Attractor nonequilibrium stationary states in perturbed long-range interacting systems

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Isolated long-range interacting particle systems appear generically to relax to nonequilibrium states ("quasistationary states" or QSSs) which are stationary in the thermodynamic limit. A fundamental open question concerns the "robustness" of these states when the system is not isolated. In this paper we explore, using both analytical and numerical approaches to a paradigmatic one-dimensional model, the effect of a simple class of perturbations. We call them "internal local perturbations" in that the particle energies are perturbed at collisions in a way which depends only on the local properties. Our central finding is that the effect of the perturbations is to drive all the very different QSSs we consider towards a unique QSS. The latter is thus independent of the initial conditions of the system, but determined instead by both the long-range forces and the details of the perturbations applied. Thus in the presence of such a perturbation the long-range system evolves to a unique nonequilibrium stationary state, completely different from its state in absence of the perturbation, and it remains in this state when the perturbation is removed. We argue that this result may be generic for long-range interacting systems subject to perturbations which are dependent on the local properties (e.g., spatial density or velocity distribution) of the system itself.

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I. INTRODUCTION

Systems of large numbers of interacting particles are subject in their physical analysis to a fundamental distinction based on whether they are short range or long range, depending on the rapidity of the decay with separation of the two body interaction potential. The distinction in its canonical form arises from the presence or absence of the property of *additivity* of the macroscopic energy, which plays a fundamental role in equilibrium statistical mechanics. While most of familiar laboratory systems studied in physics are short range—notably any system constituted of neutral atoms or molecules—there are numerous examples also of long-range systems, ranging from self-gravitating systems in astrophysics and cosmology, to vortices in turbulent fluids, laser cooled atoms, and even biological systems (for a review, see, e.g., [1,2]). Study of various isolated long-range systems has shown that they evolve from generic initial conditions to microscopical non-Boltzmann equilibria, known as "quasistationary states" (QSSs) because they evolve towards the system's true statistical equilibrium on time scales which diverge with the number of particles (see, e.g., [3–8]). The evolution to such states appears not to be characteristic of all long-range interactions, but only of the subclass of these interactions for which the pair force (rather than pair potential) is nonintegrable at large distances (see [7,9]). As these systems remain in a QSS indefinitely in the thermodynamic limit, these states can be considered to be the fundamental relevant macroscopic equilibria of such systems, just as Maxwell-Boltzmann (MB) equilibria are for short-range systems. Theoretically they are understood to be stationary solutions of the Vlasov equation which in principle describes these systems in the relevant thermodynamic limit. Unlike MB equilibria, they are infinitely numerous at given values of the global conserved quantities, and the actual equilibrium reached depends strongly on the initial condition of the system.

A basic question which arises about QSSs in long-range interacting systems concerns the "robustness" of such states. They are strictly defined only for isolated Hamiltonian systems and the question is whether they continue to exist when the system is not exactly isolated, or exactly Hamiltonian, or both. For attempts to observe these intriguing equilibria in laboratory systems, which are necessarily perturbed by and coupled to the external world in some way, it is essential to know whether these states can be expected to survive. When the external perturbation is infinitesimal, it is possible to apply linear response theory for long-range interacting systems in QSSs [10–12], or even nonlinear response theory [13], but such approaches do not allow one to conclude about the effect of a coupling to an environment which persists indefinitely in time. Studies of toy models coupled to a thermal bath (see, e.g., [14]) show, unsurprisingly, that such a coupling sends the system to its thermal equilibrium, on a time scale which depends on the coupling. However, much more generally, it has been suggested on the basis of study of the one dimensional HMF model (see, e.g., [15,16]) that QSSs will disappear in the presence of any generic stochastic perturbation to the dynamics. A study of the effect of external stochastic fields with spatial correlation applied to the same model (see, e.g., [17,18]) shows however that interesting nonequilibrium steady states can be obtained in this case.

In this paper we explore the question of the robustness of QSS in long-range systems to *weak* perturbations using a paradigmatic toy model of long-range interactions—a one dimensional self-gravitating system—subjected to a particular

class of perturbations, which we refer to as "local internal perturbations." The perturbations to the purely self-gravitating dynamics occur when particles collide, and are described by simple collision rules for the colliding particles, which may be stochastic or deterministic. The perturbations we study are weak in the sense that we consider that the regime in which the characteristic time scales on which the macroscopic evolution of the long-range system is modified by the collisional effects is long compared to the characteristic time of the mean-field dynamics of the long-range interaction itself. The perturbations can thus be considered to model physical effects at very small scales which come into play *on such time scales*, e.g., due to very short scale forces and/or internal degrees of freedom. Different from the study of [17,18] there are therefore no external forces, and indeed we will build our collision rules so that they conserve total momentum. The choice of the two specific models we study is then guided by simplicity: once momentum conservation is imposed, a nontrivial collision rule in one dimension cannot conserve energy (as an elastic collision gives rise simply to an exchange of particle velocities). As we wish to focus here on the effect of the perturbations on QSSs, which have fixed energy, we choose to define collision rules which can conserve energy on average and lead (in principle) to a steady state. The two models we study are then simple choices with this property, corresponding to collision rules drawn from the literature on granular gases arising from simple considerations of energy balance. In the one dimensional self-gravitating model these collision rules are both very simple to implement numerically, and, as we will see, also admit a straightforward analytical development of kinetic theory in an appropriate mean-field limit.

Our analytical results on kinetic theory allow us to determine a well defined large N (mean field) limit for both models, and to determine the parameter range in which the perturbations are indeed weak (i.e., in which the evolution of the phase space density due to the inelastic collisions is on a time-scale long compared to the mean-field time scale for the long-range forces). Further it provides us with calculable analytical predictions for the short time evolution of certain macroscopic parameters away from an initial QSS. Our numerical study confirms that both models indeed show an evolution consistent with the scalings with N predicted by the kinetic theory, and in good agreement with the predicted short time behavior. Further our numerical results show evolution, from different initial conditions, through a family of states, until a truly stationary state is reached. The evolution is through QSSs and the final state is also a QSS: if the perturbation is removed the system remains in this state. Further the final stationary state in each case appears to be an attractor for the perturbed long-range system; i.e., starting from different initial conditions which evolve to very different QSSs the perturbation drives them all finally to the same nonequilibrium state. This "attractor" QSS is thus determined essentially by the detailed nature of the small perturbation and the long-range force itself.

The paper is organized as follows. We first define the two models we study of self-gravitating particles perturbed in a specific manner when particles collide. In the following section we describe analytical approaches to these models which are valid in an appropriate mean-field limit. These give rise to kinetic equations which allow us to determine a well defined large N limit. They also provide predictions for the early time evolution of the system. In the next section we present results of numerical studies of the two models. In our conclusions we compare our results with other relevant works in the literature, and comment on the possible generality of the behaviors we observe in a broad class of perturbed long-range systems.

II. MODELS

A. Sheet model

We consider a system of identical particles of mass m moving in a one dimensional space and interacting by a force independent of their separation, i.e., the force on a particle i due to a particle j is

$$F_{ij} = gm^2 \mathrm{sgn}(x_j - x_i), \tag{1}$$

where g is the interaction strength. The model is known as the "sheet model" because these particles in one dimension are equivalent to infinite, infinitely thin, parallel sheets moving in three dimensions interacting by Newtonian gravity, in which case $gm = 2\pi\Sigma G$, where G is Newton's constant and Σ is the mass per unit surface area of the sheets. This model dates back at least to the early study of Camm [19] and has been studied quite extensively by numerous authors since (see, e.g., [20–22] and references therein).

For a finite system of N particles, the total force acting on the ith particle at any time is simply given as

$$F_i = gm^2[N_i^+ - N_i^-], \tag{2}$$

where N_i^+ and N_i^- denote the numbers of particles on the right and on the left of the ith particle, respectively.

The dynamics of this system has been extensively studied (see, e.g., [20–22]) and shows that the time evolution of the system displays long-lived QSSs before reaching equilibrium. More precisely the relaxation time associated with QSSs is diverging with the system size $\tau \sim AN$, where the coefficient A depends strongly on the initial states [22].

We consider now two variants of this model, in which the particle collision still conserves the momentum but not the total energy. On the other hand, as we wish the system to be able to attain a stationary state at constant energy, we constrain the exact collision rules to allow this. Both collision laws are taken from simple models of granular systems which have been studied in the literature (see references below).

B. Model A: Collisions with random coefficients of restitution

Let us denote $v_{ij} = v_i - v_j$ the relative velocity of particles i and j which undergo a collision with precollisional velocities v_i and v_j. We adopt the rule that the postcollisional velocities, v_i^* and v_j^*, are given by

$$v_i^* = v_j + \frac{1-c}{2} v_{ij}, \quad v_j^* = v_i - \frac{1-c}{2} v_{ij}, \tag{3}$$

where the coefficient of restitution c is a *non-negative* random variable. Equivalently it corresponds to momentum conservation combined with the rule

$$v_{ij}^* = -c v_{ij}. \tag{4}$$

Granular models of this kind, incorporating a random coefficient of restitution, have been introduced by [23] in order to include the effect of energy injection in a vibrated two-dimensional granular gas.

In each collision the change of the kinetic energy is given by

$$\delta K = \frac{m}{4}[(v_{ij}^*)^2 - (v_{ij})^2] = \frac{c^2-1}{4}(v_{ij})^2, \tag{5}$$

i.e., the collision is inelastic if $0 \leqslant c < 1$ and superelastic if $c > 1$. In order that the applied perturbation may admit stationary states, we choose c from a bimodal probability distribution in which c takes two values, c_A and $\tilde{c}_A$, with equal weight, with $0 < c_A < 1$ and

$$\tilde{c}_A = \sqrt{2 - c_A^2}. \tag{6}$$

The latter relation imposes that, for a collision at the same initial relative velocity, the energy lost with $c = c_A$ is the same as the energy gained when $c = \tilde{c}_A$. Thus, in the ensemble of realizations of the stochastic perturbations, the average energy is constant, with the energy loss of the inelastic collisions ($0 < c_A < 1$) balanced by the energy gain in superelastic collisions ($\tilde{c}_A > 1$). We expect that in this case the system may be able to reach a stationary state with constant energy (modulo finite N fluctuations).

