Relaxation of plasma waves in Fermi-degenerate quantum plasmas

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Plasma waves in a Fermi-degenerate quantum plasma are studied in the framework of the Vlasov-Poisson self-consistent-field theory. A complete time-dependent analytical solution of the initial-value problem is obtained for a multistream model both by stationary-wave and Laplace-transform methods. In the continuum limit, the excitation spectrum can be expressed by the imaginary part of the response function to the initial perturbations. The relaxation of plasma waves is discussed for one-dimensional systems with both Fermi and Maxwellian statistics. Apart from the usual exponential Landau damping, regimes of sub- and superexponential damping can be identified due to the phase relaxation of single-particle excitations. In addition, beat waves and echoes are discussed.

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I. INTRODUCTION

An ideal quantum plasma consists in the simplest case of a Fermi-degenerate electron gas with a charge-neutralizing homogeneous ion background. Its dynamics can be described in the framework of kinetic theory by the well-known Vlasov-Maxwell theory [1]. For classical plasmas, both the Vlasov-Maxwell theory and the computational particle-in-cell (PIC) method [2–4] are widely used, in particular for the analysis of plasma-based accelerators [5] and short-pulse beam-plasma interactions [6,7]. For quantum plasmas, the Vlasov approach is computationally less well established. A general framework is provided by the multistream approach approximating the one-particle statistical operator by an ensemble of representative quantum states [8,9]. In this work, we wish to address the basic question of how the many-particle problem can be approximated in the framework of the Vlasov theory by a relatively small number of representative states. How do collective behavior and Landau damping arise from individual electron streams? For this purpose, attention will be restricted to small-amplitude perturbations. In a first step, the linear multistream model will be treated by analytical methods. In a second step, the continuum limit is performed and compared with the well-known results for the degenerate electron gas. It will be shown explicitly that the excitation spectrum of the electron gas can be well-approximated by the multistream model. Exponential Landau damping of collective modes as well as nonexponential relaxation processes of single-particle excitations can be demonstrated in this framework. These results are in good agreement with a previous outline of the relationship between single-particle and collective excitations in finite-size systems [10]. Specific numerical results for the evolution of initial perturbations may also provide a useful benchmark test for general computational methods in this field.

We first give a brief summary of customary quantum-statistical methods for electron gases. The Vlasov-Maxwell theory is a mean-field approach that neglects exchange interactions and correlations. The many-particle theory can be reduced to the Vlasov approximation by applying the famous Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy to the density matrix [1]. The Vlasov theory of plasma oscillations is also confirmed by other approaches using quantum-field theory [11], temperature Green's functions, and diagrammatic expansion methods [12,13]. More general kinetic equations include correlations by a collision term. The quantum Boltzmann equation for bosons and fermions has been derived by Uehling and Uhlenbeck [14]. Other extensions accounting for exchange interactions and external laser fields have also been considered [15,16]. Related self-consistent-field theories are the Hartree theory and the Hartree-Fock theory, which are applied mostly in atomic and nuclear physics as well as in molecular chemistry. The standard time-independent Hartree-Fock theory for pure quantum states can also be generalized to thermal [17] and time-dependent systems [18].

Computational methods for quantum plasmas are often based on *ab initio* molecular dynamics. Classical molecular dynamics for ions is combined with density-functional theory (DFT) for electrons. Minimization of the Hohenberg-Kohn density functional [19] leads to the Kohn-Sham equations for single-particle orbitals with exchange-correlation potentials [20]. The physical density can be expanded in terms of Kohn-Sham orbitals. The basic formulation of DFT can also be extended to thermal DFT [21–24] and to time-dependent DFT [25]. Many of these computational challenges are presently explored in the fields of warm dense matter (WDM) and high-energy density plasmas (HEDPs) [26]. In this context, general computational methods for nonequilibrium electron gases are of basic importance.

Within the framework of the Vlasov approximation, grid-based computational methods are quite expensive and therefore mostly limited to one-dimensional (1D) systems [27–29]. For this reason, particle-in-cell methods (PIC) have become the standard approach. However, these methods cannot be adopted to quantum systems without making restrictive semiclassical approximations. Such semiclassical procedures consist, e.g., in the use of pseudopotentials [30–32] or of parametrized wave packets [33,34]. In a previous work we proposed a computational approach, namely the carrier-envelope-wave (CEW) method [9], to overcome these semiclassical limitations. The CEW method treats the electrons by an ensemble of representative quantum states, more specifically by a

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set of carrier-envelope waves, without using any classical particle trajectories. The Wigner function of the ensemble automatically satisfies the quantum Vlasov equation. The CEW method is a variant of the hydrodynamic multistream approach for plasmas that has been used in the past by various authors [8,35–37]. Single-stream models governed by effective nonlinear Schrödinger equations have been widely studied [38], and two-stream or three-stream models have been investigated occasionally in the framework of the quantum streaming instability [8,39–42]. For more general distribution functions, multistream models have received less attention, although they have been investigated to some extent for weak nonlinearities [43] and for strong nonlinearities at wave-breaking [9].

In Sec. II, the multistream model is introduced. The wave functions for each stream are expressed by hydrodynamic variables for the density and velocity. A linear perturbation analysis yields a set of ordinary differential equations for the wave amplitudes in momentum representation.

In Sec. III, the linearized multistream model is treated analytically by a stationary-wave method. The complete time evolution of initial perturbations is analyzed, including generalized stationary-wave solutions for degenerate eigenvalues. This method is closely related to the stationary-wave method of van Kampen [44] and Case [45] for classical plasmas. It is noted that it was recognized by van Kampen that a complete solution of the initial-value problem leads to a variety of stationary solutions that are not subject to a dispersion relation. Van Kampen modes and their relation to collective plasma modes have been a topic of wide interest in classical plasma theory; see, e.g., [35,45–48]. There is also a large literature on nonlinear stationary waves, known as Bernstein-Greene-Kruskal (BGK) modes [49]. While van Kampen modes have been widely explored in classical plasmas, their presence in quantum plasmas apparently has only been considered recently [43].

In Sec. IV, the initial-value problem is treated by the more conventional Laplace-transform method, which was introduced in the theory of classical plasmas by Landau [50]. Both the stationary-wave and the Laplace-transform method are shown to lead to identical results provided that the dielectric function of the discrete multistream model is used and also that generalized stationary-wave solutions for degenerate eigenvalues are included.

In Sec. V, the continuum limit is considered. In the continuum theory, plasma oscillations are described by the well-known Lindhard dielectric function [51]. It has been derived with different methods by various authors [52–58]. A generalization to collisional plasmas is widely known as the Lindhard-Mermin dielectric function [59]. In the present work, the continuum limit of the multistream model is treated following an approach by Dawson [35]. From this limiting procedure, a dielectric function can be obtained that agrees with the Lindhard function in the upper complex plane and with its complex conjugate in the lower complex plane. The general solution for the electrostatic potential can be expressed by this dielectric function. Collective plasma oscillations and Landau damping can be shown to arise from the imaginary part of a response function, in close analogy to the fluctuation-dissipation theorem [60,61].

The theoretical results are illustrated in Secs. VI and VII by numerical examples for 1D plasmas with Fermi and Maxwellian statistics. For a 1D quantum plasma with Fermi statistics, the standard dielectric theory predicts only one single undamped collective mode for all wave numbers [51,61,62]. Using the stationary-wave method, we find the undamped mode at small wave numbers, however with increasing wave numbers a broad spectrum of stationary waves leads to unexpected physical phenomena such as subexponential phase relaxation, beat-wave oscillations, and echoes. In contrast, for a 1D plasma with Maxwellian statistics one obtains conventional exponential Landau damping at intermediate wave numbers, and superexponential damping at large wave numbers. Accordingly, the relaxation dynamics depends strongly on the electron distribution function and is basically different for degenerate and nondegenerate electron gases.

II. MULTISTREAM MODEL

The multistream model is based on significant simplifications of the many-body problem for quantum plasma. The ions are treated as a homogeneous positive charge density. This approximation is often useful for collisionless plasmas with negligible ion motion. The electrons are described by a statistical ensemble of single-electron wave functions $\psi_s(\boldsymbol{r},t)$ occurring with probabilities w_s. This approach neglects correlations and exchange interactions among the electrons and is generally considered valid in the high-density regime of weak coupling at low temperatures when the Fermi energy exceeds the temperature. Restricting attention to the finite volume $V = L^3$ of a cube of edge length L and assuming periodic boundary conditions, the single electron states are quantized and can be labeled by a discrete quantum number s. The multistream model restricts the quantum numbers further to a finite set $s = 1,\ldots,N$. In the present work, it is our goal to describe plasma oscillations within the framework of the multistream model by linear perturbation theory. In the linear approximation, one can obtain explicit analytical solutions of initial-value problems for comparison with more advanced nonlinear computational methods. It is also feasible to perform the continuum limit $N \to \infty, L \to \infty$ at constant density $n_0 = N/V = \text{const}$ for comparison with continuum models based, e.g., on the Vlasov equation for the Wigner function. Such a treatment appears to be essential to validate the use of the multistream model for macroscopic plasmas, but it also leads to further insight into the relaxation behavior in collisionless quantum plasmas.

The wave functions $\psi_s(\boldsymbol{r},t)$ and the electrostatic potential $\phi(\boldsymbol{r},t)$ are determined self-consistently by the Schrödinger-Poisson equations,

$$i\,\partial_t \psi_s = -\frac{1}{2}\Delta\psi_s - \phi\psi_s, \tag{1a}$$

$$\Delta\phi = \sum_s w_s|\psi_s|^2 - 1, \tag{1b}$$

written in terms of the dimensionless variables

$$\psi_s = \frac{\tilde{\psi}_s}{\sqrt{n_0}}, \quad \phi = \frac{e\tilde{\phi}}{\hbar\omega_p}, \quad \boldsymbol{r} = \sqrt{\frac{m\omega_p}{\hbar}}\tilde{\boldsymbol{r}}, \quad t = \omega_p\tilde{t}.$$

Dimensional quantities are expressed in Gaussian cgs units and are denoted by a tilde, $h = 2\pi\hbar$ is Planck's constant, e is the elementary charge, m is the electron mass, $\omega_p = \sqrt{4\pi e^2 n_0/m}$ is the plasma frequency, and n_0 is the equilibrium electron density. The probability densities $|\psi_s|^2$ are normalized to 1 within the unit volume. The statistical operator

$$\rho = \sum_{s=1}^{N} w_s |\psi_s\rangle\langle\psi_s| \tag{2}$$

of the quantum states $|\psi_s\rangle$ related to the wave functions $\psi_s = \langle \boldsymbol{r}|\psi_s\rangle$ satisfies the quantum Vlasov equation

$$i\,\partial_t \rho = [H,\rho], \quad H = -\tfrac{1}{2}\Delta - \phi. \tag{3}$$

Any solution of the Schrödinger-Poisson system is therefore also a solution of the Vlasov-Poisson system.

