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Exponential energy growth due to slow parameter oscillations in quantum mechanical systems

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It is shown that a periodic emergence and destruction of an additional quantum number leads to an exponential growth of energy of a quantum mechanical system subjected to a slow periodic variation of parameters. The main example is given by systems (e.g., quantum billiards and quantum graphs) with periodically divided configuration space. In special cases, the process can also lead to a long period of cooling that precedes the acceleration, and to the desertion of the states with a particular value of the quantum number.

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Introduction. It is well known [1–4] that the state of a quantum mechanical system with slowly varied parameters changes adiabatically: For a system in a given energy eigenstate the transition amplitude to other energy levels due to a slow change of parameters is small. More precisely, if the dimensions are scaled such that the gap between the neighboring energy levels is of order 1 and the parameters of the system change periodically with the speed of order ε, then the system in a state with definite instantaneous energy will, after each period, return to the $O(\varepsilon)$ vicinity of the same initial state (with a possible phase shift [5]). This continues for at least $O(\varepsilon^{-1})$ periods, with probability close to 1. For the modern theory of this phenomenon and proofs, see, e.g., Refs. [6,7]. In this paper, we describe a mechanism of the energy level crossing, for which the system's response to the slow variation of parameters is still adiabatic (i.e., starting with a definite energy state, the system, with probability close to 1, closely follows a state of definite instantaneous energy for a long time); however, the new state after each period of the parameter oscillations is, typically, different from the initial one, and the averaged energy gain per period is positive.

This is surprising; the fact that the operator of the adiabatic evolution over one period does not need to commute with the instantaneous Hamiltonian, i.e., a perfectly adiabatic evolution of a periodically driven system can lead to a fast climb up the energy spectrum, seems to be not discussed in previous works. The construction opens a peculiar way of controlling the state of a quantum system and can possibly be used in quantum computing, etc. As the spectral properties of the Schrödinger equation translate to the spectral properties of the wave equation, the theory can be applied to acoustical or optical problems as well. In particular, it is plausible that a similar construction can be used for a significant increase in the optical frequency of the laser signal due to a properly chosen protocol of a time-periodic modulation of the resonator shape.

Level crossing is usually associated with symmetries in the system, e.g., with the integrability of the corresponding classical system [3,4,8], but our construction is different. It is based on a periodic emergence and destruction of an additional quantum number in the slowly perturbed system. This can be achieved in a variety of ways, e.g., by imposing a magnetic field whose spatial coherence properties depend on time, as in Eq. (4). Our basic example is a system with a periodically disconnected configuration space. It may be a quantum billiard—a free particle confined to a bounded domain $D \subset R^d$ (see Refs. [8–12] for references on quantum billiards and Refs. [13,14] on quantum graphs). If $d \geqslant 2$, the domain D can be slowly deformed in such a way that at some moment two boundary arcs touch, after which D is divided into two parts, D_1 and D_2 [15]. The two domains then evolve separately until they reconnect again, and the process repeats periodically. In general, there is no symmetry between D_1 and D_2, so at the separation moment the energy eigenstates are divided into two groups. The eigenstates from group I are eigenfunctions of the Laplacian in D_1 and identically zero in D_2; the eigenstates from group II are eigenfunctions of the Laplacian in D_2 and zero in D_1; cf. Ref. [1]. With time, the shape of $D_{1,2}$ changes and, typically, there is no level crossing within each group, but the levels from different groups can cross (see Fig. 1). As we show below, this can lead the system at the moment of reconnection to an energy level that is different from the initial one. Moreover, the corresponding energy values recorded at the beginning of each period grow, on average, exponentially with time.

One extends the class of examples with the divided configuration space by adding a (possibly time-dependent) potential inside the domain D. The same scheme is also applicable in the one-dimensional case: One can consider a time-dependent quantum graph whose connectivity changes adiabatically with time. This can be achieved by cutting some edges in an adiabatic manner [e.g., by introducing the semipenetration boundary conditions such as in Eq. (2) below]. If the graph is periodically divided into disconnected parts and reconnected again, then one should, in general, expect the particle in such a graph to experience an exponential acceleration.