C. Model B: Inelastic collisions with energy injection

In this model the self-gravitating particles undergo collisions specified by the following rule:

$$\begin{aligned} v_i^* &= v_j + \frac{1-c_B}{2}v_{ij} - \epsilon_{ij}\Delta, \\ v_j^* &= v_i - \frac{1-c_B}{2}v_{ij} + \epsilon_{ij}\Delta, \end{aligned} \tag{7}$$

where the constant c_B has a fixed value in the range $0 < c_B < 1$ (i.e., as for an inelastic collision), Δ is a positive constant (with dimensions of velocity) and $\epsilon_{ij} = \mathrm{sgn}(v_{ij})$. The collision manifestly still conserves total momentum, and corresponds to

$$v_{ij}^* = -\mathrm{sgn}(v_{ij})[c_B|v_{ij}| + 2\Delta], \tag{8}$$

and therefore the energy change is

$$\delta K = m\left[\frac{c_B^2-1}{4}v_{ij}^2 + c_B\Delta|v_{ij}| + \Delta^2\right]. \tag{9}$$

The term in Δ in the collision rule thus leads to an energy injection, which can be smaller or larger than the energy loss due to the inelastic term: more precisely, the collision leads to an energy gain if $|v_{ij}| < v_0$, and an energy loss if $|v_{ij}| > v_0$, where

$$v_0 = \frac{2\Delta}{1-c_B} \tag{10}$$

is the value of the relative velocity for which the collision is elastic.

This collision rule is the one dimensional version of that introduced in two dimensions by [24] in a phenomenological model of quasi-two-dimensional experiments of agitated granular particles: the particles are confined between two horizontal plates, and the vibrating bottom plate transfers the kinetic energy to the particles by collisions [25–27]. In this quasi-two-dimensional geometry, the period of the vertical vibrations is much shorter than the typical time scale of the horizontal dynamics. The collision rule, Eq. (7), then represents a time coarse-grained description of the energy transfer of particle-bottom plate collisions to horizontal particle-particle collisions.

In this paper we have chosen this collision rule simply because it provides a simple way, quite different to that in the first model, to obtain a nontrivial two body collision rule which can be expected to lead to a stationary state. More specifically if the particle velocities at collisions are assumed to be uncorrelated, the kinetic energy of the system gives a direct measure of the typical relative velocity of colliding particles: $\langle (v_i - v_j)^2 \rangle = 2\langle v_i^2 \rangle$. The kinetic energy would then be expected to be driven towards a value of order Nmv_0^2, as above this energy scale energy will be dissipated while below it energy will be injected. Indeed in the case in which gravity is turned off, and the particles are enclosed in a box with reflecting walls, if all particles have velocity $\pm v_0/2$ all collisions are elastic and the velocity distribution does not evolve at all.

Model B is in fact microscopically deterministic, while model A is explicitly stochastic. One other notable difference is that the phase space volume occupied by particles involved in a collision strictly contracts in model B, while it can contract or increase in model A depending on whether the collision is inelastic or elastic. Indeed for a collision with coefficient of restitution c in either model we have $dv_i^* dv_j^* = c dv_i dv_j$ which is always a contraction in model B. This property leads to distinctive features of the long time behavior of this model which we observe below.

III. KINETIC THEORY

A. Mean field limit without collisions

For the purely self-gravitating model the dynamics in the appropriate large N mean field limit is described by the Vlasov equation [9,28,29]. This limit is obtained by taking $N \to \infty$ at fixed values of the total system mass M and energy E. Denoting the mass density in phase space, $f(x,v,t)$, the Vlasov equation reads

$$\partial_t f(x,v,t) + J_V[f] = 0, \tag{11}$$

where $J_V[f]$, the Vlasov operator, is

$$J_V[f] = v\partial_x f(x,v,t) + \bar{a}(x,t)\partial_v f(x,v,t), \tag{12}$$

where $\bar{a}(x)$ is the mean field acceleration given by $\bar{a}(x,t) = g\int \mathrm{sgn}(x-x')f(x',v',t)dx'dv'$. The mass density obeys the normalization condition

$$\iint dxdvf(x,v,t) = M, \tag{13}$$

where M is the total mass of the system.

QSSs are interpreted as stable stationary solutions of Eq. (11). There is an infinite number of such solutions, including as a particular case the statistical equilibrium of this model (see below).

B. Model A

1. Collision operator in Boltzmann approximation

The Vlasov equation can be derived starting from the BBGKY hierarchy and making the approximation that the two point correlations can be neglected. The collisions in our model can be treated in the same approximation, and are then described by a canonical Boltzmann operator. We thus expect our model in the mean field limit to be described by a kinetic equation

$$\partial_t f(x,v,t) + v.\partial_x f(x,v,t) + \bar{a}[f](x,t).\partial_v f(x,v,t) = \sum_q P(q) J_q[f,f](x,v,t), \tag{14}$$

where, for convenience, we introduce the parameter $q = \frac{1-c}{2}$ to characterize binary collisions with a coefficient of restitution equal to c, and $J_q[f,f](x,v,t)$ is a collision operator accounting for such collisions which are assumed to occur independently with a probability $P(q)$. Initially we will leave $P(q)$ undetermined and then replace it with the specific bimodal form for Model A at the appropriate point below.

Assuming the particles to be pointlike, the collision operator is a homogeneous Boltzmann operator accounting for binary collisions, which is the sum of two contributions:

$$J_q[f,f](x,v,t) = G_q(x,v,t) - L(x,v,t)\,, \tag{15}$$

where G_q is the gain term corresponding to collisions where a particle has a postcollisional velocity equal to v,

$$G_q(x,v,t) = \frac{N}{M}\iint dv'dv''|v'-v''|f(x,v',t)f(x,v'',t) \times \delta[v - qv' - (1-q)v''], \tag{16}$$

and $L(x,v,t)$ is the loss term corresponding to collisions where a particle with a velocity v undergoes a collision at time t,

$$L(x,v,t) = \frac{N}{M} f(x,v,t)\int dv'|v'-v|f(x,v',t). \tag{17}$$

Note that the loss term does not depend explicitly on the coefficient of restitution, and indeed we can write

$$J_q[f,f](x,v,t) = G_q(x,v,t) - G_0(x,v,t). \tag{18}$$

Let us introduce a series expansion of the δ function in terms of the parameter q:

$$\delta[v - qv' - (1-q)v''] = \sum_{n\geqslant 0}\frac{[q(v''-v')]^n}{n!}\delta^{(n)}(v-v''), \tag{19}$$

where $\delta^{(n)}$ denotes the nth derivative of the δ function.

The Boltzmann operator is then expressed as

$$J_q[f,f](x,v,t) = \frac{N}{M}\sum_{n\geqslant 1}\iint dv'dv''|v'-v''|\frac{q^n(v''-v')^n}{n!} \times \delta^{(n)}(v-v'')f(x,v',t)f(x,v'',t). \tag{20}$$

To determine whether the parameters characterizing the collisions can be rescaled with N so that the collision term remains well defined (and nontrivial) in the mean field (Vlasov) limit, we consider the limit $q \to 0$ of the model, i.e., the *quasielastic* limit. Physically this is clearly the relevant limit: to obtain an N independent evolution in presence of the collisions on time scales characterizing the mean field dynamics (e.g., the time a particle typically takes to cross the system) one must clearly "compensate" the effect of the divergent growth of the number of collisions with N by making the effect of each collision arbitrarily weak.

2. Expansion of kinetic equation

Inserting Eq. (20) in the right hand side of Eq. (14) and integrating by parts term by term we obtain (following [30–32]) the collision operator given as a series of differential operators:

$$J_A[f] = \frac{N}{M}\sum_{n\geqslant 1}\left[\langle q^n\rangle\partial_v^n(f(x,v,t)a_n[f](x,v,t))\right], \tag{21}$$

with

$$a_n[f](x,v,t) = \int |v'-v|\frac{(v'-v)^n}{n!}f(x,v',t)dv', \tag{22}$$

where the brackets $\langle\ldots\rangle$ indicate an average over the probability distribution $P(q)$. Note that $a_1[f](x,v,t)$ has a simple physical meaning as the opposite of the average effective force due to the collisions [33,34].

Let us consider now the specific $P(q)$ of model A:

$$P(q) = \tfrac{1}{2}\delta(q-q_A) + \tfrac{1}{2}\delta(q-\tilde{q}_A), \tag{23}$$

where $q_A = \frac{1-c_A}{2}$ and, from the average energy conserving condition Eq. (6),

$$\tilde{q}_A = \frac{1-\sqrt{1+4(1-q_A)q_A}}{2}. \tag{24}$$

Expanding in q_A (as $q_A \to 0$ in the quasielastic limit) we have

$$\tilde{q}_A = -q_A + 2q_A^2 - 4q_A^3 + O\left(q_A^4\right), \tag{25}$$

and thus, to leading order in powers of q_A we obtain

$$\langle q^n\rangle = n q_A^{n+1} \quad n \text{ odd}$$
$$\langle q^n\rangle = q_A^n \quad n \text{ even}.$$

Defining now

$$\gamma_A \equiv q_A\sqrt{N} = \frac{(1-c_A)\sqrt{N}}{2} \tag{26}$$

we have, at leading order in $1/N$,

$$\langle q\rangle = \langle q^2\rangle = \frac{\gamma_A^2}{N},$$
$$\langle q^3\rangle = 3\langle q^4\rangle = \frac{3\gamma_A^4}{N^2},$$

while, for $n > 4$, $\langle q^n\rangle$ decreases with N more rapidly than $1/N^2$. Thus taking the mean-field limit $N \to \infty$ at constant γ_A, the full expansion of the collision term reduces to the sum of the two first derivatives of $f(x,v,t)$:

$$J_A[f] = \frac{\gamma_A^2}{M}[\partial_v(f(x,v,t)a_1[f](x,v,t)) + \partial_v^2(f(x,v,t)a_2[f](x,v,t))]. \tag{27}$$

Note that the nonlinear structure of the integral collision operator Eq. (21) is conserved because the velocity-dependent functions $a_n(v)$ are functionals of $f(x,v,t)$.

We note that the crucial relation leading to the result (27) for this model is $\langle q \rangle = \langle q^2 \rangle$, which is simply the condition of average energy conservation. Indeed from Eq. (5) it follows that the energy change in a collision at any given relative velocity is proportional to $q - q^2$. Thus the same mean-field limit for the kinetic theory will be obtained for any variant of this model in which $P(q)$ is such that energy is conserved on average.

We note further that in this derivation we have assumed implicitly that $f(v)$ has the convergence properties required for the validity of the Taylor expansion, which requires clearly sufficiently rapid decay of $f(v)$ at large v to ensure the finiteness of the coefficients. Indeed we see that while the finiteness of Eq. (16) requires only that $f(v)$ be integrable at large $|v|$ [i.e., $f(v) \sim 1/|v|^\alpha$ with $\alpha > 2$], the definiteness of the expression Eq. (27) requires $\alpha > 4$. As we will discuss below the latter assumption turns out to break down at longer times in the model.

