

Minority-spin dynamics in the nonhomogeneous Ising model: Diverging time scales and exponents

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We investigate the dynamical behavior of the Ising model under a zero-temperature quench with the initial fraction of up spins $0 \leq x \leq 1$. In one dimension, the known results for persistence probability are verified; it shows algebraic decay for both up and down spins asymptotically with different exponents. It is found that the conventional finite-size scaling is valid here. In two dimensions, however, the persistence probabilities are no longer algebraic; in particular for $x \leq 0.5$, persistence for the up (minority) spins shows the behavior $P_{\min}(t) \sim t^{-\gamma} \exp[-(t/\tau)^\delta]$ with time t , while for the down (majority) spins, $P_{\text{maj}}(t)$ approaches a finite value. We find that the timescale τ diverges as $(x_c - x)^{-\lambda}$, where $x_c = 0.5$ and $\lambda \simeq 2.31$. The exponent γ varies as $\theta_{2d} + c_0(x_c - x)^\beta$ where $\theta_{2d} \simeq 0.215$ is very close to the persistence exponent in two dimensions; $\beta \simeq 1$. The results in two dimensions can be understood qualitatively by studying the exit probability, which for different system size is found to have the form $E(x) = f[(\frac{x-x_c}{x_c})L^{1/\nu}]$, with $\nu \approx 1.47$. This result suggests that $\tau \sim L^{\bar{z}}$, where $\bar{z} = \frac{\lambda}{\nu} = 1.57 \pm 0.11$ is an exponent not explored earlier.

DOI: [10.1103/PhysRevE.93.052113](https://doi.org/10.1103/PhysRevE.93.052113)**I. INTRODUCTION**

Dynamical phenomena associated with the ordering process of spin systems have been studied for a long time. While the domain growth and behavior of correlation functions were the original issues of interest [1,2], other phenomena such as persistence and its variants have been extensively studied more recently [3,4]. It is indeed astonishing to note that even after decades of research on these topics, newer features have been revealed. For example, in the zero-temperature quenching dynamics of the Ising model, frozen states [5,6] and blinkers [7–10] as nonequilibrium steady states have been shown to occur in dimensions greater than one. Recent studies of the zero-temperature quenching dynamics in the two-dimensional Ising model indicated that the domain growth behavior can be explained in the framework of percolation theory [11–13]. Different ways of employing the zero-temperature quench have also been explored in some very recent works in the Ising model [14–16]. Models with more complicated interactions with binary spin variables have often indicated the existence of dynamical universality classes other than the simple Ising model even in one dimension [17].

Prior to the discovery of the persistence phenomena [3] in the Ising and other spin models, it was believed that there is only one dynamical exponent that governs the behavior of the system during the ordering dynamics. The persistence probability showed a power-law decay with an exponent that could not be related to other known static or dynamic exponents. This indicated that there is a second independent dynamic exponent. Since the domain size and persistence show power-law behavior, there is essentially no timescale associated with these quantities. Physically this means that there are finite probabilities existing even at very large timescales. On the other hand, diverging timescales can be associated with dynamical behavior close to the critical point. The typical relaxation behavior of the order parameter m as a function of time t is exponentially close to the critical temperature, $m(t) \sim \exp(-t/\tau)$, with τ diverging as the critical temperature is approached.

In the present work, we consider the zero-temperature single-spin flip Glauber dynamics of Ising model with nearest neighbor interaction when the initial configuration is non-

homogeneous, i.e., the fraction of up and down spins are different. x is the fraction of up spins ranging between 0 and 1. $x = 0.5$ is the homogeneous case and due to up- and down-spin symmetry, it is sufficient to take $x \leq 0.5$. The spins are otherwise uncorrelated. Three types of persistence behavior are studied: P_+ (persistence for up spins), P_- (persistence for down spins), and P_{total} (persistence for the total system). The exact expression for the variation of the above-mentioned quantities are known in one dimension in the thermodynamic limit. We have numerically evaluated the persistence probabilities for finite systems and checked that the familiar finite-size scaling [18,19] is valid for all values of x . Our main results are for the two-dimensional model, here we have numerically obtained the behavior of P_+ , P_- , and P_{total} and find the presence of a diverging timescale associated with P_+ or P_- , whichever corresponds to the initial minority.

Another quantity associated with the ordering process that has been studied quite intensely in recent years is the exit probability. As a function of x (the initial fraction of up spins), the exit probability $E(x)$ is defined as the probability that the final state comprises all up spins. The behavior of the exit probability helps in understanding qualitatively the persistence probability in the Ising model. Hence, we have also conducted a detailed study of the exit probability in two dimensions to gain more insight into the problem.

