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Inhomogeneity induces relay synchronization in complex networks

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Relay synchronization is a collective state, originally found in chains of interacting oscillators, in which uncoupled dynamical units synchronize through the action of mismatched inner nodes that relay the information but do not synchronize with them. It is demonstrated herein that relay synchronization is not limited to such simple motifs, rather it can emerge in larger and arbitrary network topologies. In particular, we show how this phenomenon can be observed in networks of chaotic systems in the presence of some mismatched units, the relay nodes, and how it is actually responsible for an enhancement of synchronization in the network.

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From physiological rhythmic processes to communication, from technological to natural networks, synchronization of interacting units is the foundation that leads a system to operate collectively [1,2]. A lot is already known about several forms of synchronization emerging as the result of specific coupling configurations (unidirectional or bidirectional, instantaneous or delayed, continuous or discontinuous in time) between two (or more) chaotic units [3]: *complete synchronization* (when the states of two identical systems overlap fully), *phase synchronization* (when the adjustment of two nonidentical oscillators leads to a bounded phase difference between them) [4], *lag synchronization* (when the systems' states are shifted in time with a lag [5]), and *generalized synchronization* (when the evolution of different coupled systems is established by a functional, possibly invertible, dependence of the state of one system to the other [6]).

As soon as more than two units are networking, other forms of collective dynamics may emerge, whose distinguishing feature is that synchronization *does not* occur at the macro-scale of the system. Examples are partial synchronization [7], cluster synchronization [8] (where nodes group into clusters of distinct synchronized dynamics), chimera states [9] (i.e., symmetry-broken structures of coherent and incoherent domains), and remote and relay synchronization [10,11]. These latter, in particular, refer to indirect coupling configurations between outer units realized by means of mismatched relay nodes, which eventually induce complete synchronization of the identical oscillators. More specifically, remote synchronization has been observed in networks of periodic (Stuart-Landau) oscillators with an amplitude dynamics that plays a central role in allowing synchronization of not directly connected nodes [12], while relay synchronization (RS) is proper of uncoupled chaotic systems that synchronize under the action of an intermediate node displaying generalized synchronization with them [13]. While chimera states and cluster, partial, and remote synchronization have been observed in complex networks, the arousal of RS has been observed so far only in regular topologies, such as, e.g., open chains of mutually coupled oscillators [13–16].

RS has been the subject of a stream of theoretical and experimental studies [14], and dynamical relaying has been proposed, for instance, as the main mechanism allowing synchronization in distant cortical areas of the brain [15]. RS's theoretical relevance is linked to the fact that it actually may be accompanied by the enhancement of synchronization: the critical coupling strength for complete synchronization of the outer nodes may even be lowered by the presence of lag synchronization between the central mismatched node and the outer oscillators [16]. Furthermore, RS has been experimentally observed, for instance, in optical systems [11].

In this paper, we show that RS is by no means limited to chains of mutually coupled oscillators, but it can also emerge in complex networks. Toward that end, and starting from the earlier studies in the simplest case of three connected nodes, we identify the main criteria to investigate RS in large graphs with arbitrary topologies. As a major result, we find that the presence of inhomogeneity is indeed fundamental to induce dynamical relaying effects. Furthermore, we show that relay nodes, either already existing or intentionally added to the graph, lead to an enhancement of synchronization in the whole network, or in a subpart of it.

For the sake of exemplification, and without lack of generality, we start by considering N Rössler oscillators [17], diffusively coupled to each other within a network. This being the case, the equations of motion read

$$\begin{aligned} \dot{x}_i &= -y_i - z_i, \\ \dot{y}_i &= x_i + a y_i + \sigma \sum_{j=1}^{N} a_{ij}(y_j - y_i), \\ \dot{z}_i &= b + z_i(x_i - c_i), \end{aligned} \tag{1}$$

where $\mathbf{x}_i = (x_i, y_i, z_i)^T$ is the three-dimensional vector state of the ith node, dots denote temporal derivative, σ is the coupling strength ($\sigma = 0$ corresponds to the uncoupled limit), and $i = 1, \ldots, N$. $\mathcal{A} = \{a_{ij}\}$ is the network's adjacency matrix. In analogy with the traditional case of a three-node open-chain configuration (where the outer nodes interact with a mismatched inner unit [13,15,16]), here we set a framework in which some of the network oscillators have different parameters with respect to the rest of the units. We choose a bimodal (fully polarized) distribution of the parameter c_i, and

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we focus on the case in which a fraction f of the network nodes (those having higher values of the degree k) displays a value of c_i equal to $c_i = c_0$ while the rest of the network has $c_i = c$. We call the nodes with $c_i = c_0$ relay nodes in anticipation of their role in mediating for synchronization. The other two parameters (a and b) are equal for all the graph's units, and they are set in a way such that all oscillators, when uncoupled, exhibit a chaotic dynamics.