In equilibrium, the charge density and the potential both vanish and the wave functions can be chosen as plane waves satisfying periodic boundary conditions,

$$\psi_s^{(0)} = e^{i\varphi_s}, \quad \varphi_s = \boldsymbol{p}_s\cdot\boldsymbol{r} - E_s t,$$
$$E_s = \frac{1}{2}p_s^2, \quad \boldsymbol{p}_s = \frac{2\pi}{L}(s_x,s_y,s_z). \tag{4}$$

For convenience of notation, the integers $s_x,s_y,s_z = 0,\pm1,\pm2,\pm3,\ldots$ for the three coordinate directions are summarized by the collective index s.

In nonequilibrium, the plane waves are conveniently generalized to carrier-envelope waves [9],

$$\psi_s = A_s(\boldsymbol{r},t)e^{i(\boldsymbol{p}_s\cdot\boldsymbol{r} - \frac{1}{2}p_s^2 t)}. \tag{5}$$

In the framework of linear perturbation theory, each envelope is written in the form $A_s(\boldsymbol{r},t) = 1 + \chi_s(\boldsymbol{r},t)$, where $\chi_s(\boldsymbol{r},t)$ describes a small-amplitude perturbation, subject to the linearized Schrödinger-Poisson system,

$$i(\partial_t + \boldsymbol{p}_s\cdot\nabla)\chi_s = -\frac{1}{2}\Delta\chi_s - \phi, \tag{6a}$$
$$\Delta\phi = \sum_{r=1}^{N} w_r(\chi_r + \chi_r^*). \tag{6b}$$

The real and imaginary parts of $\chi_s(\boldsymbol{r},t)$ can be further related to the density perturbation $n_s = |\psi_s|^2 - 1 = 2\,\mathrm{Re}\{\chi_s\}$ and to a source strength $u_s = -i\nabla\cdot\boldsymbol{v}_s = -i\,\Delta\,\mathrm{Im}\{\chi_s\}$ of a velocity perturbation defined by $\boldsymbol{v}_s = \nabla\,\mathrm{Im}\{\chi_s\}$. Using these hydrodynamic variables, one obtains the set of equations

$$i(\partial_t + \boldsymbol{p}_s\cdot\nabla)n_s = u_s, \tag{7a}$$
$$i(\partial_t + \boldsymbol{p}_s\cdot\nabla)u_s = \Delta\left(\phi + \frac{1}{4}\Delta n_s\right), \tag{7b}$$
$$\Delta\phi = \sum_{r=1}^{N} w_r n_r. \tag{7c}$$

The first equation is the equation of continuity for the density perturbation, while the second one is the divergence of the equation of motion for an electron moving with velocity $\boldsymbol{v}_s$. In addition to the classical electrostatic potential energy $q\phi = -\phi$, it includes the quantum potential $-\frac{1}{4}\Delta n_s$. The latter is the linearized form of the well-known quantum potential $-\frac{1}{2|\psi_s|}\Delta|\psi_s|$ [63,64]. These equations form the (linearized) multistream model [8]. Quantum effects are expressed by the quantized momenta, the quantum potentials, and the probabilities w_s of a degenerate equilibrium state.

Performing a spatial Fourier transform, one obtains from (7) for the Fourier amplitudes

$$\hat{f}(\boldsymbol{k},t) = \int dV f(\boldsymbol{r},t)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{8}$$

a set of ordinary differential equations,

$$i\,\partial_t\hat{n}_s - \boldsymbol{k}\cdot\boldsymbol{p}_s\hat{n}_s = \hat{u}_s, \tag{9a}$$
$$i\,\partial_t\hat{u}_s - \boldsymbol{k}\cdot\boldsymbol{p}_s\hat{u}_s = \frac{k^4}{4}\hat{n}_s + y, \tag{9b}$$
$$y = -k^2\hat{\phi} = \sum_{r=1}^{N} w_r\hat{n}_r. \tag{9c}$$

III. STATIONARY-WAVE APPROACH

The momentum representation of the linearized multistream model (9) has the form of a first-order system of differential equations with constant coefficients,

$$i\,\partial_t \boldsymbol{X}(t) = \boldsymbol{L}\cdot\boldsymbol{X}(t), \tag{10}$$

where $\boldsymbol{X}(t)$ is a $2N$-dimensional vector whose components are given by the set of $\hat{n}_s$ and $\hat{u}_s$ variables, and $\boldsymbol{L}$ is a time-independent nonsymmetric matrix with real coefficients. Particular stationary solutions $\boldsymbol{v}(t) = \boldsymbol{v}e^{-i\omega t}$ of such systems are readily obtained from the time-independent eigenvalue problem

$$\boldsymbol{L}\cdot\boldsymbol{v} = \omega\boldsymbol{v}. \tag{11}$$

First, we restrict attention to the case in which all eigenvalues are nondegenerate. To each nondegenerate eigenvalue ω_α there exists one eigenvector $\boldsymbol{v}_\alpha$. Then the stationary waves form a complete fundamental system of solutions. However, since $\boldsymbol{L}$ is a non-Hermitian operator, the right-hand eigenvectors are in general not orthonormal to each other. It is therefore useful to determine a dual basis of left-hand eigenvectors,

$$\boldsymbol{u}\cdot\boldsymbol{L} = \mu\boldsymbol{u}. \tag{12}$$

Defining $\boldsymbol{M} = \boldsymbol{L} - \omega\boldsymbol{I}$, we obtain the following identity:

$$0 = \boldsymbol{u}\cdot\boldsymbol{M}\cdot\boldsymbol{v} = \mu\boldsymbol{u}\cdot\boldsymbol{v} - \omega\boldsymbol{u}\cdot\boldsymbol{v} = (\mu-\omega)\boldsymbol{u}\cdot\boldsymbol{v}. \tag{13}$$

For eigenvectors with different eigenvalues $\mu \neq \omega$, we obtain the orthogonality relation $\boldsymbol{u}\cdot\boldsymbol{v} = 0$. Therefore, the dual basis $\{\boldsymbol{u}_\alpha\}$ can be chosen orthonormal to the original basis $\{\boldsymbol{v}_\beta\}$ by choosing a convenient normalization of $\boldsymbol{v}_\beta$ and by setting $\boldsymbol{u}_\alpha\cdot\boldsymbol{v}_\beta = \delta_{\alpha\beta}$. Then the complete solution $\boldsymbol{X}(t)$ to the initial value $\boldsymbol{X}(0) = \boldsymbol{C}$ can be expanded in the stationary-wave basis as

$$\boldsymbol{X}(t) = \sum_\alpha c_\alpha\boldsymbol{v}_\alpha e^{-i\omega_\alpha t}, \quad c_\alpha = \boldsymbol{u}_\alpha\cdot\boldsymbol{C}. \tag{14}$$

The stationary-wave approach was originally introduced by van Kampen and Case [44,45] for the corresponding classical continuum problem. In the present work, this method is applied to quantum systems with discrete momenta, and it

will now also be extended for degenerate eigenvalues. As will be seen below, degeneracy occurs generically for 1D Fermi distributions.

The general solution of non-Hermitian first-order systems is well known from the theory of ordinary differential equations [65]. It can be based on the generalized eigenvectors of $\boldsymbol{L}$. A generalized eigenvector $\boldsymbol{v}_l$ of rank l is defined by the condition

$$\boldsymbol{M}^l \cdot \boldsymbol{v}_l = 0 \quad \text{and} \quad \boldsymbol{M}^{(l-1)} \cdot \boldsymbol{v}_l \neq 0. \tag{15}$$

Ordinary eigenvectors are generalized eigenvectors of rank $l = 1$. Formally, the time evolution $\boldsymbol{X}(t)$ of any initial vector $\boldsymbol{X}(0) = \boldsymbol{C}$ given at time $t = 0$ can be expressed as $\boldsymbol{X}(t) = U(t) \cdot \boldsymbol{C}$ with the time-evolution operator

$$\boldsymbol{U}(t) = e^{-i\boldsymbol{L}t}. \tag{16}$$

The time evolution becomes particularly simple if the initial vector corresponds to a generalized eigenvector of $\boldsymbol{L}$. For each generalized eigenvector $\boldsymbol{v}_l$ of rank l with eigenvalue ω, a specific solution of (10) is given by

$$\begin{aligned}\boldsymbol{v}_l(t) &= \boldsymbol{U}(t) \cdot \boldsymbol{v}_l = e^{-i\omega t} e^{-i\boldsymbol{M}t} \cdot \boldsymbol{v}_l \\ &= e^{-i\omega t} \sum_{k=0}^{l-1} (-it\boldsymbol{M})^k \cdot \boldsymbol{v}_l. \end{aligned} \tag{17}$$

The series representation of the operator exponential function can be terminated at the power $l - 1$ due to the definition (15) of generalized eigenvectors. The time dependence of these solutions is given by stationary waves with polynomial coefficients. According to a well-known mathematical theorem, there exist g linearly independent generalized eigenvectors for each g-fold degenerate eigenvalue ω of the matrix $\boldsymbol{L}$. As a consequence, the time evolution of the generalized eigenvectors forms a complete fundamental system of solutions of the differential equation.

The generalized eigenvectors and their dual basis can be easily generated if there exists only one eigenvector to each eigenvalue, and we will restrict attention to this case. For each eigenvalue ω_α with a degree of degeneracy g_α, there exists a basis of generalized eigenvectors $\boldsymbol{v}_{\alpha l}$, $l = 1, \ldots, g_\alpha$, and one can determine an orthonormal dual basis of generalized left-hand eigenvectors $\boldsymbol{u}_{\alpha l}$ with $\boldsymbol{u}_{\alpha k} \cdot \boldsymbol{v}_{\beta l} = \delta_{\alpha\beta}\delta_{kl}$ (Appendix). The expansion of the initial vector $\boldsymbol{X}(0) = \boldsymbol{C}$ is then generalized to

$$\boldsymbol{C} = \sum_\alpha \sum_{l=1}^{g_\alpha} c_{\alpha l} \boldsymbol{v}_{\alpha l}, \quad c_{\alpha l} = \boldsymbol{u}_{\alpha l} \cdot \boldsymbol{C}, \tag{18}$$

and the time evolution of the solution vector assumes the form

$$\begin{aligned}\boldsymbol{X}(t) &= \sum_\alpha \sum_{l=1}^{g_\alpha} c_{\alpha l} \boldsymbol{v}_{\alpha l}(t) \\ &= \sum_\alpha \left(\sum_{l=1}^{g_\alpha} \sum_{k=0}^{l-1} c_{\alpha l} (-it\boldsymbol{M})^k \boldsymbol{v}_{\alpha l} \right) e^{-i\omega_\alpha t}. \end{aligned} \tag{19}$$

In this manner, the complete solution of the initial-value problem for the multistream model can be obtained.