In the next two sections, the results for one- and two-dimensional Ising models are discussed, respectively. In the last section, summary and some discussions are presented.

II. RESULTS IN ONE DIMENSION AND FINITE-SIZE SCALING

For systems with nonuniform initial condition, i.e., with unequal initial fraction of up and down spins, the exact result for the Ising model in one dimension shows that the density of persistent spins decays algebraically as [20]

$$P_{\pm}(t) \sim t^{-\theta_{\pm}}. \quad (1)$$

The exponents are different for the up and down spins,

$$\theta_{\pm} = \theta(x_{\pm}) = \frac{2}{\pi^2} \left[\cos^{-1} \left(\sqrt{2}x_{\pm} - \frac{1}{\sqrt{2}} \right) \right]^2 - \frac{1}{8}, \quad (2)$$

where x_+ (x_-) is the initial concentration of up (down) spins. For equal initial fraction of up and down spins (i.e., for $x_{\pm} = 0.5$) we have $\theta_+ = \theta_- = \frac{3}{8}$ [3]. In this paper we follow the notation $x = x_+$ unless otherwise specified.

During the coarsening, the typical domain size D shows a power-law growth with time,

$$D(t) \propto t^{1/z}, \quad (3)$$

z being the domain growth exponent. For the Ising model, $z = 2$ in all dimensions [1].

While the persistence exponents are x -dependent, the domain growth exponent z is identical for all x . This is easy to explain. Consider, e.g., the growth phenomena for $x = 0.5$. Immediately after the time evolution starts, x will attain a different value. One might as well consider this to be the initial state, hence $z = 2$ for all x [1,2].

In one dimension, it is known that for the homogeneous case, the persistence probability obeys finite-size scaling. The behavior of the persistent probability $P(t, L)$ in a system with linear dimension L can be summarized as [18]

$$P(t, L) = L^{-z\theta} f(t/L^z) = L^{-\alpha} f(t/L^z), \quad (4)$$

with $\alpha = z\theta$. We check whether this is also valid when $x \neq 0.5$. Obviously, if valid, the associated exponent α should vary with x , since z is constant. Finite-size scaling analysis in one dimension was performed for all three variants of the persistence probability for $x \leq 0.5$. Simulations were made for system sizes $L \leq 1000$ taking average over at least 1000 configurations. In each Monte Carlo step, L spins are chosen randomly and updated. Asynchronous updating rule is used and periodic boundary condition imposed. We find that indeed the finite-size scaling Eq. (4) is valid, as the curves for different values of L collapsed with proper choice of α , keeping $z \simeq 2$ (Fig. 1). It is checked that these values of α are consistent with the relation $\alpha = z\theta$. In Table I we have summarized the

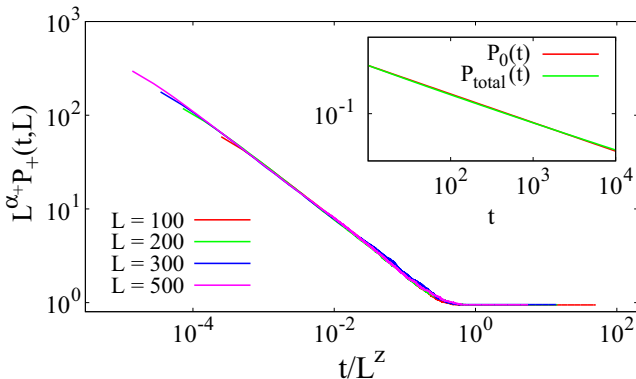


FIG. 1. Data collapse for persistence probability of up spins for systems with $x = 0.3$ in the one dimensional Ising model. This particular collapse was obtained with $\alpha_+ = 1.0$ and $z = 1.95$. Similar data collapse can be obtained for P_- and P_{total} for other values of x . Inset shows variation of $P_0(t)$ and $P_{\text{total}}(t)$ for $x = 0.4$; these calculations were made for a system size $L = 1000$.

TABLE I. Persistence and other exponents obtained for Ising model in one dimension using numerical simulation.

x	θ_0	θ_+	θ_-	α_0	α_+	α_-
0.3	0.227	0.562	0.212	0.42	1.0	0.40
0.4	0.318	0.463	0.285	0.64	0.94	0.57

values of θ and α obtained for two different values of x . The values of θ_+ and θ_- differ from the exact values by less than 0.05%. As for the up, down, and total spins θ is different, the corresponding values α_+ , α_- , and α_0 also turn out to be different.