Synchronization in the network is monitored by the error between two oscillators i and j, $e_{ij} = \langle \|\mathbf{x}_i - \mathbf{x}_j\| \rangle_t$, where $\| \cdot \|$ stands for the Euclidean norm, and $\langle \cdot \rangle_t$ indicates time averaging over a predefined (sufficiently long) observation window. Two nodes are taken as synchronized if $e_{ij} < \delta$ (δ being a small constant threshold), and a distinction is made between (i) nodes that synchronize due to the effect of a physical link between them, or a path of synchronized nodes connecting them, and (ii) nodes that synchronize due to the effect of only one relay node (or more). In other words, two nodes i and j are said to *synchronize by dynamical relaying* if they are synchronized, and they are neither connected by a direct link (i.e., $a_{ij} = 0$) nor by a path of synchronized nodes. To systematically quantify the extent of RS, the number of pairs of nodes that are synchronized by dynamical relaying is counted, and indicated by l_r. Furthermore, as in three-node open chains it has been demonstrated that the outer nodes and the inner one are in generalized synchronization [13], the synchronization points percentage (SPP) indicator [18] is also monitored. Namely, to assess whether two phase-space configurations, one within the domain and the other within the codomain subsystem, are locally synchronized, the existence of a local continuous function from one point to the other is asserted by analyzing their respective neighborhoods with a given statistical confidence level. The ratio between the number of locally synchronized configurations and the total number of available points m is the SPP: values of the index close to 1 indicate the existence of a global mapping from one subsystem to the other, i.e., they give evidence of generalized synchronization [18].

For our analysis, we fix the parameter mismatch by assigning $c_i = c_0 = 9$ if $k_i > k^*$, and $c_i = c = 7$ otherwise. These parameter values generate in the hub and in the outer nodes distinct chaotic attractors. Provided that this condition is satisfied, other settings may be used with similar results, included the case in which the mismatch is implemented by varying other parameters of the Rössler system, such as a and b. The value of k^* is set so that the percentage of relay nodes is less than the 15% of the total number of nodes in the network (i.e., $f \leqslant 0.15$). A more systematic analysis on the effect of the number of relay nodes is discussed below. Furthermore, we consider two different network types, namely the scale-free (SF [19]) and the Erdös-Rényi (ER [20]) network's topologies, and we report the results obtained for $N = 100$ and average degree $\langle k \rangle = 4$. We focus first on a single SF network, whose structure is shown in Fig. 1. The inset is a magnification of two small parts of the network, which we will use later for a better illustration of our observations.

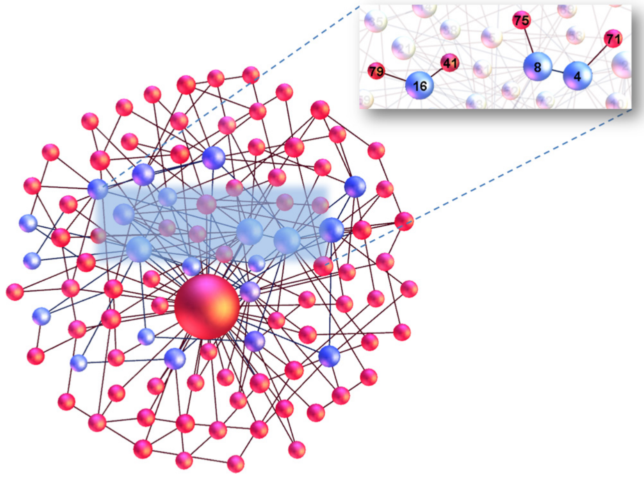


FIG. 1. Sketch of the considered scale-free network with $N = 100$, $\langle k \rangle = 4$, $k^* = 7$. The higher-degree nodes are colored in blue (the size of the nodes being proportional to their degree), and the others are in red. The inset zooms on a portion of the graph containing two different sets of nodes that are in relay configurations.