The exponential Fermi acceleration in periodically driven classical billiards was discovered in Ref. [16]; the same phenomenon for classical billiards with periodically divided configuration space was demonstrated in Ref. [15]. It was shown in Refs. [17–23] that this is a partial case of a general phenomenon: A slow periodic variation of the parameters of a nonergodic Hamiltonian (classical) system of an arbitrary nature leads, generically, to an exponential growth of energy. A quantum mechanical analog of this principle would be that if at high enough energies the gaps between energy levels become

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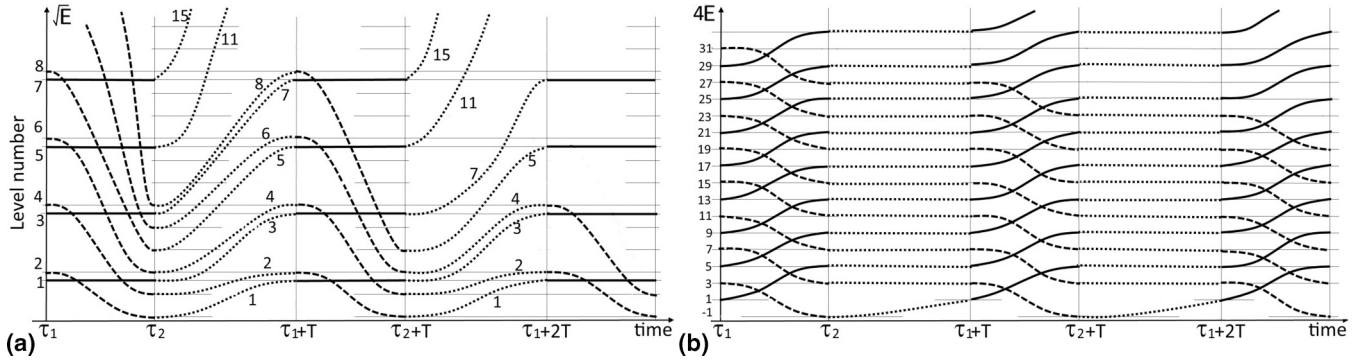

FIG. 1. Schematic diagram for the adiabatic evolution on lowest energy levels. (a) Particle in a periodically divided segment, Eq. (1), for $a(\tau_1)=1$ and $a(\tau_2)=3$. During the separation phase (from τ_1 to τ_2) the nth left state (solid lines) has constant energy $E=-1+\pi^2n^2$, while the mth right state (dashed lines) cools from $E=\pi^2m^2$ down to $\pi^2m^2/9$. During the reconnection phase (from τ_2 to τ_1+T) there is no division to left and right states, and the instantaneous energy levels (dotted lines) do not cross. This completely defines the new energy level number k at the next separation moment τ_1+T (the ground state remains the ground state, etc.); the process repeats with period T. One can observe the fast energy increase for the left states and desertion of the right states with time. (b) Nonrelativistic spin-$\frac{1}{2}$ particle in a spatially inhomogeneous, strongly oscillating magnetic field, Eq. (4), for $B(\tau_1)=-1/4$ and $B(\tau_2)=3/4$. The field is spatially homogeneous for the time intervals $[\tau_1,\tau_2]\bmod T$, so the instantaneous energy eigenstates separate into spin-up (dashed lines) and spin-down (solid lines) states. During the separation phase the energy of the spin-up states decreases from $E=m-1/4$ to $E=m-5/4$, while the energy of the spin-down states increases from $E=n-3/4$ to $E=n+1/4$. At the reconnection phase the field is inhomogeneous, which destroys the division of the energy eigenstates (dotted lines) into two groups, so the level crossing does not happen. As a result, the level number of the spin-down states at the separation moments $\tau_1 \bmod T$ increases by 2 with each period. The level number of the spin-up states decreases by 2 with each period until the lowest energy spin-up state $4E=3$ is reached. Then, after one more period T, it evolves to the spin-down ground state $4E=1$, from which the energy growth starts.

all small of order ε (in order to ensure nonvanishing amplitudes of the transition between the levels), then the quantum behavior should, probably, mimic the classical one: A fast energy growth should, typically, be expected if the classical limit system is nonergodic. We do not know to which extent this conjecture is true. Moreover, there exist systems for which the spectral gaps do not uniformly tend to zero as the energy grows (this happens, e.g., for quantum two-dimensional billiards, as follows from the Weyl law $E_k\sim k$, which implies that the gap $E_{k+1}-E_k$ between the consecutive energy levels cannot asymptotically vanish as the level number k increases [24]). The quantum acceleration construction presented in this paper does not rely on the classical dynamics properties in the high energy limit and can ensure the exponential energy growth starting even with the lowest energy state.

