otimes \mathrm{diag}\{\beta, 1-\beta\} = \mathrm{diag}\{\alpha\beta, \alpha(1-\beta), (1-\alpha)\beta, (1-\alpha)(1-\beta)\}$ having the same Hamiltonian as in Eq. (16). Consider the case $\beta < \alpha < \tfrac{1}{2}$. The states ρ_i can also be written as $\tfrac{1}{2}[\mathbf{I} + \vec{r}_i \cdot \vec{\sigma}]$ with $|\vec{r}_1| = 1 - 2\alpha$ and $|\vec{r}_2| = 1 - 2\beta$. The passive state can then be written as $\lambda_{00} = (1-\alpha)(1-\beta)$, $\lambda_{01} = (1-\beta)\alpha$, $\lambda_{10} = \beta(1-\alpha)$, and $\lambda_{11} = \alpha\beta$. To transform the state, one must apply the unitary

that has the action $|00\rangle \to |11\rangle$, $|01\rangle \to |01\rangle$, $|10\rangle \to |10\rangle$, and $|11\rangle \to |00\rangle$. Clearly, this is an entangling unitary and cannot be realized locally. In this case, the EG turns out to be

$$W_{\text{EG}} = (\alpha - \beta)\big(e_2^0 - e_1^0\big) + (\alpha + \beta)e_1^1 + (\beta - \alpha)e_2^1. \quad (19)$$

In the following, we consider a few special classes of correlated states of a two-qubit system.

(a) Mixture of Bell states. The general form of this class is given by

$$\varrho_{\text{Bell}} = \sum_{i=1}^{4} p_i |\mathcal{B}_i\rangle\langle\mathcal{B}_i|,$$

where $\{|\mathcal{B}_i\rangle\}_{i=1}^4$ are four Bell states (one singlet and three triplets). As one can see, these states are already diagonal in the Bell basis with spectral values $\{p_i\}_{i=1}^4$. A suitable global unitary can be considered such that the populations $\{p_i\}_{i=1}^4$ can be arranged in descending order as follows: $\{p_{\max}, p'_{\max}, p''_{\max}, p_{\min}\}$, where $p_{\max}$ is the maximum of $\{p_i\}_{i=1}^4$, $p'_{\max}$ being the second maximum and so on such that $p_{\max} \geqslant p'_{\max} \geqslant p''_{\max} \geqslant p_{\min}$.

Since the marginal states are completely mixed, we have

$$W_{\text{EG}}(\varrho_{\text{Bell}}) = e_1^0\big(\tfrac{1}{2} - p_{\max} - p'_{\max}\big) + e_2^0\big(\tfrac{1}{2} - p_{\max} - p''_{\max}\big)$$
$$+ e_1^1\big(\tfrac{1}{2} - p''_{\max} - p_{\min}\big) + e_2^1\big(\tfrac{1}{2} - p'_{\max} - p_{\min}\big).$$

For the case $e_1^0 = e_2^0 = 0$ and $e_1^1 = e_2^1 = 1$, it takes the simpler form $W_{\text{EG}}(\varrho_{\text{Bell}}) = p_{\max} - p_{\min}$.

(b) Werner class of states. The generic form of this class is given by

$$\varrho_w = p|\psi^-\rangle\langle\psi^-| + (1 - p)\frac{\mathbf{I}}{2} \otimes \frac{\mathbf{I}}{2}.$$

The spectral values are $(\frac{1+3p}{4}, \frac{1-p}{4}, \frac{1-p}{4}, \frac{1-p}{4})$. For this entire class of states, the completely mixed marginals imply

$$W_{\text{EG}}(\varrho_w) = \tfrac{1}{2} p\big\{\big(e_1^1 - e_1^0\big) + \big(e_2^1 - e_2^0\big)\big\}, \quad (20)$$

which for the case $e_1^0 = e_2^0 = 0$ and $e_1^1 = e_2^1 = 1$ takes the value $W_{\text{EG}}(\varrho_w) = p$. It is known that the Werner class of states contains quantumness in correlations for all values of p except $p = 0$, which implies a nonvanishing ergotropic gap for all values of p except $p = 0$.

