Turbulent Prandtl number in the A model of passive vector admixture

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Using the field theoretic renormalization group technique in the second-order (two-loop) approximation the explicit expression for the turbulent vector Prandtl number in the framework of the general A model of passively advected vector field by the turbulent velocity field driven by the stochastic Navier-Stokes equation is found as the function of the spatial dimension $d > 2$. The behavior of the turbulent vector Prandtl number as the function of the spatial dimension d is investigated in detail especially for three physically important special cases, namely, for the passive advection of the magnetic field in a conductive turbulent environment in the framework of the kinematic MHD turbulence ($A = 1$), for the passive admixture of a vector impurity by the Navier-Stokes turbulent flow ($A = 0$), and for the model of linearized Navier-Stokes equation ($A = -1$). It is shown that the turbulent vector Prandtl number in the framework of the $A = -1$ model is exactly determined already in the one-loop approximation, i.e., that all higher-loop corrections vanish. At the same time, it is shown that it does not depend on spatial dimension d and is equal to 1. On the other hand, it is shown that the turbulent magnetic Prandtl number ($A = 1$) and the turbulent vector Prandtl number in the model of a vector impurity ($A = 0$), which are essentially different at the one-loop level of approximation, become very close to each other when the two-loop corrections are taken into account. It is shown that their relative difference is less than 5% for all integer values of the spatial dimension $d \geqslant 3$. Obtained results demonstrate strong universality of diffusion processes of passively advected scalar and vector quantities in fully symmetric incompressible turbulent environments.

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I. INTRODUCTION

The problems related to the behavior of various admixtures in turbulent environments are some of the most studied in the framework of the turbulent fluid dynamics. Typical examples are scalar and vector impurities in given turbulent systems, e.g., the temperature in the problem of thermal diffusion or the problem of an impurity with internal structure in turbulent environment [1,2], as well as the weak magnetic field in a turbulent plasma (electrically conductive turbulent fluid) described by the turbulent magnetohydrodynamics (MHD) [3,4].

It is well known that one of the most important characteristics of diffusion processes in fluids is the corresponding Prandtl number, i.e., the ratio of the coefficient of kinematic viscosity to the coefficient of diffusivity of a given admixture. The numerical values of the Prandtl numbers usually depend strongly on the microscopic structure of the fluids when the motion of the fluids is characterized by low values of the Reynolds number (Re). However, when fluids are in the regime of fully developed turbulence, i.e., in the regime with very high values of the Reynolds number (in the ideal case, $\mathrm{Re} \to \infty$), one obtains universal values of the Prandtl numbers which are known as the effective or turbulent Prandtl numbers [1,3,4].

An effective approach for theoretical analysis of universal properties of various models of fully developed turbulence is the renormalization group (RG) method [2,5–7]. This technique can be also used for the analysis of diffusion processes in turbulent environments. In this respect, in Refs. [8,9] the calculation of the turbulent Prandtl number in the model of a passive scalar field advected by the Navier-Stokes turbulence was performed by using the field theoretic RG technique in the second-order approximation (the two-loop approximation in the quantum field theory terminology). It was shown that, although the corresponding field theoretical model represents a model with a strong coupling constant, the two-loop corrections to the one-loop value of the turbulent Prandtl number are extremely small and are less than 2% of its leading one-loop value. It means that the turbulent Prandtl number related to the passive scalar admixture is surprisingly very stable under the perturbation theory, at least up to the second-order approximation in the corresponding perturbation expansion. It is also important to stress that the obtained two-loop theoretical value of the turbulent Prandtl number $\mathrm{Pr_t} = 0.7051$[1] is in good agreement with its experimental estimations (see, e.g., Refs. [1,10,11]).

Among the models of passively advected vector fields maybe the most interesting from a phenomenological point of view are the models which describe diffusion of a weak magnetic field in the turbulent conductive environment with prescribed statistics of the velocity field. Here, the importance of the theoretical investigation of possible values of the turbulent magnetic Prandtl number is related to the fact that it plays a significant role in many physical processes and their simulations, such as the turbulent dynamo problem (see, e.g., Refs. [12–17] and references cited therein), astrophysical MHD turbulence phenomena (see, e.g., Refs. [18–21] and references cited therein), and MHD simulations and calculations (see, e.g., Refs. [22,23]). Therefore, without doubt, to have fundamentally determined the turbulent magnetic Prandtl number on the level of the well-defined microscopic model of fully developed MHD turbulence is very important.

[1]Note that, in fact, the real two-loop turbulent Prandtl number in the problem of scalar admixture is a little bit different and has a value $\mathrm{Pr_t} = 0.7040$. The difference is related to the precision of the numerical calculations of some integrals as it is discussed, e.g., in Ref. [39].

In this respect, the RG technique was used for the estimation of the value of the turbulent magnetic Prandtl number in Ref. [24] by using the Wilson's recursion relations technique. Later, in Ref. [25], the turbulent magnetic Prandtl number was also calculated in a more general stochastic MHD model by using the field theoretic RG technique. The results obtained in Refs. [24,25], as well as the later result obtained in Ref. [26], were calculated in the leading first-order (one-loop) approximation in the framework of the corresponding perturbation expansions and only quite recently the two-loop calculations were performed in Ref. [27], where the two-loop RG value of the turbulent magnetic Prandtl number was found in the framework of the kinematic MHD turbulence. It was shown in Ref. [27] that there is no difference between the turbulent Prandtl number of a passive scalar quantity advected by the Navier-Stokes turbulence and the turbulent magnetic Prandtl number of a passive weak magnetic field in the framework of the kinematic MHD turbulence. This intriguing result means that the internal (tensor) structure of a passively advected quantity has no impact on the turbulent diffusion coefficient. Note, however, that the exact universality of these two diffusion processes is valid only when fully symmetric incompressible turbulent environments are considered because it is destroyed, e.g., when the spatial parity violation is present in the corresponding turbulent systems as was discussed in detail in Ref. [28].

Another important model of a passive vector field advected by a given turbulent environment is the so-called $A = 0$ model in which the so-called "stretching term," which is present in the kinematic MHD, is omitted (see, e.g., Refs. [29,30] for details). This model resembles the model of passively advected scalar quantity and the main reason for its theoretical investigation is related to the fact that some features of the anomalous scaling of the correlation functions of the vector field in the framework of various $A = 0$ models are similar to the properties of the anomalous scaling of the correlation functions of the velocity field in genuine Navier-Stokes turbulence. In this respect, the $A = 0$ model of the passive vector advection was investigated in various Gaussian turbulent velocity fields [29–36] and quite recently also in the non-Gaussian turbulent velocity field governed by the Navier-Stokes equation [37,38].

The turbulent vector Prandtl number in the framework of the $A = 0$ model of a passively advected vector quantity by the Navier-Stokes turbulent velocity field was recently calculated in the two-loop RG approximation in Ref. [39] where it was shown that the relative difference between the turbulent vector Prandtl number in the $A = 0$ model and the turbulent magnetic Prandtl number in the framework of the kinematic MHD turbulence is less than 4% for the spatial dimension $d = 3$, i.e., that, at least for the spatial dimension $d = 3$, the diffusion processes in the framework of the $A = 0$ model of a passive vector quantity advected by the Navier-Stokes turbulence and in the framework of the kinematic MHD turbulence are also very similar. However, having no three-loop results for the turbulent Prandtl numbers yet, to confirm this conclusion more strongly it is necessary to show that the same is also true for other, at least, integer values of the spatial dimension $d \geqslant 3$, i.e., that this result is independent of the value of the spatial dimension, because otherwise the above discussed similarity of the diffusion processes of various passive admixtures can be considered with high probability only as an incidental fact related to the two-loop calculations. Let us remind once more that we consider here only fully symmetric turbulent systems because, as was shown in Ref. [40], the presence of a symmetry breaking, e.g., the presence of the spatial parity violation, can again change the situation radically.

In addition, there exists one more physically interesting model of a passively advected vector quantity, namely, the model of linearized Navier-Stokes equation, which, e.g., describes weak perturbations of prescribed background field [30]. Here, the question of the value of the corresponding turbulent vector Prandtl number also immediately arises. Another intriguing question is whether the corresponding two-loop turbulent vector Prandtl number is also close to the other turbulent Prandtl numbers discussed above, i.e., whether the diffusion processes of passively advected scalar and all discussed vector quantities are similar in fully symmetric incompressible turbulent environments described by the stochastic Navier-Stokes equation.

To answer all these questions, in the present paper we consider the general so-called A model of a passive vector quantity advected by the turbulent velocity field driven by the stochastic Navier-Stokes equation which describes simultaneously all above mentioned physically important models of passive vector admixtures, namely, the passive advection of the magnetic field in a conductive turbulent environment in the framework of the kinematic MHD turbulence ($A = 1$), the passive admixture of a vector impurity by the Navier-Stokes turbulent flow ($A = 0$), and the model of linearized Navier-Stokes equation ($A = -1$). Using the field theoretic RG technique we shall find the general one- and two-loop expressions for the turbulent vector Prandtl numbers as the functions of the spatial dimension $d > 2$ and the continuous parameter A, i.e., which are valid simultaneously for all above discussed models, and the complete analysis of their properties will be performed.

We shall show that the turbulent magnetic Prandtl number in the framework of the kinematic MHD turbulence ($A = 1$ model) and the turbulent vector Prandtl number in the model of vector impurity ($A = 0$ model) behaves very similarly as the functions of d at the two-loop level of approximation although their simultaneous behavior at the one-loop level of approximation is radically different. At the same time, the relative difference between their two-loop values are less than 5% for all integer values $d \geqslant 3$ of the spatial dimension. This result could signalize that in fact there is no significant difference between the diffusion processes in the model of passively advected scalar quantity by the Navier-Stokes turbulence, in the model of passively advected magnetic field in the framework of kinematic MHD turbulence (let us remind once more that, as was discussed above, the corresponding turbulent Prandtl numbers are equal to each other [27]), and in the model of passively advected vector quantity by the Navier-Stokes turbulence. On the other hand, it is shown that the turbulent vector Prandtl number in the framework of the model of linearized Navier-Stokes equation ($A = -1$) is exactly given already at the one-loop level of approximation, i.e., that all two- and higher-loop corrections to this turbulent Prandtl number vanish. At the same time, it does not depend on the spatial dimension and is exactly equal to 1. It means that from the point of view of diffusion processes the $A = -1$ model is rather specific in comparison with diffusion processes in all other physically interesting models.

In the end, let us also note that the general A model of passive vector admixtures stirred by various turbulent environments has become quite interesting at present from the point of view of their scaling properties (see, e.g., Refs. [41,42]). However, these questions are out of scope of the present paper.

The paper is organized as follows. In Sec. II, the general A model of a passively advected vector field is discussed in detail. In Sec. III, the field theoretic formulation of the model is given. In Sec. IV, the ultraviolet (UV) renormalization and the two-loop RG analysis of the model is performed. In Sec. V, the two-loop expression for the turbulent vector Prandtl number is obtained and its dependence on the parameter A and on the spatial dimension is discussed. Obtained results are briefly reviewed and discussed in Sec. VI.

