Linear response approximation in effective field theory for the calculation of elastically mediated interactions in one dimension

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The linear response approximation, used within effective field theory to calculate mediated interactions between inclusions, is studied for an exactly solvable one-dimensional model. We show that it works poorly in the case of inclusions imposing absolute deformations to the field, while it works well for massless theories in the case of inclusions imposing relative deformations to the field.

I. INTRODUCTION

Particles that deform an elastic, correlated medium, or alter its fluctuations, experience mediated interactions [1]. A famous example is the Casimir effect by which two metal plates in vacuum attract each other due to the constraints they impose on the quantum fluctuations of the electromagnetic field [2]. Mediated interactions also occur between surfaces, colloids, or proteins in soft-matter media such as critical binary mixtures [3,4], liquid crystals [5,6], capillary interfaces [7,8], and biomembranes [9–11]. The calculation of these interactions is usually made difficult by the extended character of the inclusions deforming the medium by the imposition of constraints along their boundary. One needs to compute the medium deformation that matches the boundary conditions (BCs) and the distribution of the fluctuations around this average deformation [9,12].

A few years ago, Deserno and Rothstein have imported from high-energy physics a powerful method that provides an alternative to such calculations: the effective field theory (EFT) [13–18]. In general, in order to determine the mediated interaction between embedded inclusions, one needs to calculate the free energy of the field describing the medium's deformation, in the presence of the inclusions, and to extract the dependence in their separation R. Here, we shall consider, to simplify, that the system is Gaussian (this is a standard approximation in soft matter when working in the small deformation limit). For Gaussian systems, the interaction free energy can be decomposed exactly into a mean-field contribution $F_{\mathrm{mf}}(R)$, obtained by minimizing the Hamiltonian $\mathcal{H}[\phi]$ functional of the field describing the medium's deformation subject to the BCs imposed by the embedded inclusions, and a fluctuation-induced contribution $F_{\mathrm{C}}(R)$, often called Casimir-like contribution. To compute the mean-field configuration $\Phi(x)$, that coincides with the average profile $\langle \phi(x) \rangle$ in a Gaussian model, one needs to solve the linear equation $\delta \mathcal{H}/\delta \phi(x)|_{\Phi} \equiv \mathcal{L}\Phi(x) = 0$ in the body of the medium, subject to the BCs set by the inclusions. The idea of EFT [19] is to replace the embedded inclusions by an equivalent set of pointlike charges and polarizabilities, the amplitude of which are called Wilson coefficients. To determine the latter, one first considers a background mean-field configuration $\Phi_{\mathrm{bg}}(x)$ that minimizes the Hamiltonian for some set of distant boundary conditions (DBCs) applied far away from the region of interest. We thus have $\mathcal{L}\Phi_{\mathrm{bg}}(x) = 0$ subject to the DBCs. These DBCs are taken arbitrarily, in the most general way, in order to

explore all the possible background distortions. When an extended inclusion is placed at the origin, and the same DBCs are kept, the field that matches the inclusion's BCs is $\Phi_{\mathrm{tot}} = \Phi_{\mathrm{bg}} + \Phi_r$, with $\Phi_r(x)$ the mean-field response. It satisfies $\mathcal{L}[\Phi_{\mathrm{bg}}(x) + \Phi_r(x)] = 0$ subject to the BCs and the DBC's. By virtue of linearity, and since $\mathcal{L}\Phi_{\mathrm{bg}}$ vanishes, the response field satisfies

$$\mathcal{L}\Phi_r(x) = 0, \tag{1}$$

subject on the one hand to the BCs for the total field and on the other hand to the condition that it must vanish at the distant boundary. The actual BCs for Φ_r depend therefore on Φ_{bg}, and one has thus to calculate the response Φ_r as a function of the DBCs. This is in general feasible since there is only one inclusion.

The idea of EFT is to determine equivalent pointlike particles that yield the same Φ_r for all the possible DBCs, and to deduce their Wilson coefficients (to be used further in complex situations where two or more inclusions are present). Calling α, β, etc., the Wilson coefficients, the EFT equation describing the response produced by the effective pointlike inclusion is of the form $\mathcal{L}\Phi(x) = [\alpha + \beta \Phi(x) + \gamma \Phi^2(x)]\delta(x) + \ldots$, which needs to be solved subject to the DBCs only. The equation equivalent to Eq. (1), in EFT, is thus

$$\mathcal{L}\Phi_r(x) = [\alpha + \beta \Phi_{\mathrm{tot}}(x) + \gamma \Phi_{\mathrm{tot}}^2(x)]\delta(x) + \ldots, \tag{2}$$

subject to the sole condition that Φ_r must vanish at the distant boundary. Replacing Φ_{tot} in the above equation by $\Phi_{\mathrm{bg}} + \Phi_r$, one sees that the equation for the response Φ_r is nonlinear.

If the response field is sufficiently small with respect to the background field, it may be justified to replace Φ_{tot} by Φ_{bg} in the right-hand side of the equation:

$$\mathcal{L}\Phi_r(x) \simeq [\alpha + \beta \Phi_{\mathrm{bg}}(x) + \gamma \Phi_{\mathrm{bg}}^2(x)]\delta(x) + \ldots. \tag{3}$$

The equation for Φ_r thus becomes linear. This linear response approximation (LRA), which makes the calculations much easier, has been successfully used in the literature [13,15–18]. The real conditions of its validity, however, are unclear. It is the aim of this work to examine this issue.

Our paper is organized as follows. In Sec. II, we introduce a generic one-dimensional (1D) Gaussian model and we consider either inclusions that impose the value of the field (absolute deformation, type A inclusions) or inclusions that impose a jump of the field (relative deformation, type B inclusions). We calculate their interaction exactly. The latter

can be decomposed into a mean-field contribution and a Casimir-like, fluctuation-induced contribution. In Sec. III, we apply the nonlinear EFT method, thereby determining the Wilson coefficients for both types of inclusions. We compute the interaction between the EFT particles and we show that there is a perfect agreement with the results of the exact treatment, both for inclusions of type A and inclusions of type B. In Sec. IV, we apply the EFT method within the linear response approximation (LRA) and we compare our results with the exact results. We find that the LRA doesn't work in the case of inclusions of type A, nor does it work in general for inclusions of type B, except when the theory is massless. In Sec. V, we summarize our results. We try to characterize the conditions that make the LRA successful and we briefly discuss the 2D situation.

II. DEFINITIONS AND EXACT TREATMENT

Let us consider a 1D generic elastic medium whose deformation is described by a field $\phi(x)$. We assume that the associated Hamiltonian is Gaussian, given by

$$\mathcal{H}_0[\phi] = \int_{-\infty}^{\infty} dx \left(\frac{m}{2}\phi^2 + \frac{K}{2}\phi'^2 \right). \tag{4}$$

Then the correlation length of the field is $\xi = \sqrt{K/m}$ and $\epsilon = \sqrt{mK} = K/\xi$ has the dimension of an energy.