B. Multiparticle-multilevel systems

Here we generalize the calculation of a two-qubit state for arbitrary states of multiparty systems. Consider an N-particle state $\varrho_{A_1,\dots,A_N}$ and the Hamiltonian H, which is of the form of Eq. (1). Since there is no interaction term in the Hamiltonian, the expression of local ergotropy as in Eq. (5) turns out to be

$$W_{\text{opt}}^L = \max_{U \in \mathcal{LU}} \text{Tr}\big[\big(\varrho_{A_1,\dots,A_n} - U\varrho_{A_1,\dots,A_n} U^\dagger\big)H\big]$$

$$= \max_{U \in \mathcal{LU}} \text{Tr}\left[\big(\varrho_{A_1,\dots,A_n} - U\varrho_{A_1,\dots,A_n} U^\dagger\big)\sum_{i=1}^{N} H_i \bigotimes_{k\in\bar{i}} \mathbf{I}_k\right]$$

$$= \text{Tr}\left[\varrho_{A_1,\dots,A_n}\sum_{i=1}^{N} H_i \bigotimes_{k\in\bar{i}} \mathbf{I}_k\right]$$

$$- \min_{U \in \mathcal{LU}} \text{Tr}\left[\big(U\varrho_{A_1,\dots,A_n} U^\dagger\big)\sum_{i=1}^{N} H_i \bigotimes_{k\in\bar{i}} \mathbf{I}_k\right]$$

$$= \sum_{i=1}^{N} \text{Tr}\big[\varrho_{A_i} H_i\big]$$

$$- \min_{U \in \mathcal{LU}} \text{Tr}\left[\varrho_{A_1,\dots,A_n} U^\dagger\left(\sum_{i=1}^{N} H_i \bigotimes_{k\in\bar{i}} \mathbf{I}_k\right)U\right], \quad (21)$$

where $\varrho_{A_i} = \text{Tr}_{\bar{i}}(\varrho_{A_1,\dots,A_n})$ is the normalized reduced state of the ith subsystem; here $\text{Tr}_{\bar{i}}$ denotes the partial trace over all parties except i. The second term on the right-hand side of Eq. (21) can be written as

$$\min_{U \in \mathcal{LU}} \text{Tr}\left[\varrho_{A_1,\dots,A_n} U^\dagger\left(\sum_{i=1}^{N} H_i \bigotimes_{k\in\bar{i}} \mathbf{I}_k\right)U\right]$$

$$= \min_{\bigotimes_{i=1}^{N} U_i} \text{Tr}\left[\varrho_{A_1,\dots,A_n} \bigotimes_{i=1}^{N} U_i^\dagger\left(\sum_{i=1}^{N} H_i \bigotimes_{k\in\bar{i}} \mathbf{I}_k\right)\bigotimes_{i=1}^{N} U_i\right]$$

$$= \min_{\bigotimes_{i=1}^{N} U_i} \text{Tr}\left[\varrho_{A_1,\dots,A_n}\left(\sum_{i=1}^{N} U_i^\dagger H_i U_i \bigotimes_{k\in\bar{i}} \mathbf{I}_k\right)\right]$$

$$= \min_{U_i} \sum_{i=1}^{N} \text{Tr}\big[\varrho_{A_i}(U_i^\dagger H_i U_i)\big]$$

$$= \sum_{i=1}^{N} \min_{U_i} \text{Tr}\big[\varrho_{A_i}(U_i^\dagger H_i U_i)\big]. \quad (22)$$

Putting the expression of Eq. (22) into Eq. (21), we get

$$W_{\text{opt}}^L = \sum_{i=1}^{N}\big[\text{Tr}\big(\varrho_{A_i} H_i\big) - \min_{U_i} \text{Tr}\big(U_i\varrho_{A_i} U_i^\dagger H_i\big)\big]. \quad (23)$$

From the above expression, it is clear that local ergotropy is the sum of optimal work extracted by each party individually by applying a local unitary. Obviously, to extract optimal work, each party applies suitable unitaries that transform their reduced density matrix ϱ_{A_i} to the corresponding local passive state $\varrho_{A_i}^{\text{passive}}$. Therefore, we have

$$W_{\text{opt}}^L = \sum_{i=1}^{N}\big[\text{Tr}\big(\varrho_{A_i} H_i\big) - \text{Tr}\big(\varrho_{A_i}^{\text{passive}} H_i\big)\big]. \quad (24)$$

Substituting Eqs. (2) and (24) in Eq. (6), we have

$$W_{\text{EG}} = \sum_{i=1}^{N} \text{Tr}\big[\varrho_{A_i}^{\text{passive}} H_i\big] - \text{Tr}\big[\varrho_{A_1,\dots,A_N}^{\text{passive}} H\big]. \quad (25)$$

Thus for any state $\varrho_{A_1,\dots,A_N}$ the ergotropic gap is quantified by the difference of the internal energy of the global passive state from the sum of internal energies of the passive state of each parties' reduced state.