As the setting of synchronization in the network depends on the value of σ, we first analyze the scenario at $\sigma = 1$ and later on follow the evolution of synchronization for coupling strengths that gradually increases from 0 to 1. The $N \times N$ matrix reported in Fig. 2 summarizes the scenario emerging at $\sigma = 1$. In the matrix, the elements are color-coded according to the final (asymptotic) state of the corresponding pairs of nodes. To be specific, black pairs are those pairs of nodes that come out to evolve asynchronously, orange pairs are those pairs of synchronized nodes that, however, are directly connected in the network either by physical links or paths, and white pairs are those nodes that evolve synchronously thanks to the unique effect of dynamical relay. As is seen, the incidence of relay synchronization is significant, leading to the presence of a consistent amount of pairs of nodes that synchronize through some relay nodes. A better illustration is gathered

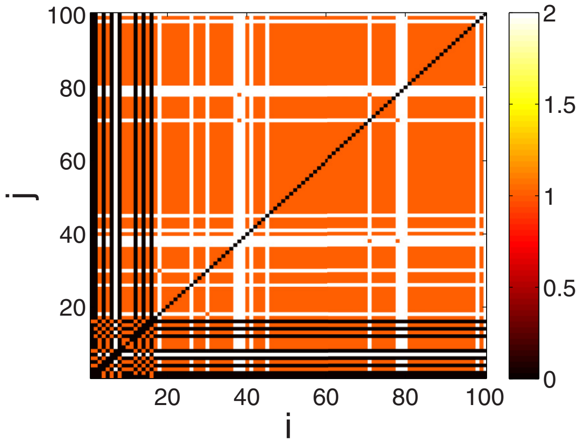
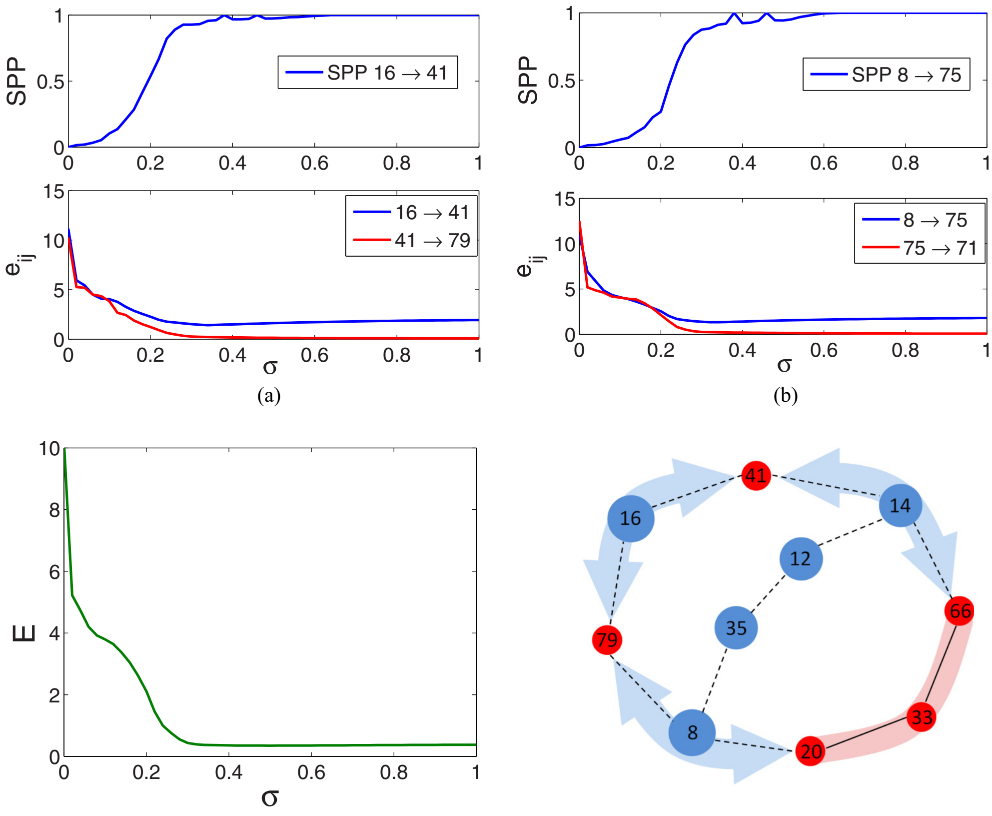


FIG. 2. $N \times N$ matrix summarizing the synchronization scenario in the network of Fig. 1 at $\sigma = 1$. Color codes are used with the following stipulations: black corresponds to pairs of unsynchronized nodes, orange to those pairs of synchronized nodes directly connected in the network either by physical links or paths, and white to pairs of nodes whose asymptotic behavior is synchronous thanks to the effect of dynamical relay. The vertical and horizontal blue lines denote furthermore the nodes with $c_i = c_0 = 9$.