In the present case, the matrix $\boldsymbol{L}$ is a real nonsymmetric matrix. The eigenvalues therefore are either real or they occur in complex-conjugate pairs. For complex-conjugate pairs, one of the solutions is exponentially growing, corresponding to an instability of the system. An example is the two-stream instability that occurs in a two-stream model. It is also found that for a one-dimensional Fermi distribution, the ordinary eigenvectors are not complete. There exist generically twofold-degenerate eigenvalues that require one generalized eigenvector of rank 2.

We now specifically consider the set of Eqs. (9) of the multistream model. Rather than transforming these equations to a first-order system, a simple elimination procedure proves more convenient. Looking first for ordinary stationary-wave solutions of the form $\boldsymbol{X} \propto e^{-i\omega t}$ and eliminating $\hat{u}_s$, one obtains a set of equations for the densities and the potential,

$$P_s(\omega, \boldsymbol{k}) \hat{n}_s = y, \quad y = \sum_r w_r \hat{n}_r, \tag{20}$$

with the definitions

$$y \equiv -k^2 \hat{\phi}, \quad P_s(\omega, \boldsymbol{k}) \equiv (\omega - \boldsymbol{k} \cdot \boldsymbol{p}_s)^2 - \frac{k^4}{4}.$$

First, consider the special case of a vanishing potential, $y = 0$. Such solutions correspond to neighboring equilibrium solutions with different populations of the streams. Obviously, one can change the equilibrium solution (4) in a straightforward way by transferring the population between any pair of waves with momenta $\boldsymbol{p}_s$ and $\boldsymbol{p}_{s'} = \boldsymbol{p}_s + \boldsymbol{k}$ in such a way that the total population is conserved. The corresponding solutions will be called ballistic modes. For a nonvanishing density perturbation to exist for $y = 0$, the first equation of (20) requires $P_s(\omega, \boldsymbol{k}) = 0$. This condition determines the excitation frequencies

$$\omega_{s1} = \boldsymbol{k} \cdot \left(\boldsymbol{p}_s + \tfrac{1}{2}\boldsymbol{k} \right), \quad \omega_{s2} = \boldsymbol{k} \cdot \left(\boldsymbol{p}_s - \tfrac{1}{2}\boldsymbol{k} \right). \tag{21}$$

For a physical explanation of these frequencies, it is noted that a momentum change $\boldsymbol{p}'_s = \boldsymbol{p}_s \pm \boldsymbol{k}$ of a free particle with momentum $\boldsymbol{p}_s$ and energy $E_s(\boldsymbol{p}_s) = \frac{p_s^2}{2}$ leads to excitation energies $E'_s(\boldsymbol{p}_s + \boldsymbol{k}) - E_s(\boldsymbol{p}_s) = \omega_{s1}$ and $E_s(\boldsymbol{p}_s) - E'_s(\boldsymbol{p}_s - \boldsymbol{k}) = \omega_{s2}$, respectively. We now consider a pair of carrier waves with momenta $\boldsymbol{p}_s$ and $\boldsymbol{p}'_s = \boldsymbol{p}_s + \boldsymbol{k}$. Noting that $\omega_{s1} = \omega_{s'2}$, the condition $P_s(\omega, \boldsymbol{k}) = 0$ can be satisfied for both waves, and the condition of zero potential requires

$$y = w_s \hat{n}_s + w_{s'} \hat{n}_{s'} = 0. \tag{22}$$

These pair excitations just exchange the population between two carrier waves without disturbing the total density or the potential.

We now consider solutions of (20) with nonzero potential, $y \neq 0$, which will be called electrostatic modes. The sum over carrier momenta $\boldsymbol{p}_s$ can be simplified by using coordinates parallel and perpendicular to the momentum vector $\boldsymbol{k}$. According to the first equation, the density perturbations of electrostatic modes do not depend on the perpendicular momenta. Introducing new weights for the parallel momenta by summing over the perpendicular momenta

$$W_{s_\parallel} = \sum_{s_\perp} w_s, \tag{23}$$

and denoting the parallel momenta again by the labels s, r and their total number by N, one arrives at a reduced system for

the set of parallel momenta,

$$P_s(\omega,\boldsymbol{k})\hat{n}_s = y, \quad y = \sum_{r=1}^{N} W_r \hat{n}_r. \tag{24}$$

Electrostatic solutions have by definition a nonzero potential, and the fluid variables can then be expressed as

$$\hat{n}_s = \frac{1}{P_s(\omega,\boldsymbol{k})} y, \quad \hat{u}_s = \frac{\omega - \boldsymbol{k}\cdot\boldsymbol{p}_s}{P_s(\omega,\boldsymbol{k})} y. \tag{25}$$

Eliminating the densities from (20) and (25), one arrives at a single equation for the electrostatic potential,

$$y = \sum_s W_s \hat{n}_s = \sum_s \frac{W_s}{P_s(\omega,\boldsymbol{k})} y. \tag{26}$$

Accordingly, electrostatic modes with $y \neq 0$ can only exist if the solubility condition

$$D_N(\omega,\boldsymbol{k}) \equiv 1 - \sum_{s=1}^{N} \frac{W_s}{P_s(\omega,\boldsymbol{k})} = 0 \tag{27}$$

is satisfied. The function $D_N(\omega,\boldsymbol{k})$ will be called the dispersion function of the N-stream model. The zeros ω_α of $D_N(\omega,\boldsymbol{k})$ are eigenfrequencies of the electrostatic modes that correspond to the eigenvalues ω_α of $\boldsymbol{L}$. The eigenvectors are given by the components of (25) evaluated for $\omega = \omega_\alpha$.

The denominators in (27) can be factorized as $P_s(\omega,\boldsymbol{k}) = (\omega - \omega_{s1})(\omega - \omega_{s2})$, and an expansion into a partial fraction then yields

$$D_N(\omega,\boldsymbol{k}) = 1 - \frac{1}{k^2}\sum_s \frac{W_s}{\omega - \omega_{s1}} - \frac{W_s}{\omega - \omega_{s2}}. \tag{28}$$

Setting $p_s = s\Delta p$, $k = n\Delta p$, $\Delta p = 2\pi/L$, and n even, one can rewrite the dispersion function in the form

$$D_N(\omega,\boldsymbol{k}) = 1 - \frac{1}{k^2}\sum_{s=-(N+n)/2}^{(N+n)/2} \frac{W_{s-n/2} - W_{s+n/2}}{\omega - p_s k}. \tag{29}$$

Each denominator increases the order of the characteristic polynomial by 1, yielding altogether $N+n$ electrostatic solutions. The missing $N-n$ solutions are ballistic modes. For each quantum number within $-(N-n)/2 \leqslant s \leqslant (N-n)/2$, there exists a ballistic mode with the frequency $\omega_s = kp_s$. For these frequencies one has $P_{s\pm n/2} = 0$, and (20) can be satisfied by setting

$$\begin{aligned} \hat{n}_{s\pm n/2} &= \pm W_{s\mp n/2}, \\ \hat{u}_{s\pm n/2} &= -(k^2/2) W_{s\mp n/2} \end{aligned} \tag{30}$$

with $\hat{n}_r = 0$ and $\hat{u}_r = 0$ for $r \neq s \pm n/2$. These ballistic modes just exchange the population $W_{s+n/2}W_{s-n/2}$ between any two carrier waves satisfying the momentum relation $p_{s+n/2} - p_{s-n/2} = k$. In a macroscopic description, the populations of the ensemble states are not necessarily limited by the Pauli exclusion principle, and we therefore include these modes for mathematical consistency. Our main results concerning large wave numbers are not affected by this choice.

There is one more special case to be considered. If $W_{s-n/2} - W_{s+n/2} = 0$, the dispersion function (29) for electrostatic modes has a zero at the frequency $\omega = kp_s$ of a ballistic mode. In this case, the frequency $\omega = kp_s$ becomes twofold degenerate and there exists a generalized eigenvector of rank 2 with a nonzero electrostatic potential. Note that all $N-n$ ballistic frequencies are twofold degenerate for a 1D Fermi distribution, and there remain $N+n-(N-n) = 2n$ nondegenerate electrostatic solutions.

The electrostatic potential has been calculated by this stationary-wave approach for the initial conditions

$$\hat{n}_s(0) = \hat{n}_0, \quad \hat{u}_s(0) = \hat{u}_0. \tag{31}$$

Choosing the same initial perturbation for all streams is just one way of realizing a superposition of many modes. Later it will be seen from (57) that the spectral weights of the modes are primarily influenced by the dielectric function. The resulting potential is given by

$$y = \sum_\alpha c_\alpha e^{-i\omega_\alpha t} \tag{32}$$

with

$$c_\alpha = \frac{\hat{n}_0}{\frac{\partial D_N}{\partial\omega}|_{\omega_\alpha}}\left(\omega_\alpha - \sum_s \frac{W_s \boldsymbol{k}\cdot\boldsymbol{p}_s}{P_s(\omega_\alpha)}\right) + \frac{\hat{u}_0}{\frac{\partial D_N}{\partial\omega}|_{\omega_\alpha}} \tag{33}$$

for nondegenerate and

$$c_\alpha = \frac{k^4 W_{s-n/2}\hat{n}_0}{k^4 D_N'(\omega_\alpha) + 2W_{s-n/2}} \tag{34}$$

for degenerate eigenvalues. D_N' denotes the dispersion function (29) without the contribution $s = \alpha$ of the degenerate frequency ω_α.

IV. LAPLACE-TRANSFORM METHOD

The Laplace-transform method is another convenient approach to obtain the solution of the initial-value problem (9). It was introduced by Landau for classical plasmas, leading to the famous result of Landau damping. We therefore first show that the Laplace-transform method leads to the same result as the stationary-wave approach for the discrete multistream model. In the following section, we then use the Laplace-transform approach to derive Landau damping in the continuum limit.