3. Evolution of moments of velocity distribution

We now discuss some properties of the collision operator by considering the evolution of the moments of the velocity distribution. Multiplying both sides of the kinetic equation by v^n, and integrating over v, we obtain

$$\frac{d}{dt}[\rho(x,t)\overline{v^n}(x,t)] + \int dv\, v^n J_V[f] = \int dv\, v^n J_A[f], \quad (28)$$

where $\rho(x,t) = \int dv f(x,v,t)$ is the spatial mass density, and $\overline{v^n}(x,t)$ is the nth moment of the velocity distribution at x, i.e.,

$$\overline{v^n}(x,t) = \int dv\, v^n p_x(v,t), \quad (29)$$

where

$$p_x(v,t) = \frac{f(x,v,t)}{\rho(x,t)}.$$

It is straightforward to show, either directly from the exact collision operator, or for each of the two terms in Eq. (28), that

$$\int dv J_A[f] = 0, \quad \int dv\, v J_A[f] = 0, \quad (30)$$

which express, respectively, the conservation of particle number and conservation of momentum in the collisions. Indeed it is straightforward to show that the left-hand side of Eq. (28) corresponds for $n = 0$ to the continuity equation, and $n = 1$ to the Euler equation.

For the case $n = 2$, integration by parts using Eq. (27) gives

$$\int dv\, v^2 J_A[f] = 2\frac{\gamma_A^2}{M} \int dv (v a_1[f] + a_2[f]) f(x,v,t), \quad (31)$$

from which it follows using Eq. (22) that

$$\int dv\, v^2 J_A[f] = 0, \quad (32)$$

which expresses the conservation of the kinetic energy by the collisions. Thus the local pressure $\rho\overline{v^2}$ can change only due to the mean field gravitational force (through the Vlasov flow term $J_V[f]$). Note that while the vanishing of the zero and first moments hold for any $P(q)$, it can be verified from Eq. (21) that the second moment vanishes only if $\langle (q - q^2) \rangle = 0$ which is, as noted above, just the condition of average energy conservation.

For any $n \geqslant 2$ it is straightforward to show that

$$\begin{aligned} &\int dv\, v^n J_A[f] \\ &= \frac{n\gamma_A^2}{4M} \int dv \int dv' |v - v'|(v - v') \\ &\quad \times [(n-3)(v^{n-1} - v'^{n-1}) - (n-1)vv'(v^{n-3} - v'^{n-3})] \\ &\quad \times f(x,v,t) f(x,v',t), \end{aligned} \quad (33)$$

from which we recover the previous result for $n = 2$, and further find that the first nonzero moment is for $n = 4$, and it has the simple expression

$$\int dv\, v^4 J_A[f] = \frac{\gamma_A^2}{M} \int dv \int dv' |v - v'|^5 f(x,v,t) f(x,v',t). \quad (34)$$

The fact that the right hand of this expression is strictly positive has an important consequence: if this kinetic equation is valid, the system cannot reach a stationary state. Or, conversely, if the system reaches a stationary state, it must be such that the assumptions necessary for the derivation of the kinetic equation break down. As noted above we will see that our numerical study shows that the system generically evolves to such a regime. In fact, we will see that when the system reaches a stationary state it is characterized by a non-Gaussian velocity distribution with tails decaying as a power law. Indeed the estimated exponent of the velocity distribution is such that the fourth moment is not defined and thus the above equation is not applicable in this state.

On the other hand if the system is prepared in an initial state which is a QSS, and which does satisfy the conditions necessary for the validity of the derivation leading to Eq. (27), we can use Eq. (34) to infer nontrivial information about the temporal evolution at sufficiently short times. Indeed in this case

$$\frac{d}{dt}(\rho(x,t)\overline{v^4}(x,t)) = \frac{\gamma_A^2}{M} \int dv \int dv' |v - v'|^5 f(x,v,t) f(x,v',t). \quad (35)$$

In practice we measure the integrated quantity, i.e., the rescaled fourth moment of velocity (the kurtosis), defined by

$$\beta_2(t) = \frac{M \int dx \int dv\, v^4 f(x,v,t)}{(\int dx \int dv\, v^2 f(x,v,t))^2}, \quad (36)$$

i.e., the fourth rescaled moment of the full velocity distribution $P(v,t) = \frac{1}{M} \int dx f(x,v)$. For a Gaussian distribution, the value of β_2 is constant, independent of the temperature of the system and equal to 3. The time evolution $\beta_2(t)$ characterizes deviation of the distribution from a Gaussian distribution.

Integrating Eq. (35) over x and, assuming that the time evolution of the second moment (i.e., of the total kinetic

energy) can be neglected, we have

$$\frac{d\beta_2(t)}{dt} \simeq \frac{\gamma_A^2 \int dx \int dv \int dv' |v-v'|^5 f(x,v,t) f(x,v',t)}{(\int dx \int dv\, v^2 f(x,v,t))^2}. \tag{37}$$

We will test this prediction below for the case where the initial state is the statistical equilibrium of the system.

C. Model B

Following exactly the same approach we write the kinetic equation, in the Boltzmann and mean field approximations, for this model as

$$\partial_t f(x,v,t) + J_V[f] = J_B[f,f](x,v,t), \tag{38}$$

where $J_B[f,f](x,v,t)$ is the collision operator with the same structure as Eq. (15), but with the gain operator now given by

$$G_q(x,v,t) = \frac{N}{M} \iint dv' dv'' |v'-v''| f(x,v',t) f(x,v'',t) \times \delta[v - q v' - (1-q)v'' - \mathrm{sgn}(v'-v'')\Delta], \tag{39}$$

where $q = \frac{1-c_B}{2}$ is a fixed positive parameter less than unity.

Following the same arguments as above, it is clear that to obtain a collision operator $J_B[f,f](x,v,t)$ which is independent of N in the mean field limit, we must consider a quasielastic limit, with $q \to 0$ and $\Delta \to 0$ as $N \to \infty$. Further if the evolution to a stationary state is to be described in such a limit, the energy of this state, which we have inferred must be $\sim M v_0^2$, must be extensive (like the energy in the mean field limit), and therefore v_0 must be taken independent of N. Now, since $\Delta = q v_0$, holding v_0 fixed and taking $q \to 0$ indeed defines a quasielastic limit.

Proceeding as in the previous case, we perform again an expansion of the Boltzmann operator in powers of q about $q = 0$. This gives

$$J_B[f,f] = \frac{N}{M} \sum_{n \geqslant 1} q^n \partial_v^n [a_n(x,v,t) f(x,v,t)], \tag{40}$$

where

$$a_n[f](x,v,t) = \frac{1}{n!} \int |v'-v| [v'-v-\mathrm{sgn}(v'-v)v_0]^n \times f(x,v',t) dv'. \tag{41}$$

Defining now

$$\gamma_B \equiv qN = \frac{(1-c_B)N}{2}, \tag{42}$$

and taking $N \to \infty$ at constant γ_B, we obtain a finite limit for the collision operator which corresponds to the mean field limit. In this case only the leading linear term of the expansion contributes, and the effect of collisions corresponds to the presence of an effective velocity dependent force $-a_1(x,v)$ $a_1(x,v)$:

$$J_B[f] = \frac{\gamma_B}{M} [\partial_v (f(x,v,t) a_1[f](x,v,t))]. \tag{43}$$

We note that the diffusive term which was nonzero in the mean-field limit of model A thus vanishes for model B.

As for model A, we can calculate the velocity moments of the collision operator $J_B[f]$. The first two moments again vanish as a consequence of conservation of particle number and momentum, while

$$\int dv v^2 J_B[f] = 2\frac{\gamma_B}{M} \int dv v a_1[f] f(x,v,t). \tag{44}$$

Different from model A, this is not zero, in general: indeed the model does not necessarily conserve energy on average. On the other hand we expect the system to be able to reach a stationary state in which the collision operator is zero, and in this case Eq. (44) will vanish.

IV. NUMERICAL RESULTS

A. Simulation method and units

1. Code

The molecular dynamics of a one dimensional self-gravitating model is conveniently simulated using an event-driven algorithm as between particle collisions trajectories can be calculated explicitly. Such an algorithm is exact up to the machine rounding error in computing the solutions of quadratic equations giving the collision times (see [20,22,35] and references therein). Further the algorithm may be sped up using a "heap structure" [36] and by updating at each step only the positions of particles involved in each collision. It is straightforward to modify this algorithm to implement the simple collision rules of our two models instead of elastic collisions (equivalent to particle crossings). We use a modified version of the code described in [32] (and greater detail in [35]) [37]. Model A is characterized by the choice of the parameter $c_A < 1$, and each collision is then chosen with probability 0.5 to be inelastic (with $c = c_A$) or superelastic (with $c = \tilde{c}_A$). Model B is characterized fully by the values of c_A and Δ.

2. Initial conditions

For both models we study evolution starting from two kinds of initial conditions:

(i) "Rectangular waterbag": particles are randomly distributed with uniform probability in a rectangular region of phase space, $[-L_0/2, L_0/2] \times [-V_0/2, V_0/2]$. For the case of gravity only, which has no characteristic length scale, this is a one parameter family of initial conditions which may be conveniently characterized fully by the initial virial ratio R_0 (and the particle number N), where the virial ratio R (at any time) is defined as

$$R = \frac{2K}{U}, \tag{45}$$

where K is the kinetic energy and U is the potential energy. A virialized system thus has $R = 1$.

(ii) Thermal equilibrium: the statistical equilibrium of the purely self-gravitating system in the microcanonical and canonical ensemble has been derived for any finite N by [38] and its mean field limit (derived earlier by [19]) is

$$f(x,v) = \frac{M}{2\sqrt{\pi}\sigma\Lambda} e^{-v^2/\sigma^2} \mathrm{sech}^2\left(\frac{x}{\Lambda}\right), \tag{46}$$

with $\sigma^2 = \frac{4E}{3M}$, $\Lambda = \frac{4E}{3gM^2}$, and E is the total energy.

3. Units

For our study the only dimensional parameters of relevance are the time, and, in model B, the velocity (because of the parameter Δ). A natural choice of units for both are those characteristic of the mean field dynamics. For the time unit we choose

$$\tau_{dyn} = \frac{1}{\sqrt{g\rho_0}}, \tag{47}$$

where ρ_0 is the initial mass density of the system, and for the velocity

$$v_{dyn} = \sqrt{\frac{2E_0}{3M}}, \tag{48}$$

where E_0 is the initial energy. With this definition v_{dyn}^2 is the velocity dispersion of a virialized system with energy E_0.