This Hamiltonian is classical in the description of soft matter. It may describe the interface profile between two immiscible fluids due to the combined effects of gravity and surface tension [7]. In this case K is the surface tension and m is the gravitational field multiplied by the difference of the fluids densities. It may also describe a semiflexible polymer, with ϕ the angle that the monomers make with a fixed direction [20]. In this case K is the bending stiffness and m originates from an applied tension.

In the following we shall define two types of inclusions that will locally affect the field: inclusions of type A fixing the value of the field and inclusions of type B imposing a jump of the field. We shall calculate the elastic deformation that they produce together with the associated mediated interaction. For instance, inclusions of type A correspond in the capillary case to particles that pin the meniscus [21]. Inclusions of type B correspond in the polymer case to adsorbed proteins that locally bend the polymer. Similar types of Hamiltonians and inclusions are also used in the description of biological membranes [17,22–24].

A. Type A inclusions imposing a field value

We define the inclusions of type A as objects of length $2a$, located at $x = x_i$, that impose the value $\phi(x) = \phi_i$ on their boundary:

$$\phi(x_i - a) = \phi(x_i + a) = \phi_i. \tag{5}$$

Note that the field is not defined in the interval $[x_i - a, x_i + a]$, i.e., the inclusions effectively expel the field. We show below that the field-mediated interaction free energy $F(R) = \epsilon \tilde{F}(R)$ between two such inclusions, separated by a distance R, is
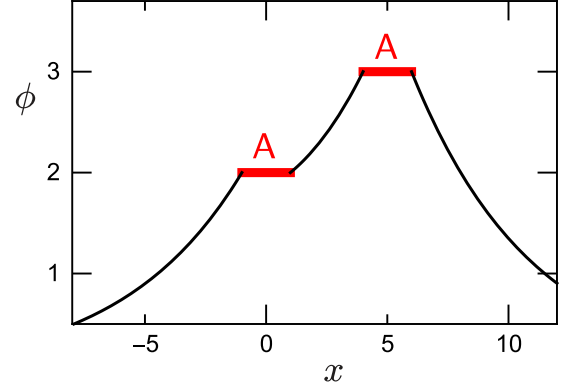

FIG. 1. Average elastic deformation produced by two inclusions imposing the value of the field (type A). The parameters are $\phi_1 = 2, \phi_2 = 3, a = 1, R = 5$, and $\xi = 5$. The inclusions are drawn in red.

given exactly by

$$\tilde{F}(\tilde{R}) = \frac{(\phi_1 + \phi_2)^2}{2} \coth \tilde{R} - \phi_1 \phi_2 \coth(\tilde{R}/2)$$
$$+ \frac{k_B T}{2\epsilon} \ln(1 - e^{-2\tilde{R}}), \tag{6}$$

where $\tilde{R} = (R - 2a)/\xi$ is the dimensionless separation between the inclusions. This interaction is strongly repulsive at short separations and it is asymptotically attractive (repulsive) when $\phi_1 \phi_2 > 0$ ($\phi_1 \phi_2 < 0$). Note that it includes the effects of the thermal fluctuations (term proportional to $k_B T$).

Let the first inclusion be placed at $x = 0$ and the second one at $x = R$. Because the field is Gaussian, the average profile $\langle \phi(x) \rangle$ coincides with the mean-field profile $\Phi(x)$ that minimizes the Hamiltonian. The latter is solution of the Euler-Lagrange equation $\delta \mathcal{H}_0 / \delta \phi(x)|_\Phi = \mathcal{L}\Phi(x) = 0$, where

$$\mathcal{L} = d^2/dx^2 - \xi^{-2} \tag{7}$$

is the Euler-Lagrange operator. In the whole paper, capital Greek letters such as Φ will denote average, mean-field profiles, and lowercase Greek letters such as ϕ will denote fluctuating fields. Solving Eq. (7) subject to the boundary conditions $\Phi(a) = \phi_1$ and $\Phi(R - a) = \phi_2$ yields, in between the inclusions:

$$\Phi(x) = \langle \phi(x) \rangle = \phi_1 \frac{\sinh(\tilde{R} - \tilde{x})}{\sinh \tilde{R}} + \phi_2 \frac{\sinh \tilde{x}}{\sinh \tilde{R}}, \tag{8}$$

where $\tilde{x} = (x - a)/\xi$ is the dimensionless distance to the left inclusion's boundary, and a simple exponential decay in the outer regions (Fig. 1).

The energy of the mean-field profile $H_0[\Phi]$ between the inclusions can be directly calculated from Eq. (4), by integrating in the interval $[a, R - a]$, and yields the first two terms in the interaction $\tilde{F}(R)$ given by Eq. (6). Note that the deformation in the regions $[-\infty, -a]$ and $[R + a, \infty]$ do not depend on the separation between the inclusion and therefore do not participate to the interaction.

Because the model is Gaussian, the total interaction free energy $F(R)$ can be exactly decomposed into the above mean-field contribution $H_0[\Phi]$ and a Casimir-like contribution $F_C(R)$ arising from the fluctuations around the average profile. Indeed, writing the field as $\phi(x) = \Phi(x) + \psi(x)$, it is easy

to show that $H_0[\phi] = H_0[\Phi] + H_0[\psi]$, and thus we have $F(R) = H_0[\Phi] - k_B T \ln Z_\psi$, with Z_ψ the partition function of the field ψ. Let us compute the latter Casimir-like contribution to the interaction free energy.

The eigenfunctions of the Hamiltonian $H_0[\psi]$ in the interval between the inclusions are solutions of the equation $\xi^{-2}\Psi_n(x) - \Psi_n''(x) = \lambda_n \Psi_n(x)$ subject to the boundary conditions $\Psi_n(0) = \Psi_n(L) = 0$, where $L = R - 2a$ is the separation between the boundaries of the inclusions. The solutions are quantified sine waves, and using them as a basis we can write $\psi(x) = L^{-1/2} \sum_n \psi_n \sin(n\pi x/L)$, which yields

$$\mathcal{H}_0[\psi] = \frac{K}{4} \sum_{n=1}^{\infty} \lambda_n \psi_n^2, \quad \lambda_n = \xi^{-2} + \left(\frac{n\pi}{L}\right)^2. \quad (9)$$

Hence,

$$F_C = -k_B T \ln Z_\psi = \frac{1}{2} k_B T \sum_{n=1}^{\infty} \ln[\beta K \lambda_n/(4\pi)]. \quad (10)$$

To extract the R-dependent, regular part of this diverging series, we integrate with respect to x^2 the quantity $S(x) = \sum_{n=1}^{\infty} [x^2 + (n\pi/L)^2]^{-1} = L \coth(Lx)/(2x) - 1/(2x^2)$ obtained by differentiating the series with respect to x^2. Setting then $x = 1/\xi$, multiplying by $k_B T/2$ and removing unimportant contributions, we obtain the third term of Eq. (6).