The Laplace transform of the function $f(t)$ will be defined by

$$\tilde{f}(\omega) = \int_0^\infty dt f(t) e^{i\omega t}. \tag{35}$$

Here it is assumed that $\tilde{f}(\omega)$ converges above the line $\mathrm{Im}\{\omega\} = c$ in the upper complex ω plane for some constant $c > 0$. Using the well-known derivative rule

$$\widetilde{\partial_t f}(\omega) = -f(0) - i\omega\tilde{f}(\omega), \tag{36}$$

the Laplace transform of the multistream model (9) is given by

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{p}_s)\tilde{n}_s = \tilde{u}_s + i\hat{n}_{s0}, \tag{37a}$$

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{p}_s)\tilde{u}_s = \frac{k^4}{4}\tilde{n}_s + \tilde{y} + i\tilde{u}_{s0}, \tag{37b}$$

$$\tilde{y} = \sum_r W_r \tilde{n}_r, \tag{37c}$$

with the initial values $\hat{n}_{s0} = \hat{n}_s(\boldsymbol{k},0)$ and $\hat{u}_{s0} = \hat{u}_s(\boldsymbol{k},0)$. Eliminating $\tilde{u}_s$ leads to an inhomogeneous equation for $\tilde{n}_s$,

$$P_s(\omega,\boldsymbol{k})\tilde{n}_s = \tilde{y} + i[\hat{u}_{s0} + (\omega - \boldsymbol{k}\cdot\boldsymbol{p}_s)\hat{n}_{s0}], \tag{38}$$

where the definition of $P_s(\omega,\boldsymbol{k})$ given in (20) is now extended to the upper ω plane. Since $P_s(\omega,\boldsymbol{k}) \neq 0$ above $\mathrm{Im}(\omega) = c > 0$, the Laplace transform of the electrostatic potential can be obtained in the form

$$\tilde{y} = i\frac{S_N(\omega,\boldsymbol{k})}{D_N(\omega,\boldsymbol{k})} \tag{39}$$

with

$$D_N(\omega,\boldsymbol{k}) = 1 - \sum_s \frac{W_s}{P_s(\omega,\boldsymbol{k})},$$

$$S_N(\omega,\boldsymbol{k}) = \sum_s \frac{W_s}{P_s(\omega,\boldsymbol{k})}[\hat{u}_{s0} + (\omega - \boldsymbol{k}\cdot\boldsymbol{p}_s)\hat{n}_{s0}].$$

The Laplace transform of the potential has now been expressed by the dispersion function $D_N(\omega,\boldsymbol{k})$ of the N-stream model and by a source function $S_N(\omega,\boldsymbol{k})$ describing the dependence on the initial conditions.

The time-dependent potential is obtained by the inverse Laplace transform

$$\hat{y}(\boldsymbol{k},t) = \int_{-\infty+is}^{+\infty+is} \frac{d\omega}{2\pi}\tilde{y}(\omega,\boldsymbol{k})e^{-i\omega t}. \tag{40}$$

The integration contour can be closed by parallel lines to the imaginary axis for $\mathrm{Im}\{\omega\} \geqslant 0$ and $\mathrm{Re}\{\omega\} \to \infty$ and by a semicircle in the lower ω plane with radius $r \to \infty$. Along the parallel lines, the integral vanishes since $S_N(\omega,\boldsymbol{k}) \to 0$ for $\mathrm{Re}\{\omega\} \to \infty$. Along the semicircle, the integral vanishes because $|e^{-i\omega t}| \to 0$ for $\omega_i t < 0$. According to the residue theorem, the integral along the closed contour becomes a sum over the residue of the integrand within the contour. Specifically, for a negatively oriented contour and simple poles $\omega = \omega_\alpha$, one has

$$\hat{y}(\boldsymbol{k},t) = (-1)2\pi i \sum_\alpha i \lim_{\omega\to\omega_\alpha} \frac{(\omega-\omega_\alpha)S_N(\omega,\boldsymbol{k})}{2\pi D_N(\omega,\boldsymbol{k})}e^{-i\omega t}$$

$$= \sum_\alpha \lim_{\omega\to\omega_\alpha} \frac{(\omega-\omega_\alpha)S_N(\omega,\boldsymbol{k})}{D_N(\omega,\boldsymbol{k})}e^{-i\omega t}. \tag{41}$$

First consider the poles that arise from the zeros of $D_N(\omega_\alpha,\boldsymbol{k}) = 0$. Obviously, these zeros coincide with the frequencies of the electrostatic stationary waves, and their contribution to the integral becomes

$$\hat{y}(\boldsymbol{k},t) = \sum_\alpha \frac{S_N(\omega_\alpha,\boldsymbol{k})}{\partial_\omega D_N(\omega_\alpha,\boldsymbol{k})}e^{-i\omega_\alpha t}. \tag{42}$$

For the initial conditions (31) used in the present work, the source function becomes particularly simple at the zeros $\omega = \omega_\alpha$ of $D_N(\omega,\boldsymbol{k})$,

$$S_N(\omega_\alpha,\boldsymbol{k}) = \sum_s \frac{W_s}{P_s(\omega_\alpha,\boldsymbol{k})}[\hat{u}_0 + (\omega_\alpha - \boldsymbol{k}\cdot\boldsymbol{p}_s)\hat{n}_0]$$

$$= \hat{u}_0 + \left(\omega_\alpha - \sum_s \frac{\boldsymbol{k}\cdot\boldsymbol{p}_s W_s}{P_s(\omega_\alpha,\boldsymbol{k})}\right)\hat{n}_0. \tag{43}$$

The result (42) evaluated with the initial condition (43) is in complete agreement with the stationary-wave solution for nondegenerate eigenvalues (33).

In addition, contributions arise from the poles of $S_N(\omega,\boldsymbol{k})$ at the zeros of $P_s(\omega,\boldsymbol{k})$. These zeros correspond to ballistic modes and degenerate electrostatic modes. Only the latter give rise to an electrostatic potential, and we therefore consider this case. As previously discussed, these modes can arise if two carrier momenta $\boldsymbol{p}_q$ and $\boldsymbol{p}_{q'}$ with equal weights $W_q = W_{q'}$ are separated by the wave momentum $\boldsymbol{p}_{q'} - \boldsymbol{p}_q = \boldsymbol{k}$. Then the ballistic mode frequencies (21) become

$$\omega_{q1} = \omega_{q'2} = \boldsymbol{k}\cdot\left(\boldsymbol{p}_q + \tfrac{1}{2}\boldsymbol{k}\right),$$

$$\omega_{q2} = \omega_{q1} - k^2, \quad \omega_{q'1} = \omega_{q1} + k^2. \tag{44}$$

Defining S' as the sum S without the two terms $s = q$ and $s = q'$ and setting $c_s = W_s[\hat{u}_{s0} + (\omega - \boldsymbol{k}\cdot\boldsymbol{p}_s)\hat{n}_{s0}]$, one can separate the pole at $\omega = \omega_{q1}$ as

$$S - S' = \frac{1}{\omega-\omega_{q1}}\left(\frac{c_q}{\omega-\omega_{q1}+k^2} + \frac{c_{q'}}{\omega-\omega_{q1}-k^2}\right). \tag{45}$$

For the present initial conditions, the corresponding residue is given by

$$\mathrm{Res}S = \lim_{\omega\to\omega_{q1}}(\omega-\omega_{q1})S_N(\omega,\boldsymbol{k}) = \frac{c_q - c_{q'}}{k^2} = \hat{n}_0 W_q. \tag{46}$$

Choosing the same notation for the restricted sum in $D(\omega,\boldsymbol{k})$, one obtains

$$D_N - D'_N = -\frac{W_q}{\omega-\omega_{q1}}\left(\frac{1}{\omega-\omega_{q1}+k^2} + \frac{1}{\omega-\omega_{q1}-k^2}\right)$$

$$= \frac{2W_q}{k^4 - (\omega-\omega_{q1})^2}. \tag{47}$$

Using (46) and (47) in (41), one finds for each degenerate ballistic wave the contribution

$$\hat{y}(\boldsymbol{k},t) = \frac{W_q k^4}{D'_N k^4 + 2W_q}\hat{n}_0 e^{-i\omega_\alpha t}, \tag{48}$$

which is in complete agreement with the result (34) of the generalized stationary-wave method for degenerate eigenvalues. It is therefore concluded that both approaches are completely equivalent for the solution of the initial-value problem of the discrete multistream model for both nondegenerate and degenerate eigenvalues.

V. CONTINUUM LIMIT

For a large number of stationary waves, the multistream model is expected to converge to a unique continuum limit, and thereby conventional Landau damping should be recovered from the phase relaxation of the stationary waves. In this section, the continuum limit is performed explicitly, and thereby a collective mode subject to conventional Landau damping can be obtained.

We first consider the continuum limit of the dispersion function $D_N(\omega,\boldsymbol{k})$ for $N \to \infty$. The corresponding limit of the classical multistream model has been discussed by Dawson [35], and we partly follow these lines for the present quantum treatment. It is advantageous to represent the dispersion

function in the form of (29),

$$D_N(\omega,k) = 1 - \frac{1}{k^2} \sum_{s=-(N+n)/2}^{(N+n)/2} \frac{W_{s-n/2} - W_{s+n/2}}{\omega - kp_s}. \quad (49)$$

The momenta are taken on an equidistant grid as $p_s = s\delta$, $s = 0 \pm 1, \pm 2, \ldots, \pm N/2$, and we perform the limit $N \to \infty$, $\delta \to 0$. The probability density of states is defined as a continuous function $\Omega(p)$, and $W_{s\pm n/2} = \Omega(p_s \pm k/2)\delta$ is the probability of states within the momentum interval δ. For simplicity of notation, we define $Z(p_s) = \Omega(p_s - k/2) - \Omega(p_s + k/2)$ to obtain

$$D_N(\omega,k) = 1 - \frac{1}{k^2} \sum_{s=-(N+n)/2}^{(N+n)/2} \frac{Z(p_s)\delta}{\omega - kp_s}. \quad (50)$$

To pass to the continuum limit, one can remove the singularity in the sum by writing

$$D_N(\omega,k) = 1 - \frac{\delta}{k^2} \sum_{s=-(N+n)/2}^{(N+n)/2} \frac{Z(p_s) - Z(\omega/k)}{\omega - kp_s} - \frac{Z(\omega/k)\delta}{k^2} \sum_{s=-(N+n)/2}^{(N+n)/2} \frac{1}{\omega - kp_s}. \quad (51)$$