II. GENERAL A MODEL OF PASSIVE VECTOR ADMIXTURE

Let us consider a general passive solenoidal vector field $\mathbf{b} \equiv \mathbf{b}(x)$ $[x \equiv (t,\mathbf{x})]$, described by the following stochastic advection-diffusion equation:

$$\partial_t \mathbf{b} = \nu_0 u_0 \triangle \mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + A(\mathbf{b}\cdot\partial)\mathbf{v} - \partial Q + \mathbf{f}^{\mathbf{b}}, \qquad (1)$$

advected by the solenoidal turbulent velocity field $\mathbf{v} \equiv \mathbf{v}(x)$ driven by the stochastic Navier-Stokes equation

$$\partial_t \mathbf{v} = \nu_0 \triangle \mathbf{v} - (\mathbf{v}\cdot\partial)\mathbf{v} - \partial P + \mathbf{f}^{\mathbf{v}}. \qquad (2)$$

In Eqs. (1) and (2) we have used the following notation: $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\triangle \equiv \partial^2$ is the Laplace operator, ν_0 is the viscosity coefficient (in what follows, a subscript 0 will denote bare parameters of the unrenormalized theory), $\nu_0 u_0$ is the bare coefficient of diffusivity (where we have already extracted dimensionless reciprocal general vector Prandtl number u_0 for convenience), $P \equiv P(x)$ and $Q \equiv Q(x)$ represent corresponding pressures, A is an arbitrary parameter which will be discussed below, and $\mathbf{f}^{\mathbf{b}} = \mathbf{f}^{\mathbf{b}}(x)$ and $\mathbf{f}^{\mathbf{v}} = \mathbf{f}^{\mathbf{v}}(x)$ represent random noises. Their statistical properties will be specified a little bit later. As was already mentioned, both fields $\mathbf{v}$ and $\mathbf{b}$ are solenoidal (divergence-free) vector fields (owing to the assumption of incompressibility), i.e., in what follows, we always suppose that $\partial\cdot\mathbf{v} = \partial\cdot\mathbf{b} = 0$.

Parameter A in front of the so-called "stretching term" in Eq. (1), which is not fixed by the Galilean symmetry, allows us to describe and study simultaneously three different physically interesting situations [30], namely (a) the passive advection of the magnetic field in a conductive turbulent environment in the framework of the kinematic MHD model ($A = 1$); (b) the passive admixture of a vector impurity (an impurity with internal structure) by the Navier-Stokes turbulent flow ($A = 0$), which represents direct analogy of the corresponding model of the passively advected scalar impurity in the given turbulent flow; and (c) the model of the linearized Navier-Stokes equation, which describes weak perturbations of a prescribed background field ($A = -1$).

The energy pumping given by a transverse Gaussian random noise $\mathbf{f}^{\mathbf{b}} = \mathbf{f}^{\mathbf{b}}(x)$ in Eq. (1) with zero mean and the correlation function

$$D^b_{ij}(x;x') \equiv \langle f^b_i(x) f^b_j(x')\rangle = \delta(t-t')C_{ij}(|\mathbf{x}-\mathbf{x}'|/L) \qquad (3)$$

represents the source of the fluctuations of the vector field $\mathbf{b}$ and, in what follows, its main role is to maintain the steady state of the system. Here, L is an integral scale related to the corresponding stirring, and C_{ij} is a function finite in the limit $L\to\infty$. In what follows, the detailed form of the function C_{ij} is unimportant, the only condition which must be satisfied is that C_{ij} decreases rapidly for $|\mathbf{x}-\mathbf{x}'| \gg L$. If C_{ij} depends on the direction of the vector $\mathbf{r} \equiv \mathbf{x}-\mathbf{x}'$ and not only on its modulus $r = |\mathbf{r}|$ then it can be considered as a source of the large-scale anisotropy [43].

On the other hand, the explicit form of the transverse random force per unit mass $\mathbf{f}^{\mathbf{v}} = \mathbf{f}^{\mathbf{v}}(x)$ in Eq. (2) is important in what follows. It must be taken in a form which simulates the energy pumping into the system on large scales. We assume that its statistics is also Gaussian with zero mean and the following explicit form of the pair correlation function:

$$D^v_{ij}(x;x') \equiv \langle f^v_i(x) f^v_j(x')\rangle = \delta(t-t')\int \frac{d^d\mathbf{k}}{(2\pi)^d} D_0 k^{4-d-2\varepsilon} P_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \qquad (4)$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ is the ordinary transverse projector, d denotes the spatial dimension of the system, $D_0 \equiv g_0 \nu_0^3 > 0$ is the positive amplitude, and the physical value of formally small parameter $0 < \varepsilon \leqslant 2$ is $\varepsilon = 2$. It plays an analogous role as the parameter $\epsilon = 4-d$ in the theory of critical behavior and the introduced parameter g_0 plays the role of the coupling constant of the model. It is a formal small parameter of the ordinary perturbation theory and is related to the characteristic ultraviolet (UV) momentum scale Λ (or inner length $l \sim \Lambda^{-1}$) by the relation $g_0 \simeq \Lambda^{2\varepsilon}$. In Eq. (4) the needed infrared regularization is reached by a restriction of the integration from below, namely, it is supposed that $k \geqslant m$, where m corresponds to another integral scale. In addition, in what follows, we always suppose that $L \gg 1/m$.

Note that the correlation function (4) is chosen in the form which is suitable for description of the real infrared energy pumping to the system because for $\varepsilon \to 2$ the function $D_0 k^{4-d-2\varepsilon}$ is proportional to $\delta(\mathbf{k})$ for the appropriate choice of the amplitude factor D_0, which corresponds to the injection of energy to the system through interaction with the largest turbulent eddies. Besides, the powerlike form of the correlation function (4) allows us to apply the standard RG technique in the analysis of the problem (see, e.g., Refs. [5–7] for details).

III. FIELD THEORETIC FORMULATION OF THE MODEL

Using the well-known theorem [44] the stochastic problem (1)–(4) can be reformulated into the corresponding field theoretic model of the double set of fields $\Phi = \{\mathbf{v},\mathbf{b},\mathbf{v}',\mathbf{b}'\}$ with the following action functional:

$$\begin{aligned} S(\Phi) = {}& \frac{1}{2}\int dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2} \big[v'_i(x_1) D^v_{ij}(x_1;x_2) v'_j(x_2) \\ & + b'_i(x_1) D^b_{ij}(x_1;x_2) b'_j(x_2)\big] \\ & + \int dt\, d^d\mathbf{x} \{\mathbf{v}'[-\partial_t + \nu_0\triangle - (\mathbf{v}\cdot\partial)]\mathbf{v} \\ & + \mathbf{b}'[-\partial_t \mathbf{b} + \nu_0 u_0 \triangle\mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + A(\mathbf{b}\cdot\partial)\mathbf{v}]\}, \end{aligned} \qquad (5)$$

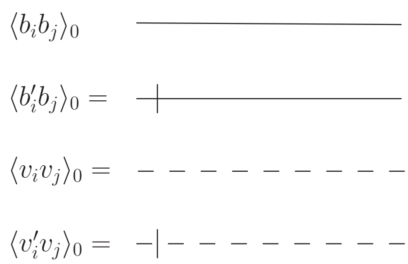

FIG. 1. Graphical representation of the propagators of the model. The end with a slash in the propagators $\langle b'_i b_j\rangle_0$ and $\langle v'_i v_j\rangle_0$ corresponds to the fields $\mathbf{b}'$ and $\mathbf{v}'$, respectively, and the end without a slash corresponds to the fields $\mathbf{b}$ and $\mathbf{v}$, respectively.

where $x_i = (t_i,\mathbf{x_i})$, $i = 1,2$, $\mathbf{v}'$ and $\mathbf{b}'$ are added auxiliary transverse fields which have the same tensor properties as fields $\mathbf{v}(x)$ and $\mathbf{b}(x)$, D^b_{ij}, D^v_{ij} are given in Eqs. (3) and (4), respectively, and required summations over dummy indices are assumed.

Note that the pressure terms ∂Q and ∂P in Eqs. (1) and (2) are omitted in action (5) owing to the fact that the auxiliary vector fields $\mathbf{b}'(x)$ and $\mathbf{v}'(x)$ are also transverse. Therefore, using the integration by parts they vanish.

The standard Feynman diagrammatic perturbation theory of the field theoretic model given by action functional (5) is based on the following set of bare propagators written in the frequency-momentum representation:

$$\langle b'_i b_j\rangle_0 = \langle b_i b'_j\rangle_0^* = \frac{P_{ij}(\mathbf{k})}{i\omega + \nu_0 u_0 k^2}, \tag{6}$$

$$\langle v'_i v_j\rangle_0 = \langle v_i v'_j\rangle_0^* = \frac{P_{ij}(\mathbf{k})}{i\omega + \nu_0 k^2}, \tag{7}$$

$$\langle b_i b_j\rangle_0 = \frac{C_{ij}(\mathbf{k})}{|-i\omega + \nu_0 u_0 k^2|^2}, \tag{8}$$

$$\langle v_i v_j\rangle_0 = \frac{g_0 \nu_0^3 k^{4-d-2\varepsilon} P_{ij}(\mathbf{k})}{|-i\omega + \nu_0 k^2|^2}, \tag{9}$$

where $C_{ij}(\mathbf{k})$ is the Fourier transform of function $C_{ij}(|\mathbf{x}-\mathbf{x}'|/L)$ in Eq. (3). The graphical representation of the set of bare propagators (6)–(9) is shown in Fig. 1. On the other hand, the triple interaction vertices $b'_i(-v_j\partial_j b_i + A\, b_j\partial_j v_i) = b'_i v_j V_{ijl} b_l$ and $-v'_i v_j \partial_j v_i = v'_i v_j W_{ijl} v_l/2$, where $V_{ijl} = i(k_j\delta_{il} - A k_l \delta_{ij})$ and $W_{ijl} = i(k_l\delta_{ij} + k_j\delta_{il})$ (in the momentum-frequency representation), are present in Fig. 2, where momentum $\mathbf{k}$ is flowing into the vertices via the auxiliary fields $\mathbf{b}'$ and $\mathbf{v}'$, respectively.

Note that the main advantage of the formulation of the stochastic problem given in Eqs. (1)–(4) through action functional (5) is given by the fact that it allows one to use the well-defined field theoretic means, e.g., the RG technique, to analyze the problem. In the framework of the field theoretic approach the statistical averages of random quantities in the stochastic problem are replaced with the corresponding functional averages with weight $\exp S(\Phi)$ (see, e.g., Ref. [7] for details).

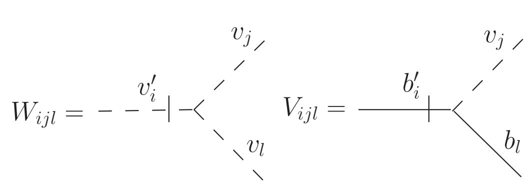

FIG. 2. The graphical representation of the interaction vertices of the model.

IV. RENORMALIZATION GROUP ANALYSIS

Let us discuss briefly the RG analysis of the field theoretic model described by action functional (5). All details can be found, e.g., in Refs. [5–7].

Standardly, the RG analysis of a field theoretic model is based on the analysis of UV divergences of the model. On the other hand, the analysis of UV divergences is based on the canonical dimensional analysis. In addition, it is also necessary to bear in mind that our dynamical model described by action functional (5) belongs to the class of the so-called two-scale models [5–7]. For a model from this class the canonical dimension of a quantity Q is given by two numbers: the momentum dimension d_Q^k and the frequency dimension d_Q^ω which are obtained by the requirement that each term of the corresponding action functional must be dimensionless separately with respect to the momentum and frequency together with the standard definitions (normalization conditions) $d_k^k = -d_x^k = d_\omega^\omega = -d_t^\omega = 1$. In our case the total canonical dimension d_Q is then defined as $d_Q = d_Q^k + 2d_Q^\omega$ [it is related to the fact that $\partial_t \propto \partial^2$ in the free action (5) with the choice of zero canonical dimensions for ν_0 and u_0]. The total canonical dimensions then play the same role in the renormalization theory of our dynamical model as the simple momentum dimensions do in static models.