B. Type B inclusions imposing a field jump

We define the inclusions of type B as objects of length $2a$, located at $x = x_i$, that impose a jump of the field on their boundary:

$$\phi(x_i + a) - \phi(x_i - a) = \alpha_i. \quad (11)$$

These inclusions constrain the field less than the previous ones since they impose only one condition instead of two. In particular, the average of the boundary fields is free. Again, the field is not defined within the inclusion, i.e., in the interval $[x_i - a, x_i + a]$. We show below that the dimensionless interaction free energy $\tilde{F}(R) = F(R)/\epsilon$ between two such inclusions, separated by a distance R, is given exactly by

$$\tilde{F}(\tilde{R}) = \frac{\alpha_1 \alpha_2}{2} e^{-\tilde{R}}, \quad (12)$$

where $\tilde{R} = (R - 2a)/\xi$. It is attractive (repulsive) for $\alpha_1 \alpha_2 < 0$ ($\alpha_1 \alpha_2 > 0$).

Placing the two inclusions as previously, we compute the mean-field, average profile $\Phi(x)$, by minimizing the Hamiltonian $\mathcal{H}_0$. We thus need to solve the equation $\mathcal{L}\Phi(x) = 0$ within the intervals $]-\infty, -a], [a, R-a]$ and $[R+a, \infty[$. We need six BCs: two are provided by the asymptotic vanishing of $\Phi(x)$ as $|x| \to \infty$, two are provided by the BCs set by the inclusions, i.e., $\Phi(a) - \Phi(-a) = \alpha_1$ and $\Phi(R+a) - \Phi(R-a) = \alpha_2$, and the last two are obtained from minimizing the total energy with respect to $\frac{1}{2}[\Phi(a) + \Phi(-a)]$ and $\frac{1}{2}[\Phi(R+a) + \Phi(R-a)]$. It is straightforward to show that the latter conditions are equivalent to $\Phi'(-a) = \Phi'(a)$ and $\Phi'(R-a) = \Phi'(R+a)$. We thus obtain

$$\Phi(x) = \langle \phi(x) \rangle = \frac{\alpha_1}{2} e^{-\tilde{x}} - \frac{\alpha_2}{2} e^{\tilde{x} - \tilde{R}}, \quad (13)$$
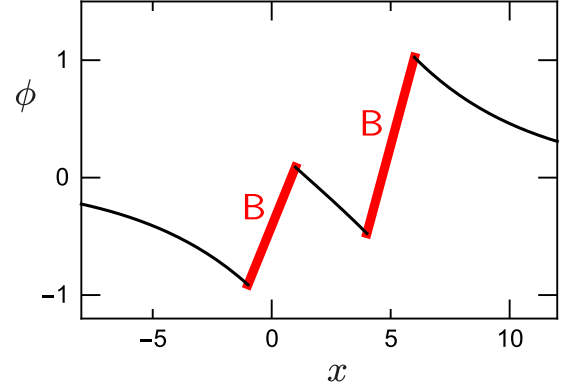

FIG. 2. Average elastic deformation produced by two inclusions imposing a jump of the field (type B). The parameters are $\alpha_1 = 1, \alpha_2 = 1.5, a = 1, R = 5$, and $\xi = 5$. The inclusions are drawn in red.

where $\tilde{x} = (x - a)/\xi$, and simple exponential decays in the outer regions (Fig. 2).

As in the preceding section, because the model is Gaussian, the total interaction free energy is composed of the energy of the mean-field profile and a Casimir-like correction. Calculating $\mathcal{H}_0[\Phi]$ within the intervals outside the inclusions yields the interaction term given in Eq. (12). As for the contribution arising from the fluctuations, i.e., the Casimir-like contribution, it actually vanishes for inclusions of type B. Indeed, as shown in the preceding section, this contribution is given by $-k_B T \ln Z_\psi$, with Z_ψ the partition function of the field $\psi = \phi - \Phi$. Because Ψ satisfies the BCs given by Eq. (11), the remaining BCs applying to ψ are merely continuity condition across the inclusions. As a consequence, the inclusions play no physical role for ψ; removing them and joining together the boundary of the inclusions, ψ becomes a free field fluctuating in $]-\infty, \infty[$ and its free energy is independent of R.

III. NONLINEAR EFFECTIVE FIELD THEORY TREATMENT

Let us apply the EFT method sketched in Sec. I to the particles of type A and B. We shall design pointlike particles of type A$'$ and B$'$ that mimic the particles of type A and B. The most general form of the Hamiltonian associated to a pointlike particle placed at $x = x_i$ is given by

$$\mathcal{H}_i = \int_{-\infty}^{\infty} dx f[\phi(x), \phi'(x), \ldots] \delta(x - x_i), \quad (14)$$

where f is an arbitrary function. In high-energy physics, $\mathcal{H}_i$ is called a worldline [19]. Because our theory is Gaussian, we keep only the terms that are linear or quadratic in ϕ. Besides, we limit ourselves in the expansion of f to derivatives of order less than two: theories eliminating derivatives that would be redundant at the saddle point, i.e., for $\phi'' = -\xi^{-2}\phi$ in our case, are termed on-shell and are standard [17]. We therefore consider for f a linear combination of the five monomials $\phi(x), \phi'(x), \phi^2(x), \phi'^2(x)$ and $\phi(x)\phi'(x)$. Note that in higher space dimensions, the set of Wilson coefficients is in general infinite, even for on-shell theories [17].

A. Effective pointlike inclusions of type A′

Since the type A inclusions are invariant under the transformation $x \to -x$, the Hamiltonian of the type A′ inclusions must be of the form:

$$\frac{\mathcal{H}_i}{2\epsilon} = \int dx \left[c_1 \phi(x) + \frac{c_3}{2} \phi^2(x) + \frac{c_5 \xi^2}{2} \phi'(x)^2 \right] \delta(x - x_i),$$

(15)

where, defined in such a way, the Wilson coefficients c_i are dimensionless. Let us consider an arbitrary background deformation $\Phi_{\text{bg}}(x)$ produced by some distant boundary conditions (DBCs):

$$\Phi_{\text{bg}}(\pm L) = \phi^\pm.$$

(16)

Being solution of $\mathcal{L}\Phi_{\text{bg}} = 0$, it is given by $\Phi_{\text{bg}}(x) = \phi^+ \sinh(\tilde{x} + \tilde{L}) / \sinh(2\tilde{L}) - \phi^- \sinh(\tilde{x} - \tilde{L}) / \sinh(2\tilde{L})$, with now $\tilde{x} = x/\xi$ and $\tilde{L} = L/\xi$. Then, we add an inclusion at the origin and we require that the same DBCs (or same asymptotic profile for large L) be satisfied for the total deformation. The latter may be written as $\phi_{\text{tot}} = \Phi_{\text{bg}} + \phi_r$, which defines the response field ϕ_r, of average, mean-field profile Φ_r. The DBCs for the total and the response fields are thus $\Phi_{\text{tot}}(\pm L) = \Phi^\pm$ and thus

$$\Phi_r(\pm L) = 0.$$

(17)

Identifying, for all possible backgrounds, the response produced by a real type A inclusion to that produced by its effective type A′ counterpart fixes the Wilson coefficients (Fig. 3).