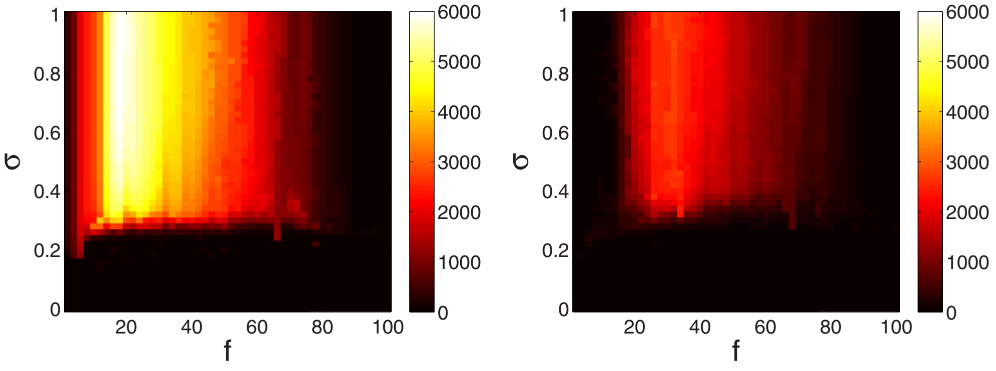
(a) (b)

(c) (d)

FIG. 3. Relay synchronization in the SF network of Fig. 1. (a),(b) SPP (upper plots, see text for definition) and synchronization error e_{ij} (lower plots) vs σ (a) between nodes 16 and 41 (blue line) and between nodes 41 and 79 (red line), and (b) between nodes 8 and 75 (blue line) and between nodes 75 and 71 (red line). (c) Global synchronization error E vs σ. (d) Schematic representation of three paths from node 41 to 79. The network edges have been redrawn as continuous (dashed) lines if they connect (do not connect) two nodes that are synchronized. Light blue arrows (light red boxes) indicate synchronization through dynamical relaying (through direct links).

by selecting two specific pairs of nodes, and reporting for them the evolution of the synchronization error versus the coupling coefficient [lower plots of Figs. 3(a) and 3(b)]. Such pairs of nodes are the ones appearing in the inset of Fig. 1 along with the relay nodes mediating for their synchronization; they are nodes 41 and 79 (which are synchronized through the relay action of node 16) and nodes 71 and 75 (which are synchronized through the action of nodes 8 and 4). In fact, nodes 4, 8, and 16 are all of high degree and, therefore, they have $c_i = c_0$. Along with the synchronization error, the upper plots of Figs. 3(a) and 3(b) report the trend of the SPP, confirming that the microscopic mechanism leading to synchronization is that of RS. Indeed, for the first case [nodes 41-16-79, Fig. 3(a)] the synchronization between the outer nodes is allowed thanks to the passage of information through the central node. The synchronization between nodes 41 and 79 occurs at $\sigma_c = 0.28$, while $e_{16,41}$ remains greater than zero (analogously $e_{16,79} > 0$). From the SPP behavior of Fig. 3(a) we also observe a transition at $\sigma_c = 0.28$ from low values to values close to 1, indicating the existence of generalized synchronization. In the second case, the relay nodes are two (nodes 4 and 8). As shown in Fig. 3(b), when nodes 71 and 75 synchronize, the SPP takes values close to 1, indicating the presence of generalized synchronization between node 75 (or 71) and either one of the two relays [Fig. 3(b), for example, reports the SPP between nodes 75 and 8, but the same holds for the other pairs]. Figure 3(c) reports the global synchronization error $E = \frac{1}{N(N-1)} \sum_{i,j} e_{ij}$, illustrating a clear dynamical transition of relay synchronization, occurring at the same critical value $\sigma_c = 0.28$ that yields synchronization at the microscopic scale.