Previous studies (see, e.g., [22]) of evolution from this first class of initial conditions for the self-gravitating system show that the system evolves, in a time of order 10–100 τ_{dyn}, to QSSs of which the properties depend strongly on R_0. At longer times, of order $(10^2$–$10^3)N\tau_{dyn}$, the different QSSs all relax to thermal equilibrium [4].

4. Additional macroscopic observables

To monitor in a simple way the evolution of the global properties of the system, we measure in addition to the energy and the virial ratio that of a global parameter which is a simple measure of the "phase space entanglement" of the state of the system:

$$\phi_{11} = \frac{\langle |x\, v| \rangle}{\langle |x| \rangle \langle |v| \rangle} - 1. \tag{49}$$

As shown in [22] the only stationary solution of the Vlasov equation which is a separable function of position and velocity is that corresponding to thermal equilibrium. Thus if ϕ_{11} is constant and nonzero this indicates that the system is in a QSS distinct from thermal equilibrium, and its amplitude can be taken roughly as a measure of "proximity" to the latter. We also monitor the evolution of the kurtosis β_2 as defined in Eq. (36).

B. Model A

As seen in Sec. III B, the relevant parameter characterizing the perturbations due to collisions in the mean field limit is γ_A defined in Eq. (26). If this limit describes accurately the dynamics of the system, the term arising from the perturbations is proportional to γ_A^2, and thus the time scale on which they are expected to modify the evolution of the system is $\sim\tau_{dyn}/\gamma_A^2$. In order to study the desired range of weak perturbation, and assess the validity of the mean-field limit, we will thus consider small values of γ_A and vary N keeping γ_A fixed. We report here results for $\gamma_A = 0.03$ and $\gamma_A = 0.1$, and for N in the range $N = 128$ to $N = 1024$. We average our results over a large number of realizations in each case.

1. Evolution of energy and virial ratio

We have constructed this model so that the fluctuations in energy of the system should become arbitrarily small for sufficiently large N. Indeed in the mean field limit we have derived above the energy is exactly conserved, and this limit evidently thus does not describe effects associated with the energy fluctuations at finite N. In our numerical study, at finite N, we therefore need to check whether, on the time scale simulated, the energy fluctuations are indeed small. Figure 1 shows the evolution as a function of time of the mean energy, for a model with $\gamma_A = 0.03$, in an ensemble of realizations starting from waterbag initial conditions with the different indicated N, over a time scale roughly an order of magnitude greater than τ_{dyn}/γ_A^2. We see, on these time scales, that in all cases, the ensemble averaged energy is indeed very close to constant, but (lower panel) the variance of the normalized energy (i) grows almost linearly in time, as indicated by the dashed straight lines and (ii) monotonically decreases as N increases. Thus, as we would expect, taking N sufficiently large at any given time, we can in principle converge to arbitrarily precise conservation of the energy in any single realization. In our numerical simulations at (relatively small) finite N, we have, however, significant finite N fluctuations developing in all cases at times a few times τ_{dyn}/γ_A^2, so we might anticipate that such effects may begin to play a significant role on these time scales.

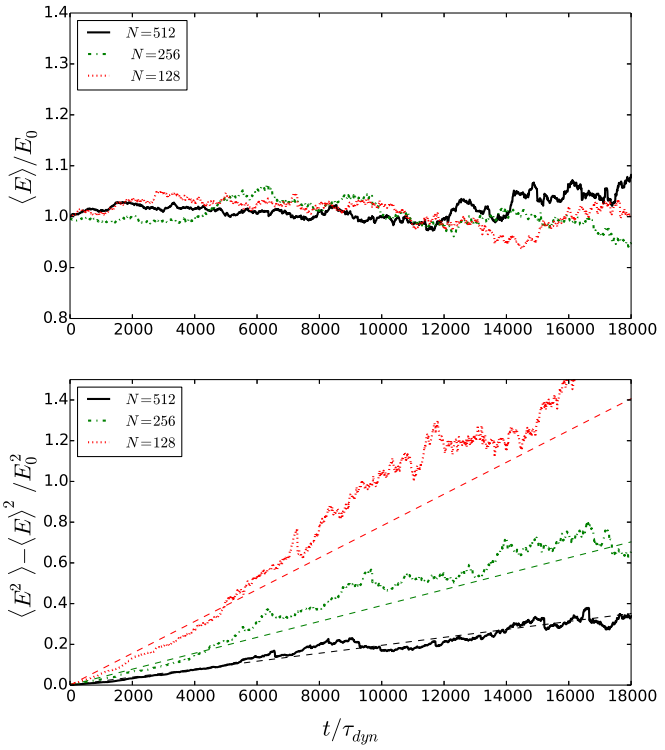

FIG. 1. Model A. Top panel: Dimensionless energy $\langle E\rangle/E_0$ (E_0 is the initial energy). Bottom panel: dimensionless variance $(\langle E^2\rangle - \langle E\rangle^2)/E_0^2$ vs dimensionless time t/τ_{dyn} averaged over 100 realizations with $\gamma_A = 0.03$, of rectangular waterbag initial conditions for $N = 128,256,512$ particles.

Figure 2 shows the evolution of the mean (upper panel) and standard deviation (lower panel) of the virial ratio in the same ensemble of simulations as in the previous figure. Because of the equilibrium initial conditions, the system remains always, as we would expect, very close to virialized, with only finite

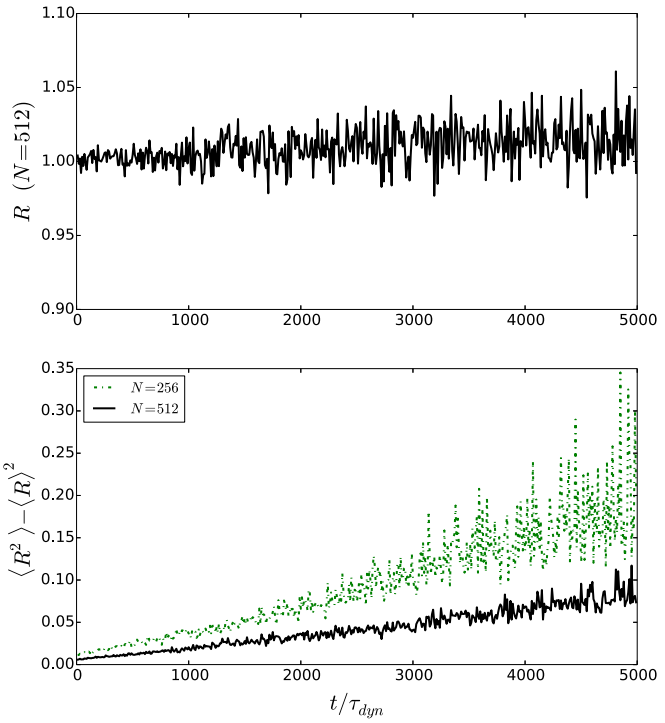

FIG. 2. Model A. Top panel: virial ratio R as a function of time t/τ_{dyn} averaged over 100 realizations with equilibrium initial conditions and for $\gamma_A = 0.03$ and $N = 512$. Bottom panel: variance of the virial ratio for $N = 512$ and $N = 256$ with the same initial conditions.

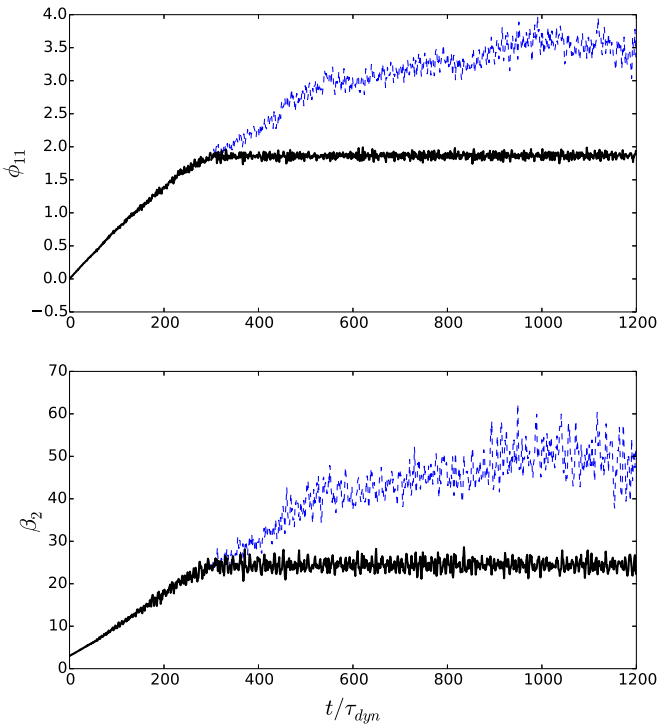

FIG. 3. Model A: Evolution of the entanglement parameter ϕ_{11} (top panel) and of the dimensionless kurtosis β_2 averaged over 100 realizations with $\gamma_A = 0.1$ and equilibrium initial conditions for $N = 1024$. The black curves correspond to a modified simulation in which the perturbation is switched off for $t \geqslant 300\tau_{dyn}$.

small N fluctuations which (lower panel) clearly decrease monotonically as N increases. We will see below that the nontrivial time dependence of these fluctuations are a reflection of the macroscopic evolution of the system on the same time scales.

2. *Macroscopic evolution due to perturbation*

Figure 3 shows the time evolution of the entanglement parameter ϕ_{11} and the rescaled kurtosis β_2 (both defined above), averaged over 100 realizations of equilibrium initial conditions with $N = 1024$ particles. As expected, at $t = 0$, $\phi_{11} = 0$, and $\beta_2 = 3$: the equilibrium state phase space density is a separable function of space and velocity, and the velocity distribution has a Maxwell-Boltzmann shape. The single curve plotted up to $t = 300\tau_{dyn}$, and the upper (dashed) curve starting from this time, represent the evolution in model A with $\gamma_A = 0.1$. Note that in line with what would be expected from mean field theory, the characteristic time for the evolution is about ten times shorter than in the data in the previous figures, for the case $\gamma_A = 0.03$. The lower (filled) curve corresponds to the case in which the perturbation of model A is "switched off" at the time $t = 300\tau_{dyn}$, i.e., it corresponds to the evolution of a purely self-gravitating system starting from this time. The stabilization of the value of the parameters—which is likewise observed at any time when the perturbation is switched off—indicate that the evolution induced by the perturbation is through a continuum of QSS, i.e., at all times the system remains very close to a stationary *and stable* state of the Vlasov equation. This is indeed what we would anticipate because the perturbation modifies the system macroscopically on a time scale which is long compared to τ_{dyn}. Further we note that both time scales are also short compared to the time scale [of order $(10^2 - 10^3)\, N\tau_{dyn}$, see [4]] on which finite N corrections to the Vlasov dynamics play a role, causing the system to evolve to statistical equilibrium. What is particularly noteworthy of the observed behaviors is the following:

(i) The effect of the small perturbation is not, at sufficiently long times, perturbative: the system is clearly progressively driven very far from its initial state. This is qualitatively similar to the effect of finite N effects on QSSs of the purely self-gravitating system, but contrasts strongly to the behavior of familiar short-range systems subjected to a small perturbation which evolve to an out of equilibrium state close to the initial one (in the sense that all macroscopic quantities are changed perturbatively).