The terms of the first sum are regular at $\omega = kp_s$. Setting $\epsilon_s = \omega - kp_s$, $p_s = \frac{\omega - \epsilon_s}{k}$, and expanding the nominator about ω/k, one obtains

$$\frac{Z(p_s) - Z(\omega/k)}{\omega - kp_s} = \frac{Z'(\omega/k)\left(-\frac{\epsilon_s}{k}\right) + O\left(\epsilon_s^2\right)}{\epsilon_s} = -\frac{Z'(\omega/k)}{k} + O(\epsilon) = O(1). \quad (52)$$

Passing to the continuum limit, the first sum can be replaced by an integral in the usual manner,

$$\sum_{s=-\infty}^{\infty} \delta \frac{Z(p_s) - Z(\omega/k)}{\omega - kp_s} \to \int_{-\infty}^{+\infty} dp \frac{Z(p) - Z(\omega/k)}{\omega - kp}.$$

Noting that the Cauchy principal part of the integral of $1/x$ is zero, one has

$$\int_{-\infty}^{+\infty} dp \frac{Z(p) - Z(\omega/k)}{\omega - kp} = \mathrm{P}\int_{-\infty}^{+\infty} dp \frac{Z(p) - Z(\omega/k)}{\omega - kp} = \mathrm{P}\int_{-\infty}^{+\infty} dp \frac{Z(p)}{\omega - kp}. \quad (53)$$

The second sum can be seen to be the expansion of $\cot(z)$ in partial fractions. This expansion assumes the well-known form [66]

$$\pi \cot(\pi x) = \sum_{s=-\infty}^{+\infty} \frac{1}{x - s} = \frac{1}{x} + \sum_{s=1}^{+\infty} \frac{2x}{x^2 - s^2}. \quad (54)$$

The second form of the expansion is simply obtained by writing the term with $s = 0$ separately and combining the sums over negative and positive s values. Using this expansion yields

$$D_N(\omega,k) = 1 - \frac{1}{k^2}\mathrm{P}\int_{-\infty}^{+\infty} dp \frac{Z(p)}{\omega - kp} - \frac{Z(\omega/k)}{k^3}\pi \cot\left(\frac{\pi\omega}{k\delta}\right).$$

One still has to pass to the limit $\delta \to 0$ in the last part of the formula. For real frequencies ω and $Z(\omega/k) \neq 0$, there will be infinitely many zeros of $D_N(\omega,k)$ for $N \to \infty$, since each branch of $\cot x$ passes from $+\infty$ to $-\infty$ and there will be a zero within each branch. The solution of the initial-value problem then is just a superposition of these modes with real frequencies. However, if the dispersion function is evaluated for complex frequencies above (or below) the real axis, the summation of the partial fraction series has to be taken formally with a positive (negative) imaginary part $\beta = \mathrm{Im}\{\frac{\pi\omega}{k\delta}\}$. For $\delta \to 0$ it approaches $\beta \to \pm\infty$, where the $+$ sign corresponds to an integration below and the $-$ sign to an integration above the pole. Substituting $\frac{\pi\omega}{k\delta} \to x + i\beta$ yields the limits

$$\lim_{\beta\to\pm\infty} \cot(x + i\beta) = i \lim_{\beta\to\pm\infty} \frac{e^{ix-\beta} + e^{-ix+\beta}}{e^{ix-\beta} - e^{-ix+\beta}} \to \mp i. \quad (55)$$

The continuum limit of the dispersion function for complex frequencies is therefore

$$D^{\pm}(\omega,k) \equiv 1 - \frac{1}{k^2}\mathrm{P}\int_{-\infty}^{+\infty} dp \frac{Z(p)}{\omega - kp} \pm i\pi \frac{Z(\omega/k)}{k^3} \quad (56)$$

for $\mathrm{Im}\{\omega\} \gtrless 0$, respectively. The upper sign corresponds to Landau's integration contour, and it will be shown to determine the asymptotic behavior of the initial perturbation.

We now consider the continuum limit of the solution (40) for the time-dependent potential. In general, there are contributions to this solution from isolated pairs of complex conjugate zeros of the dispersion function. These complex roots can be simply added and will not be considered further. In addition, the dispersion function has infinitely many zeros on the real axis, which are no longer isolated in the continuum limit. The contribution of these poles can be found by choosing a closed rectangular integration contour consisting of two parallel lines immediately above and below the real axis with infinitesimally small vertical boundaries at infinity. In the continuum limit, the dielectric function above and below the real axis is given by (56). Defining $D(\omega) = D^{\pm}$ and similarly $S(\omega) = S^{\pm}$ for $\mathrm{Im}\{\omega\} \gtrless 0$, respectively, the contour integral around the poles on the real axis assumes the form

$$\hat{y}(\boldsymbol{k},t) = i\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\left(\frac{S^+}{D^+} - \frac{S^-}{D^-}\right)e^{-i\omega t}, \quad (57)$$

where the integration is now performed along the real ω axis. This representation corresponds to the superposition of stationary waves with real frequencies in the multistream model. The continuous excitation spectrum can be defined by

$$c(\omega) \equiv \frac{i}{2\pi}\left[\frac{S^+}{D^+} - \frac{S^-}{D^-}\right] = -\frac{1}{\pi}\mathrm{Im}\left[\frac{S^+}{D^+}\right]. \quad (58)$$

It is found proportional to the imaginary part of the response function S^+/D^+, in close analogy to the fluctuation-dissipation theorem [60,61]. Setting $D^{\pm} = D_r \pm i D_i$ and $S^{\pm} = S_r \mp i S_i$, the excitation spectrum can be expressed by the real and imaginary parts of $D^{\pm}$ and $S^{\pm}$ as

$$c(\omega) = \frac{1}{\pi}\frac{S_r D_i + D_r S_i}{D_r^2 + D_i^2}. \quad (59)$$

As a specific example, we consider a uniform velocity perturbation $\hat{u}_s = \hat{u}_0, \hat{n}_s = 0$. In this case, the source function can be

expressed by the dispersion function itself, $S^{\pm} = (1 - D^{\pm})\hat{u}_0$, and the excitation spectrum then reduces to a simpler form with response function $1/D^{+}$,

$$c(\omega) = \frac{i}{2\pi}\left[\frac{1-D^{+}}{D^{+}} - \frac{1-D^{-}}{D^{-}}\right]\hat{u}_0 = \frac{i}{2\pi}\left[\frac{1}{D^{+}} - \frac{1}{D^{-}}\right]\hat{u}_0 = \frac{1}{\pi}\frac{D_i}{D_r^2 + D_i^2}\hat{u}_0. \quad (60)$$

This spectrum is further discussed in Sec. VI and compared to the weights of the discrete multistream model in Fig. 3.

In general, the superposition of modes leads to exponential Landau damping, as can be seen in the following way. According to the Landau theory, the complex frequencies of the collective modes are given by the zeros $\omega_\alpha = \omega_{\alpha,r} + i\omega_{\alpha,i}$ of $D^{+}(\omega) = 0$. For small imaginary parts, the real and imaginary parts of these frequencies are determined by

$$D_r(\omega_{\alpha,r}) = 0, \quad \partial_\omega D_r(\omega)\big|_{\omega=\omega_{\alpha,r}}\omega_{\alpha,i} + D_i = 0. \quad (61)$$

The contribution of this pole to the integral (57) can be evaluated by setting $D_r(\omega) \approx \partial_\omega D_r(\omega)(\omega - \omega_{\alpha,r})$ near the zero $\omega_{\alpha,r}$, where ω now denotes the real integration variable. In the nominator, one can approximate $|D_r S_i| \ll |S_r D_i|$, assuming that $S_r = O(1)$ for $D_r \to 0$ and that the imaginary parts D_i and S_i are of the same order of magnitude. Substituting (61), one finds the pole spectrum,

$$c_{\rm pole}(\omega) \approx \frac{1}{\pi}\frac{D_i S_r}{D_r^2 + D_i^2} \approx \frac{1}{\pi}\frac{-\omega_{\alpha,i}}{(\omega-\omega_{\alpha,r})^2 + \omega_{\alpha,i}^2}\frac{S_r(\omega,\boldsymbol{k})}{\partial_\omega D_r(\omega,\boldsymbol{k})}. \quad (62)$$

Assuming that $S_r(\omega)/\partial_\omega D_r(\omega)$ varies only weakly over the pole region, the pole spectrum is given by a Lorentzian profile. Its Fourier transform leads to a damped oscillation with the Landau plasmon frequency and damping constant as given by (56) and (61),

$$\hat{y}_{\rm pole}(\boldsymbol{k},t) = \frac{S_r(\omega,\boldsymbol{k})}{\partial_\omega D_r(\omega,\boldsymbol{k})}\bigg|_{\omega=\omega_{\alpha,r}} e^{-i\omega_{\alpha,r}t - \gamma_\alpha t}, \qquad \gamma_\alpha = -\omega_{\alpha,i} = \frac{D_i(\omega,\boldsymbol{k})}{\partial_\omega D_r(\omega,\boldsymbol{k})}\bigg|_{\omega=\omega_{\alpha,r}}. \quad (63)$$

Although exponential decay has been recovered in this way, it is noted that the pole approximation is not always applicable, and one then has to go back to the complete excitation spectrum (58). The relaxation of plasma waves in a 1D Fermi-degenerate electron gas provides one such example, which will be discussed in the following section.

VI. FERMI STATISTICS

In this section, we consider as a specific example a multistream model with N streams for a 1D Fermi-degenerate electron gas with equal statistical weights $W_s = 1/N$ for all representative states. We have chosen a momentum grid $p_s = -p_F + (s - 0.5)\Delta p$, $s = 1, \ldots, N$ with a Fermi momentum $p_F = 1.5$ and a momentum step $\Delta p = 2p_F/N$. The wave number of the perturbation is also chosen on the grid, $k = n\Delta p$. Setting $\Delta p = 2\pi/L$, all waves satisfy periodic boundary conditions at the end points of a spatial interval of length L.

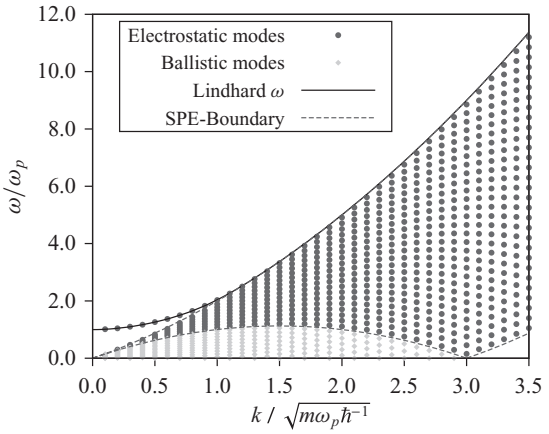

FIG. 1. Eigenfrequencies of electrostatic and ballistic modes. The mode frequencies are compared with the Lindhard dispersion relation (solid line) and the boundaries of the SPE continuum (dashed lines). Parameters: $N = 30$, $p_F = 1.5$.