Toy model. We begin with a very simple model, where all the computations can be done explicitly. It corresponds to a particle confined in a segment of a straight line; this is the simplest case of both a quantum billiard and a quantum graph. Let a particle stay in a segment $[-1,a]$ whose end point a moves slowly and periodically in time, and for a portion of the period the segment is divided into two parts, so the particle cannot penetrate from one part to another. This is described by the Schrödinger equation

$$i\psi_t=-\psi_{xx}+V(x)\psi, \tag{1}$$

with the boundary conditions $\psi(-1)=\psi(a)=0$, where $a(\varepsilon t)>0$ is a T-periodic function and $\varepsilon>0$ is sufficiently small. We introduce a barrier at $x=0$ by allowing a discontinuity of the first derivative at zero and introducing the following time-dependent boundary condition:

$$\alpha(\varepsilon t)\psi(0)+[1-\alpha(\varepsilon t)][\psi_x(+0)-\psi_x(-0)]=0. \tag{2}$$

At $\alpha=0$ the barrier is absent, while at $\alpha=1$ the segments $x<0$ and $x>0$ are completely separated. It is easy to check that the operator $\psi\to\psi_{xx}$ with this boundary condition remains self-adjoint at each moment of time. We do not insist on any physical meaning behind this separation mechanism; our goal is just to be sure that the adiabatic separation of a segment into two disjoint parts is mathematically feasible.

We assume that there is a total separation for a slow-time interval $[\tau_1,\tau_2]\subset(0,T)$, i.e., $\alpha\equiv 1$ for $\tau:=\varepsilon t\in[\tau_1,\tau_2]$. We choose $V(x)=-1$ at $x<0$ and $V(x)=0$ at $x>0$ (this just helps us to make the computations explicit). During the separation interval, at each moment of τ there are two groups of energy eigenfunctions: the "left" states $\psi_n^-(\tau)$ $(n\geqslant 1)$ concentrated completely at $x\in[-1,0]$ and the "right" states $\psi_m^+(\tau)$ $(m\geqslant 1)$ concentrated at $x\in[0,a(\tau)]$. The corresponding energy levels are $E_n^-=-1+\pi^2n^2$ and $E_m^+=(\frac{\pi m}{a(\tau)})^2$. During the slow motion of the boundary, the energy levels within each group do not intersect, so we may assume that the system evolves adiabatically, i.e., if the system is in an eigenstate $\psi_n^-(\tau)$ or $\psi_m^+(\tau)$ at $\tau=\tau_1$, then it remains in this state at $\tau=\tau_2$. Thus, we neglect small (at most of the order ε) amplitudes of the transition between the eigenstates [4].

After the reconnection at $\tau=\tau_2$ the division into the two groups becomes meaningless. We order the eigenstates by their energy; it is obvious that the state $\psi_n^-(\tau_2)$ acquires the number $k=n+\left[a(\tau_2)\sqrt{n^2-\frac{1}{\pi^2}}\right]$ in this total order (i.e., k equals n plus the number of the right states with the

energies smaller than E_n^-), while the state $\psi_m^+(\tau_2)$ acquires the number $k = m + \left[\sqrt{(m/a(\tau_2))^2 + \frac{1}{\pi^2}}\right]$ (here the square brackets denote the integer part; note also that we assume a generic choice of $a(\tau)$, so that $E_n^- \neq E_m^+$ for any m and n at the moments of the reconnection and separation). Between the reconnection and the next separation at $\tau = \tau_1 + T$ the system changes adiabatically, and we may assume, by the avoided crossing theorem [3], that for a generic choice of $\alpha(\tau)$ and $a(\tau)$ the instantaneous energy levels do not cross for this time interval. Therefore, the level number k is conserved between the reconnection and the next separation. At the separation moment the eigenstate with the number k is the right state with the number m if $k = m + \left[\sqrt{(m/a(\tau_1))^2 + \frac{1}{\pi^2}}\right]$ and the left state with the number n if $k = n + [a(\tau_1)\sqrt{n^2 - \frac{1}{\pi^2}}]$.