The canonical dimensions of the model under consideration are present in Table I, where also the canonical dimensions of the renormalized parameters are already shown (see below). It follows from Table I that the coupling constant of the model g_0 is dimensionless at $\varepsilon = 0$. It means that the model is the so-called logarithmic for $\varepsilon = 0$ and, in the framework of the minimal subtraction (MS) scheme [45] which is always used in what follows, all possible UV divergences in the correlation functions of the model have the form of poles in ε. Finally, using the general expression for the total canonical dimension of an arbitrary one-irreducible Green's function $\langle\Phi\cdots\Phi\rangle_{1-ir}$ (it plays the role of the formal index of the UV divergence) together with the symmetry properties of the model, one finds that, for spatial dimensions $d > 2$, the superficial UV divergences are present only in the one-irreducible Green's functions $\langle v'_i v_j\rangle_{1-ir}$ and $\langle b'_i b_j\rangle_{1-ir}$. Besides, the action functional (5) has all necessary tensor structures to remove divergences multiplicatively (see, e.g., Refs. [7,45]) and all

TABLE I. Canonical dimensions of the fields and parameters of the model under consideration.

Q	$\mathbf{v}$	$\mathbf{v}'$	$\mathbf{b}$	$\mathbf{b}'$	m,Λ,μ	ν_0,ν	g_0	g,u_0,u
d_Q^k	-1	$d+1$	0	d	1	-2	2ε	0
d_Q^ω	1	-1	$-1/2$	$1/2$	0	1	0	0
d_Q	1	$d-1$	-1	$d+1$	1	0	2ε	0

of them can be removed by the counterterms of the forms $\mathbf{v}'\Delta\mathbf{v}$ and $\mathbf{b}'\Delta\mathbf{b}$. It is explicitly expressed in the multiplicative renormalization of the bare parameters g_0, u_0, and ν_0 in the following form:

$$\nu_0 = \nu Z_\nu, \quad g_0 = g\mu^{2\varepsilon} Z_g, \quad u_0 = u Z_u, \tag{10}$$

where the dimensionless parameters g, u, and ν are the renormalized counterparts of the corresponding bare ones and μ is the so-called renormalization mass (a scale setting parameter), an artefact of the dimensional regularization. Quantities $Z_i = Z_i(g,d;\varepsilon)$ for $i = \nu, g$ and $Z_u = Z_u(g,u,A,d;\varepsilon)$ are the corresponding renormalization constants which contain poles in ε.

Using all these facts the renormalized action functional can be written in the following form:

$$\begin{aligned} S_R(\Phi) = \frac{1}{2}\int & dt_1\, d^d\mathbf{x_1}\, dt_2\, d^d\mathbf{x_2}\big[v_i'(x_1)D^v_{ij}(x_1;x_2)v_j'(x_2) \\ & + b_i'(x_1)D^b_{ij}(x_1;x_2)b_j'(x_2)\big] \\ & + \int dt\, d^d\mathbf{x}\{\mathbf{v}'[-\partial_t + \nu Z_1\triangle - (\mathbf{v}\cdot\partial)]\mathbf{v} + \mathbf{b}'[-\partial_t\mathbf{b} \\ & + \nu u Z_2\triangle\mathbf{b} - (\mathbf{v}\cdot\partial)\mathbf{b} + A(\mathbf{b}\cdot\partial)\mathbf{v}]\}, \end{aligned} \tag{11}$$

where Z_1 and Z_2 are new renormalization constants which are defined through the renormalization constants introduced in Eq. (10) as follows:

$$Z_\nu = Z_1, \quad Z_g = Z_1^{-3}, \quad Z_u = Z_2 Z_1^{-1}. \tag{12}$$

It means that the present model is renormalized through two independent renormalization constants Z_1 and Z_2 which have the following explicit forms in the framework of the MS scheme:

$$Z_1(g,d;\varepsilon) = 1 + \sum_{n=1}^{\infty} g^n \sum_{j=1}^{n} \frac{z_{nj}^{(1)}(d)}{\varepsilon^j}, \tag{13}$$

$$Z_2(g,u,A,d;\varepsilon) = 1 + \sum_{n=1}^{\infty} g^n \sum_{j=1}^{n} \frac{z_{nj}^{(2)}(u,A,d)}{\varepsilon^j}. \tag{14}$$

In Eqs. (13) and (14), the coefficients $z_{nj}^{(1)}$ and $z_{nj}^{(2)}$, which are independent of ε, are directly determined by the requirement that the one-irreducible Green's functions $\langle v_i' v_j\rangle_{1-ir}$ and $\langle b_i' b_j\rangle_{1-ir}$ must be UV finite when written in the renormalized variables (they have no singularities in the limit $\varepsilon \to 0$).

Technically, the coefficients $z_{nj}^{(1)}$ and $z_{nj}^{(2)}$ in Eqs. (13) and (14) are determined through the Dyson equations which relate one-irreducible Green's functions $\langle v_i' v_j\rangle_{1-ir}$ and $\langle b_i' b_j\rangle_{1-ir}$ to the self-energy operators $\Sigma^{v'v}$ and $\Sigma^{b'b}$, which are calculated perturbatively by analyzing the corresponding Feynman diagrams. In the frequency-momentum representation the Dyson equations of the present model can be written in the following form:

$$\langle v_i' v_j\rangle_{1-ir} = [i\omega - \nu_0 p^2 + \Sigma^{v'v}(\omega,\mathbf{p})]P_{ij}(\mathbf{p}), \tag{15}$$

$$\langle b_i' b_j\rangle_{1-ir} = [i\omega - \nu_0 u_0 p^2 + \Sigma^{b'b}(\omega,\mathbf{p})]P_{ij}(\mathbf{p}). \tag{16}$$

The requirement that the UV divergences are canceled in Eqs. (15) and (16) after the substitution $e_0 = e\mu^{d_e} Z_e$ for $e = \{g,u,\nu\}$ determines Z_1 and Z_2 up to a UV finite contribution. This freedom is subsequently fixed by the choice of the scheme of the renormalization. As was already mentioned, we use the MS scheme in our calculations in the framework of which the renormalization constants have the form 1 + poles in ε.

In what follows, our aim is to analyze the model at the second-order level of approximation (the two-loop approximation in the language of the Feynman diagrams), therefore we are interested in coefficients $z_{11}^{(i)}$, $z_{21}^{(i)}$, and $z_{22}^{(i)}$ for $i = 1,2$ in series (13) and (14). As for the coefficients $z_{11}^{(1)}$, $z_{21}^{(1)}$, and $z_{22}^{(1)}$ of the two-loop expansion of the renormalization constant Z_1 related to the one-irreducible Green's function $\langle v_i' v_j\rangle_{1-ir}$, they are known for a long time. The one-loop coefficient $z_{11}^{(1)}$ was obtained, e.g., in Ref. [46] and has the following simple form:

$$z_{11}^{(1)} = -\frac{S_d}{(2\pi)^d}\frac{(d-1)}{8(d+2)}, \tag{17}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface area of the d-dimensional unit sphere. Here, $\Gamma(x)$ is the standard Euler's Γ function. On the other hand, the two-loop corrections $z_{21}^{(1)}$ and $z_{22}^{(1)}$ were calculated in Ref. [47]. The coefficient $z_{22}^{(1)}$ is directly related to the one-loop result $z_{11}^{(1)}$ by the simple relation $z_{22}^{(1)} = -(z_{11}^{(1)})^2$ and the explicit integral form of the coefficient $z_{21}^{(1)}$ was found in Ref. [47]. In Ref. [47], the coefficient $z_{21}^{(1)}$ was found in the form of a double integral which is rather huge and relatively complicated. There exists, however, a much simpler representation of $z_{21}^{(1)}$ in the form of a single integral which reads

$$\begin{aligned} z_{21}^{(1)} = & \frac{S_d S_{d-1}}{(2\pi)^{2d}}\frac{1}{256d(d-1)(d+2)} \\ & \times \int_0^1 dx \frac{(1-x^2)^{(d-1)/2}}{x^5}(k_0 + k_1 Y_1 + k_2 Y_2), \end{aligned} \tag{18}$$

where

$$\begin{aligned} k_0 = & 2[192dx + 20d(4d-19)x^3 + 2(2d^3 - 4d^2 \\ & + 97d - 24)x^5 - (10d^3 + 87d^2 - 109d - 156)x^7 \\ & + 2(15d^2 - 78d - 66)x^9 \\ & + 8(4d+3)x^{11}]/[(1-x^2)(4-x^2)], \end{aligned} \tag{19}$$

$$\begin{aligned} k_1 = & 2d[-96 - 2(20d - 121)x^2 - 3(2d^2 - 17d + 67)x^4 \\ & + 2(2d^2 - 17d + 38)x^6 + 4(23d - 14)x^8 \\ & - 32(2d-1)x^{10}]/(1-x^2)^{3/2}, \end{aligned} \tag{20}$$

$$\begin{aligned} k_2 = & 8d[192 + 4(20d - 67)x^2 + 6(2d^2 - 2d + 19)x^4 \\ & + (8d^2 - 10d + 9)x^6 - (2d^2 + 17d + 24)x^8 \\ & + 4(d+1)x^{10}]/(4-x^2)^{3/2}, \end{aligned} \tag{21}$$

and

$$Y_1 = \arctan\left(\frac{1+x}{\sqrt{1-x^2}}\right) - \arctan\left(\frac{1-x}{\sqrt{1-x^2}}\right), \tag{22}$$

$$Y_2 = \arctan\left(\frac{2+x}{\sqrt{4-x^2}}\right) - \arctan\left(\frac{2-x}{\sqrt{4-x^2}}\right). \tag{23}$$

In Eq. (18), the integration variable x is the cosine of the angle between two independent momenta $\mathbf{k}$ and $\mathbf{q}$ over which the integration is taken in the two-loop case, i.e., $x = \mathbf{k}\cdot\mathbf{q}/(|\mathbf{k}||\mathbf{q}|)$.

On the other hand, as for the renormalization constant Z_2 in Eq. (14) related to the one-irreducible Green's function $\langle b_i' b_j\rangle_{1-ir}$ of the general vector field $\mathbf{b}$ in the framework of the studied A model, it is not known, up to now, either to the second or even to the first order of the perturbation theory. The one- and two-loop coefficients $z_{11}^{(2)}$, $z_{21}^{(2)}$, and $z_{22}^{(2)}$ are known only for two special cases of the present general model, namely, for the kinematic MHD model (the $A = 1$ model) [27,48] and for the model of a passively advected vector field (the $A = 0$ model) [39,40]. Thus, our first aim is to find an explicit form of the one-loop coefficient $z_{11}^{(2)}$ and the two-loop coefficients $z_{21}^{(2)}$ and $z_{22}^{(2)}$ in the Z_2 series defined in Eq. (14) in the framework of the general A model given by action functional (5). To this end, the corresponding analysis of the structure of the self-energy operator $\Sigma^{b'b}$ given in the Dyson equation (16) must be done.