The magnitudes of the eigenfrequencies (occurring in positive-negative pairs) of a multistream model with $N = 30$ streams are illustrated in Fig. 1. The frequency of one electrostatic mode is clearly separated from all other frequencies for small wave numbers. The other electrostatic-mode frequencies lie between the cutoffs $\omega_1 = E_{p_F+k} - E_{p_F} = k(p_F + k/2)$ and $\omega_2 = E_{p_F} - E_{p_F-k} = k(p_F - k/2)$ corresponding to the maximum and minimum energies of single-particle momentum excitations above the Fermi level. The range of energies between these cutoffs is called the single-particle excitation (SPE) continuum. The isolated frequency above the SPE continuum cannot be explained by single-particle excitations, and therefore it corresponds to a collective mode. In accordance with this interpretation, it is found to be in excellent agreement with the plasmon frequency determined by the Lindhard dispersion relation. For large wave numbers, the plasmon frequency approaches asymptotically the SPE continuum. The frequencies of the ballistic modes are located beneath the lower cutoff of the SPE continuum.

Figure 2 shows the electrostatic potential y resulting from an initial density perturbation $\hat{n}_s(0) = \delta(k - k_0)$, $\hat{u}_s(0) = 0$, and the weights $c_\alpha/\Delta\omega$ per frequency interval $\Delta\omega = k\Delta p$ of the various modes for different values of the wave number k_0. For the isolated frequency outside the SPE continuum, the weight c_α itself is shown. For small wave numbers, the collective mode is dominant, resulting in an undamped oscillation with the Lindhard frequency. With increasing wave number, the weights $c_\alpha/\Delta\omega$ of the other modes increase. The electrostatic potential then shows additional features. A beat wave and nonexponential damping of the oscillation is observed. In addition, the potential shows echoes after time intervals $T_{\rm echo} = 2\pi/(k\Delta p) = L/k$. The echo period is the propagation time of the wave with momentum k over the periodicity length L and is therefore specific for finite-size systems. The initial decrease of the potential as well as the weights $c_\alpha/\Delta\omega$ per frequency interval converge rapidly with the number of streams.

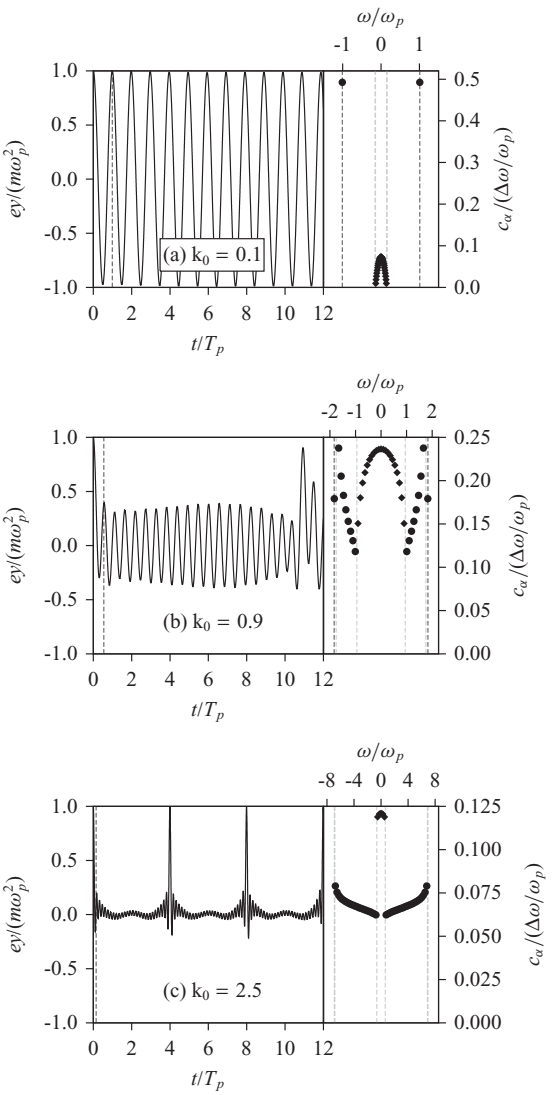

FIG. 2. Electrostatic potential $y = -k^2\hat{\phi}$ and weights $c_\alpha/\Delta\omega$ per frequency interval for (a) $k_0 = 0.1$, (b) $k_0 = 0.9$, and (c) $k_0 = 2.5$. Vertical dashed lines mark the frequency given by the Lindhard dispersion relation (dark gray) and the cutoff frequencies of the SPE continuum (light gray). For $k_0 = 2.5$, the upper cutoff coincides with the Lindhard frequency. Parameters: $N = 30$, $p_F = 1.5$, $\hat{n}_s(0) = \delta(k - k_0)$, and $\hat{u}_s(0) = 0$.

In Fig. 3, the specific weights $c_\alpha/\Delta\omega$ of the multistream model are compared with the excitation spectrum (60) in the continuum limit. In this case, the results are already nicely converged for a moderately large number of streams ($N = 300$). In summary, echoes and discrete momenta are specific features of finite-size systems, while beat waves, decay, and the specific weights of the excitation spectrum are common features of infinite-size continuum systems.

The maximum of the weights occurs near the edge of the Fermi distribution. Evidently, such an edge maximum is only poorly represented by the Lorentzian distribution (62), indicating nonexponential decay. To analyze relaxation for a 1D Fermi distribution in more detail, we consider the limit of large wave numbers. The electrostatic potential for large wave numbers can be derived in the free-particle approximation.

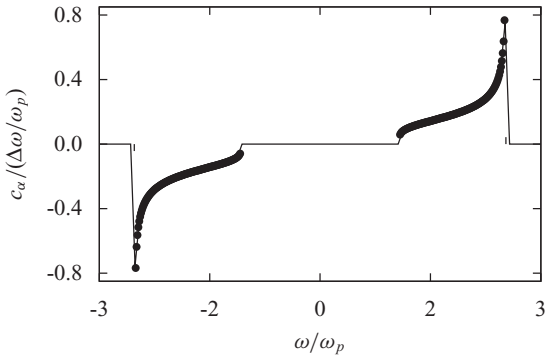

FIG. 3. Weights $c_\alpha/\Delta\omega$ per frequency interval for a velocity perturbation with $k_0 = 1.2$. The weights of the multistream model (dots) are compared with the excitation spectrum (60) of the continuum theory. Parameters: $N = 300$, $p_F = 1.5$, $\hat{n}_s(0) = 0$, and $\hat{u}_s(0) = k_0^2\delta(k - k_0)$.

For this purpose, the electrostatic potential is neglected in the equation of motion of the electrons by setting $y = 0$ in (9b). The electrostatic potential is then calculated perturbatively to first order by inserting the zeroth-order densities in the Poisson equation (9c). Choosing an initial density perturbation $\hat{n}_s(0) = \hat{n}_0$ and using the free-particle excitation frequencies (21), the evolution of the electrostatic potential is then found to be

$$\hat{\phi}(t) = -\frac{\hat{n}_0}{2k^2N}\sum_{s=1}^{N}(e^{-i\omega_{s1}t} + e^{-i\omega_{s2}t}) = -\frac{\hat{n}_0}{2k^2}\frac{\sin(\omega_1 t) + \sin(\omega_2 t)}{N\sin\left(\frac{\pi t}{T_{\text{echo}}}\right)}. \tag{64}$$

The electrostatic potential shows beat-wave oscillations with the cutoff frequencies $\omega_{1,2}$ of the SPE continuum. In addition, the electrostatic potential is modulated by an envelope function $1/\sin(\frac{\pi t}{T_{\text{echo}}})$ that explains the observed damping and the echoes occurring after time intervals T_{echo}. For intermediate wave numbers, qualitatively the same effects occur. In this case, however, the collective mode still plays a crucial role, and the beat wave and damping are less pronounced. The subexponential damping is illustrated in Fig. 4 for a

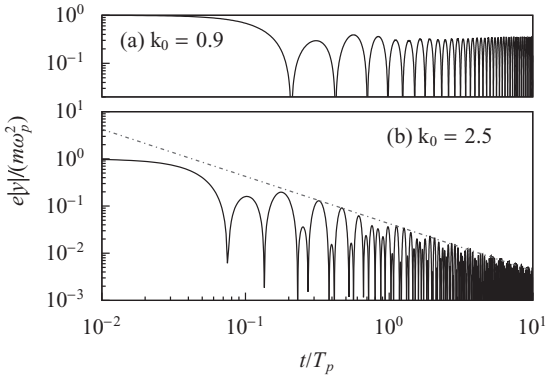

FIG. 4. Magnitude of the electrostatic potential $y = -k^2\hat{\phi}$ for (a) $k_0 = 0.9$ and (b) $k_0 = 2.5$. For $k_0 = 2.5$ it is compared to the envelope $(kp_F t)^{-1}$ (dashed line). Parameters: $N = 300$, $p_F = 1.5$, $\hat{n}_s(0) = \delta(k - k_0)$, $\hat{u}_s(0) = 0$.

multistream model with $N = 300$ streams. For large wave numbers, the thermodynamic limit $N \to \infty$, $L \to \infty$, and $p_F = \frac{\pi N}{L} = 1.5$ leads to a complete subexponential damping proportional to t^{-1} for all times despite the absence of Landau damping in the Lindhard plasmon-dispersion relation.

VII. MAXWELLIAN STATISTICS

In the following calculations, the system described in the previous section is considered for nonzero temperature under Maxwellian statistics. The statistical weights of the discrete momenta p_s are given by

$$W_s = \frac{\exp\left(-\frac{1}{2}\frac{p_s^2}{T}\right)}{\sum_{r=1}^{N}\exp\left(-\frac{1}{2}\frac{p_r^2}{T}\right)}. \tag{65}$$

We chose temperature $T = 1.5^2/2$, the momentum step between successive discrete momenta $\Delta p = 2p_{\max}/N$, and wave numbers $k = n\Delta p$. The discretization of the Maxwell distribution is cut off at the cutoff momentum $p_{\max} = 4.5$. Since $p_{\max}$ is greater than three standard deviations $\sigma = \sqrt{T}$ of the Maxwell distribution, only a negligible fraction of momenta lie above the cutoff momentum. The magnitudes of the eigenfrequencies of a multistream model with $N = 90$ streams are illustrated in Fig. 5.