One should notice that, as the graph is connected, there are obviously other paths connecting nodes 41 and 79, which could allow synchronization between them. To discriminate the genuine nature of RS between nodes 41 and 79, one has therefore to analyze in detail the structure of network paths between the two nodes. Two such paths are illustrated in Fig. 3(d) along with the one passing through node 16. In the figure, based on the results of numerical simulations with $\sigma = 1$, we have redrawn the network edges as continuous (dashed) lines if they connect two nodes that are (are not) synchronized. Furthermore, we distinguish between those subpaths formed only by synchronized nodes due to physical links (highlighted with light red boxes) and those including dynamical relays (light blue arrows). The path through the nodes 14-12-35-8 includes nodes that are not synchronized with each other (14 and 12, for instance, which are linked by a dashed edge), therefore it cannot be responsible for the synchronization of nodes 41 and 79. The two other paths are formed either by one node in dynamical relaying (node



(a) (b)

FIG. 4. l_r (see text for definition) in the parameter space (σ, f) for SF (a) and ER networks (b). Color bars report the corresponding color codes.

16) or by two (node 14, which relays 41 with the group of synchronized nodes 66, 33, and 20; and node 8, which relays the group with 79), and, hence, they have equivalent synchronization features. All other paths between nodes 41 and 79 [not reported in Fig. 3(d)] have properties similar to the ones discussed above.

ER networks also show RS in the presence of inhomogeneity. However, the overall phenomenon appears with distinguishing quantitative features. To illustrate the differences, we performed a systematic analysis of the parameter l_r as a function of the coupling strength and the impact of inhomogeneity measured by the fraction of relay nodes. Figure 4 reports the values of l_r for SF [Fig. 4(a)] and ER [Fig. 4(b)] networks. The parameter l_r is greater in SF networks than in ER topologies. In addition, the peak of RS is observed in SF networks for a number of relay nodes smaller than that in ER networks. Inhomogeneity in a small set of nodes is sufficient to induce a high level of relay synchronization, whereas the more homogeneous the structure is, the higher is the inhomogeneity required in the node dynamics to elicit RS, generally associated with lower peaks of l_r. The difference of RS in SF and ER networks originates from their topological structure. In SF networks, a few high-degree nodes act as hubs to connect many units. As many paths pass through these nodes, they can relay the information needed to synchronize many pairs of units. Therefore, SF networks require few relay nodes to induce a large number of pairs synchronized by dynamical relaying. On the contrary, in more homogeneous structures, such as in ER networks, a larger fraction of mismatched nodes is required to elicit RS. In addition, the more homogeneous topology yields the formation of a smaller number of paths passing through relay nodes, thus explaining the lower values of l_r observed in ER networks than in SF.

The appearance of RS in complex networks shows a profound difference with the phenomenon of remote synchronization [12]. Although the latter is also characterized by the synchronization of not directly connected nodes through the action of intermediate units, the microscopic mechanism underlying RS is different. In fact, RS is allowed by generalized synchronization between the hubs and the outer nodes. On the contrary, remote synchronization is ruled at the microscopic scale by the action of the amplitude dynamics, whose modulation permits the transmission of the information for synchronization. Another distinguishing trait is that remote synchronization appears as a state prior to the onset of global synchronization observed for larger coupling, whereas RS represents the strongest form of synchronization possible in the inhomogeneous network, as global synchronization is prevented by the large mismatch between the parameters of the hubs and those of the outer nodes.

We now move to show that dynamical relaying leads, in fact, to an enhancement of synchronization. Toward that end, we reconsider the SF network of Fig. 1, and we extract a subnetwork Γ of 77 nodes. The subnetwork Γ is connected, and it is formed by those nodes having all $c_i = c$ and fully synchronized at $\sigma = 1$. We then compare the synchronization properties of Γ in two distinct situations: (i) when it is isolated from the other nodes of Fig. 1, and (ii) when a variable number of relay nodes from the network of Fig. 1 are added to it. The synchronization error is calculated on all the possible pairs of nodes of the subnetwork Γ: $\langle e\rangle = \frac{1}{\sum_{(i,j)\in\Gamma} a_{ij}} \sum_{(i,j)\in\Gamma} e_{ij}$. Figure 5 shows the behavior of $\langle e\rangle$ for Γ (black line) and four