The two-loop analysis shows that the self-energy operator $\Sigma^{b'b}$ is given by the sum of singular parts of the corresponding one-irreducible one-loop and two-loop Feynman diagrams and one can write

$$\Sigma^{b'b} = \Gamma^{(1)} + \Gamma^{(2)} = \Gamma^{(1)} + \sum_{l=1}^{8} s_l \Gamma_l^{(2)}, \tag{24}$$

where $s_l, l = 1,\dots,8$ are components of the vector

$$\mathbf{s} = (1,1,1,1/2,1,1,1,1). \tag{25}$$

It represents the symmetry coefficients of all two-loop Feynman diagrams which are shown explicitly in Fig. 3. The explicit analytic form of the singular part of the one-loop contribution $\Gamma^{(1)}$ is given as follows:

$$\Gamma^{(1)} = -\frac{S_d}{(2\pi)^d}\frac{g\nu p^2}{4\varepsilon}\left(\frac{\mu}{m}\right)^{2\varepsilon} \times \frac{(d^2-3)(u+1) + A[d-u(d-2)] + A^2(1+3u)}{d(d+2)(u+1)^2}, \tag{26}$$

where m is an integral scale which provides needed IR regularization (see, e.g., Ref. [47] for details). Result (26) leads to the following explicit expression for the one-loop coefficient $z_{11}^{(2)}$ in the framework of the studied A model:

$$z_{11}^{(2)} = -\frac{S_d}{(2\pi)^d} \times \frac{(d^2-3)(u+1) + A[d-u(d-2)] + A^2(1+3u)}{4d(d+2)u(u+1)^2}. \tag{27}$$

Note that for $A = 1$ one has

$$z_{11}^{(2)} = -\frac{S_d}{(2\pi)^d}\frac{(d-1)}{4du(u+1)}, \tag{28}$$

which represents the well-known one-loop result for the model of kinematic MHD (see, e.g., Ref. [48]) and, on the other hand, for $A = 0$ one obtains

$$z_{11}^{(2)} = -\frac{S_d}{(2\pi)^d}\frac{d^2-3}{4d(d+2)u(u+1)}, \tag{29}$$

which corresponds to the one-loop result in the framework of the model of the passively advected vector field by the turbulent velocity field described by the stochastic Navier-Stokes equation [39].

Finally, let us also show explicitly the form of the coefficient $z_{11}^{(2)}$ for the third physically interesting case, namely $A = -1$, which corresponds to the linearized Navier-Stokes model. It reads

$$z_{11}^{(2)} = -\frac{S_d}{(2\pi)^d}\frac{2(u-1) + d(d-1)(u+1)}{4d(d+2)u(u+1)^2}. \tag{30}$$

On the other hand, the explicit analytic expression for the two-loop contribution $\Gamma^{(2)}$ from Eq. (24) defined by eight two-loop Feynman diagrams shown in Fig. 3 can be written in the form of a single integral, namely,

$$\Gamma^{(2)} \equiv \sum_{l=1}^{8} s_l \Gamma_l^{(2)} = \frac{S_d}{(2\pi)^{2d}}\frac{g^2\nu\, p^2}{16}\left(\frac{\mu}{m}\right)^{4\varepsilon}\frac{1}{\varepsilon} \times \left[\frac{S_d}{\varepsilon}C + S_{d-1}\int_0^1 dx\ (1-x^2)^{(d-1)/2}\ B\right], \tag{31}$$

where

$$\begin{aligned} C = \{&2A^4(9u^2-1) + 4A^3[1+3(1+d)u+3(d-2)u^2] \\ &+ A^2[-4u(11+7u) - d(-2+17u+31u^2 \\ &+ 15u^3+3u^4) + d^2(4+29u+37u^2+15u^3+3u^4)] \\ &+ A(1+u)[-12+24u+2d(-6-3u+4u^2+u^3) \\ &+ d^3(6+9u+4u^2+u^3) - d^2(-2+19u+12u^2 \\ &+ 3u^3)] + (d^2-3)(1+u)^2[-6-d(2+3u+u^2) \\ &+ d^2(4+3u+u^2)]\}/[4d^2(2+d)^2(1+u)^5] \end{aligned} \tag{32}$$

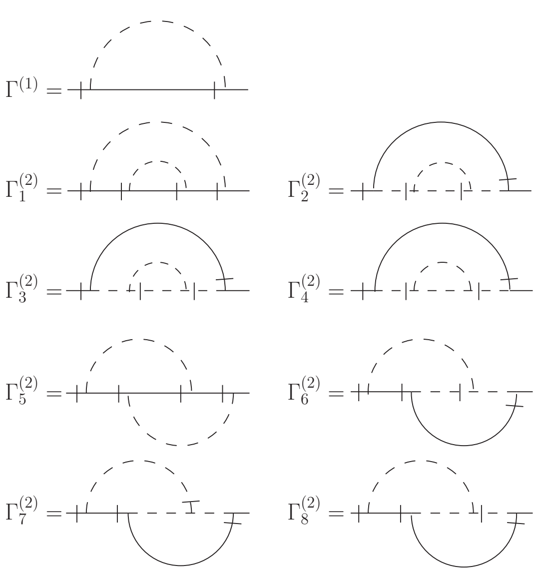

FIG. 3. The one-loop and two-loop Feynman diagrams that contribute to the self-energy operator $\Sigma^{b'b}(\omega,\mathbf{p})$ in Eq. (16).

and the explicit form of the huge coefficient B as the function of d, u, A, and x is given in the Appendix. Here, the integration variable x has the same meaning as in Eq. (18).

Now, by using the Dyson equation (16) together with the relation (24), the two-loop coefficients $z_{21}^{(2)}$ and $z_{22}^{(2)}$ in Eq. (14) for the renormalization constant Z_2 are given as follows:

$$z_{22}^{(2)} = -\frac{S_d^2}{(2\pi)^{2d}}\frac{C}{16u}, \tag{33}$$

and

$$z_{21}^{(2)} = \frac{S_d S_{d-1}}{16u(2\pi)^{2d}}\int_0^1 dx\ (1-x^2)^{(d-1)/2}\ B, \tag{34}$$

where the function C is given in Eq. (32) and, as was already mentioned, the explicit form of the function B is given in the Appendix.

Further, using the fact that all fields $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{b}$, and $\mathbf{b}'$ are not renormalized in the framework of the present model one can immediately conclude that, e.g., all renormalized connected correlation functions $W^R = \langle\Phi\ldots\Phi\rangle^R$ must be equal to their unrenormalized counterparts $W = \langle\Phi\ldots\Phi\rangle$. The only difference is in the choice of variables (renormalized or unrenormalized) and, of course, in the corresponding perturbation expansion (in g or g_0). It means that one can write

$$W^R(g,u,\nu,\mu,\ldots) = W(g_0,u_0,\nu_0,\ldots), \tag{35}$$

where the dots stand for other arguments which are not influenced by renormalization, e.g., parameter A, coordinates, and times. Further, using the fact that unrenormalized correlation functions are independent of the scale-setting parameter μ one can apply the differential operator $\mu\partial_\mu$ at fixed unrenormalized parameters on both sides of Eq. (35) which leads to the basic differential RG equation

$$[\mu\partial_\mu + \beta_g\partial_g + \beta_u\partial_u - \gamma_\nu\nu\partial_\nu]W^R(g,u,\nu,\mu,\ldots) = 0. \tag{36}$$

Here, the so-called RG functions (the β and γ functions) are given as

$$\beta_g \equiv \mu\partial_\mu g = g(-2\varepsilon + 3\gamma_1), \tag{37}$$

$$\beta_u \equiv \mu\partial_\mu u = u(\gamma_1 - \gamma_2), \tag{38}$$

$$\gamma_i \equiv \mu\partial_\mu \ln Z_i, \quad i = 1,2, \tag{39}$$

where we have used the relations among the renormalization constants (12) and Z_1 and Z_2 are given in Eqs. (13) and (14), respectively.

We are looking for the IR asymptotic scaling behavior (the scaling behavior deep inside of the inertial interval) of the correlation functions of the model which is driven by the corresponding IR stable fixed point of the RG equations. Standardly, the coordinates of a fixed point (g_*,u_*) are determined by the requirement of vanishing of the β functions of the model, namely,

$$\beta_g(g_*) = 0, \quad \beta_u(g_*,u_*) = 0. \tag{40}$$

The nontrivial fixed point with $g_* \neq 0$ and $u_* \neq 0$ within the two-loop approximation studied in the present paper has the following form:

$$g_* = g_*^{(1)}\varepsilon + g_*^{(2)}\varepsilon^2 + O(\varepsilon^3), \tag{41}$$

$$u_* = u_*^{(1)} + u_*^{(2)}\varepsilon + O(\varepsilon^2), \tag{42}$$

where

$$g_*^{(1)} = \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}, \tag{43}$$

$$g_*^{(2)} = \frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}\lambda, \tag{44}$$

$$u_*^{(1)} = \frac{1}{3a_2}\left[-2a_2 - \frac{2^{1/3}b_1}{(b_2+b_3)^{1/3}} + \frac{(b_2+b_3)^{1/3}}{2^{1/3}}\right], \tag{45}$$

$$u_*^{(2)} = \frac{2(d+2)}{d[1+2u_*^{(1)}]}\left[\lambda - \frac{128(d+2)^2}{3(d-1)^2}\mathcal{B}(u_*^{(1)})\right]. \tag{46}$$

In Eq. (45) we have used the following notation:

$$b_1 = a_2(3a_1 - 4a_2), \tag{47}$$

$$b_2 = a_2^2(-27a_0 + 18a_1 - 16a_2), \tag{48}$$

$$b_3 = \sqrt{4b_1^3 + b_2^2}, \tag{49}$$

and

$$a_0 = -2[d^2 - 3 + A(A+d)], \tag{50}$$

$$a_1 = 6[1 - A^2 - 2A(d-2) - d(d+1)] \tag{51}$$

$$a_2 = d(d-1). \tag{52}$$

In addition, the coefficient λ in Eqs. (44) and (46) is related to the coefficient $z_{21}^{(1)}$ in Eq. (18) by the following equation:

$$\lambda = \frac{2}{3}\frac{(2\pi)^{2d}}{S_d^2}\left[\frac{8(d+2)}{d-1}\right]^2 z_{21}^{(1)}, \tag{53}$$

and the coefficient $\mathcal{B}(u_*^{(1)})$ in Eq. (46), which is an explicit function of $u_*^{(1)}$, is given by the coefficient $z_{21}^{(2)}$ in Eq. (34) by substitution $u \to u_*^{(1)}$ as follows:

$$\mathcal{B}(u_*^{(1)}) = \frac{(2\pi)^{2d}}{S_d^2} z_{21}^{(2)}(u_*^{(1)}). \tag{54}$$

Note that, for convenience, we have used the same notation as in Refs. [8,47].

It can be shown by numerical analysis that the fixed point given in Eqs. (41)–(46) is IR stable for $\varepsilon > 0$, i.e., that the real parts of both eigenvalues of the following matrix of the first derivatives:

$$\Omega_{ij} = \begin{pmatrix} \partial\beta_g/\partial g & \partial\beta_g/\partial u \\ \partial\beta_u/\partial g & \partial\beta_u/\partial u \end{pmatrix}, \tag{55}$$

calculated at the point (g_*,u_*), are positive. It also means that the correlation functions of the model exhibit scaling behavior deep inside of the inertial interval with given critical dimensions but we shall not discuss this question here.