Local ergotropy is the sum of the optimal works extracted by each party, locally. As discussed earlier, switching on the suitable time-dependent local interacting field $V_i(t)$, each party can transform its initial reduced state to a local passive

state. However, as no global interaction is allowed, the global state $\varrho_{A_1,\ldots,A_N}$ does not in general evolve into its passive form $\varrho_{A_1,\ldots,A_N}^{\text{passive}}$, rather the global state $\varrho_{A_1,\ldots,A_N}$ evolves into a state $\eta_{A_1,\ldots,A_N}$, where $\text{Tr}_{\bar{i}}(\eta_{A_1,\ldots,A_N}) = \varrho_{A_i}^{\text{passive}} \forall \, i$. At this point, one can further ask whether the state $\eta_{A_1,\ldots,A_N}$ can be considered as a resource for work extraction under the constraint that no global interaction field among different parties can be applied. Interestingly, given many copies of the state $\eta_{A_1,\ldots,A_N}$, the answer is yes. This is because the composition of many copies of passive systems may not remain passive and exhibit a form of activation [22,23]. It has been shown that the only completely passive state is the Gibbs state (thermal state) from where no additional work can be extracted even with many (unbounded) copies [22]. Therefore, using the activation process, work can be extracted until each local passive state transforms into a completely passive state, i.e., a thermal state. In the following, we focus on global and local work extraction from a correlated quantum system when it is no longer isolated but rather put in contact with a thermal bath.

V. EXTRACTING WORK IN THE PRESENCE OF A THERMAL BATH

Consider the scenario in which the system is no longer isolated but each of the subsystems is put in contact with local baths of the same temperature β^{-1}. In this scenario, one can again be interested in the maximal work that can be extracted via global unitaries acting jointly on the system and the bath and also the amount of maximal extracted work via local unitaries acting jointly on the subsystems and the local bath. In such a scenario, it is well known that the extractable work is upper-bounded by the difference between initial and final (thermal) free energies [6,26,27], i.e.,

$$W_{\text{opt}}^{(\text{cb})} = F(\sigma) - F(\sigma^{\text{thermal}}),$$

where $F(\sigma) = \text{Tr}(H\sigma) - \beta^{-1} S(\sigma)$ and $\sigma^{\text{thermal}} = \frac{\exp(-\beta H)}{\mathcal{Z}}$, with $\mathcal{Z} = \text{Tr}[\exp(-\beta H)]$ being the partition function. The superscript "(cb)" indicates that work is extracted in contact with a bath. Thus we have

$$W_{\text{opt}}^{G(\text{cb})} = F(\varrho_{A_1,\ldots,A_N}) - F(\varrho_{A_1,\ldots,A_N}^{\text{thermal}}),$$

$$W_{\text{opt}}^{L(\text{cb})} = \sum_{i=1}^{n} \left[F(\varrho_{A_i}) - F(\varrho_{A_i}^{\text{thermal}}) \right].$$

In such a scenario, the difference between global and local extractable work is therefore

$$\Delta W^{(\text{cb})} = W_{\text{opt}}^{G(\text{cb})} - W_{\text{opt}}^{L(\text{cb})}. \tag{26}$$

Since we have $H = \sum_{i=1}^{n} H_i$, which further gives $\varrho_{A_1,\ldots,A_n}^{\text{thermal}} = \bigotimes_i \varrho_{A_i}^{\text{thermal}}$, we have

$$\Delta W^{(\text{cb})} = \beta^{-1} \left[\sum_{i=1}^{n} S(\varrho_{A_i}) - S(\varrho_{A_1,\ldots,A_n}) \right]. \tag{27}$$

For a two-particle system, the above expression reduces to the well-known *quantum mutual information* [48]. It is worthwhile to mention that this quantity can be nonzero for classically correlated states.

VI. DISCUSSIONS

Quantumness in correlations is a topic of fundamental research interest. It captures a more general type of correlations than the entanglement. As discussed earlier, for a bipartite scenario, *discord* is one of the quantities that measures the quantumness present in a correlation. The concepts of quantum correlations can easily be extended to a multiparticle scenario in the sense that a multipartite quantum state contains quantum correlations if it cannot be written as a convex combination of any orthonormal product basis of the subsystems pertaining to the whole system. In this work, we show that this nonclassical feature of correlations has an interesting manifestation in thermodynamics, particularly in work extraction from isolated systems. In this direction, we have proven that the presence of quantum correlations in a multipartite state sufficiently implies a nonzero difference between global and local ergotropy, which we call the ergotropic gap. To motivate local ergotropy, we have considered a situation in which the spatially separated parties are unable to implement any global interaction field. This leads to the concept of extracting optimal work by transforming the reduced states to corresponding passive states, applying local unitaries independently. As a future area of research, the following topics can be explored: First, it would be interesting to classify the states for which the ergotropic gap is nonzero. Second, it would be worthwhile to explore the concept of ergotropy in situations in which different parties are allowed to come together and global unitaries can be applied.

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