(ii) The parameter ϕ_{11}, which in the absence of the perturbation would remain stable at its initial value $\phi_{11} = 0$, evolves to a final value $\phi_{11} \approx 2$–3 depending on N, i.e., towards a state in which the correlation between position and velocity is ever stronger. The perturbation clearly drives the system away from statistical equilibrium.

3. *Validity of mean field kinetic theory*

Let us now consider the degree to which the evolution in our simulations of the system are described well by the mean-field limit of the kinetic equations derived above. The most basic prediction of this theory is that, when we adopt the

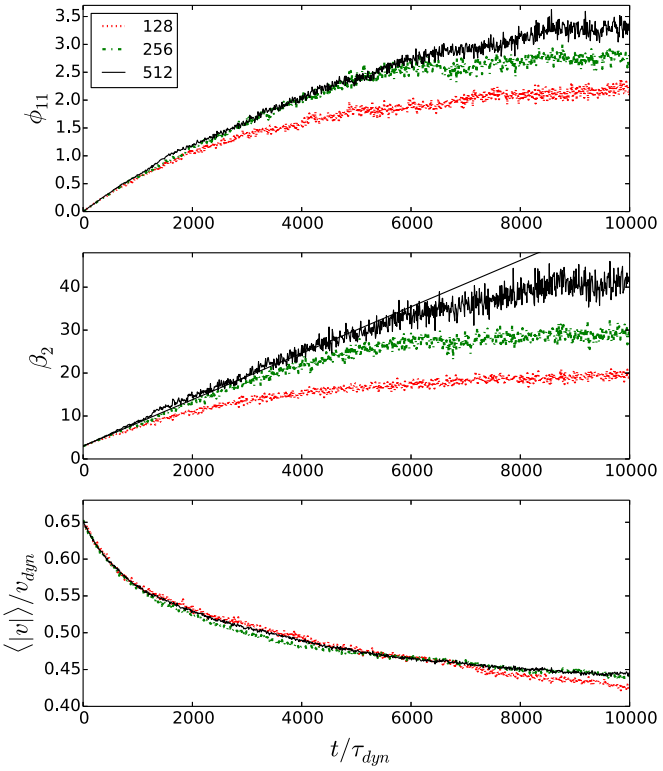

FIG. 4. Model A: Entanglement parameter ϕ_{11}, dimensionless kurtosis of the velocity distribution β_2 and $\langle|v|\rangle/v_{dyn}$, averaged over 300 realizations, as functions of time t/τ_{dyn} for $\gamma_A = 0.03$ and $N = 128,256,512$ (from top to bottom).

associated scalings of the parameters with N, we observe an evolution which is independent of N. The upper two panels of Fig. 4 show respectively the evolution of ϕ_{11} and β_2, in each case averaged over 300 realizations of the model with $\gamma_A = 0.03$ starting from equilibrium initial conditions, and for the different particle numbers indicated: $N = 128,256,512$. In both cases we observe that at sufficiently early times there is a nontrivial evolution of the system which is very well superimposed for the different N, indicating the validity of the mean field theory. Further the evolution of β_2 at these times agrees well with that predicted by the mean field kinetic theory at early time, shown as a straight line obtained using Eq. (37) with $f(x,v)$ taken equal to the initial thermal equilibrium phase space density Eq. (46). At longer times however we see that the evolution changes: for each N, the evolution breaks away, at a time scale which appears roughly to increase with N, from the common behavior, and shows on a similar time scale a tendency to reach a plateau, indicating in principle the attainment of a stationary state. By measuring the velocity and spatial distributions below we will verify that this is indeed the case, and in so doing also find the explanation for the N dependence and the noisiness of the evolution of ϕ_{11} and β_2 at longer times: the stationary state to which the system evolves is in fact one for which these particular macroscopic variables become ill defined in the mean field limit. This is the case because these states are characterized by velocity and spatial distributions which have slowly decaying power-law tails at long distances, for which both $\langle|x|\rangle$ and $\langle v^4\rangle$ diverge. Their values in a finite simulation are then regulated by the cutoff due to the finite particle number, and are thus highly fluctuating. Shown in the lower panel of Fig. 4 is the evolution of $\langle|v|\rangle$, which, in contrast, is a well defined quantity in the final state. In this case we see that the evolution for different N in the mean-field scaling agrees well, and fluctuates little, right up to the time at which the stationary state is attained. The small deviation at the latest times for $N = 128$ is a result of the large fluctuations of the total energy in this case (see Fig. 1).

4. Dependence on initial conditions

Figure 5 shows the evolution of ϕ_{11} and $\langle|v|\rangle/v_{dyn}$ for $\gamma_A = 0.03$ and $N = 512$ and with two different initial conditions: thermal equilibrium, and two rectangular waterbags, $R_0 = 1$ and $R_0 = 0.01$. The number of realizations are 300, 100, 200 respectively. At long times, these quantities evolve towards the same mean value, independently of the initial conditions. Note that for ϕ_{11}, fluctuations increase with time and are associated with the long tails of position and velocity distributions.

In the presence of the perturbation, the system goes at long times to a nonequilibrium stationary state which is an attractor of the dynamics, i.e., the perturbed dynamics of this long-range system has an "attractor" stationary state.

5. Properties of the final state

Let us consider now the properties of this apparently stationary state, and check in particular whether the phase space distribution is indeed stationary and the same in the different cases.

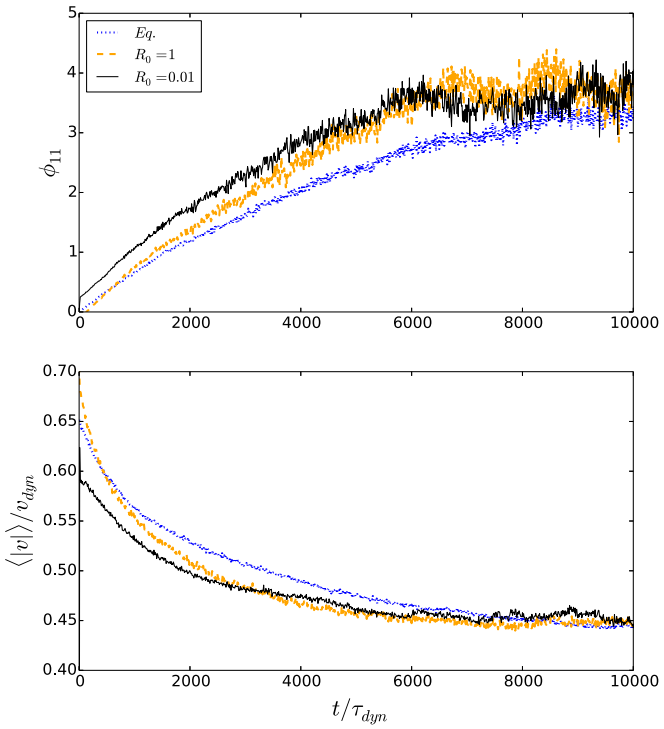

FIG. 5. Model A: ϕ_{11} (top) and of $\langle|v|\rangle/v_{dyn}$ (bottom) as a function of time t/τ_{dyn}, with $\gamma_A = 0.03$ and $N = 512$ for different initial conditions: thermal equilibrium (averaged over 300 realizations), rectangular waterbags with $R_0 = 1$ (100 realizations), and $R_0 = 0.01$ (200 realizations).

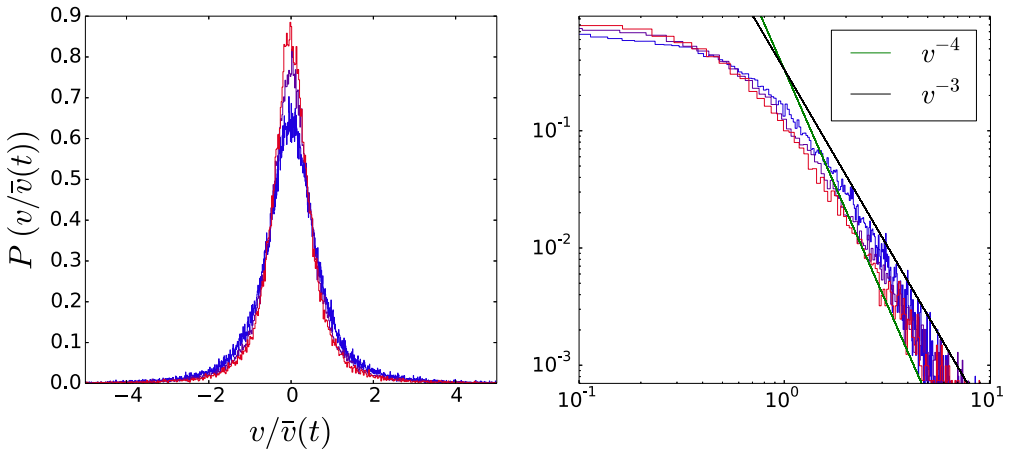

FIG. 6. Model A: Velocity distribution at different times ($t = 500, 1500, 3500\tau_{dyn}$) averaged over 100 realizations for $\gamma_A = 0.1$, $N = 1024$ with initial thermal distribution. Velocities are normalized by $\bar{v}(t)$, the standard deviation of the velocity distribution. The left panel is a linear plot and the right panel is a log-log plot. The two straight lines indicate the range of the exponents of power law fits to the tails of the distribution.