1. Matching between inclusions of types A and A′

The response Φ_r of a real, type A inclusion fixing the value ϕ_0 of the field (Fig. 3, thick gray line) satisfies $\mathcal{L}\Phi_r = 0$ with the boundary conditions $\Phi_r(\pm L) = 0$ and $\Phi_r(\pm a) =$
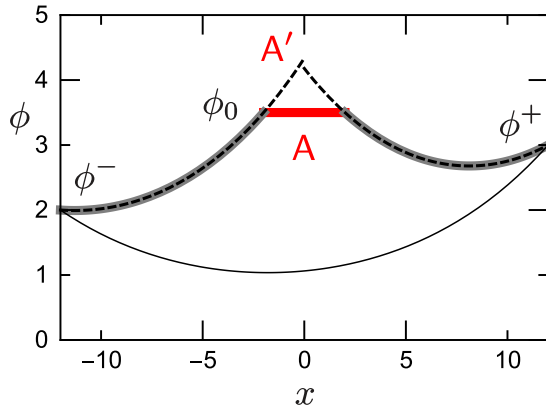

FIG. 3. In a background (thin black line) set by the boundary values $\phi^\pm$ separated by a distance $2L$, either a real inclusion of type A or its effective counterpart of type A′ are placed at the origin. The Wilson coefficients of the effective inclusion are such that the deformation produced by the type A inclusion (thick gray line) and that produced by the type A′ inclusion (dashed black line) match whatever the background. The discontinuity of the dashed line at the origin can be much larger for other parameter values. The parameters are $\phi^- = 2, \phi^+ = 3, L = 12, \phi_0 = 3.5, a = 2$, and $\xi = 8$. The inclusions are drawn in red.

$\phi_0 - \Phi_{\text{bg}}(\pm a)$. The solution is of the form

$$\Phi_r(x) = \begin{cases} A_1 \sinh(\tilde{x} + \tilde{L}), & \text{for } x \in [-L, -a], \\ A_2 \sinh(\tilde{x} - \tilde{L}), & \text{for } x \in [a, L], \end{cases}$$

(18)

where again $\tilde{x} = x/\xi$ and $\tilde{L} = L/\xi$. Satisfying the boundary conditions yields the constants $A_{1,2}$ in the form $A_i = A_{i0} + A_{i+}\phi^+ + A_{i-}\phi^-$, which we shall not explicate.

In the case of an effective pointlike inclusion of type A′ placed at $x = 0$ (Fig. 3, dashed line), the Euler-Lagrange equation associated with the minimization of $\mathcal{H}_0 + \mathcal{H}_1$, with $\mathcal{H}_1$ given by Eq. (15), is

$$\mathcal{L}\Phi_r(x) = 2[c_1 + c_3 \Phi_{\text{tot}}(0)]\xi^{-1}\delta(x) - 2c_5\xi \Phi'_{\text{tot}}(0)\delta'(x).$$

(19)

The solution has the same form as Eq. (18), with new constants A'_i instead of A_i, and intervals now corresponding to $[-L, 0[$ and $]0, L]$. Because $\Phi_{\text{tot}}(x)$ and $\Phi'_{\text{tot}}(x)$ are discontinuous in $x = 0$, we replace them by their average value at the discontinuity. To determine the constants A'_i, we integrate the differential Eq. (19) around the singularity, either directly or after multiplying both sides by x. This yields the system

$$(A'_2 - A'_1)\cosh \tilde{L} = 2c_1 + \frac{c_3(\phi^+ + \phi^-)}{\cosh \tilde{L}} - c_3(A'_2 - A'_1)\sinh \tilde{L},$$

(20)

$$(A'_2 + A'_1)\sinh \tilde{L} = \frac{c_5(\phi^+ - \phi^-)}{\sinh \tilde{L}} + c_5(A'_2 + A'_1)\cosh \tilde{L}.$$

(21)

We thus obtain the constants $A'_{1,2}$ in the form $A'_i = A'_{i0} + A'_{i+}\phi^+ + A'_{i-}\phi^-$, which again we shall not explicate.

Matching the profiles for all values of $\phi^\pm$ corresponds to solving for $\{c_1, c_3, c_5\}$ the system $\{A'_{i0} = A_{i0}, A'_{i+} = A_{i+}, A'_{i-} = A_{i-}\}$, which reads after some simplifications

$$\frac{\phi_0}{\sinh(\tilde{a} - \tilde{L})} = \frac{c_1}{c_3 \sinh \tilde{L} + \cosh \tilde{L}},$$

(22)

$$\frac{1 + c_3 c_5}{(c_5 - \tanh \tilde{L})(1 + c_3 \tanh \tilde{L})\cosh \tilde{L}} = 0,$$

(23)

$$\frac{2\sinh(\tilde{L} + \tilde{a})}{\sinh(2\tilde{L})\sinh(\tilde{L} - \tilde{a})} = \frac{c_3 \tanh \tilde{L} + c_5 \coth \tilde{L}}{(\cosh \tilde{L} + c_3 \sinh \tilde{L})(\sinh \tilde{L} - c_5 \cosh \tilde{L})},$$

(24)

where $\tilde{a} = a/\xi$. Although this system is nonlinear, it is quite easy to determine its exact solution. We obtain

$$c_1 = \frac{\phi_0}{\sinh \tilde{a}},$$

(25)

$$c_3 = -\coth \tilde{a},$$

(26)

$$c_5 = \tanh \tilde{a}.$$

(27)

Note that these Wilson coefficients do not depend on the distance L at which the DBCs are applied. This is probably

a peculiarity of the 1D character of this problem. Note also that the value of the field fixed by the inclusion, i.e., ϕ_0, only appears in c_1. The whole matching process is illustrated in Fig. 3.

2. Interaction free energy between inclusions of type A′

Let us now determine the interaction free-energy between two inclusions of type A′. We assume that they mimic two type-A inclusions of size $2a$ and imposed field values ϕ_1 and ϕ_2. We thus have the exact counterpart of the problem treated in Sec. II A. As previously, we place the first inclusion at $x = 0$ and the second one at $x = R$. From Eqs. (25)–(27), we see that the two inclusions have the same c_3 and c_5 coefficients, but their first coefficients are different, equal to $c_1 = \phi_1/\sinh \tilde{a}$ and $c_1' = \phi_2/\sinh \tilde{a}$, respectively.

We need to calculate the free-energy $F(R) = -k_B T \ln Z(R)$ associated with the total Hamiltonian $\mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2$, where the latter two terms correspond to the contributions given by Eq. (15). However, because the mean-field profile $\Phi(x)$ will be discontinuous at $x = 0$ and $x = R$, just as in the case of Fig. 3, we cannot proceed as previously and calculate separately the mean-field part of the interaction and the Casimir-like correction. Indeed, the quantity $\Phi'(x)^2$ that would appear in $\mathcal{H}_0$ contains a squared Dirac distribution, which is not integrable. Let us emphasize that given the form of the Hamiltonian, the fields $\phi(x)$ sampled by the fluctuations are necessarily continuous. It is thus only the average $\langle \phi(x) \rangle$ that will develop discontinuities (in the limit, actually, where the ultraviolet cutoff is sent to infinity).