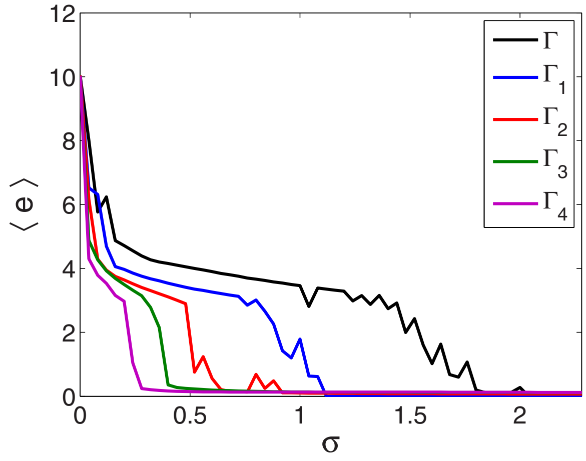


FIG. 5. $\langle e\rangle$ (see text for definition) vs σ for the isolated subnetwork Γ (black line) and when an increasing number of relay nodes is considered (colored lines, color codes in the inset). Dynamical relays enhances synchronization of Γ by lowering the synchronization threshold.

other networks (color lines), which are obtained by adding 2 (Γ_1), 4 (Γ_2), 6 (Γ_3), or 8 (Γ_4) relay nodes. It is seen that the threshold for synchronization is progressively reduced, as an increasing number of relay nodes is added.

In the previous example, the relay nodes have been added according to their connectivity in the SF network of Fig. 1, that is, each relay node is added along with its links in the SF network. By iterating the procedure, the original SF network can therefore be fully retrieved. However, we have found that the enhancement by dynamical relaying is a much more general phenomenon. Let us indeed consider here a different scenario, in which a number of relay nodes with totally arbitrary links to the already existing nodes is added to a given pristine network. A remarkable result is that such relay nodes *generically* favor the onset of synchronization. In particular, we take into account a SF network with $N = 100$ identical nodes with $c_i = c$ (the pristine network) and we analyze the behavior of the synchronization error $\langle e \rangle$ when we add two relay nodes (having $c_j = c_0$) with an increasing number of links to the original network. The result is shown in Fig. 6, where the pristine network has again been named Γ, and the networks Γ_a, Γ_b, Γ_c, and Γ_d are obtained with the inclusion of the two relays with different node degrees. The introduction of only two dynamical relays in the network led to an enhancement of the synchronization of the pristine network, which becomes more significant as the number of links increases. In networks of coupled oscillators, synchronization is made possible by the links that enable the interactions between the units. We have found that dynamical relaying provides a further mechanism to effectively couple the dynamics of two nodes, which eventually are not connected by a physical edge. Consequently, adding mismatched nodes able to act as dynamical relays generally results in a lower threshold for the overall synchronization of the network.

In summary, we presented evidence of relay synchronization in a generic complex network. Analogously to what was observed in a simple chain configuration, we have demonstrated that the underlying mechanism through which relay nodes act is generalized synchronization. In particular, we have shown that the presence of relay synchronization is induced by the inhomogeneity of the node dynamics, and such synchronization appears in SF and ER networks with distinct features, requiring in the former case a lower fraction of mismatched nodes with respect to the latter. We have also shown that relay synchronization is generically associated with an enhancement of synchronization. Remarkably, when some mismatched nodes are randomly added to a generic pristine network, their inclusion (provided that their degree is high enough) favors synchronization by lowering the threshold for complete synchronization.

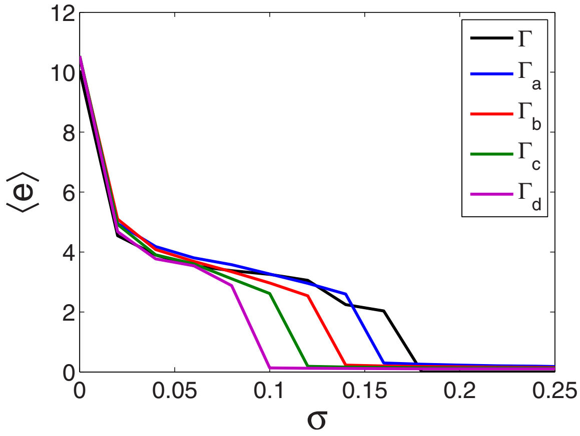


FIG. 6. $\langle e \rangle$ vs σ in a SF pristine network, Γ, of $N = 100$ identical nodes with $c_i = c$ (black line), compared with the case when two relay nodes with $c_j = c_0$ are added to this network with a different degree: in Γ_a (blue line) each relay node forms 10 links with the pristine networks, in Γ_b (red line) it forms 20 links, in Γ_c (green line) 30 links, and in Γ_d (purple line) 40 links.

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