Note also that the form of β_g and β_u in Eqs. (37) and (38) does not depend on the order of the perturbation expansion, i.e., it is exactly given by the one-loop approximation without higher-loop corrections. This fact leads to the exact values for the anomalous dimensions γ_1 and γ_2 at the IR stable fixed point (g_*,u_*), namely,

$$\gamma_1^* = \gamma_2^* = \frac{2\varepsilon}{3}. \tag{56}$$

V. TURBULENT VECTOR PRANDTL NUMBER IN GENERAL *A* MODEL OF VECTOR ADMIXTURE

Let us start the analysis of the turbulent vector Prandtl number in the framework of the general A model with the investigation of its properties in the one-loop approximation. The one-loop value of the turbulent vector Prandtl number in the A model is directly given by the inverse value of the expression for $u_*^{(1)}$ in Eq. (45), i.e.,

$$\mathrm{Pr}_{\mathrm{A,t}}^{(1)} = 1/u_*^{(1)}, \tag{57}$$

which is an explicit function of the parameter A as well as of the spatial dimension $d > 2$ (we remind once more that our RG approach is valid only for $d > 2$). The general expression (57) for the one-loop turbulent vector Prandtl number contains three physically interesting special cases. Namely, for $A = 1$ one obtains the well-known one-loop result for the turbulent magnetic Prandtl number $\mathrm{Pr}_{\mathrm{m,t}}^{(1)}$ in the framework of the kinematic MHD turbulence (see, e.g., Ref. [27]),

$$\mathrm{Pr}_{\mathrm{m,t}}^{(1)} = 2\Big/\left(-1+\sqrt{\frac{9d+16}{d}}\right). \tag{58}$$

For $A = 0$ one gets the one-loop expression for the turbulent vector Prandtl number $\mathrm{Pr}_{\mathrm{v,t}}^{(1)}$ in the model of passively advected vector quantity [40]

$$\mathrm{Pr}_{\mathrm{v,t}}^{(1)} = 2\Big/\left[-1+\sqrt{1+\frac{8(d^2-3)}{d(d-1)}}\right], \tag{59}$$

and, finally, for $A = -1$ one obtains an expression for the turbulent vector Prandtl number $\mathrm{Pr}_{\mathrm{l,t}}^{(1)}$ in the framework of the linearized Navier-Stokes model as a function of the spatial dimension d:

$$\mathrm{Pr}_{\mathrm{l,t}}^{(1)} = 3\Bigg/\left\{\frac{\sqrt[3]{3\sqrt{3}[3(d-1)d-2]\sqrt{-(d-1)^3d^3[(d-1)d+16]}+10(d-1)^2[(d-1)d-9]d^2}}{(d-1)d}\right.$$
$$\left.+\frac{7(d-1)d+12}{\sqrt[3]{3\sqrt{3}[3(d-1)d-2]\sqrt{-(d-1)^3d^3[(d-1)d+16]}+10(d-1)^2[(d-1)d-9]d^2}}-2\right\}. \tag{60}$$

However, although it is not evident at the first sight, the numerical analysis of the expression (60) shows that

$$\mathrm{Pr}_{\mathrm{l,t}}^{(1)} = 1, \tag{61}$$

for arbitrary $d > 2$, i.e., the turbulent Prandtl number of the linearized Navier-Stokes model does not depend on the value of the spatial dimension [although, from a purely mathematical point of view, the expression (61) which follows from Eq. (60) can also be valid for $d \leqslant 2$, we consider here only spatial dimensions $d > 2$ since the present RG analysis is valid only in this case].

The behavior of the one-loop turbulent vector Prandtl number in the A model is demonstrated in Figs. 4 and 5. In Fig. 4, the dependence of the one-loop turbulent vector Prandtl number on the parameter A and on the spatial dimension $d > 2$ is shown explicitly. In addition, in Fig. 5, the behavior of the one-loop turbulent vector Prandtl numbers for three physically interesting special cases $A = -1,0,1$ is shown as the function of the spatial dimension d. It is evident from Fig. 5 that, at the one-loop level of approximation, there exist rather significant differences in the behavior of the turbulent vector Prandtl numbers of the models with $A = -1,0,1$ as for their dependence on the spatial dimension. While the one-loop turbulent vector Prandtl number is a constant function of d in the framework of the $A = -1$ model, the one-loop turbulent magnetic Prandtl number ($A = 1$) is a concave function of d and the one-loop vector Prandtl number of passively advected vector quantity ($A = 0$) is a convex function of d, respectively. At the same time, there are also rather significant differences

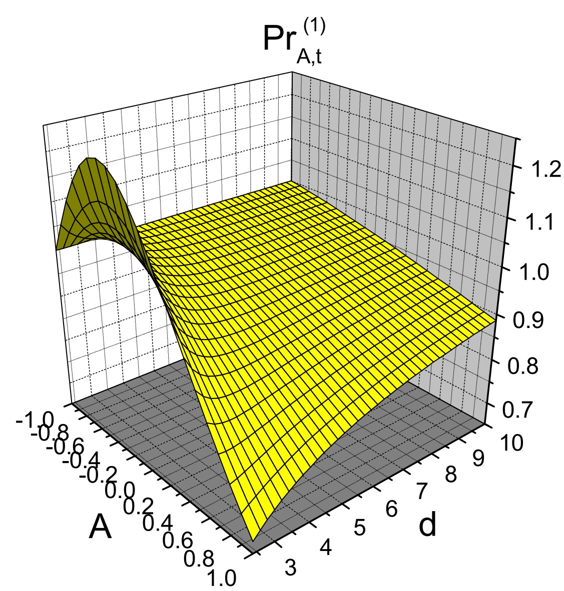

FIG. 4. The dependence of the one-loop turbulent vector Prandtl number $\mathrm{Pr}_{\mathrm{A,t}}^{(1)}$ in the general A model on the parameter A as well as on the spatial dimension d.

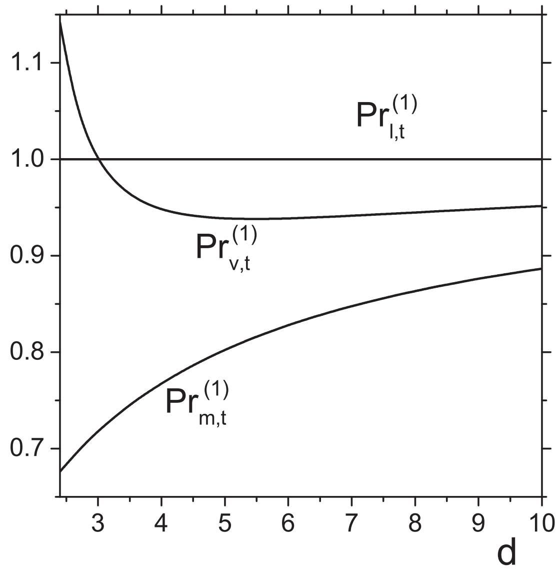

FIG. 5. The behavior of the one-loop turbulent vector Prandtl numbers for three physically interesting special cases, namely, for $A=-1$ ($\mathrm{Pr}^{(1)}_{\mathrm{l,t}}$), for $A=0$ ($\mathrm{Pr}^{(1)}_{\mathrm{v,t}}$), and for $A=1$ ($\mathrm{Pr}^{(1)}_{\mathrm{m,t}}$), is shown explicitly as functions of the spatial dimension d.

among their numerical values as it follows from Fig. 5 as well as from Table II, where the values of the one-loop turbulent vector Prandtl numbers of the models with $A=-1,0,1$ for various spatial dimensions d are shown explicitly (The first three lines in Table II). The only exceptions are the values of $\mathrm{Pr}^{(1)}_{\mathrm{l,t}}$ and $\mathrm{Pr}^{(1)}_{\mathrm{v,t}}$ for the most interesting case, $d=3$, for which $\mathrm{Pr}^{(1)}_{\mathrm{l,t}}=\mathrm{Pr}^{(1)}_{\mathrm{v,t}}=1$. On the other hand, it is also evident from Fig. 5 that if one considers only the integer values of the spatial dimension d then the largest difference between the one-loop turbulent magnetic Prandtl number $\mathrm{Pr}^{(1)}_{\mathrm{m,t}}$ and the other two physically interesting one-loop turbulent vector Prandtl numbers ($\mathrm{Pr}^{(1)}_{\mathrm{l,t}}$ and $\mathrm{Pr}^{(1)}_{\mathrm{v,t}}$) is again for the physical spatial dimension $d=3$. In this respect, in Table II the relative difference $\epsilon^{(1)}_{\mathrm{m,v}}$ between the one-loop values of the turbulent magnetic Prandtl number ($A=1$) and the turbulent vector Prandtl number ($A=0$) with respect to the one-loop value of the last one, i.e.,

$$\epsilon^{(1)}_{\mathrm{m,v}}=\left|\frac{\mathrm{Pr}^{(1)}_{\mathrm{m,t}}-\mathrm{Pr}^{(1)}_{\mathrm{v,t}}}{\mathrm{Pr}^{(1)}_{\mathrm{v,t}}}\right|, \tag{62}$$

is shown.

Now, let us turn our attention to the two-loop analysis of the turbulent vector Prandtl number in the general A model of passively advected vector fields. The technique for derivation of the two-loop expression for the turbulent (effective) inverse Prandtl number which holds within the inertial interval and, at the same time, does not depend on the renormalization scheme was introduced in Ref. [8] [see Eq. (33) in Ref. [8]] in the framework of the model of the passively advected scalar field by the turbulent velocity field driven by the stochastic Navier-Stokes equation. By using this formula the second-order corrections to the turbulent Prandtl number were calculated and it was shown that the two-loop corrections are surprisingly very small (they are less than 2% of the leading one-loop result) [8,9].

Using the same procedure as in Ref. [8] the corresponding two-loop formula for the calculation of the turbulent inverse vector Prandtl number u_{eff} in the framework of the general A model of the passively advected vector quantity (studied in the present paper) can also be derived and written in a similar form as in Ref. [8], namely,

$$u_{\mathrm{eff}}=u_*^{(1)}\left(1+\varepsilon\left\{\frac{1+u_*^{(1)}}{1+2u_*^{(1)}}\left[\lambda-\frac{128(d+2)^2}{3(d-1)^2}\mathcal{B}(u_*^{(1)})\right]+\frac{(2\pi)^d}{S_d}\frac{8(d+2)}{3(d-1)}[a_v-a_b(u_*^{(1)})]\right\}\right), \tag{63}$$

where $u_*^{(1)}$, λ, and $\mathcal{B}(u_*^{(1)})$ are given in Eqs. (45), (53), and (54), respectively, and quantities a_v and $a_b(u_*^{(1)})$ are given by the corresponding expansions to the leading order in ε of the scaling functions of response functions $\langle vv'\rangle$ and $\langle bb'\rangle$ of the velocity field and the advected vector field, respectively (see Ref. [8] for details). Their explicit form in our general model is the following:

$$\begin{aligned} a_v=&\frac{S_{d-1}}{4(d-1)(2\pi)^d}\int_0^\infty dk\int_{-1}^1 dx\,(1-x^2)^{(d-1)/2}\\ &\times\left\{\frac{2k[2k^3x-(d-3)k^2-(d-1)(2kx-1)]}{(2k^2+2kx+1)(k^2+2kx+1)}-\frac{\theta(k-1)(2kx-6x^2-d+3)}{k}\right\}\end{aligned} \tag{64}$$

and

$$\begin{aligned} a_b(u)=&-\frac{S_{d-1}}{2u(d-1)(2\pi)^d}\int_0^\infty dk\int_{-1}^1 dx\,(1-x^2)^{(d-1)/2}\\ &\times\left(\frac{k\{k^3xA(1-A)+k^2[x^2(1-A^2)+A+d-2]+2kx(d-1)+d-1\}}{(k^2+2kx+1)[(1+u)k^2+2ukx+u]}\right.\\ &\left.-\frac{\theta(k-1)\{kA(1-A)(1+u)x+A^2(1+3u)x^2+A[1+u-2(1+2u)x^2]+(1+u)(x^2+d-2)\}}{k(1+u)^2}\right),\end{aligned} \tag{65}$$

where $\theta(y)$ is the well-known Heaviside step function.