We need, thus, to compute the partition function

$$Z(R) = \int \mathcal{D}[\phi] \, e^{-\beta[\mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2]}, \tag{28}$$

where $\beta = 1/(k_B T)$ is the inverse temperature. Making use of the Green's function $\Gamma(x) = (2\epsilon)^{-1} e^{-|x|/\xi}$, we may write $\mathcal{H}_0 = \frac{1}{2} \int dx \, dy \, \phi(x) \Gamma^{-1}(x,y) \phi(y)$. Then calling $\bar{c}_1 = 2\epsilon c_1, \bar{c}_1' = 2\epsilon c_1', \bar{c}_3 = 2\epsilon c_3$, and $\bar{c}_5 = 2\epsilon \xi^2 c_5$, we have

$$\mathcal{H}_1 + \mathcal{H}_2 = S + \int dx [\bar{c}_1 \delta(x) + \bar{c}_1' \delta(x - R)] \phi(x), \tag{29}$$

where

$$S = \frac{\bar{c}_3}{2} \phi^2(0) + \frac{\bar{c}_3}{2} \phi^2(R) + \frac{\bar{c}_5}{2} \phi'^2(0) + \frac{\bar{c}_5}{2} \phi'^2(R). \tag{30}$$

Applying four Hubbard-Stratonovich transformations to linearize the four quadratic terms in S, i.e.,

$$\exp\left[-\frac{\beta}{2} \bar{c}_3 \phi(0)^2 \right] \propto \int dk_1 \exp\left[-\frac{k_1^2}{2\beta \bar{c}_3} + i k_1 \phi(0) \right], \tag{31}$$

and so on, we can integrate out the field ϕ, which yields

$$Z = \int d^4k \exp\left[-\frac{k_1^2 + k_2^2}{2\beta \bar{c}_3} - \frac{k_3^2 + k_4^2}{2\beta \bar{c}_5} \right. $$
$$\left. + \frac{\beta}{2} \int dx \, dy \, T(x) \Gamma(x - y) T(y) \right], \tag{32}$$

with

$$T(x) = \left(\bar{c}_1 - \frac{i}{\beta} k_1 \right) \delta(x) + \frac{i}{\beta} k_3 \delta'(x)$$
$$+ \left(\bar{c}_1' - \frac{i}{\beta} k_2 \right) \delta(x - R) + \frac{i}{\beta} k_4 \delta'(x - R). \tag{33}$$

Note that we shall systematically discard in Z any multiplicative constants independent of R. Ordering the terms, we obtain

$$Z = \int d^4k \exp\left(\beta \bar{c}_1 \bar{c}_1' \Gamma(R) + k_i N_i - \frac{1}{2\beta} k_i M_{ij} k_j \right), \tag{34}$$

with

$$M = \begin{pmatrix} \frac{1}{\bar{c}_3} + \Gamma(0) & \Gamma(R) & 0 & \Gamma'(R) \\ \Gamma(R) & \frac{1}{\bar{c}_3} + \Gamma(0) & -\Gamma'(R) & 0 \\ 0 & -\Gamma'(R) & \frac{1}{\bar{c}_5} - \Gamma''(0) & -\Gamma''(R) \\ \Gamma'(R) & 0 & -\Gamma''(R) & \frac{1}{\bar{c}_5} - \Gamma''(0) \end{pmatrix}, \tag{35}$$

and

$$N = -i \begin{pmatrix} \bar{c}_1 \Gamma(0) + \bar{c}_1' \Gamma(R) \\ \bar{c}_1' \Gamma(0) + \bar{c}_1 \Gamma(R) \\ -\bar{c}_1' \Gamma'(R) \\ \bar{c}_1 \Gamma'(R) \end{pmatrix}. \tag{36}$$

Note that several components of M vanish due to the even character of $\Gamma(x)$. We therefore obtain

$$F(R) = -k_B T \ln Z$$
$$= -\frac{1}{2} N_i M_{ij}^{-1} N_j - \bar{c}_1 \bar{c}_1' \Gamma(R) + \frac{k_B T}{2} \ln \det M, \tag{37}$$

which yields $\tilde{F}(R) = F(R)/\epsilon$ in the form

$$\tilde{F}(R) = \tilde{F}_{\text{mf}} + \tilde{F}_{\text{C}}, \tag{38}$$

where $\tilde{F}_{\text{C}}$ corresponds to the third term of Eq. (37), i.e., the Casimir-like contribution, and $\tilde{F}_{\text{mf}}$ to the first two terms of Eq. (37), i.e., the mean-field contribution.

Taking into account the precise form of $\Gamma(x)$ given above, and calculating the determinant of M, we obtain after some simplifications:

$$\tilde{F}_{\text{C}}(R) = \frac{k_B T}{2\epsilon} \ln \left(1 - \frac{(c_3 + c_5)^2}{(c_3 + 1)^2 (c_5 - 1)^2} e^{-2R/\xi} \right). \tag{39}$$

Now, replacing the Wilson coefficients by their value given by Eqs. (25)–(27), we recover exactly the Casimir-like contribution to the interaction between two real inclusions of type A, i.e., the third term of Eq. (6).

Concerning the mean-field part, the calculations yield a form similar in structure to the first two terms of Eq. (6):

$$\tilde{F}_{\text{mf}}(R) = (c_1 + c_1')^2 \tilde{f}_1(R) - 2 c_1 c_1' \tilde{f}_2(R), \tag{40}$$

$$\tilde{f}_1(R) = \frac{(c_5 - 1)(c_3 + c_5)}{(c_3 + 1)(c_3 + c_5)^2 - (c_3 + 1)^3 (c_5 - 1)^2 e^{2R/\xi}}, \tag{41}$$

$$\tilde{f}_2(R) = \frac{c_5 - 1}{(c_3 + 1)[c_3 + c_5 + (c_3 + 1)(c_5 - 1)e^{R/\xi}]}. \tag{42}$$

Again, replacing the Wilson coefficients by their values given by Eqs. (25)–(27), with $c_1 = \phi_1/\sinh \tilde{a}$ and $c_1' = \phi_2/\sinh \tilde{a}$,

we recover exactly the mean-field contribution to the interaction between two real inclusions of type A, i.e., the first two terms of Eq. (6).

The EFT method thus captures exactly—in the nonlinear framework—the interaction free energy of the type-A particles.

B. Effective pointlike inclusions of type B'

Since the type-B inclusions are invariant under the simultaneous transformations $x \to -x$ and $\phi \to -\phi$, the Hamiltonian of the type-B' inclusions must be of the form:

$$\frac{\mathcal{H}_i}{2\epsilon} = \int dx \left[c_2 \xi \phi'(x) + \frac{c_3}{2} \phi^2(x) + \frac{c_5 \xi^2}{2} \phi'(x)^2 \right] \delta(x - x_i),$$
(43)

where, again, the Wilson coefficients c_i are dimensionless.