$P_{\text{total}}(t)$, the total persistence is expected to be a weighted average of P_+ and P_- . We calculate the quantity

$$P_0(t) = x_+ P_+(t) + x_- P_-(t) \quad (5)$$

to check whether $P_0(t)$ and $P_{\text{total}}(t)$ are equal and find a very good agreement indeed (inset of Fig. 1). Even though Eq. (5) is valid, it is interesting to note that the exponent θ_0 associated with the total persistence probability is different from the $\min[\theta_+, \theta_-]$, albeit close to it (see Table I).

Qualitatively, the algebraic decay of P_+ and P_- can be explained from the behavior of the exit probability $E(x)$, which in the one-dimensional Ising model is given by $E(x) = x$. This indicates that there is a finite probability of ending up with either all up or all down spins even for values of $x \neq 0.5$ (except for $x = 0$ and 1).

III. RESULTS IN TWO DIMENSIONS

In the ordering dynamics of the two-dimensional Ising model, the three variants of persistence were studied for various values of x less than 0.5 on $L \times L$ square lattices with $L \leq 256$. Simulations were performed for 1000 configurations. Here also one Monte Carlo time step comprises L^2 updates and random asynchronous updating rule is used. Helical boundary condition has been imposed for the simulation in two dimensions. In Fig. 2, the results for x close to 0.5 are shown. As x deviates from 0.5, P_+ shows a faster decay while both P_- and P_{total} saturate at higher values. No appreciable finite-size dependence is found for these two persistence probabilities (inset of Fig. 2).

By the present convention, P_- is the persistence probability for the down spins, which is the initial majority. Generalizing the notation for initial majority and minority spins, we have P_{maj} saturating to a finite value and P_{min} going to zero. To explain why P_{maj} does not show system-size dependence, we can argue like this: had it been decreasing with size, it would ultimately go to zero. On the other hand, had it been increasing for larger sizes, it would approach 1. Both the possibilities are unrealistic and hence no system-size dependence is observed. Since P_{total} is basically a weighted average of P_+ and P_- , its nature is dominated by that of P_- , the majority spins, and hence it does not show finite-size dependence also.

The persistence probabilities do not show algebraic behavior unlike in the one-dimensional Ising model. The persistence probability for majority spins as well as for the total spins do not show much interesting behavior as they decay to rather

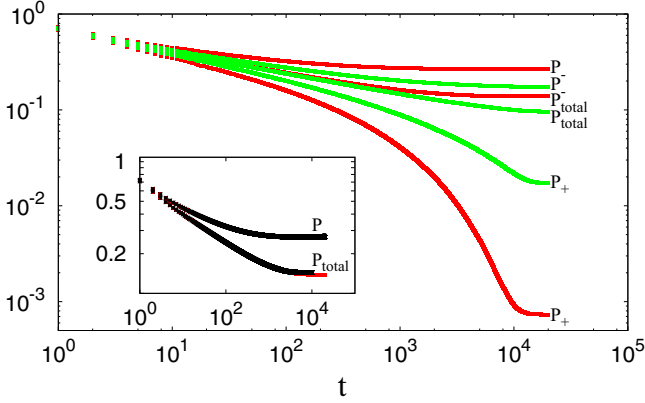


FIG. 2. Behavior of three types of persistence probabilities for two-dimensional Ising model for two different initial fraction of up spins with $L = 256$. The darker (red) curves correspond to $x = 0.482$ and the lighter (green) ones to $x = 0.492$. Inset shows persistence probabilities for total spins and down spins for $x = 0.482$ for two different system sizes. The lighter (red) curves correspond to $L = 256$ and the darker (black) ones to $L = 128$.

high saturation values and show no finite-size dependence. However, in the case of the minority spins, finite-size effect is apparent and a number of interesting features are revealed on further analysis. The decay of the persistence probability P_{\min} can be fitted to the form

$$P_{\min}(t) \sim t^{-\gamma} \exp[-(t/\tau)^\delta], \quad (6)$$

which is a combination of an algebraic decay accompanied by a stretched exponential cutoff (Fig. 3). We find that this fit is valid almost up to the time where P_{\min} saturates to a finite value. The fits become more accurate as x approaches 0.5. It is clear that P_{\min} extrapolates to a zero value for $L \rightarrow \infty$. However, the P_{\min} versus t curves for different values of L (shown in the inset of Fig. 3) shows that the finite-size dependence is evident only at times beyond which P_{\min} saturates. This is very similar to what happens for $x = 0.5$. Thus, it is sufficient to use Eq. (6) for the maximum size ($L = 256$) only.