TABLE II. The one-loop and the two-loop values of the turbulent vector Prandtl numbers for passively advected vector fields by the Navier-Stokes velocity field for models with $A = 1$ ($\mathrm{Pr}^{(1)}_{\mathrm{m,t}}$ and $\mathrm{Pr}_{\mathrm{m,t}}$), with $A = 0$ ($\mathrm{Pr}^{(1)}_{\mathrm{v,t}}$ and $\mathrm{Pr}_{\mathrm{v,t}}$), and with $A = -1$ ($\mathrm{Pr}^{(1)}_{\mathrm{l,t}}$ and $\mathrm{Pr}_{\mathrm{l,t}}$) for various values of spatial dimension $d > 2$. The relative differences $\epsilon^{(1)}_{m,v}$ and $\epsilon_{\mathrm{m,v}}$ between the one-loop values $\mathrm{Pr}^{(1)}_{\mathrm{m,t}}$ and $\mathrm{Pr}^{(1)}_{\mathrm{v,t}}$ and between the two-loop values $\mathrm{Pr}_{\mathrm{m,t}}$ and $\mathrm{Pr}_{\mathrm{v,t}}$ are also present as the functions of the spatial dimension d.

d	2.4	2.6	2.8	3	3.2	3.6	4	5	6	8	10	100	$d \to \infty$
$\mathrm{Pr}^{(1)}_{\mathrm{m,t}}$	0.6761	0.6914	0.7052	0.7179	0.7295	0.75	0.7676	0.8023	0.8279	0.8633	0.8866	0.9869	1
$\mathrm{Pr}^{(1)}_{\mathrm{v,t}}$	1.1418	1.0701	1.0274	1	0.9815	0.9595	0.9483	0.9387	0.9387	0.9448	0.9515	0.9935	1
$\mathrm{Pr}^{(1)}_{\mathrm{l,t}}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathrm{Pr}_{\mathrm{m,t}}$	0.4572	0.5472	0.6300	0.7040	0.7686	0.8705	0.9417	1.0350	1.0684	1.0804	1.0747	1.0093	1
$\mathrm{Pr}_{\mathrm{v,t}}$	0.5449	0.6135	0.6756	0.7307	0.7789	0.8559	0.9115	0.99	1.0234	1.0434	1.0449	1.0068	1
$\mathrm{Pr}_{\mathrm{l,t}}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$\epsilon^{(1)}_{\mathrm{m,v}} \times 10^2$ (%)	40.8	35.4	31.4	28.2	25.7	21.8	19.1	14.5	11.8	8.6	6.8	0.66	0
$\epsilon_{\mathrm{m,v}} \times 10^2$ (%)	16.1	10.8	6.7	3.7	1.3	1.7	3.3	4.5	4.4	3.5	2.9	0.25	0

The quantities λ and a_v in Eq. (63), which are given in Eqs. (53) and (64), respectively, are common for all models of passively advected scalar [8,9] or vector [27,39] fields advected by the turbulent velocity field driven by the stochastic Navier-Stokes equation (2) with the correlator (4). On the other hand, quantities a_b and $\mathcal{B}(u_*^{(1)})$ depends on the value of A, i.e., on the concrete model of passively advected vector field. It is also important to note that, at least as for the turbulent Prandtl number, the model of a passively advected scalar quantity [8,9] is completely equivalent to the problem of passively advected weak magnetic field in the framework of the kinematic MHD turbulence [27], i.e., to the $A = 1$ model in the formulation given in the present paper. It means that formula (63) for $A = 1$ gives also the two-loop value of the turbulent Prandtl number in the problem of the scalar admixture (see discussion in Ref. [27] for details).

From general equation (65) one immediately obtains the corresponding expressions for the three most interesting physical cases. Namely, for $A = 1$ one comes to the result given in Eq. (45) in Ref. [27] and for $A = 0$ one has the result given in Eq. (47) in Ref. [40], therefore we shall not present their explicit form here. In the end, the explicit expression for the third interesting case $A = -1$ is given as follows:

$$a_b(u) = -\frac{S_{d-1}}{2u(d-1)(2\pi)^d}\int_0^\infty dk \int_{-1}^1 dx\,(1-x^2)^{(d-1)/2} \times \left\{ \frac{k[-2k^3x + k^2(d-3) + 2kx(d-1) + d - 1]}{(k^2+2kx+1)[(1+u)k^2 + 2ukx + u]} - \frac{\theta(k-1)[(1+u)(-2kx+d-3) + 4x^2(2u+1)]}{k(1+u)^2} \right\}. \tag{66}$$

The behavior of the two-loop turbulent vector Prandtl number in the framework of the general A model and for the physical value $\varepsilon = 2$ (see Sec. II for details), which is given as the inverse value of u_{eff}, i.e.,

$$\mathrm{Pr}_{\mathrm{A,t}} = 1/u_{\mathrm{eff}}, \tag{67}$$

is demonstrated in Figs. 6 and 7.

In Fig. 6, the dependence of the two-loop value of the general turbulent vector Prandtl number $\mathrm{Pr}_{\mathrm{A,t}}$ on the parameter A and on the spatial dimension $d > 2$ is shown explicitly. By comparison of Fig. 4 with Fig. 6 one finds that, in general, the two-loop corrections are rather significant. The only exception is the $A = -1$ model for which the two-loop value of the turbulent Prandtl number is again equal to 1 for arbitrary $d > 2$, i.e., there are no two-loop corrections to the one-loop value of this turbulent Prandtl number:

$$\mathrm{Pr}_{\mathrm{l,t}} = \mathrm{Pr}^{(1)}_{\mathrm{l,t}} = 1. \tag{68}$$

Although, at first sight, this result looks rather intriguing, it has the following explanation. Using the fact that the one-loop value of the inverse turbulent Prandtl number $u_*^{(1)}$ is independent of the value of spatial dimension d in the $A = -1$ model and, at the same time, is equal to 1, one finds that $a_b \equiv a_v$ in this case and the corresponding term in Eq. (63) proportional to $a_v - a_b$ vanishes, i.e., it gives no contribution to the two-loop value of the inverse turbulent Prandtl number (63) in this case.

Further, it is easy to check that a similar situation also takes place for the two-loop contributions to the inverse turbulent Prandtl number (63) given by the expressions for λ and $\mathcal{B}(u_*^{(1)})$.

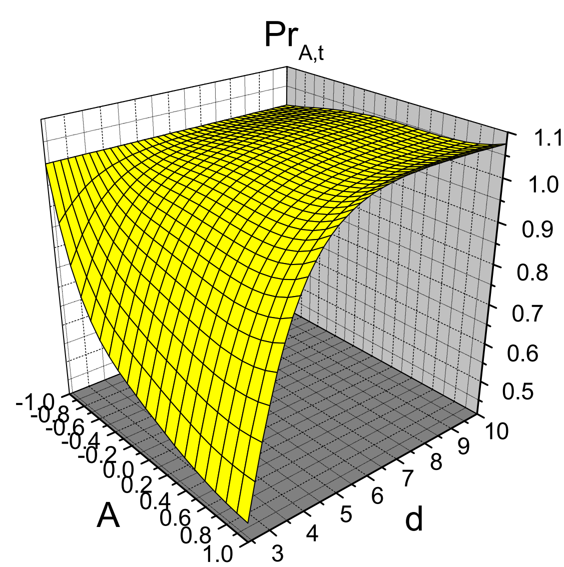

FIG. 6. The dependence of the two-loop turbulent vector Prandtl number $\mathrm{Pr}_{\mathrm{A,t}}$ in general A model on the parameter A as well as on the spatial dimension d.

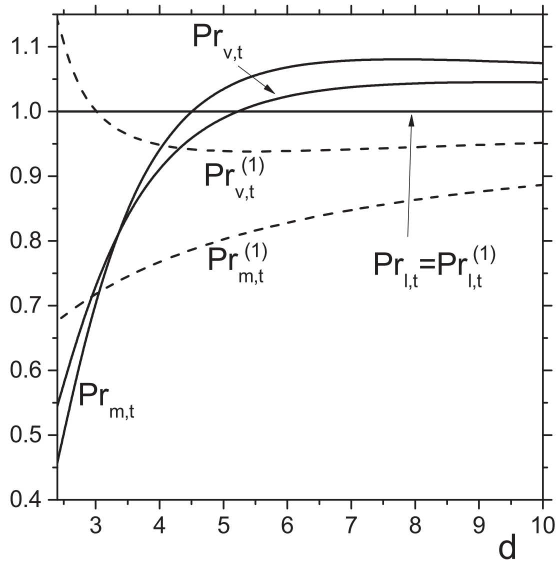

FIG. 7. The behavior of the two-loop turbulent vector Prandtl numbers for three physically interesting special cases, namely, for $A=-1$ ($\mathrm{Pr}_{\mathrm{l,t}}$), for $A=0$ ($\mathrm{Pr}_{\mathrm{v,t}}$), and for $A=1$ ($\mathrm{Pr}_{\mathrm{m,t}}$), is shown explicitly as functions of the spatial dimension d. In addition, the corresponding one-loop curves for $\mathrm{Pr}_{\mathrm{l,t}}^{(1)}$, $\mathrm{Pr}_{\mathrm{v,t}}^{(1)}$, and $\mathrm{Pr}_{\mathrm{m,t}}^{(1)}$ are also shown for comparison.

Namely, the following relation holds for the model with $A=-1$:

$$\mathcal{B}(u_*^{(1)})|_{u_*^{(1)}=1} = \lambda \frac{3(d-1)^2}{128(d+2)^2}, \tag{69}$$

where we have again used the fact that $u_*^{(1)}=1$ in this case. It means that, in this special case, the second contribution to the two-loop expression for the inverse turbulent vector Prandtl number given in Eq. (63) vanishes too. However, it also means that there exist no two-loop contributions to the inverse turbulent vector Prandtl number in the $A=-1$ model and the corresponding turbulent vector Prandtl number $\mathrm{Pr}_{\mathrm{l,t}}$ is exactly equal to 1 (see Table II).

This result has a theoretical explanation given by the mathematical structure of the $A=-1$ model together with the existence of the only one infrared stable fixed point for the parameter u in the one-loop approximation, namely, $u_*^{(1)}=1$.