1. Matching between inclusions of types B and B'

As previously, we consider an arbitrary background $\Phi_{\text{bg}}(x)$ satisfying $\mathcal{L}\Phi_{\text{bg}} = 0$ together with the DBCs $\Phi_{\text{bg}}(\pm L) = \phi^\pm$, and we place a real inclusion of type B at the origin, imposing a jump α of the field. Calling again $\Phi_{\text{tot}} = \Phi_{\text{bg}} + \Phi_r$, we solve $\mathcal{L}\Phi_r = 0$ with the boundary conditions $\Phi_r(\pm L) = 0$ and

$$\Phi_{\text{tot}}(a) - \Phi_{\text{tot}}(-a) = \alpha,$$
(44)

$$\Phi'_{\text{tot}}(-a) = \Phi'_{\text{tot}}(a).$$
(45)

The solution is still of the form (18) but with coefficients B_1 and B_2.

In the case of an effective pointlike inclusion of type B' placed at $x = 0$, the equation for the response field Φ_r becomes

$$\mathcal{L}\Phi_r(x) = 2c_3 \Phi_{\text{tot}}(0)\xi^{-1}\delta(x) - 2[c_2 + c_5 \xi \Phi'_{\text{tot}}(0)]\delta'(x).$$
(46)

The solution satisfying $\Phi_r(\pm L) = 0$ has again the same form as Eq. (18), however with constants B'_i instead of A_i. Matching the two profiles for all values of $\phi^\pm$ yields a nonlinear system for the unknowns $\{c_2, c_3, c_5\}$, the exact solution of which is

$$c_2 = -\frac{\alpha}{2 \cosh \tilde{a}},$$
(47)

$$c_3 = -\tanh \tilde{a},$$
(48)

$$c_5 = \tanh \tilde{a},$$
(49)

with, again, $\tilde{a} = a/\xi$. The matching process is illustrated in Fig. 4.

2. Interaction free energy between inclusions of type B'

The calculations proceed exactly as in the case of the inclusions of type A', yielding

$$Z = \int d^4k \exp\left(-\beta \bar{c}_2 \bar{c}'_2 \Gamma''(R) + k_i N_i - \frac{1}{2\beta} k_i M_{ij} k_j \right).$$
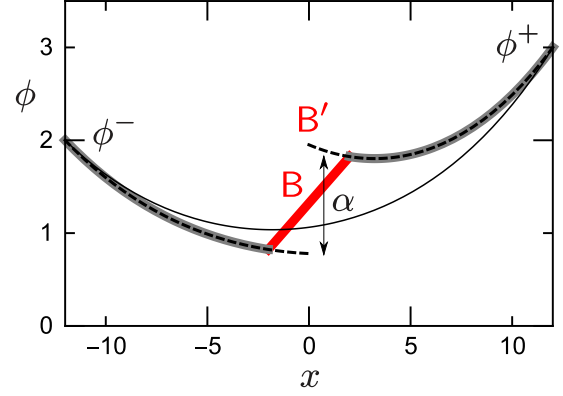(50)

FIG. 4. In a background (thin black line) set by the boundary values $\phi^\pm$ separated by a distance $2L$, either a real inclusion of type B or its effective counterpart of type B' are placed at the origin. The Wilson coefficients of the effective inclusion are such that the deformation of the type-B inclusion (thick gray line) and the deformation of the type-B' inclusion (dashed black line) match whatever the background. Here, the parameters are $\phi^- = 2, \phi^+ = 3, L = 12, \alpha = 1, a = 2$, and $\xi = 8$. The inclusions are drawn in red.

The matrix M is unchanged, while N is now given by

$$N = i \begin{pmatrix} -\bar{c}'_2 \Gamma'(R) \\ \bar{c}_2 \Gamma'(R) \\ \bar{c}_2 \Gamma''(0) + \bar{c}'_2 \Gamma''(R) \\ \bar{c}'_2 \Gamma''(0) + \bar{c}_2 \Gamma''(R) \end{pmatrix},$$
(51)

where $\bar{c}_2 = 2\epsilon \xi c_2$ and $\bar{c}'_2 = 2\epsilon \xi c'_2$. This yields now

$$F(R) = -k_B T \ln Z$$
$$= -\frac{1}{2} N_i M_{ij}^{-1} N_j + \bar{c}_2 \bar{c}'_2 \Gamma''(R) + \frac{k_B T}{2} \ln \det M, \quad (52)$$

and thus the normalized interaction free energy is

$$\tilde{F}(R) = \tilde{F}_{\text{mf}} + \tilde{F}_{\text{C}},$$
(53)

with the Casimir-like contribution $\tilde{F}_{\text{C}}$ corresponding to the third term of Eq. (52) and the mean-field contribution $\tilde{F}_{\text{mf}}$ corresponding to the first two terms of Eq. (52).

It follows that the Casimir-like energy is still formally given by Eq. (39), but since we have now $c_3 = -c_5$, it vanishes, just as in the case of the real type-B inclusions.

In a way similar to Eq. (40), we obtain for the mean-field part the result:

$$\tilde{F}_{\text{mf}}(R) = (c_2 + c'_2)^2 \tilde{g}_1(R) - 2c_2 c'_2 \tilde{g}_2(R),$$
(54)

$$\tilde{g}_1(R) = \frac{(c_3 + 1)(c_3 + c_5)}{(c_5 - 1)(c_3 + c_5)^2 - (c_3 + 1)^2 (c_5 - 1)^3 e^{2R/\xi}}, \quad (55)$$

$$\tilde{g}_2(R) = \frac{c_3 + 1}{(c_5 - 1)(c_3 + c_5) - (c_3 + 1)(c_5 - 1)^2 e^{R/\xi}}. \quad (56)$$

Because $c_3 = -c_5$ for type-B' inclusions, we see at once that $\tilde{g}_1$ vanishes; hence the interaction is going to be proportional to $c_2 c'_2$, that is to $\alpha_1 \alpha_2$. Replacing the Wilson coefficients by their values given by Eqs. (47)–(49), with $c_2 = -\alpha_1/(2 \cosh \tilde{a})$ and $c'_2 = -\alpha_2/(2 \cosh \tilde{a})$, we recover exactly the mean-field contribution to the interaction between two real inclusions of type B, i.e., Eq. (12).

Again, the EFT method captures exactly—in the nonlinear framework—the interaction free energy of the type-B particles.

IV. EFFECTIVE FIELD THEORY TREATMENT IN THE LINEAR RESPONSE APPROXIMATION

We now start again the construction of the EFT pointlike inclusions mimicking the inclusions of type A and B, but we apply this time the linear response approximation (LRA). We shall denote the resulting inclusions by A'' and B''.