No exact result is known for the persistence exponent even for $x = 0.5$ and one depends only on approximate estimates,

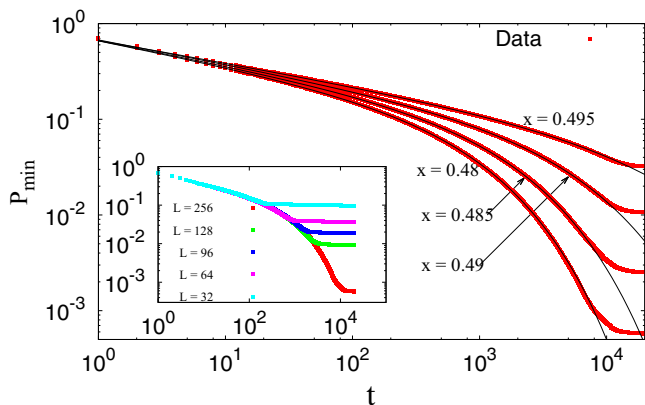


FIG. 3. $P_{\min}(t)$ against time t for several values of x along with the best fit curves according to Eq. (6) are shown. Inset shows persistence probabilities for $x = 0.48$ for five different system sizes.

most of which are numerical [4,21–23]. However, the behavior of the persistence probability has been found to be strictly a power law even for finite system sizes. The resultant exponent has been shown to have some finite time dependence on closer examination [23]. In the present case, therefore, we have varied the range of time when Eq. (6) is used. Noting there is not much variation in the values of τ and δ for different ranges, we have used the average values for further analysis.

In order that Eq. (6) is valid for $x \rightarrow 0.5$, τ must diverge at that limit. Also, the associated exponent should be close to 0.20 according to the more recent estimates of the persistence exponent in two dimensions. To check this, we analyzed the behavior of γ and τ as a function of $(x_c - x)$, where $x_c = 0.5$.

The variation of τ with $(x_c - x)$ is found to be of the form

$$\tau \sim (x_c - x)^{-\lambda}, \quad (7)$$

thus showing the divergence at $x \rightarrow 0.5$ (Fig. 4). The value of λ is numerically equal to 2.31 ± 0.16 . On the other hand, the data for γ shows some fluctuations but apparently has a linear variation with $x - x_c$. In general, γ can be written as $\theta_{2d} + c$, where

$$c = c_0(x_c - x)^\beta. \quad (8)$$

Putting $\beta = 1$, we obtain $\theta_{2d} = 0.215 \pm 0.004$, which is indeed very close to the persistence exponent in two dimensions; $c_0 = 4.11 \pm 0.37$. The data is shown in Fig. 4 (inset). The value of θ_{2d} is quite insensitive to the value of β , which may vary between 0.8 to 1.2 due to the scatter in the data. Thus, we find the presence of two quantities τ and c , which diverge and vanish, respectively, as $x \rightarrow 0.5$. The value of δ is $\mathcal{O}(10^{-1})$ and it also shows a slow decrease with x . However, it remains finite (≈ 0.4) even at values of x very close to 0.5.

The nature of the persistence probabilities in the two-dimensional Ising model can also be explained qualitatively by studying the exit probability. It is known that exactly at $x = 0.5$, the system may not reach the all spins up or down configuration always due to freezing [7–9]. Since for finite sizes it is possible that such frozen states may persist at small deviation from $x = 0.5$ as well, we considered only those configurations that led to the true ground states while calculating $E(x)$. $E(x)$ for the two-dimensional Ising model

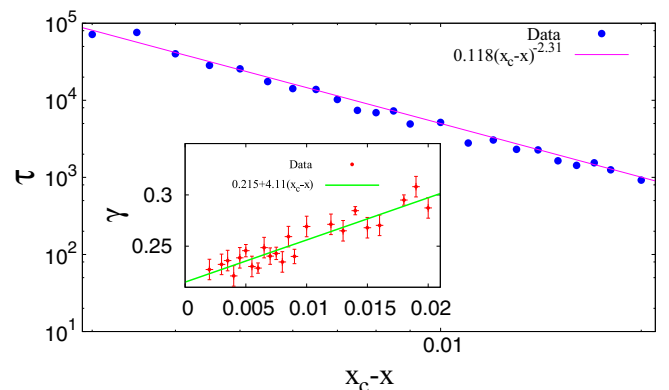


FIG. 4. Divergence of τ as $(x_c - x)^{-\lambda}$ with $\lambda = 2.37$. Inset shows variation of γ with $(x_c - x)$ as $\theta_{2d} + c_0(x_c - x)^\beta$, with $\theta_{2d} = 0.215$, $c_0 = 4.11$, and $\beta = 1$.