These formulas take a particularly simple form if we, for example, choose $a(\tau)$ such that $a(\tau_1) = 1$ and $a(\tau_2) = 3$. Then the left state ψ_n^- has the number $k = 4n - 1$ at the reconnection moment and the number $k = 2n - 1$ at the separation moment, while the right state ψ_m^+ has the number $k = 2m$ at the separation and the number $k = m + [m/3] < 2m$ at the reconnection [see Fig. 1(a)]. Thus, the states with odd k become left states after the separation and, after the reconnection, they acquire a higher, and still odd, value of $k_{\text{new}} = 2k_{\text{old}} + 1$. The states with even k become right, and acquire a strictly lower value of k. As we see, even though the system changes adiabatically all the time, and absolutely no jumps between the energy levels is assumed, the level's number k changes with each period of the adiabatic oscillations. In our example, if we start with a right state, it will, eventually, become left; we also have that the left states gain energy exponentially (the left level's numbers double with each period). Therefore, if we start with a superposition of a finitely many eigenstates, then after finitely many periods we will, with accuracy of order ε, have a superposition of only left states, and the energy of the system will start growing exponentially with time.

Genericity of the exponential growth. Let us show that the exponential energy growth does not disappear if we change the details of the above construction, and is a robust phenomenon for a general class of systems with a periodically divided configuration space. Even more generally, we consider a quantum system with parameters which oscillate slowly enough (so we assume that jumps between different energy levels do not occur), and let for a part $[\tau_1, \tau_2]$ of the oscillation period an additional quantum number emerge, so the energy eigenstates are divided into two groups, I and II, such that the transition between group I and group II states is forbidden at $\tau \in [\tau_1, \tau_2]$. At each moment of time we order the eigenstates by their energy E_k, $k = 1,2,\ldots$, and introduce the indicators $\sigma_j(k)$ as follows: $\sigma_j(k) = 1$ if the eigenstate ψ_k belongs to group I at $\tau = \tau_j$, and $\sigma_j(k) = -1$ if ψ_k belongs to group II at $\tau = \tau_j$ $(j = 1,2)$. The two indicator sequences $\sigma_{1,2}$ completely determine the energy evolution in the adiabatic approximation. Indeed, at $\tau = \tau_j$, if the state ψ_k is group I with the number m, then there are exactly m group I and $(k - m)$ group II states with the energies not exceeding E_k, so $S_j(k) := \sigma_j(1) + \cdots + \sigma_j(k) = 2m - k$; if ψ_k is a group II state with the number n, then there are n group I and $(k - n)$ group II states with the energies not exceeding E_k, so $S_j(k) = k - 2n$. Thus, after the separation moment $\tau = \tau_1$, the state ψ_k becomes the group I [if $\sigma_1(k) = 1$] or group II [if $\sigma_1(k) = -1$] state with the number n or m equal to $[k + \sigma_1(k)S_1(k)]/2$. After the reconnection at $\tau = \tau_2$ this state acquires the new number $\bar{k}$ such that

$$\bar{k} + \sigma_2(\bar{k})S_2(\bar{k}) = k + \sigma_1(k)S_1(k) \text{ and } \sigma_2(\bar{k}) = \sigma_1(k). \quad (3)$$

By construction, this formula completely determines the change in the energy level after each period of the parameter oscillations: The new level number $\bar{k}$ is a function of the previous level number k, and vice versa. Thus, given an initial energy level number k_0, the iteration of the rule (3) provides a uniquely defined trajectory k_s—the sequence of the values of k at the beginning of each period (the same is true for backward iterations).

There are only two logically possible types of trajectories for the energy level number: loops, when the system returns to the same energy level after a finite number of periods, and unbounded trajectories, when the level number k_s tends to infinity as the number of periods s grows (in the latter case the level number will tend to infinity also backwards in time). What happens for a given initial state of a particular system depends on the detailed structure of the group I and II energy spectra at the moments of separation and reconnection. This structure can be essentially arbitrary: While the asymptotic behavior at large energies can be restricted by a Weyl type formula [24], it is easy to build, for any given N, a potential in any given domain such that the first N energy eigenvalues for a particle in this potential would take any given values; cf. Ref. [25]. Thus, if the class of systems under consideration is sufficiently large, we may think of the energy spectra of the group I and II states as random, i.e., the indicator sequences σ_1 and σ_2 can be viewed as realizations of a certain random process (at least until the energy level number reaches a sufficiently large value N).