Looking at the interaction vertices $V_{ijl}=i(k_j\delta_{il}-Ak_l\delta_{ij})$ and $W_{ijl}=i(k_l\delta_{ij}+k_j\delta_{il})$ of the present model (see Sec. III for details) it is evident that their mathematical structure is identical for the $A=-1$ model. At the same time, the sets of the one-loop and the two-loop Feynman diagrams for determining the coefficients $z_{11}^{(1)}, z_{21}^{(1)}$, and $z_{22}^{(1)}$ and $z_{11}^{(2)}, z_{21}^{(2)}$, and $z_{22}^{(2)}$, respectively, are completely the same too. The only difference is that some bare propagators $\langle v_i' v_j\rangle_0$ are replaced by propagators $\langle b_i' b_j\rangle_0$ (note that the mathematical structure of the interaction vertices remains the same). But the difference between these two propagators is only in the presence of the bare parameter u_0 (the bare inverse Prandtl number) in the propagator $\langle b_i' b_j\rangle_0$ [see Eqs. (6) and (7) in Sec. III] and because the one-loop fixed point value of u_0 is $u_*^{(1)}=1$ then we have $z_{11}^{(1)}=z_{11}^{(2)}$ (one-loop result) as well as $z_{21}^{(1)}=z_{21}^{(2)}$ and $z_{22}^{(1)}=z_{22}^{(2)}$ (two-loop results). It then means that $u_*^{(2)}=0$ [see Eq. (46)] as well as $u_{\mathrm{eff}}=1$ [see Eq. (63)] and the two-loop turbulent Prandtl number $\mathrm{Pr}_{\mathrm{l,t}}$ is also equal to 1.

Here, however, we can go further in our analysis because the same logical conclusions are also valid for arbitrary higher-loop approximation in the framework of the $A=-1$ model. Having $u_*^{(1)}=1$ and $u_*^{(2)}=0$, together with the fact that the sets of the three-loop Feynman diagrams for determining the coefficients $z_{31}^{(1)}, z_{32}^{(1)}$, and $z_{33}^{(1)}$ and $z_{31}^{(2)}, z_{32}^{(2)}$, and $z_{33}^{(2)}$, respectively, are again completely the same as for their structure, it is easy to show that the three-loop corrections to the parameter u are also equal to 0, i.e., $u_*^{(3)}=0$. In the same way it can be shown that all other potential corrections to the three-loop turbulent inverse Prandtl number in the framework of the $A=-1$ model vanish too. But it also means that the corresponding three-loop turbulent vector Prandtl number $\mathrm{Pr}_{l,t}$ remains to be equal to 1. Continuing in the same manner, one concludes that the turbulent vector Prandtl number $\mathrm{Pr}_{\mathrm{l,t}}$, i.e., the turbulent vector Prandtl number in the framework of the $A=-1$ model, has no higher-loop corrections and is exactly equal to 1, i.e., is exactly determined by the one-loop analysis.

On the other hand, the role of two-loop corrections to the corresponding turbulent Prandtl numbers for two other physically interesting models with $A=0$ and $A=1$ is significant which is demonstrated in Fig. 7 where the corresponding one-loop and two-loop turbulent Prandtl numbers are compared and shown as the functions of the spatial dimension d. In addition, the two-loop values of the turbulent Prandtl numbers $\mathrm{Pr}_{\mathrm{m,t}}$ ($A=1$) and $\mathrm{Pr}_{\mathrm{v,t}}$ ($A=0$) for various values of the spatial dimension d are shown in Table II. As follows from Fig. 7 as well as from Table II, the turbulent Prandtl numbers $\mathrm{Pr}_{\mathrm{m,t}}$ and $\mathrm{Pr}_{\mathrm{v,t}}$ behave very similar as functions of d at the two-loop level of approximation. At the same time, their values are very close to one another not only for $d=3$, as was shown in Ref. [39], but for all values of d. This behavior is radically different from the corresponding behavior which one can see at the one-loop level of approximation (see Figs. 5 and 7). To demonstrate quantitatively the closeness of the two-loop results for the turbulent Prandtl numbers $\mathrm{Pr}_{\mathrm{m,t}}$ and $\mathrm{Pr}_{\mathrm{v,t}}$ we have again calculated the relative differences $\epsilon_{\mathrm{m,v}}$ of the two-loop values of the turbulent magnetic ($A=1$) with respect to the two-loop values of the turbulent vector ($A=0$) Prandtl numbers, i.e.,

$$\epsilon_{\mathrm{m,v}} = \left| \frac{\mathrm{Pr}_{\mathrm{m,t}} - \mathrm{Pr}_{\mathrm{v,t}}}{\mathrm{Pr}_{\mathrm{v,t}}} \right|. \tag{70}$$

The corresponding results for various spatial dimensions d are shown in the last line in Table II. It is evident from Table II that the relative difference between the corresponding two-loop values of the turbulent Prandtl numbers $\mathrm{Pr}_{\mathrm{m,t}}$ and $\mathrm{Pr}_{\mathrm{v,t}}$ are less than 5% for all integer values $d \geqslant 3$ of the spatial dimension. It is a nontrivial result which means that, at least at the two-loop level of approximation, there is no significant difference between the diffusion processes in the model of passively advected magnetic field in the framework of kinematic MHD turbulence and in the model of passively advected vector quantity by the Navier-Stokes turbulence.

At the same time, as was already mentioned, in Ref. [27] it was shown that the turbulent magnetic Prandtl number in the model of a passively advected magnetic field in the framework of the kinematic MHD turbulence ($A = 1$ model) is the same as the turbulent Prandtl number in the corresponding model of a passive scalar advection [8,9]. It means that the diffusion processes of the passive weak magnetic field advected by the incompressible isotropic turbulent environment driven by the stochastic Navier-Stokes equation in the framework of the kinematic MHD turbulence have the same properties as the corresponding diffusion processes of passive scalar quantities advected by the same stochastic environment, at least, up to the second-order approximation. Using this fact together with the results obtained in the present paper one can conclude that diffusion processes are very similar in all relevant models of passively advected scalar or vector quantities, at least, if the two-loop approximation results are taken into account. The only exception is the mathematically as well as physically specific $A = -1$ model in the framework of which the corresponding turbulent Prandtl number is exactly equal to 1. It means that the turbulent viscosity is always numerically equal to the corresponding coefficient of diffusivity in the $A = -1$ model.

Here, of course, a few questions immediately arise. The first nontrivial question is whether these results will also be valid if the higher-loop corrections are taken into account. Although the two-loop RG calculations in the framework of quantum field theory models are usually enough to demonstrate, at least, general tendencies we must be careful making general conclusions because working in the models with strong coupling constants it would be fine to confirm our conclusions by three-loop calculations.

The second nontrivial question is the question about the role of various breaking of symmetries, such as the presence of an anisotropy, of the helicity (the spatial parity violation), or the compressibility in the turbulent systems, on the diffusion processes in the above discussed models. In this respect, some results were obtained in this direction recently in Refs. [39,40], where the influence of the helicity on the various turbulent Prandtl numbers was discussed at the two-loop level of approximation and it was shown that such symmetry breaking has a rather significant impact on the diffusion processes in turbulent systems.

The third interesting question is related to the two-loop values of the turbulent Prandtl numbers in the two-dimensional case, which is out of scope of the present paper. Here, additional divergences appear in the renormalization process which must be correctly taken into account. This question is also open for now.

Finally, let us also show explicitly the dependence of the one-loop and the two-loop turbulent general vector Prandtl numbers as the functions of the parameter A for the physically most interesting value of the spatial dimension $d = 3$ which demonstrates the importance of the two-loop corrections for various models with a given value of the parameter A (see Fig. 8). As it follows from Fig. 8, the models with $A = -1$ and $A = 1$ are very well described already at the one-loop level of approximation (let us remind that the one-loop result for the $A = -1$ model is actually exact). On the other hand, the two-loop corrections to the turbulent vector Prandtl number are significant especially for models with values of the parameters A deep from the interval $-1 < A < 1$. It demonstrates once more the importance of the two-loop corrections to the turbulent vector Prandtl number in the $A = 0$ model of a passively advected vector field.

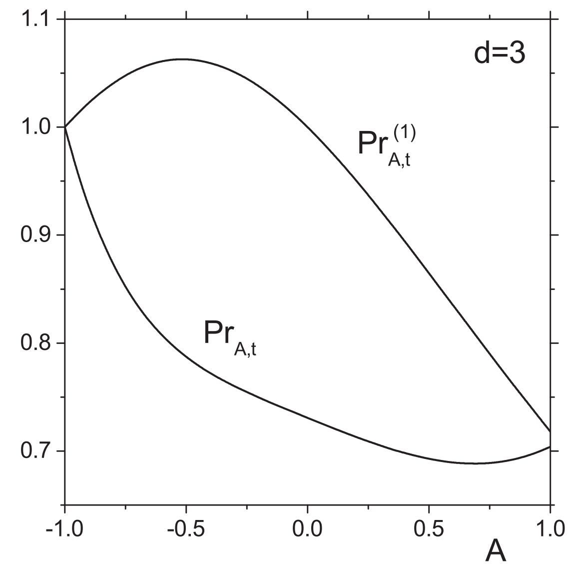

FIG. 8. The dependence of the one-loop ($\mathrm{Pr}^{(1)}_{A,t}$) and the two-loop ($\mathrm{Pr}_{A,t}$) turbulent vector Prandtl numbers on the parameter A for the spatial dimension $d = 3$.

VI. CONCLUSION

In the present paper, by using the field theoretic RG technique in the second-order approximation (the two-loop approximation) we have calculated and analyzed properties of the turbulent vector Prandtl number in the framework of the general A model of a passive vector field advected by the Navier-Stokes turbulent environment. Special attention is devoted to the three physically most interesting cases, namely, to the passive advection of the magnetic field in an conductive turbulent environment in the framework of the kinematic MHD turbulence ($A = 1$), to the passive admixture of a vector impurity (an impurity with the internal structure) by the Navier-Stokes turbulent flow ($A = 0$), which is an analog of the corresponding model of the passively advected scalar impurity in the given turbulent flow, and to the model of the linearized Navier-Stokes equation, which describes, e.g., weak perturbations of prescribed background field ($A = -1$).

It is shown that the turbulent vector Prandtl number $\mathrm{Pr}_{l,t}$ in the framework of the $A = -1$ model is exactly given already at the one-loop level of approximation, i.e., that all higher-loop corrections to this turbulent Prandtl number vanish, and it is identically equal to 1 for all values of the spatial dimension $d > 2$. This nontrivial result, which is explained by the mathematical structure of the model, means that, in this case, the coefficient of turbulent diffusivity is numerically exactly equal to the coefficient of turbulent viscosity.

In addition, it is shown that the two-loop turbulent Prandtl numbers $\mathrm{Pr}_{m,t}$ ($A = 1$) and $\mathrm{Pr}_{v,t}$ ($A = 0$) behave very similar as the functions of the spatial dimension d (see Fig. 7). They are both increasing functions of d and both go to 1 in the limit $d \to \infty$. At the same time, the relative difference between two-loop values of the turbulent Prandtl numbers $\mathrm{Pr}_{\mathrm{m,t}}$ and $\mathrm{Pr}_{\mathrm{v,t}}$ are less than 5% for all integer values $d \geqslant 3$ (see Table II). As follows from Fig. 7, this behavior is radically different from the corresponding behavior which is seen at the one-loop level of approximation. This behavior indicates that the diffusion processes in all relevant models of passively advected scalar and vector quantities are very similar, at least at the two-loop level of approximation, because, as was shown in Ref. [39], the behavior of the turbulent magnetic Prandtl number $\mathrm{Pr}_{\mathrm{m,t}}$, i.e., the turbulent Prandtl number in the framework of the $A = 1$ model, is completely the same as for the turbulent Prandtl number in the model of passively advected scalar quantity by the velocity field driven by the stochastic Navier-Stokes equation. The only exception is the turbulent Prandtl number in the case of the $A = -1$ model in the framework of which it does not depend on the spatial dimension $d > 2$ and is identically equal to 1.