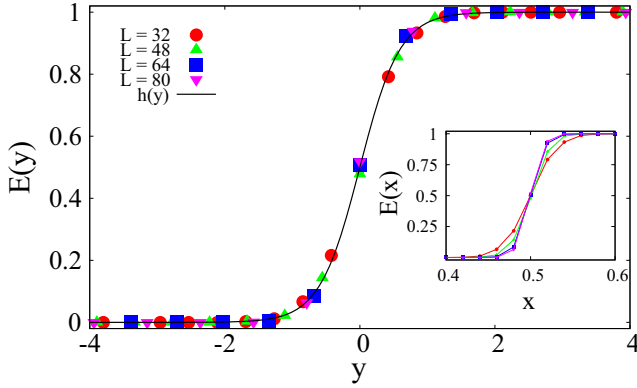


FIG. 5. The data collapse for exit probability using $\nu = 1.47$ for different system sizes in two-dimensional Ising model. The collapsed plot was fitted according to $h(y) = [\tanh(ay) + 1]/2$, where $y = \frac{(x-x_c)}{x_c}L^{1/\nu}$ with $a = 1.67$. Inset shows the unscaled data against initial concentration x of up spins.

shows strong finite-size effect and the data indicate that there is a step-function behavior in the thermodynamic limit. Finite-size scaling analysis of $E(x)$ was done using the scaling form valid for such a behavior [24],

$$E(x, L) = h\left[\frac{(x - x_c)}{x_c}L^{1/\nu}\right], \quad (9)$$

where $h(y) \rightarrow 0$ for $y \ll 0$ and $h(y) = 1$ for $y \gg 0$ (i.e., a step-function-like behavior) shown in Fig. 5. This scaling argument indicates that $L^{-1/\nu}$ is basically the factor by which the width of the region, where $E(x)$ is not equal to 0 or 1, decreases. The value of ν from our data collapse is estimated to be 1.47 ± 0.05 . The scaling function h is found to fit with the form

$$h(y) = [\tanh(ay) + 1]/2, \quad (10)$$

with $a \simeq 1.67$.

Since $E(x)$ shows a step-function-like behavior in the thermodynamic limit, for an initial concentration of up spins less than 0.5, the probability to reach a configuration with all spins up is simply zero. Hence all the minority spins eventually flip state and therefore $P_{\min}(t \rightarrow \infty) = 0$.

As already noted, the scaling form Eq. (9) suggests $|x - x_c|$ scales as $L^{-1/\nu}$ and using this in Eq. (7) we get $\tau \sim L^{\tilde{z}}$, where $\tilde{z} = \frac{\tilde{z}}{\nu}$. Hence, $\tilde{z} = 1.57 \pm 0.11$ can be interpreted as a dynamic exponent connecting time and length scales. The only

other known dynamic exponent is $z = 2$ [Eq. (3)] associated with the domain growth phenomena and is clearly different from \tilde{z} .

The scaling function h in Eq. (10) has the same form as found for a class of models with dynamical rules quite different from the Ising model [25]. The value of ν , however, is completely different.

IV. SUMMARY AND DISCUSSION

In summary, we obtained the persistence probability of up, down, and total spins for Ising spin systems using the zero-temperature Glauber dynamics in both one and two dimensions. In the initial state, the up-spin density x varies between 0 and 1, while the spins are otherwise uncorrelated. In one dimension, the exact results for the persistence exponents are known and we find that the well-known finite-size scaling form is valid even in the nonhomogeneous case.

In two dimensions, the results differ drastically, no algebraic decay is observed for the three types of persistence probabilities. The persistence probability P_{\min} corresponding to the initial minority spin vanishes. P_{maj} saturates to a finite value consistent with the behavior of the exit probability. The most significant finding is the diverging timescale associated with the minority-spin persistence probability. This timescale is related to the system size through an exponent $\tilde{z} = 1.57 \pm 0.11$, not explored so far to the best of our knowledge. In addition we obtain two other exponents $\beta \simeq 1$ and $\nu \simeq 1.47$. The values of the exponents obtained in the present study are all close to multiples of 0.5 (within error bars), which suggests that these may be related to the growth exponent $z = 2$. If this can be shown directly it will lead to a very important and striking result that domain growth and persistence probability are no longer independent for $x = 0.5$. However, this a difficult proposition as exact analytical estimate of persistence has been possible in one dimension only. Hence, the existence of yet another independent dynamical exponent in the ordering process in the two-dimensional Ising model remains an open question as of now.

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