The simplest model for σ_1 and σ_2 is given by sequences of independent random variables. If β is the probability of $\sigma_1(k) = 1$, and γ is the probability of $\sigma_2(k) = 1$, then $S_1(k) \sim (2\beta - 1)k$ and $S_2(\bar{k}) \sim (2\gamma - 1)\bar{k}$. Now, Eq. (3) gives $\bar{k} \sim \frac{\beta}{\gamma}k$ with probability β [this corresponds to $\sigma_1(k) = 1$], and $\bar{k} \sim \frac{1-\beta}{1-\gamma}k$ with probability $1 - \beta$ [corresponding to $\sigma_1(k) = -1$]. It follows that

$$\rho := E(\ln \bar{k} - \ln k) \sim \beta \ln \frac{\beta}{\gamma} + (1 - \beta) \ln \frac{1 - \beta}{1 - \gamma}.$$

It is a well-known fact that if $\beta \neq \gamma$, then the right-hand side is strictly positive (cf. Ref. [19]), i.e., the average logarithmic energy gain per period is strictly positive in this case. By the law of large numbers, we have essentially linear growth of $\ln k$ with time: $\ln k_s - \ln k_0 \sim \rho s$ for a typical realization of the process under consideration. Therefore, the probability to return to the initial value k_0 falls rapidly with the number of periods s if $\rho > 0$. This means that while several short loops may exist, long loops are rare, and a typical trajectory of the energy level number is unbounded. It is natural to assume $E_k \sim k^\nu$ for some $\nu > 0$, thus the linear growth of $\ln k$ with time implies the exponential growth of the energy E_k. Obviously, the positivity of the logarithmic gain

ρ cannot be violated by small changes to the statistics of the transitions between the energy levels at the separation and reconnection moments; it will persist even if we allow for a nonzero amplitude of transition from one to several levels with a small spectral gap, provided such events are sufficiently rare. Thus, the exponential energy growth should be a generic phenomenon in the adiabatic process under consideration.

Note that $\ln k$ can be identified with the entropy of the system in the energy eigenstate ψ_k [26]. The sustained linear growth of entropy (hence, exponential growth of energy) in classical systems with periodically divided configuration space was described in Refs. [15,19]. The difference with the quantum acceleration described here is that the mechanism of the classical acceleration is the loss of the ergodicity in the phase space, which leads to the destruction of the adiabatic invariance of the entropy. This works in a universal fashion for other nonergodic classical systems [22,23]. In the quantum case, we have infinitely many adiabatic invariants—the populations I_k, $k = 1, \dots, +\infty$, of the instantaneous energy levels [$I_k = |\langle\psi(t)|\psi_k(\varepsilon t)\rangle|^2$, where $\psi(t)$ is the wave function and ψ_k are the instantaneous energy eigenstates]. So, the sustained entropy and energy growth is possible only if all of these adiabatic invariants are destroyed, and the mere ergodicity violation of the classical limit does not seem to be enough for this.

Example of a slow energy growth and a long cooling period. The exponential energy growth is not guaranteed in the special case $\rho = 0$, and more subtle effects are possible. For an example, we consider a nonrelativistic spin-$\frac{1}{2}$ particle in a time-dependent, spatially inhomogeneous, strong magnetic field. We scale the Planck constant, the mass, and the charge of the particle to 1, and consider the slowly changing vector potential $\mathbf{A}(x,y,z,\varepsilon t) = [-yB(\varepsilon t) - F(z,\varepsilon t), xB(\varepsilon t) + G(z,\varepsilon t), 0]$ and scalar electric potential $\phi = V(z) - \mathbf{A}^2/2$. As $\operatorname{div}\mathbf{A} = 0$, the Pauli equation [4] takes the form

$$i\frac{\partial}{\partial t}\psi_\pm = \frac{1}{2}(-\triangle + 2i\mathbf{A}\cdot\nabla)\psi_\pm + V(z)\psi_\pm - \frac{1}{2}\sigma\cdot\mathbf{B}(z,\varepsilon t)\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix},$$

where σ are the Pauli matrices and $\mathbf{B} = [F_z(z,\varepsilon t), G_z(z,\varepsilon t), 2B(\varepsilon t)]$ is the magnetic field. We consider only the wave functions $\psi_\pm$ which are independent of (x,y). In other words, by imposing the boundary condition $\psi = 0$ at $x^2 + y^2 = \delta^2$, we confine the particle to the δ-thin infinite cylinder around the z axis, so in the limit $\delta \to 0$ the problem becomes one dimensional, and the equation takes the form