For a Maxwellian velocity distribution, there is a continuum of electrostatic modes without any isolated frequencies for not too small wave numbers. For wave numbers that are odd multiples of Δp, there is a single degenerate ballistic eigenfrequency in the center of the SPE continuum at frequency zero. The Lindhard dispersion relation for a plasmon in a quantum plasma with Maxwellian velocity distribution lies inside the continuum of modes. The cutoff of the SPE continuum and the isolated frequency at very small wave numbers are explained solely by the artificially introduced cutoff of the momentum discretization at $p_{\max}$.

Figure 6 shows the electrostatic potential y resulting from an initial density perturbation $\hat{n}_s(0) = \delta(k - k_0)$, $\hat{u}_s(0) = 0$,

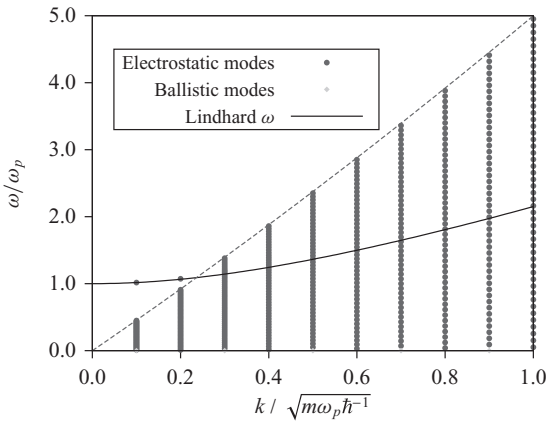

FIG. 5. Eigenfrequencies of electrostatic and degenerate ballistic modes. The mode frequencies are compared with the Lindhard dispersion relation (solid line) and the artificial boundary of the SPE continuum (dashed line). The artificial boundary of the SPE continuum is introduced by the cutoff of the Maxwell distribution at momentum $p_{\max}$. Parameters: $N = 90$, $T = 1.5$, and $p_{\max} = 4.5$.

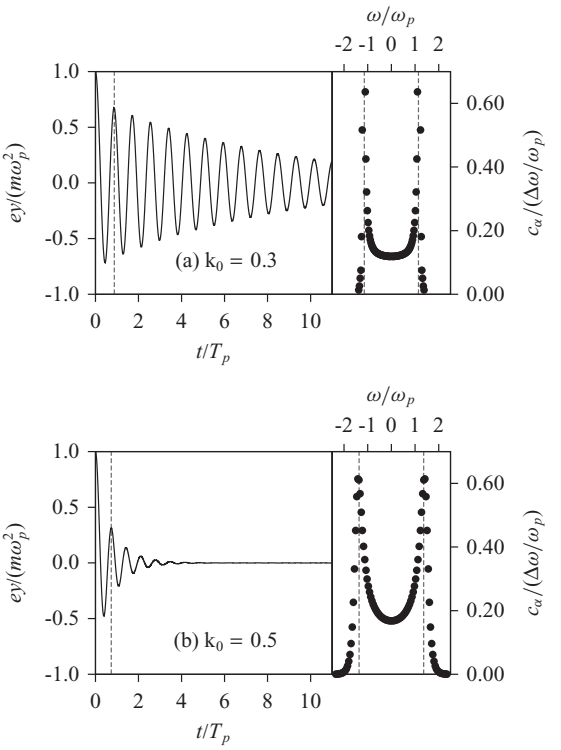

FIG. 6. Electrostatic potential $y = -k^2\hat{\phi}$ and weights $c_\alpha/\Delta\omega$ per frequency interval for (a) $k_0 = 0.3$ and (b) $k_0 = 0.5$. Vertical dashed lines mark the frequency given by the Lindhard dispersion relation (dark gray). Parameters: $N = 90$, $T = 1.5$, $\hat{n}_s(0) = \delta(k - k_0)$, and $\hat{u}_s(0) = 0$.

and the weights $c_\alpha/\Delta\omega$ per frequency interval $\Delta\omega = k\Delta p$ of the various modes for wave number $k_0 = 0.3$ and 0.5. For Maxwellian statistics, a damped collective mode is observed. The damping gets stronger with increasing wave number. The collective mode is created by a superposition of electrostatic modes. The distribution of the weight per frequency interval of the modes has a peak approximately at the Lindhard frequency and broadens with increasing wave number of the perturbation. The peak is asymmetric, however to a first approximation its shape is close to the shape of a Lorentz peak. Thus for intermediate wave numbers, the multistream model reproduces very well the Landau-damped plasma oscillations in a quantum plasma with Maxwellian velocity distribution given by the Landau-Lindhard theory.

As an example, exponential Landau damping is shown in Fig. 7 for the wave number $k_0 = 0.3$ over more than 15 plasma periods. The numerical solution of the multistream model is compared to an exponential decay law with the decay constant $\gamma = 0.019\,67$ according to the Lindhard-Landau theory. There is excellent agreement between the numerical result and the theoretical decay law, indicating that discrete multistream models can reproduce the usual Landau damping very well by phase relaxation of a sufficiently large number of streams.

For large wave numbers, the electrostatic potential can be calculated in the free-particle approximation as in (64). For a Maxwellian distribution $f(p) = \frac{1}{\sqrt{2\pi T}}e^{-\frac{p^2}{2T}}$, one obtains for the electrostatic potential caused by a density perturbation $\hat{n}_0$ in first-order perturbation theory the

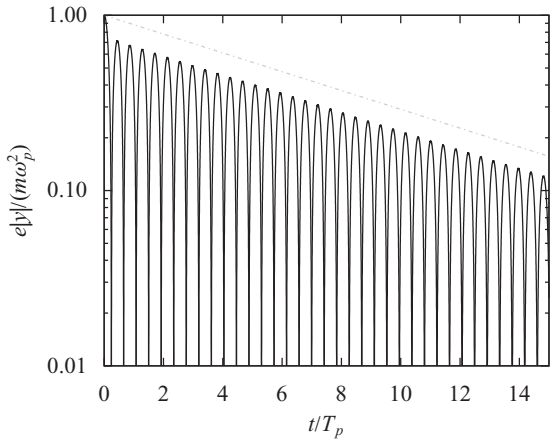

FIG. 7. Exponential Landau damping of a potential perturbation $y = -k^2\hat{\phi}(k,t)$ with wave number $k_0 = 0.3$ for a Maxwellian velocity distribution. The numerical solution of the stationary-wave method (solid line) is compared with exponential decay (dashed line) with a damping constant $\gamma = 0.019\,67$ according to the Landau-Lindhard theory. Parameters: $N = 90$, $T = 1.5^2/2$, $\hat{n}_s(0) = \delta(k - k_0)$, and $\hat{u}_s(0) = 0$.

expression

$$\hat{\phi}(t) = -\frac{\hat{n}_0}{k^2}\cos\left(\frac{k^2}{2}t\right)\int_{-\infty}^{+\infty} dp\, f(p)e^{-ikpt}$$
$$= -\frac{\hat{n}_0}{k^2}\cos\left(\frac{k^2}{2}t\right)\exp\left(-\frac{1}{2}Tk^2t^2\right). \qquad (66)$$

The electrostatic potential shows oscillations with frequency $k^2/2$ corresponding to the asymptotic single-particle energies. However, these oscillations are strongly damped with a Gaussian function of width $1/(\sqrt{T}k)$. This superexponential damping is illustrated in Fig. 8 for a multistream model with $N = 300$ streams. Note that for a longer time period than shown, echoes in the electrostatic potential occur due to the finite number of streams, as in the case of Fermi statistics considered above.

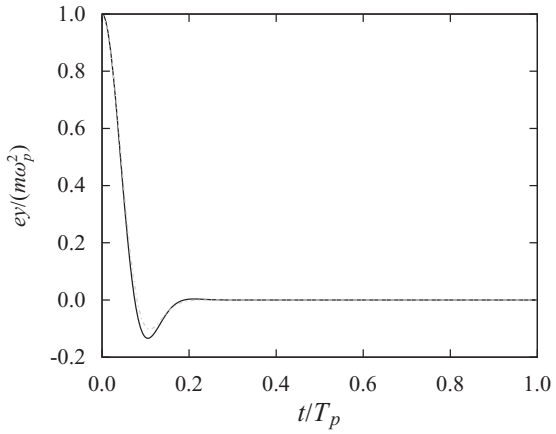

FIG. 8. Superexponential damping of a potential perturbation $y = -k^2\hat{\phi}(k,t)$ with wave number $k_0 = 2.5$ for a Maxwellian velocity distribution. The stationary-wave solution (solid line) is compared with the asymptotic approximation (66) for large wave numbers (dashed line). Parameters: $N = 90$, $T = 1.5^2/2$, $\hat{n}_s(0) = \delta(k - k_0)$, and $\hat{u}_s(0) = 0$.

VIII. CONCLUSIONS

The quantum-kinetic treatment of ideal plasmas is commonly based on the Vlasov-Maxwell SCF theory. Alternatively, one can solve the related SCF Schrödinger-Poisson system for an ensemble of representative quantum states. Its formulation in terms of hydrodynamic variables is known as the multistream model.

In the present work, the analytical properties of the multistream model have been studied in the linear approximation. The initial-value problem has been addressed both by the stationary-wave method and by the Laplace-transform method. It is found that the solution consists of a superposition of electrostatic modes whose frequencies are zeros of the dispersion function $D_N(\omega,k)$ of the N-stream model. In addition, there are contributions from degenerate electrostatic-ballistic modes determined by the zeros of $P_s(\omega,k)$ for a pair of streams coupled by the wave momentum. It is instructive to note that the same modes are obtained by both methods. The difference between collective modes of the Landau Laplace-transform method and van Kampen modes of the stationary-wave method is not present at the level of individual streams.

We then performed the continuum limit and thereby obtained analytic continuations $D^{\pm}(\omega,k)$ of the dielectric function in the upper and lower complex planes, respectively. The contribution from the continuum of modes on the real axis can be expressed in terms of the Landau-Lindhard dielectric function $D^{+}(\omega,k)$, and the spectral weights of these modes are given by the imaginary part of a response function. Near complex zeros of $D^{+}(\omega,k)$, the spectral weights can be approximated by a Lorentzian profile, and thereby exponential Landau damping is recovered.