A. Effective pointlike inclusions of type A''

Let us thus consider again the problem of replacing the inclusions of type A by pointlike inclusions. By symmetry, their Hamiltonian still has to be of the form Eq. (15). However, as explained in Sec. I, we shall neglect the feedback terms in the resulting Eq. (19). In other words we shall replace Φ_{tot} by Φ_{bg} in this equation. We thus proceed as in Sec. III A 1; the only difference is that the equation for the response field becomes

$$\mathcal{L}\Phi_r(x) = 2[c_1 + c_3\Phi_{\text{bg}}(0)]\xi^{-1}\delta(x) - 2c_5\xi\,\Phi'_{\text{bg}}(0)\delta'(x) \tag{57}$$

instead of Eq. (19). As a consequence, the term containing A'_1 and A'_2 in the right-hand side of the system (20) disappears, and the matching system becomes linear. The solutions of this system are

$$-\frac{\cosh\tilde{L}}{\sinh(\tilde{L}-\tilde{a})}\phi_0 \sim -\sqrt{\gamma}\phi_0 \equiv c_1, \tag{58}$$

$$\frac{\cosh(\tilde{L})\cosh(\tilde{a})}{\sinh(\tilde{L}-\tilde{a})} \sim \frac{1}{2}(\gamma+1) \equiv c_3, \tag{59}$$

$$\frac{\sinh(\tilde{L})\sinh(\tilde{a})}{\sinh(\tilde{L}-\tilde{a})} \sim \frac{1}{2}(\gamma-1) \equiv c_5. \tag{60}$$

where $\tilde{a} = a/\xi$ and $\gamma = \exp(2\tilde{a})$. These values converge to finite limits as $L \to \infty$, as indicated above. We shall thus naturally define the Wilson coefficients as those limits. Indeed, the idea of EFT is to match at least asymptotically any background.

At this point, it looks unlikely that the type-A'' inclusions will reproduce exactly the interaction free energy between two inclusions of type A. This is because the matching is only asymptotic, while there is a finite number of Wilson coefficients. We may however expect the correct asymptotic interaction. The response to a background of an inclusion of type A'' is illustrated in Fig. 5. It is apparent that this response is bad: the inclusions of type A'' do not respond like the type-A inclusions, contrary to the type-A' inclusions (see Fig. 3 for comparison).

Interaction free energy between inclusions of type A''

Because the Hamiltonian of the type-A'' inclusions still has the general form given by Eq. (15), all the calculations of Sec. III A 2 hold. Hence the Casimir part of the interaction free energy is still given by Eq. (39) and the mean-field part of the interaction free energy is still given by Eq. (40). Inserting the new Wilson coefficients, given by Eqs. (58)–(60), we obtain

$$\tilde{F}_{\text{C}} = \frac{k_{\text{B}}T}{2\epsilon}\ln\left(1 - \frac{16}{(\gamma^2-9)^2}e^{-2R/\xi}\right). \tag{61}$$

$$F_{\text{mf}} = \frac{16(\gamma-3)}{16(\gamma+3) - (\gamma-3)^2(\gamma+3)^3 e^{2R/\xi}}(\phi_1+\phi_2)^2$$
$$- \frac{8(\gamma-3)}{(\gamma+3)[4+(\gamma^2-9)e^{R/\xi}]}\phi_1\phi_2. \tag{62}$$

Clearly, these formulas do not match the interaction between two inclusions of type A, given by Eq. (6). The structure is similar, but the prefactors are different.

Let us look at the asymptotic behavior. From Eq. (6), we have for the interaction between two type-A inclusions $\tilde{F} \sim -k_{\text{B}}T/(2\epsilon)z^2 + (\phi_1+\phi_2)^2z^2 - 2\phi_1\phi_2 z$, with $z = \exp(-R/\xi)$. Hence, even the asymptotic behaviors differ. Even in the case $\xi \to \infty$, which corresponds to $\gamma = 1$, we obtain for the type-A'' inclusions $\tilde{F} \sim -k_{\text{B}}T/(8\epsilon)z^2 + (\phi_1 + \phi_2)^2z^2/8 - \phi_1\phi_2 z/2$ that differs from the interaction between type-A inclusions. We conclude that the LRA never provides the correct interaction for inclusions fixing the value of the field.
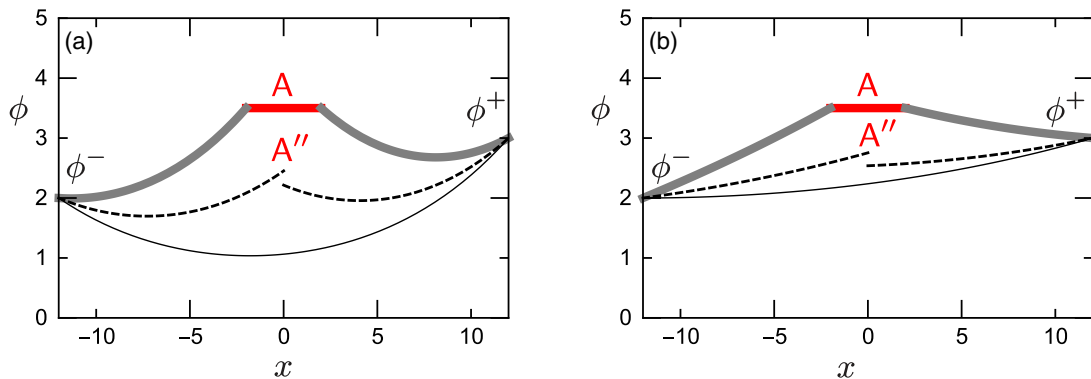

FIG. 5. Response of an inclusion of type A'' (LRA) to a background, and comparison with the response of a real type-A inclusion. (a) The parameters are $\phi^- = 2, \phi^+ = 3, L = 12, \phi_0 = 3.5, a = 2$, and $\xi = 8$, exactly as in Fig. 3. (b) Same parameters except that $\xi = 25$ instead of $\xi = 8$. In both cases the discrepancy between the real field profile (thick gray line) and the effective one (dashed line) is large.

B. Effective pointlike inclusions of type B″

We proceed exactly as in Sec. IV A. Neglecting the feedback terms in Eq. (46), the equation for the response field becomes in the LRA:

$$\mathcal{L}\Phi_r(x) = 2c_3\Phi_{\text{bg}}(0)\xi^{-1}\delta(x) - 2[c_2 + c_5\xi\,\Phi'_{\text{bg}}(0)]\delta'(x),$$
(63)

and the matching system that allows to determine the Wilson coefficients becomes linear. Its solutions are

$$-\frac{\sinh\tilde{L}}{2\sinh(\tilde{L}-\tilde{a})}\alpha \sim -\frac{1}{2}\sqrt{\gamma}\alpha \equiv c_2,$$
(64)

$$-\frac{\cosh(\tilde{L})\sinh(\tilde{a})}{\cosh(\tilde{L}-\tilde{a})} \sim \frac{1}{2}(1-\gamma) \equiv c_3,$$
(65)

$$\frac{\sinh(\tilde{L})\sinh(\tilde{a})}{\sinh(\tilde{L}-\tilde{a})} \sim \frac{1}{2}(\gamma-1) \equiv c_5,$$
(66)

where, again, $\tilde{a} = a/\xi$ and $\gamma = \exp(2\tilde{a})$. These values converge to finite limits as $L \to \infty$, as indicated above. As in the previous section, since the idea of EFT is to match at least asymptotically any background, we define the Wilson coefficients as those limits.