$$i\frac{\partial\psi_\pm}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi_\pm}{\partial z^2} + V(z)\psi_\pm - \begin{pmatrix}B(\varepsilon t) & C(z,\varepsilon t)\\ C^*(z,\varepsilon t) & B(\varepsilon t)\end{pmatrix}\psi, \tag{4}$$

where $C = (F_z - iG_z)/2$. Computations become explicit if we take $V(z) = \frac{1}{2}z^2$. Let the magnetic field oscillate periodically as a function of the slow time $\tau = \varepsilon t$, so that for some interval $[\tau_1,\tau_2]$ during the period the x and y components of the magnetic field vanish. Thus, the coefficient C in Eq. (4) vanishes and the components ψ_+ and ψ_- of the wave function evolve independently at $\tau \in [\tau_1,\tau_2]$. During the rest of the oscillation period, we assume that C is a nonconstant function of z, i.e., the magnetic field direction varies with z, so ψ_+ and ψ_- become coupled. This means that the system has an additional quantum number (spin up or spin down) at $\tau \in [\tau_1,\tau_2]$, while for the rest of the period this quantum number is destroyed.

According to the theory above, the slow time evolution of such system is described by the values the energy level number k takes at the beginning of each oscillation period; the trajectories of the level numbers are completely determined by the energy spectra at the moments of separation and reconnection, τ_1 and τ_2. In our case, the spectrum at $\tau = \tau_j$ is given by $E_m^+ = m - 1/2 - B(\tau_j)$ for the "up" states and $E_n^- = n - 1/2 + B(\tau_j)$ for the "down" states ($m,n \geqslant 1$). We take $B(\tau_1) = -1/4$ and $B(\tau_2) = 3/4$. Thus, if we order the spectrum by the increase of the energy, the up and down states will alternate. Namely, the energy levels with even number k correspond to the up states and odd k correspond to the down states, except for the ground state ($k = 1$) at the moment of reconnection $\tau = \tau_2$, which is also the up state. In terms of the indicator sequences $\sigma_j(k)$ from formula (3) this reads as $\sigma_1(k) = (-1)^k$ and $\sigma_2(1) = 1$, $\sigma_2(k) = (-1)^k$ at $k \geqslant 2$, and their sums are given by $S_1(k) = -1,0,-1,0,-1,\dots$ and $S_2(k) = 1,2,1,2,1,2,\dots$. Thus, if $\sigma_1(k) = \sigma_2(\bar{k})$, then either k and $\bar{k}$ are of the same parity, so $S_1(k) - S_2(\bar{k}) = -2$, or k is even and $\bar{k} = 1$, which gives $S_1(k) - S_2(\bar{k}) = -1$. By (3), we obtain the following law for the change of the energy level number after one period:

Spin up: $\bar{k} = k - 2$ for even $k \geqslant 4$, $\bar{k} = 1$ if $k = 2$, and $\bar{k} = k + 2$ if k is odd (spin down).

As we see, in the process under consideration, the nonhomogeneous magnetic field makes slow, large amplitude oscillations in time in such a manner that it becomes spatially homogeneous during a part of the oscillation period. This may lead to a linear growth of the particle energy for the states with a certain spin orientation (down states in our example), while in the states with the opposite spin a constant amount of energy will be lost with each period, until a minimal value of energy is reached, after which the spin orientation changes and the eternal acceleration starts [see Fig. 1(b)].

Conclusion. We have shown that a slow periodic change of parameters of a quantum mechanical system leads, typically, to an exponential growth of energy (due to an adiabatic level crossing) provided an additional quantum number is created and destroyed during the oscillation period. The basic example of such processes is given by systems with periodically divided configuration space. A slower rate of the energy growth is also possible in special cases (such as in the example of a spin-$\frac{1}{2}$ particle in a strong, oscillating, inhomogeneous magnetic field). For particular choices of the parameters of our process the energy growth may be preceded by a period of cooling. This happens when, for a group of energy states with a particular "winning" value of the quantum number, the system accelerates and, in the adiabatic approximation, the state of the system remains in this group after each oscillation period. Then a state which does not belong to this winner group has to lose energy until a certain minimal value of energy is

reached, after which the quantum number changes and the state joins the winner group. Note that this leads to the desertion of energy states with certain values of the quantum number: For an arbitrary initial superposition of states of finite energy, the system evolves with time to the superposition where most of the contribution is given by the energy states with the winning value of the quantum number only.

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