Figure 6 shows the velocity probability distribution averaged over 100 realizations, for $\gamma_A = 0.1$ and $N = 1024$ and with an initial thermal distribution (black curve in the left panel). The analogous spatial distributions are shown in Fig. 7 for the same data. In both cases the relevant variable has been normalized to the square root of its variance at the given time. Except for the initial distribution [cf. Eq. (46)] and the next time plotted ($t = 500\tau_{dyn}$), all the curves are thereafter very well superimposed. The tail of the corresponding velocity distribution is well fitted by a simple power-law behavior $\sim v^{-\kappa}$, with $\kappa = 3.5 \pm 0.5$, while that of the spatial distribution by $\sim x^{-\epsilon}$, with $\epsilon = 2 \pm 0.25$ (where the error bars are estimates of the range of compatible exponents; see Figs. 6 and 7). We find consistent behaviors for our different initial conditions, and for the different N in the range we have considered. Further we note that we find these behaviors to be stable even if we extend our analysis to data at longer times, in which the fluctuations of energy become large. Thus the state appears to be a very robust attractor even when the energy can vary considerably. As anticipated above, these asymptotic behaviors of the evolved system explain why we observed the strongly N dependent behaviors of the parameters ϕ_{11} and β_2: for such asymptotic behaviors of the velocity and space distribution these quantities are divergent, and thus in practice, when measured in a system with a finite number of particles, they are dominated by the contribution from just a few of the highest energy particles.

We note that the velocity distribution observed is very similar to that found for the original purely granular model (i.e., without gravity) [23]. As in this case one must in fact suppose that $\kappa > 3$ to ensure that the kinetic energy (proportional to the velocity dispersion) of the state is finite. Further it is interesting to note that the measured exponents are compatible with the relation $\kappa = 2\epsilon$ which can be inferred in the hypothesis that the phase space density is a function of the (mean field) energy only (as the mean field potential at large x is dominated by the monopole contribution proportional to $|x|$).

C. Model B

We have seen that the contribution from collisions in model B is characterized in the mean field limit by the dimensionless

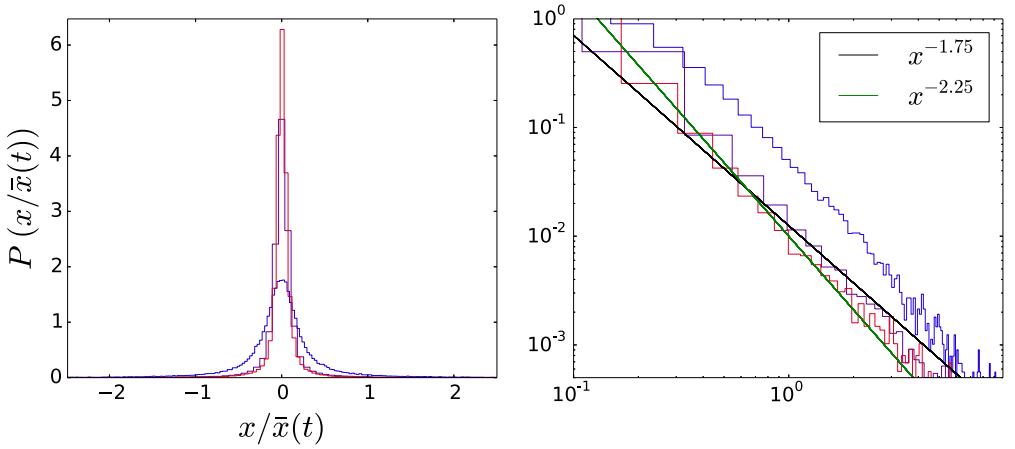

FIG. 7. Model A: Position distribution at different times ($t = 500, 1500, 3500\tau_{dyn}$) averaged over 100 realizations for $\gamma_A = 0.1$, $N = 1024$ with initial thermal distribution. Positions are normalized by $\bar{x}(t)$, the standard deviation of the position distribution. The left panel is a linear plot and the right panel is a log-log plot. The two straight lines indicate the range of the exponents of power law fits to the tails of the distribution.

parameter γ_B, and the velocity scale v_0, with both being held fixed in the mean field limit. As the mean field scaling leaves invariant also the characteristic velocity v_{dyn} defined in Eq. (48), we can define the dimensionless ratio

$$u_B = \frac{v_0}{v_{dyn}}, \tag{50}$$

which also remains fixed in the mean field limit. We then can characterize our simulations by the dimensionless parameters γ_B, u_B, and N, and the results will then be N independent at sufficiently large N if the mean field treatment is valid.

1. Macroscopic evolution due to perturbation

Figure 8 shows results for the evolution of the dimensionless energy E/E_0 (E_0 is the initial value), of the virial ratio R, ϕ_{11}, and β_2, for $N = 512$, $u_B = 1$, and rectangular waterbag initial conditions with $R_0 = 0.01$. The two curves correspond to the different values of $\gamma_B = 0.01, 0.005$ with an average over 20 realizations. We observe that, as expected, the system reaches virial equilibrium on a time scale $\sim 10\tau_{dyn}$ and remains, to an extremely good approximation, virialized thereafter. Compared to model A, the finite N fluctuations are extremely small. As we will see in further detail below, this is a result of the presence of a well defined energy scale in the model to which the system is efficiently driven. Thus the system evolves on a time scale $\sim \tau_{dyn}/\gamma_B$ as expected from the mean field kinetic theory, through a continuum of QSSs. Indeed, to test this conclusion, we have performed again simulations in which we "turn off" the perturbation at different times. We

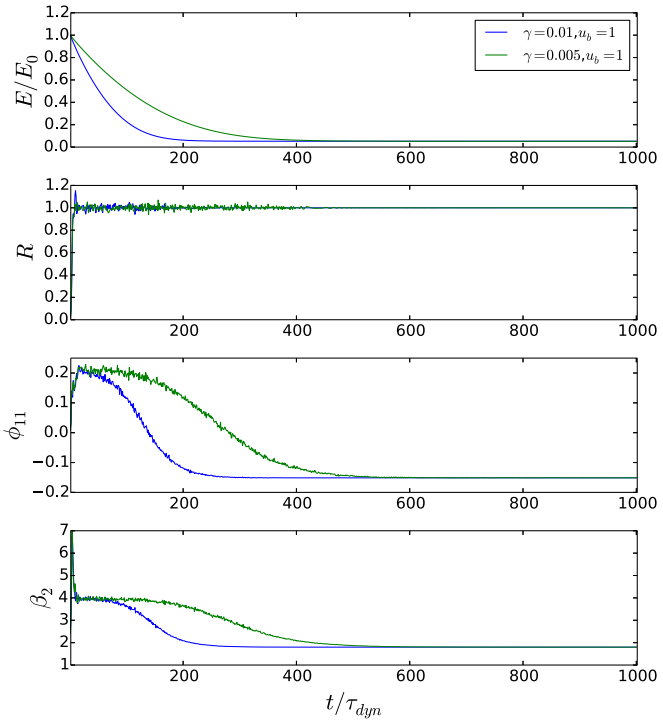

FIG. 8. Model B: Dimensionless energy E/E_0 (E_0 is the initial energy), virial ratio R, entanglement parameter ϕ_{11}, and dimensionless kurtosis β_2 vs t/τ_{dyn} for $u_B = 1$, $\gamma_B = 0.01, 0.005$, $N = 512$, and initial rectangular waterbag conditions $R_0 = 0.01$. Simulation results are averaged over 20 realizations.

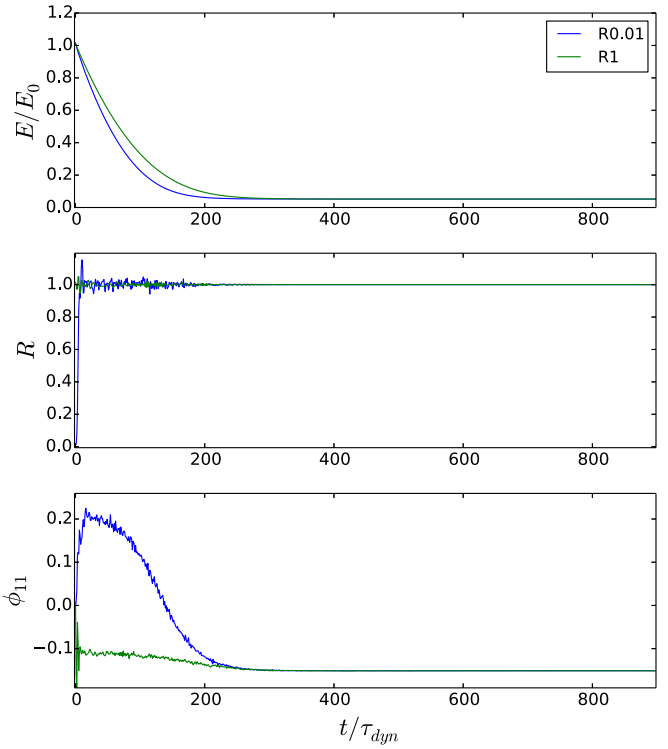

FIG. 9. Model B: E/E_0 (top) and ϕ_{11} (bottom) vs time t/τ_{dyn} for $N = 512$, $\gamma_B = 0.01$, $u_B = 1$ with rectangular waterbag initial conditions, $R_0 = 0.01, 1$.

find, as in model A, that the macroscopic parameters remain essentially frozen at their values at this time.

For the chosen value of $u_B = 1$ the simulations start with an energy which turns out to be about an order of magnitude larger than the energy in the stationary state. This means that the characteristic velocities are initially so large that most collisions are inelastic and the evolution depends little on the presence of the term depending on v_0. In this case the evolution is then well approximated by the case of purely inelastic collisions which we have studied in [32]. For smaller values of γ_B than those shown here the validity of this approximation is sufficiently extended in time so that one can see the presence of an approximate plateau in ϕ_{11} corresponding to the "scaling QSSs" derived in this work.

2. Dependence on initial conditions

Figure 9 shows the evolution of the energy E/E_0, R, and ϕ_{11} for $\gamma_B = 0.01$, $u_B = 1$, $N = 512$, and for two different rectangular waterbag initial conditions, $R = 0.01$ and $R = 1$. We see, as indicated by the behavior of ϕ_{11}, that each of the two initial conditions initially evolves to a quite different QSS, but then on the longer time scale both converge towards an identical value of ϕ_{11}. That this indeed corresponds to evolution to the same final state is confirmed, as we will detail further below, by study of the final configuration in phase space.