The response of an inclusion of type B″ is illustrated in Fig. 6(a). It is apparent that this response is bad (see Fig. 4 for comparison). It becomes however satisfying in the case where $\xi/L \gg 1$, as shown in Fig. 6(b). We might thus expect a correct interaction behavior in this limit.

Interaction free energy between inclusions of type B″

Again, because the Hamiltonian of the type-B″ inclusions still has the general form given by Eq. (43), all the calculations of Sec. III B 2 hold. Hence the Casimir part of the interaction free energy is still given by Eq. (39) and the mean-field part of the interaction free energy is still given by Eq. (54). Inserting the new Wilson coefficients, given by Eqs. (64)–(66), we obtain

$$\tilde{F}_{\text{C}} = 0,$$
(67)

$$F_{\text{mf}} = \frac{2}{(\gamma-3)^2}\alpha_1\alpha_2 e^{-\tilde{R}}.$$
(68)

First, we see that the Casimir contribution vanishes, just as in the case of the real type-B inclusions. The mean-field part, however, differs. Indeed, comparing with Eq. (12), we have a prefactor $2/(\gamma-3)^2$ instead of $1/2$. However, the mean-field contribution to the interaction becomes now exact, at all distances, in the case $\xi \to \infty$ for which the prefactor $2/(\gamma-3)^2 \to 1/2$. Note that the error in the prefactor of the interaction is of order $2a/\xi$, therefore $\xi \gg a$ is required for a good accuracy of the interaction law.

We have thus found that the type-B″ inclusions do not capture the interaction between real type-B inclusions, except in the limit $\xi \to \infty$, i.e., for massless theories, where it becomes exact at all distances.

V. SUMMARY AND DISCUSSION

In this paper, using a one-dimensional (1D) Gaussian field $\phi(x)$ with a mass term $\propto \xi^{-2}\phi^2$ and a squared gradient term $\propto \phi'^2$, we have investigated the interaction between two types of extended embedded particles of length $2a$: inclusions of type A that fix the value of the field, and inclusions of type B that fix a jump in the value of the field.

Applying the principles of effective field theory (EFT), and using an exact nonlinear formalism, we have determined pointlike inclusions of type A′ and B′ that produce the same response as the real type A and B inclusions to any underlying background field. We have verified that these effective inclusions behave exactly as the type-A and type-B inclusions and produce the same exact interaction free energy, that can be decomposed into a Casimir-like contribution and a mean-field contribution.

We have then applied the EFT principles within the linear response approximation (LRA), which is in general easier to achieve. We have found that the resulting pointlike inclusions of type A″ and B″ do not reproduce the interaction free energy of the real type-A and type-B particles—even asymptotically—except in one case: for type-B particles when the mass term goes to zero, i.e., when the correlation length ξ goes to infinity.

Regarding the interaction free energy between two inclusions, the LRA should obviously work better when one inclusion does not perturb much the background field created by the other inclusion. This is clearly not the case for the
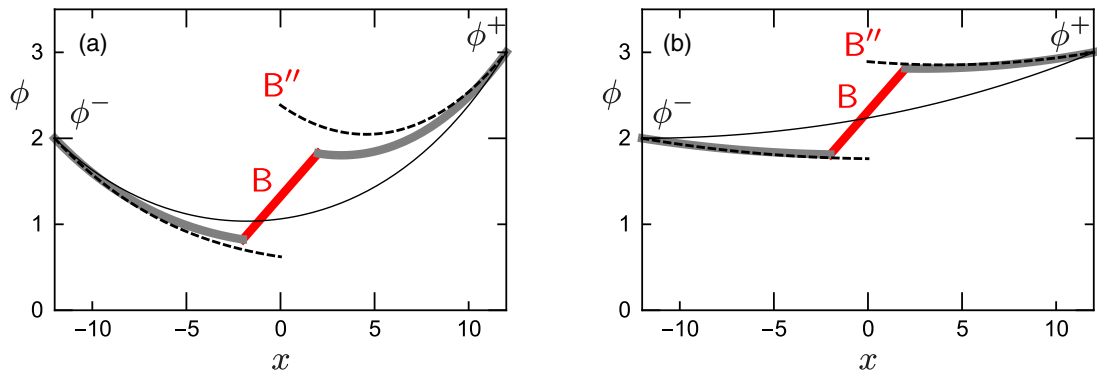

FIG. 6. Response of an inclusion of type B″ (LRA) to a background, and comparison with the response of a real type-B inclusion. (a) The parameters are $\phi^- = 2, \phi^+ = 3, L = 12, \alpha = 1, a = 2$, and $\xi = 8$, exactly as in Fig. 4. (b) Same parameters except that $\xi = 25$ instead of $\xi = 8$. In the latter case the response of the type-B″ inclusion is very good (agreement between the thick gray line and the dashed line).

type-A inclusions, since they set a value of the field that is independent of the background. One can see indeed in Fig. 5 that the type-A″ inclusion lies far away from the background (thin solid line). This is true whatever the value of ξ, as can be seen by comparing Figs. 5(a) and 5(b).

The situation is more favorable, however, for the inclusions of type B. Since they set only a difference, i.e., a jump, in the value of the field, while the average field value is free to adjust to equilibrium, these inclusions are less constraining. In order to minimize the total free energy, the type-B inclusions naturally tend to adjust to the background. One can see indeed in Fig. 6 that the type-B″ inclusion lies relatively close to the background (thin solid line), especially in the case where the correlation length is very large [Fig. 6(b)]. This is most probably the reason why the type-B″ inclusions produce the correct interaction free energy in the limit where ξ goes to infinity.

We conclude that in 1D systems, one needs to use nonlinear EFT in order to work out properly the two-body interaction, and by extension also the many-body interaction, between inclusions. As an example, this might be useful for inclusions in polymers.

In 2D systems, the EFT method is more difficult to apply, as in general an infinite series of Wilson coefficients is needed [17]. Working with the nonlinear EFT equations is very challenging. The EFT method, within the simplified LRA framework, has successfully been applied by Rothstein to lacunae in free bosonic fields [14], and by Deserno *et al.* to protein inclusions in membranes [17] and colloids in tense interfaces [15]. These inclusions are similar to the type-B inclusions discussed above, because they fix a local curvature of the membrane, or contact angle of the interface, or simple Neumann boundary conditions, without any absolute position or tilt constraints. In all cases, the theory was also massless. The EFT in the linear response approximation was shown to work very well in these cases, as verified by comparing the results with those of exact calculations up to six orders in the inverse particles separation [12,17], and to exact calculations obtained from conformal field theory [14]. The situation for 2D systems with more constraining particles, or massive fields, seems less clear to us, although it is generally believed that in two dimensions the matching of the Wilson coefficients may be performed in any way, even linear [25].

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