Finally, numerical evaluations have been performed for Fermi and Maxwell distributions to demonstrate dispersion and relaxation of plasma waves. While Maxwellian distributions lead to collective mode frequencies and exponential Landau damping in accordance with the Landau-Lindhard dispersion function, important differences arise for Fermi distributions. Because of the separation of the collective frequency from the continuum of single-particle excitations, the Lorentzian profile of the spectrum becomes a poor approximation, and as a consequence the relaxation behavior is in general nonexponential. In particular, at large wave numbers, when relaxation is determined by single-particle excitations, subexponential damping and beat-wave oscillations have been shown to occur in Fermi-degenerate plasmas. Under the same approximations, Maxwellian statistics leads to superexponential damping, thus demonstrating a basically different relaxation behavior in degenerate and nondegenerate plasmas. This discrepancy arises from the different statistical weights of the single-electron excitations. The normal distribution over an infinite range of the weights in nondegenerate plasmas gets replaced by a uniform distribution over a finite range in the degenerate case. The corresponding change of temporal relaxation is quite similar to the more familiar change of spatial screening. In degenerate plasmas, exponential Debye screening gets modified by long-range oscillations [67].

Observation of subexponential temporal relaxation of beat-wave oscillations might be an alternative interesting means to diagnose the degeneracy of the plasma state.

APPENDIX

In this Appendix, we discuss the basis of generalized right-hand eigenvectors and the dual basis of generalized left-hand eigenvectors of a non-Hermitian operator $\boldsymbol{L}$ with degenerate eigenvalues if there exists only one ordinary eigenvector for each eigenvalue. In this case, the maximum rank of the generalized eigenvectors to the eigenvalue ω corresponds to its degree of degeneracy g. For the present multistream problem, we will only need the special case $g = 2$. However, it is instructive to show that the method remains valid for arbitrary degeneracy g.

Starting from the ordinary eigenvector $\boldsymbol{v}_1$ of rank 1, the g generalized eigenvectors can be obtained by defining $\boldsymbol{M} = \boldsymbol{L} - \omega \boldsymbol{I}$ and setting recursively

$$\boldsymbol{M} \cdot \boldsymbol{v}_1 = 0, \boldsymbol{M} \cdot \boldsymbol{v}_2 = \boldsymbol{v}_1, \ldots, \boldsymbol{M} \cdot \boldsymbol{v}_g = \boldsymbol{v}_{g-1}. \tag{A1}$$

The generalized eigenvectors are not uniquely determined. If $\boldsymbol{v}_l$ is a generalized eigenvector of rank $l > 1$, one can always add an arbitrary linear combination of generalized eigenvectors of a lower rank to obtain another generalized eigenvector of rank l,

$$\boldsymbol{v}_l' = \boldsymbol{v}_l + \sum_{k=1}^{l-1} c_k \boldsymbol{v}_k. \tag{A2}$$

This follows from the fact that the defining conditions for a generalized eigenvector of rank l remain invariant under this transformation,

$$\begin{aligned} \boldsymbol{M}^l \cdot \boldsymbol{v}_l' &= \boldsymbol{M}^l \cdot \boldsymbol{v}_l = 0, \\ \boldsymbol{M}^{l-1} \cdot \boldsymbol{v}_l' &= \boldsymbol{M}^{l-1} \cdot \boldsymbol{v}_l \neq 0. \end{aligned} \tag{A3}$$

It is noted that for each rank l, the new vector (A2) satisfies the recurrence relation of (A1) with the chain of generalized eigenvectors of lower rank $m < l$ given by

$$\boldsymbol{v}_{lm}' = \boldsymbol{v}_m + \sum_{k=1}^{m-1} c_{l-m+k} \boldsymbol{v}_k. \tag{A4}$$

The ambiguity in the definition of the generalized eigenvectors can be used to obtain a set of mutually orthogonal generalized eigenvectors $\boldsymbol{v}_k' \cdot \boldsymbol{v}_l' = \delta_{kl}$. The corresponding coefficients in (A2) are simply given by $c_k = -\boldsymbol{v}_l \cdot \boldsymbol{v}_k$. These coefficients also occur in (A4), but these nonorthogonal vectors are discarded from the basis. In the following, we always use orthogonal basis vectors and omit the prime for simplicity of notation.

The left-hand eigenvalue problem, being defined by the transposed matrix of $\boldsymbol{L}$, has the same eigenvalues with the same algebraic and geometric degeneracy. For a given eigenvalue ω, the generalized eigenvectors can be obtained in the same recursive manner as

$$\boldsymbol{u}_g \cdot \boldsymbol{M} = 0, \boldsymbol{u}_{g-1} \cdot \boldsymbol{M} = \boldsymbol{u}_g, \ldots, \boldsymbol{u}_1 \cdot \boldsymbol{M} = \boldsymbol{u}_2. \tag{A5}$$

For later convenience, the left-hand eigenvectors have been labeled in reverse order. The rank of $\boldsymbol{u}_l$ is therefore $1 + g - l$. The left-hand eigenvectors will also be chosen mutually orthogonal, $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = \delta_{ij}$.

Next, we consider the transformation matrix $R_{ij} = \boldsymbol{u}_i \cdot \boldsymbol{v}_j$ between the original and the dual basis. For the first column and the last row of this matrix, one obtains the constraints

$$\boldsymbol{u}_{i-1} \cdot \boldsymbol{M} \cdot \boldsymbol{v}_1 = \boldsymbol{u}_i \cdot \boldsymbol{v}_1 = 0, \quad 1 \leqslant i < g, \tag{A6a}$$

$$\boldsymbol{u}_g \cdot \boldsymbol{M} \cdot \boldsymbol{v}_{j+1} = \boldsymbol{u}_g \cdot \boldsymbol{v}_j = 0, \quad 1 < j \leqslant g. \tag{A6b}$$

In addition, it follows that all elements along diagonals (from upper left to lower right corners) are equal, since

$$\boldsymbol{u}_i \cdot \boldsymbol{M} \cdot \boldsymbol{v}_j = \boldsymbol{u}_i \cdot \boldsymbol{v}_{j-1} = \boldsymbol{u}_{i+1} \cdot \boldsymbol{v}_j. \tag{A7}$$

According to these constraints, all matrix elements below the main diagonal are zero. The elements on the main diagonal can be chosen for convenience as $R_{ii} = \boldsymbol{u}_i \cdot \boldsymbol{v}_i = 1$. Furthermore, since both basis systems are orthogonal, the matrix has to satisfy the orthogonality conditions

$$\sum_k R_{ik} R_{jk} = \delta_{ij}. \tag{A8}$$

Applying this condition to neighboring rows, starting from the rows g and $g - 1$, it then follows that all elements above the main diagonal are also zero. In summary, the dual basis can and will be chosen to satisfy the canonical orthonormality conditions

$$\boldsymbol{u}_i \cdot \boldsymbol{v}_j = \delta_{ij}. \tag{A9}$$

Finally, it remains to be shown that the generalized dual eigenvectors $\boldsymbol{u}_{\alpha i}$ for the eigenvalue ω_α are always orthogonal to the generalized eigenvectors $\boldsymbol{v}_{\beta j}$ for a different eigenvalue ω_β. Setting $\boldsymbol{M}_{\alpha,\beta} = \boldsymbol{L}_{\alpha,\beta} - \omega_{\alpha,\beta} \boldsymbol{I}$, one has

$$\boldsymbol{M}_\alpha = \boldsymbol{M}_\beta + (\omega_\beta - \omega_\alpha) \boldsymbol{I}. \tag{A10}$$

Consider the elements of the matrix $R_{ij} = \boldsymbol{u}_{\alpha i} \cdot \boldsymbol{v}_{\beta j}$ for $\omega_\alpha \neq \omega_\beta$. From

$$\begin{aligned} 0 &= \boldsymbol{u}_{\alpha g} \cdot \boldsymbol{M}_{\boldsymbol{\beta}} \cdot \boldsymbol{v}_{\beta 1} = \boldsymbol{u}_{\alpha g} \cdot \boldsymbol{M}_{\boldsymbol{\alpha}} \cdot \boldsymbol{v}_{\beta 1} + (\omega_\alpha - \omega_\beta) R_{g1} \\ &= (\omega_\alpha - \omega_\beta) R_{g1} \end{aligned} \tag{A11}$$

one obtains $R_{g1} = 0$. Next, one can calculate the elements R_{i1} of the first column with $i < g$ from the identity

$$\begin{aligned} 0 &= \boldsymbol{u}_{\alpha i} \cdot \boldsymbol{M}_{\boldsymbol{\beta}} \cdot \boldsymbol{v}_{\beta 1} = \boldsymbol{u}_{\alpha i} \cdot \boldsymbol{M}_{\boldsymbol{\alpha}} \cdot \boldsymbol{v}_{\beta 1} + (\omega_\alpha - \omega_\beta) R_{i1} \\ &= \boldsymbol{u}_{\alpha i+1} \cdot \boldsymbol{v}_{\beta 1} + (\omega_\alpha - \omega_\beta) R_{i1} \\ &= R_{i+11} + (\omega_\alpha - \omega_\beta) R_{i1}. \end{aligned} \tag{A12}$$

Starting with $i = g - 1$ one subsequently finds $R_{i1} = 0$ for all i. The elements of the second column now follow from the substitution

$$\begin{aligned} 0 &= \boldsymbol{u}_{\alpha i} \cdot \boldsymbol{v}_{\beta 1} = \boldsymbol{u}_{\alpha i} \cdot \boldsymbol{M}_\beta \cdot \boldsymbol{v}_{\beta 2} \\ &= \boldsymbol{u}_{\alpha i} \cdot [\boldsymbol{M}_\alpha + (\omega_\alpha - \omega_\beta) \boldsymbol{I}] \cdot \boldsymbol{v}_{\beta 2} \\ &= R_{i+12} + (\omega_\alpha - \omega_\beta) R_{i2}. \end{aligned} \tag{A13}$$

Starting with $i = g$ and noting that $R_{g+12} = 0$ according to the eigenvector equation in (A5), one finds $R_{g2} = 0$. Subsequently,

there follows $R_{i2} = 0$ for $i < g$. This procedure can be continued for all columns leading to the result $R_{ij} = 0$ for all i and j. In summary, a dual basis can be chosen that is orthonormal to the original basis for all eigenvectors and all generalized eigenvectors,

$$\boldsymbol{u}_{\alpha i} \cdot \boldsymbol{v}_{\beta j} = \delta_{\alpha\beta}\delta_{ij}. \tag{A14}$$

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