3. Mean field limit of kinetic theory

To test the validity of the mean field limit derived in Sec. III B, we have run sets of simulations for the same initial conditions with fixed values of γ_B and u_B, but different values

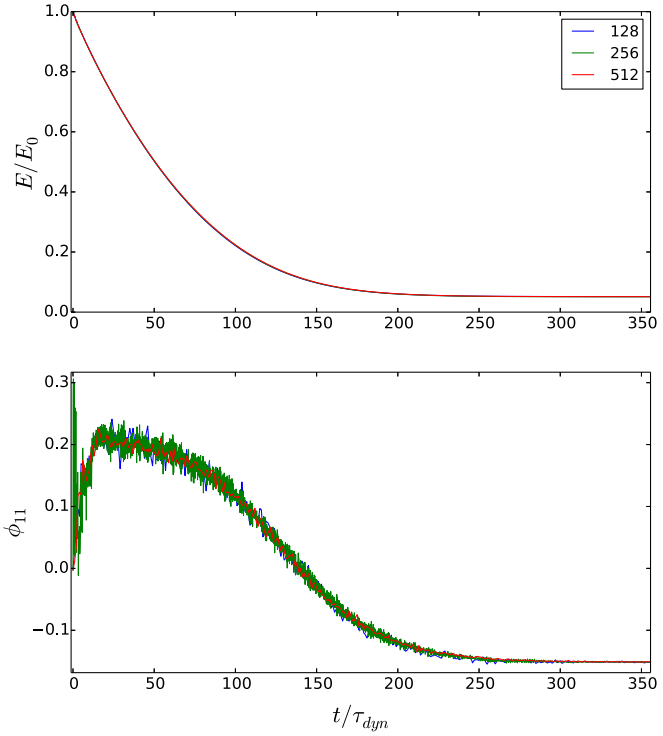

FIG. 10. Model B: E/E_0 (top) and ϕ_{11} (bottom) for $\gamma_B = 0.01$, $u_B = 1$, and for rectangular waterbag initial conditions, $R_0 = 0.01$ with different system sizes, $N = 128,256,512$.

of the particle number N. Figure 10 shows the evolution of the energy E/E_0 (top) and ϕ_{11} (bottom) for $\gamma_B = 0.01$, $u_B = 1$, and for rectangular waterbag initial conditions, $R_0 = 0.01$ with different system sizes, $N = 128,256,512$. We see in the evolution of the energy an almost perfect superposition of the curves, indicating thus an N independent evolution corresponding to the mean field limit.

4. Properties of final state

We finally consider in greater detail the properties of the apparently very well defined final state to which the system is driven very efficiently in model B. Figure 11 shows snapshots, at the indicated times, of the phase space of particle positions in dimensionless units (where $x_0 = v_0\tau_{dyn}$) in 20 realizations with $N = 128$ of $R = 0.01$ waterbag initial conditions, for a model with $\gamma_B = 0.01$ and $u_B = 1$. The phases of the evolution, already evident in the evolution of the energy and ϕ_{11} as discussed above, are again clearly visible. However, the phase space plot reveals that, from the time (here about $150\tau_{dyn}$) at which the macroscopic diagnostics indicate the establishment of the stationary states, and do not themselves appear to evolve anymore, there is a further nontrivial evolution in phase space of the *microscopic* particle distribution: the particles progressively "aggregate" onto distinct separated curves. A study of the particle energies shows that they are, to a very good approximation, constant on each curve, and take well separated values on each curve i.e., they are effectively discretized. Note that these phase space "rings" are time independent, so the time average in a narrow time window (e.g., of order τ_{dyn}) produces a plots essentially identical to Fig. 11 at longer times.

Figure 12 shows phase space configurations at two times ($t = 1000\tau_{dyn}$ on the left, $t = 3500\tau_{dyn}$ on the right) for 20 realizations of the same waterbag initial condition, and the same γ_B and u_B, as in the previous figure, for $N = 128$ (upper panels), $N = 256$ (middle panels), and $N = 512$ (bottom panels). In each plot the particle positions for a single chosen realization are also plotted as red stars. These plots show clearly that in all cases the system evolves towards a highly ordered distribution, in which the particles are not only on "shells" in phase space as noted above, but also have highly ordered positions along these shells, i.e., the relative phases of the particles' motions on the shells are fixed in time and completely coherent. Further the time scale to attain the final state appears to grow strongly with N (in units of τ_{dyn}): the $N = 128$ simulations have already attained the completely ordered state at $t = 1000\tau_{dyn}$, the $N = 256$ simulations are close to attaining it for $t = 3500\tau_{dyn}$, while the $N = 512$ systems are still evolving towards it at this later time.

This evolution of the system towards the "ordered" state thus occurs on a time scale which diverges when we take the mean field limit. Indeed it is an evolution intrinsically characteristic of the finite N system which thus cannot be described by the mean field kinetic theory we have developed in Sec. IV. Conversely, in this mean field limit the system should remain in the QSS to which it is driven by the collisions, and the phase space density of this state is in principle a stationary solution of the kinetic equations (38) and (43). Interestingly we observe that these macroscopic properties of the system in this mean field stationary state are apparently unchanged, even on the very longest times scales simulated, by the microscopic evolution, e.g., the total energy and parameters ϕ_{11} and β_2 do not evolve on average. This behavior can be contrasted with the relaxation to thermal equilibrium of QSS attained in the purely self-gravitating model. This relaxation is likewise driven by finite N effects, (not described by the Vlasov equation) but it also causes the system to evolve (on a time scale $\sim N\tau_{dyn}$) through a family of QSSs until it finally equilibrates.

As mentioned in Sec. IV C 1 above, we have verified that the intermediate states the system evolves through from the time it virializes are indeed a family of QSSs of the purely self-gravitating model: when we turn off the perturbation any time after virialization, the system's macroscopic properties do not change on mean field time scales. Thus the unique state reached from different initial conditions should be not just a stationary solution of the kinetic equations (38) and (43) but one in which both sides of the equation are zero [39]. For a few cases with $N = 128$ we have evolved the system, after turning off the inelastic collisions, and found that it evolves, as expected, towards thermal equilibrium on a time scale $\sim N\tau_{dyn}$. However, when the same experiment is performed but the inelastic collisions are turned off at a longer time at which the system has evolved to the fully "ordered" microscopic state we find an intriguing result: this ordered microscopic state remains unchanged under the purely gravitational evolution, not only on the mean field time scale (τ_{dyn} and τ_{dyn}/γ_B) but even on the time scale much greater than $N\tau_{dyn}$. Thus the microscopic state attained at long times in presence of the

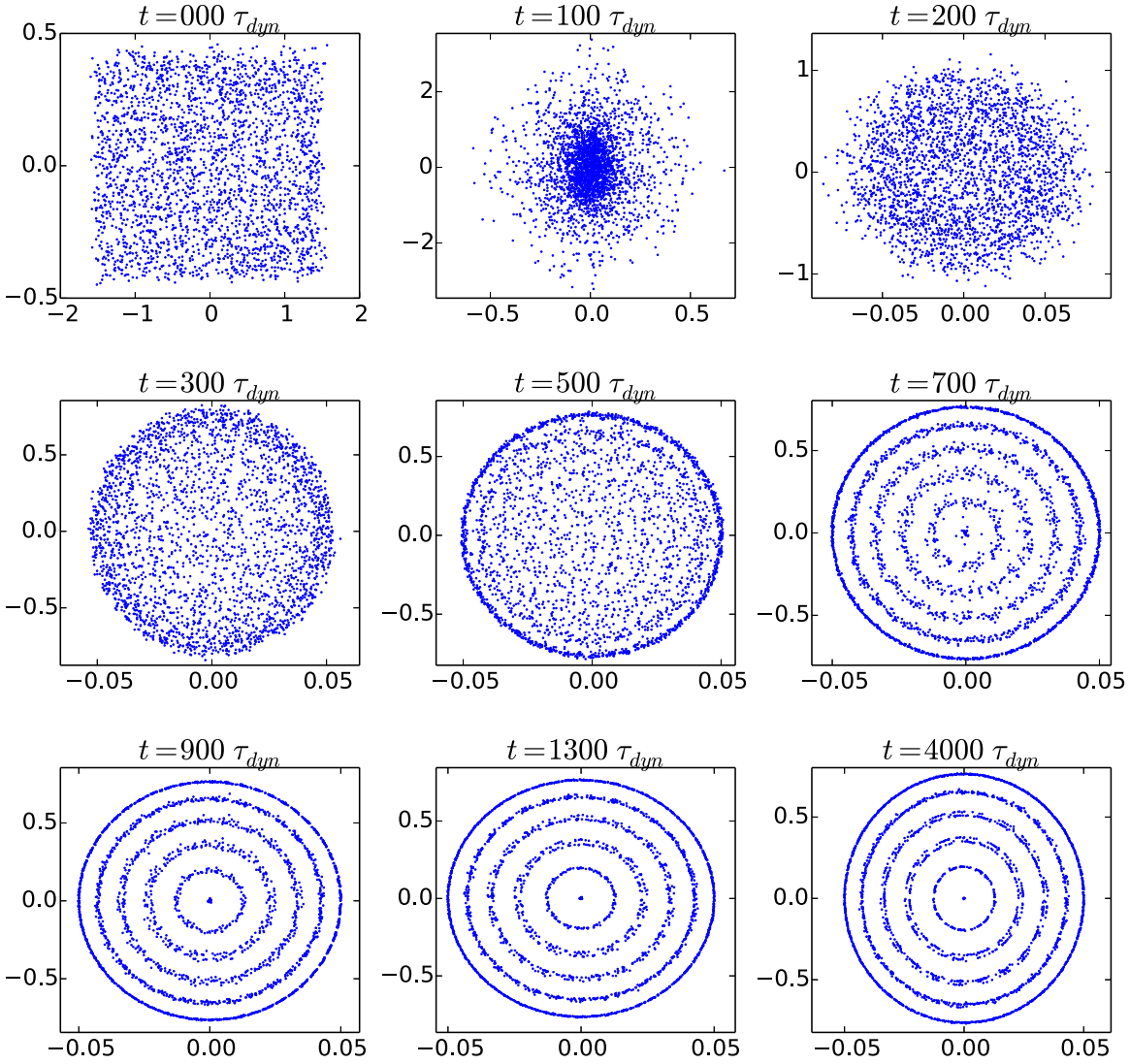

FIG. 11. Model B: Phase space snapshots in dimensionless units (x/x_0 and v/v_0) for $N = 128$, $\gamma_B = 0.01$, $u_B = 1$, rectangular waterbag initial conditions, $R_0 = 0.01$ and 20 realizations at different times t ranging from $t = 0$ to $t = 4000\ \tau_{dyn}$.

perturbation appears to be a periodic or quasiperiodic solution of the pure gravitational N-body system, and to belong to a stable island in the N-body phase space which leads to a breaking of ergodicity. We will investigate further both the dynamics giving rise to and the properties of these intriguing "ordered" states of the N-body system in future work.

V. DISCUSSION AND CONCLUSION

We have investigated the effects on the dynamics of long-range interacting systems of a class of "local internal" perturbations through the study of a canonical one dimensional toy model subjected to such perturbations. More specifically we consider two perturbations inspired by granular studies of the dynamics of a one dimensional self-gravitating system considering momentum conserving, and energy violating, collisions which are designed so that they can, nevertheless, conserve energy in an average sense. Our main focus has been on the question of how the characteristic nonequilibrium stationary states or QSSs of the long-range system are affected by these perturbations. We consider the case that these perturbations are *weak*, in the sense that the time scale on which they affect the system macroscopically are long compared to the time scales characteristic of the dynamics of the mean gravitational field.

We have derived first kinetic equations for both models which describe the system's evolution in a large N mean field and quasielastic limit. Our numerical study of the models shows that this limit describes well the macroscopic evolution due to the perturbations at sufficiently large N, but at given N we see always also at sufficiently long times evolution in both models which are finite N effects not captured by the mean field treatment. In model A such effects are manifest in large excursions of the energy at longer times, and in model B in the appearance of a highly ordered microscopic phase space distribution. Within the regime of validity of the mean field approximation both models show at longer times evolution towards an apparently unique virialized state. This state is not the thermal equilibrium of the isolated model, and indeed is typically "further away" (in terms of correlations measured by

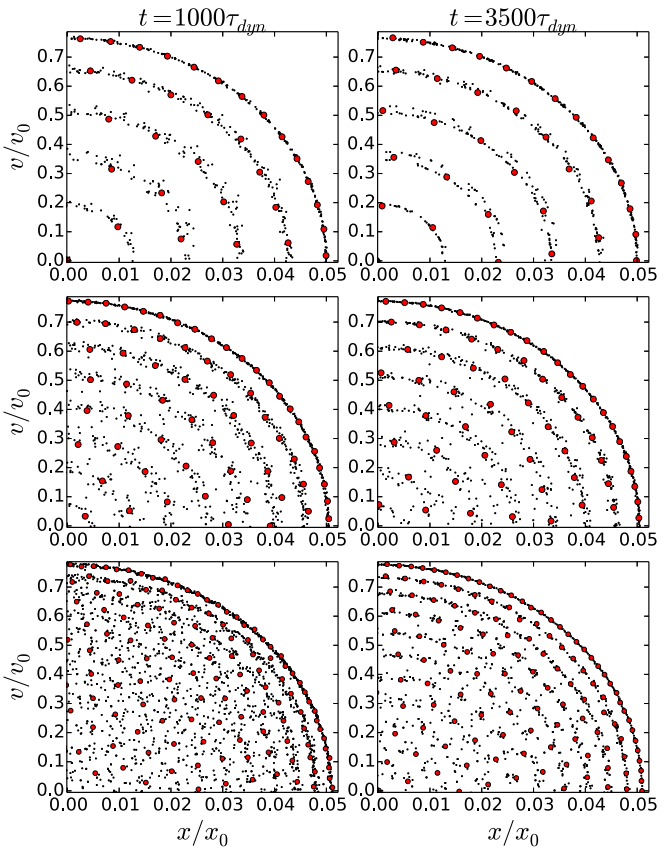

FIG. 12. Model B: Phase space snapshots in dimensionless units (x/x_0 and v/v_0) for $\gamma_B = 0.01$, $u_B = 1$, rectangular waterbag initial conditions, $R_0 = 0.01$, 20 realizations and at two different times $t = 1000\tau_{dyn}$ (left panels), $t = 3500\tau_{dyn}$ (right panels). We compare three different system sizes $N = 128,256,512$ (from top to bottom). Red dots correspond to one chosen realization, small black dots correspond to the other realizations.

ϕ_{11}) from the thermal equilibrium. Therefore we observed compelling evidence for the establishment of an attractor nonequilibrium stationary state in both models. We note that, in contrast to the HMF model with the modified stochastic dynamics described in [15], our perturbed long-range models show no tendency to relax toward thermal equilibrium.

Both perturbations, which act microscopically and locally, thus completely modify the global organization of the system: they drive the long-range system far from the QSS it is in initially (due to mean field relaxation from the initial conditions on time scales significantly shorter on which the perturbations act). In this sense the QSSs are not robust to such perturbations, and are modified macroscopically as soon as the perturbation starts to act. However, the evolution which results is through a succession of virialized states which are stable stationary solutions of the Vlasov equation. In both models the system is then driven finally to a nonequilibrium stationary state (NESS), which is itself also a QSS of the unperturbed long-range system. This final state does not depend on the initial conditions, but does depend strongly on the details of the perturbation. Indeed in the two models we have considered the final state has completely different properties, with notably in model A a power-law decaying space and velocity distribution compared to a phase space distribution with compact support for model B. Thus the perturbation, albeit apparently very weak, turns out to completely dominate and determine the behavior of the long-range system. This contrasts dramatically with the effect such a weak perturbation would have on a short-range system which relaxes efficiently to thermal equilibrium: in this case the perturbation would indeed just perturb slightly this equilibrium.

It is interesting to compare our results with related previous work in the literature. In [15,16] the stochastic perturbation applied to the long-range system (the HMF model) permutes the momenta of triplets of particles chosen randomly in the system, and drives the system to relax to thermal equilibrium efficiently. Applied to a one dimensional self-gravitating system, we have checked that we observe the same behavior. Indeed it suffices in this case to consider exchanges of the velocities of randomly chosen pairs of particles because the QSSs are inhomogeneous. This drives the system efficiently to equilibrium because it destroys directly, because of the nonlocality of the perturbation, the entanglement in space and velocity of the phase space distribution. In the models we have presented here the perturbation, as we have underlined, has the property of being *local*. This means that the instantaneous change of the velocity distribution induced by the collisions depends on the local properties, and as these typically vary in space in a nontrivial manner there is no reason to expect the system to evolve towards the same velocity distribution everywhere. On the contrary, as we have seen, the system tends to evolve towards configurations in which the space and velocity distributions are ever more strongly entangled, until a stationary state is reached in which the spatial organization induced by the long-range forces "compensates" the local modification of the velocity distributions by the perturbations.

Our study is complementary also to that of Refs. [17,18] in which the effect of an *external* stochastic force acting on a long-range system is studied. Indeed we can consider the perturbations we have introduced at particle collisions as stochastic forces, with the difference that they are *internal*, i.e., they are determined by the instantaneous microscopic state of the system itself. As we have mentioned, our models are thus appropriate to model the effects, for example, of additional short-range interactions at play in the system, while that of [40] models the effects of interactions with matter external to the system. For their treatment with kinetic theory, our models admit a considerable simplification compared to that required for the models of [17,18]: as we have seen, we obtain in both our models a nontrivial kinetic theory which includes the effect of the perturbation in the mean field limit, i.e., by neglecting two point correlations in phase space. As described in [17,18] a nontrivial large N limit for the evolution induced by the external perturbation is obtained going beyond the mean field limit, and specifically can be obtained by including nontrivial two point correlations. The reason for this difference is that the action of the external forces on the system depend crucially on the spatial correlations of these forces, and their effect on the evolution cannot be described self-consistently without incorporating the resultant correlations in the perturbations to the phase space distributions. While [17,18] can obtain a range of different behaviors from the external stochastic forces—ranging from thermalization of the system to out of equilibrium

states characterized by intermittency—our models display the simpler phenomenology of attractive nonequilibrium steady states we have described.

We have constructed our models so that they either conserve energy on average (model A, in the large N limit) or can attain states in which energy is stationary (model B). When considering perturbations to such systems, there is no reason in general to expect them to have such a property. What would we expect the effect notably of net energy dissipation or injection to be? In [32] we have considered a simple class of perturbations which dissipate energy, and found that they admit what we have called "scaling QSSs." These are states of such systems in which the dissipation of the energy leads simply to an evolution in which the phase space density remains unchanged other than to an overall rescaling of its characteristic size and velocity. This study suggests that in models like those considered here, but including a constant energy dissipation, one might expect to see established an "attractive scaling QSS," i.e., evolution to a unique phase space distribution in rescaled variables reflecting the dissipation of energy. Indeed we note that in model A extrapolated to the very long time scales where the macroscopic energy strongly fluctuates due to finite N, we have found that, in suitably rescaled coordinates, the system's velocity and space distributions remain very stable, with notably the same power law tails measured in the mean field regime.

We have studied here only two very specific and simple models, but the rich and interesting results they produce motivate further study. First, concerning these specific models, we have only explored the regime in which the perturbations introduced are weak, in the sense that the characteristic times scale of their dynamics is long compared to that of the mean-field dynamics of gravity. The regimes in which the two scales are comparable, or even that in which the long-range interaction acts only on a longer time scale, may produce interesting behaviors too. More generally it would be interesting to explore, using other models, how generic to long-range interacting systems the interesting behaviors we have observed are—in particular the evolution towards a unique stationary state which is a QSS of the unperturbed system "selected" by the perturbation. In this respect it would of course be interesting to study models analogous to those considered here in two or three dimensions, and in particular the case of gravity in either dimension. An important difference with respect to the one dimensional case is that purely elastic collisions of nongravitational origin do then have a nontrivial effect on the dynamics driving it, in principle, to statistical equilibrium (for a detailed treatment of the case of hard core collisions, see [41]). Further, in three dimensions if the system is to reach a truly stationary state (or in particular statistical equilibrium) it must be enclosed in a box. On the basis of what we have observed, we believe it is reasonable to anticipate that qualitative behaviors like those observed here may indeed be common to many long-range systems when subjected to perturbations sharing the essential characteristics of those in our models. Isolated long-range systems admit an infinite number of QSSs, and which of these states the system relaxes to on mean field time scales is determined by the initial conditions, and depends on them in general in a way which is extremely complex (see, e.g., [22,42–46]). The application of a weak perturbation to the system can be understood as providing a breaking of the degeneracy of the infinite number of QSSs which drives the system to a QSS which is invariant under its own action, i.e., in which, in our models, the collision term induced by the perturbations is zero. As the perturbation will generically violate all the conservation laws (Casimirs) of the Vlasov dynamics, there would appear to be no reason why the perturbed dynamics cannot explore the full space of QSSs accessible from a given starting energy and mass, and thus "find" the stable state starting from any initial condition. Further generically we would not expect this final state to be the thermal equilibrium of the system (which is a particular QSS): as we have underlined, unless the perturbation applied locally to the velocity distribution tends to drive the system everywhere to the same velocity distribution, we expect the long-range force to give rise to a stationary state in which the velocity and space distributions are entangled in a manner characteristic of nonequilibrium QSSs.

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