The closeness of the turbulent Prandtl numbers on the two-loop level of approximation for three standard models of passively advected scalar and vector quantities means that the corresponding values of the turbulent diffusion coefficients are also close to each other in the vicinity of 5% for arbitrary integer spatial dimension $d \geqslant 3$. It means that the internal structure of the advected fields as well as the form of the interaction in the models have negligible influence on the values of the corresponding diffusion coefficients, i.e., for the velocity of diffusion processes. This is an intriguing result and it would be quite interesting to confirm or refute it by performing the corresponding three-loop calculations. However, this work is left for the future.

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APPENDIX

The explicit form of the function B in Eq. (31) can be written as follows:

$$B = \sum_{i=0}^{6} \frac{Y_i}{X_i} \sum_{j=0}^{4} B_{ij} A^j,$$

where $Y_0 = 1$, Y_1 and Y_2 are given in Eqs. (22) and (23), respectively,

$$Y_3 = \arctan\left(\frac{1+u(1+x)}{\sqrt{1+2u+u^2(1-x^2)}}\right) - \arctan\left(\frac{1+u(1-x)}{\sqrt{1+2u+u^2(1-x^2)}}\right),$$

$$Y_4 = \arctan\left(\frac{2+x}{\sqrt{2(1+u)-x^2}}\right) - \arctan\left(\frac{2-x}{\sqrt{2(1+u)-x^2}}\right) + \arctan\left(\frac{1+u+x}{\sqrt{2(1+u)-x^2}}\right) - \arctan\left(\frac{1+u-x}{\sqrt{2(1+u)-x^2}}\right),$$

$$Y_5 = \ln\left(\frac{2}{1+u}\right),$$

$$Y_6 = \ln\left(\frac{1-x^2-x\sqrt{x^2-1}}{1-x^2+x\sqrt{x^2-1}}\right),$$

$$X_0 = 4(d-1)d(d+2)(u-1)(u+1)^4x^2(x^2-1)[u(x-1)-1](ux+u+1)(2u-x^2+2)[u^2+u(4x^2-2)+1]^3,$$

$$X_1 = 2(d-1)d(d+2)(u+1)^3x^3\sqrt{1-x^2}(u^2+4ux^2-2u+1)^2,$$

$$X_2 = (d-1)d(2+d)(u-1)^2(1+u)^2x^3\sqrt{4-x^2},$$

$$X_3 = -2(d-1)d(2+d)(u-1)^2(1+u)^5x^3(1+2u+u^2-u^2x^2)^{3/2},$$

$$X_4 = (d-1)d(d+2)(u-1)^2(u+1)^4(2u-x^2+2)^{3/2}(u^2+4ux^2-2u+1)^4,$$

$$X_5 = (d-1)d(d+2)(u-1)^2(u+1)^4(u^2+4ux^2-2u+1)^4,$$

$$X_6 = 8(d-1)d(d+2)(u+1)^3x^3(x^2-1)^{3/2}(u^2+4ux^2-2u+1)^4,$$

$$\begin{aligned}B_{00} = {} & (u+1)^2[2u^6(x^2-1)+u^5x^2(7x^2-11)+u^4(-4x^6+7x^4-17x^2+6)+u^3x^2(4x^4-7x^2+2)\\ & +u^2(-3x^4+16x^2-6)+u(9x^2-4x^4)-x^2+2]\{2d^3(u+1)x^2[u^2+u(4x^2-2)+1]^2+d^2[u^5(2-5x^2)\\ & +u^4(-28x^4+19x^2-6)+2u^3(8x^6-30x^4+5x^2+2)+2u^2(96x^8-208x^6+138x^4-43x^2+2)\\ & +u(144x^6-244x^4+91x^2-6)+24x^4-29x^2+2]-2d[u^5(4x^6-10x^4+11x^2-2)+u^4(32x^8-94x^6\end{aligned}$$

$$+121x^4-48x^2+4)+u^3x^2(64x^8-208x^6+298x^4-151x^2+27)+u^2(32x^{10}-56x^8+64x^6-11x^4+5x^2-4)$$
$$+u(16x^8-14x^6+13x^4-2x^2+2)+x^2(6x^4-10x^2+7)]+3u^5(5x^2-2)+u^4(88x^4-61x^2+18)$$
$$-2u^3(8x^6-68x^4+9x^2+6)-2u^2(256x^8-568x^6+380x^4-119x^2+6)+u(-384x^6+664x^4-253x^2+18)$$
$$-64x^4+79x^2-6\},$$

$$B_{01}=(u+1)\{2(x^2-1)[8x^4+2d^2x^2-2x^2+d(24x^6-54x^4+27x^2-4)]u^{12}+[2d^2(7x^4+x^2-12)x^2$$
$$+2(92x^8-127x^6+13x^4+6x^2+4)+d(552x^{10}-2074x^8+2627x^6-1345x^4+276x^2-8)]u^{11}$$
$$+[672x^{10}-660x^8-776x^6+580x^4+40x^2+d^2(-200x^8+610x^6-554x^4+80x^2)+d(2016x^{12}-8596x^{10}$$
$$+12\,392x^8-7227x^6+1631x^4-8)-24]u^{10}+[2d^2(-592x^8+1708x^6-1460x^4+223x^2+12)x^2$$
$$+2(320x^{12}+780x^{10}-2966x^8+1322x^6+543x^4-238x^2-4)+d(1920x^{14}-9072x^{12}+13\,364x^{10}-7066x^8$$
$$+1530x^6-61x^4+28x^2+8)]u^9-[2d^2(704x^{10}-1840x^8+428x^6+1544x^4-759x^2+122)x^2$$
$$+4(128x^{14}-1192x^{12}+1298x^{10}+2460x^8-3704x^6+1272x^4-63x^2-22)+d(1536x^{16}-8768x^{14}$$
$$+25184x^{12}-41272x^{10}+39\,974x^8-25360x^6+8583x^4-994x^2+48)]u^8+2[2d^2(256x^{12}-1376x^{10}$$
$$+3520x^8-3770x^6+1619x^4-379x^2+36)x^2-2(864x^{14}-5112x^{12}+11\,086x^{10}-9125x^8+2531x^6$$
$$+95x^4-246x^2+12)+d(640x^{16}-3552x^{14}+7688x^{12}-7434x^{10}+63x^8+5230x^6-1585x^4-652x^2+24)]u^7$$
$$+2[2d^2(32x^{10}+720x^8-1466x^6+1037x^4-401x^2+64)x^2+2(128x^{16}-1504x^{14}+5136x^{12}-8918x^{10}$$
$$+7614x^8-3923x^6+1682x^4-124x^2-28)+d(512x^{16}-2880x^{14}+10\,240x^{12}-21\,372x^{10}+19\,385x^8-7574x^6$$
$$+2943x^4-1352x^2+56)]u^6-2[2d^2(256x^{12}-1120x^{10}+2440x^8-2818x^6+1640x^4-559x^2+84)x^2$$
$$+2(128x^{16}-1312x^{14}+4384x^{12}-8978x^{10}+10\,979x^8-5924x^6+433x^4+150x^2-28)$$
$$+d(384x^{16}-480x^{14}-4552x^{12}+15\,114x^{10}-13\,687x^8+3775x^6+529x^4-684x^2+56)]u^5$$
$$+2[2d^2(320x^{10}-1832x^8+2398x^6-972x^4+200x^2-11)x^2+2(32x^{14}-632x^{12}+4674x^{10}-8266x^8+4879x^6$$
$$-533x^4-25x^2+12)+d(96x^{14}-1088x^{12}+3068x^{10}-5821x^8+4219x^6-2274x^4+1347x^2-44)]u^4$$
$$+[2d^2(512x^{10}-1568x^8+100x^6+1819x^4-859x^2+132)x^2+2(256x^{14}-2096x^{12}+6076x^{10}-5286x^8+85x^6$$
$$+1089x^4+46x^2-44)+d(1024x^{14}-4912x^{12}+7724x^{10}-2156x^8-7311x^6+5283x^4-316x^2+88)]u^3$$
$$+[2d^2(384x^8-1336x^6+1101x^4-57x^2-40)x^2+4(96x^{12}-610x^{10}+1315x^8-803x^6-99x^4+114x^2+2)$$
$$+d(512x^{12}-3236x^{10}+8334x^8-10\,329x^6+5379x^4-880x^2+24)]u^2+[2d^2(96x^6-324x^4+275x^2-36)x^2$$
$$-d(96x^{10}-436x^8+652x^6-351x^4+52x^2+24)+2(112x^{10}-612x^8+1086x^6-589x^4-6x^2+12)]u$$
$$+(x^2-2)[8(d+5)x^6+2(8d^2-13d-62)x^4+(-18d^2+21d+80)x^2+4]\},$$

$$B_{02}=-2(4x^6+7x^4-17x^2+6)u^{13}+(-92x^8-49x^6+205x^4+24x^2-52)u^{12}-2(168x^{10}-252x^8+679x^6$$
$$-1355x^4+670x^2-60)u^{11}+(-320x^{12}+3000x^{10}-12\,020x^8+17\,103x^6-6662x^4-288x^2+264)u^{10}$$
$$+2(128x^{14}+2512x^{12}-12\,184x^{10}+13\,212x^8+2500x^6-7138x^4+2239x^2-210)u^9+(1920x^{14}-4864x^{12}$$
$$-25\,536x^{10}+67\,528x^8-44\,038x^6+6467x^4+1352x^2-540)u^8-4(640x^{16}-4928x^{14}+15\,688x^{12}-17\,196x^{10}$$
$$+418x^8+8255x^6-4311x^4+1410x^2-180)u^7+2(512x^{16}+1536x^{14}-13\,984x^{12}+22\,920x^{10}-5912x^8$$
$$-7281x^6+2558x^4-1280x^2+280)u^6+2(768x^{16}-4096x^{14}+7408x^{12}-5848x^{10}+10\,748x^8-10\,220x^6$$
$$-1059x^4+1471x^2-330)u^5-(2944x^{14}-6656x^{12}+3392x^{10}-10\,796x^8+10\,765x^6+3981x^4-2088x^2$$
$$+300)u^4-2(768x^{14}-1488x^{12}+240x^{10}-440x^8-605x^6+1929x^4+190x^2-156)u^3+(-1152x^{12}+3640x^{10}$$
$$-4220x^8+3187x^6-1174x^4-608x^2+72)u^2-2(80x^{10}-248x^8+268x^6-156x^4+47x^2+30)u+8x^8$$
$$-28x^6+29x^4-8x^2-2d^2(u+1)^2(x^2-1)[2(x^2-1)^2u^{11}+(23x^6-66x^4+45x^2-2)u^{10}+(84x^8-349x^6$$
$$+309x^4-6x^2-10)u^9+(80x^{10}-628x^8+718x^6+111x^4-95x^2+10)u^8+(-64x^{12}-48x^{10}+72x^8+721x^6$$
$$-186x^4-16x^2+20)u^7+2(160x^{12}-480x^{10}+784x^8-225x^6+29x^4+5x^2-10)u^6+(-256x^{12}+752x^{10}$$
$$-572x^8+373x^6-380x^4+124x^2-20)u^5+(-48x^{10}-340x^8+278x^6-492x^4+90x^2+20)u^4$$
$$+(224x^{10}-512x^8-425x^6+360x^4-156x^2+10)u^3+(296x^8-865x^6+408x^4-55x^2-10)u^2$$
$$+(32x^8-32x^6-105x^4+58x^2-2)u+8x^6-19x^4+5x^2+2]-2d[6(4x^8-13x^6+14x^4-6x^2+1)u^{13}$$

$$
\begin{aligned}
&+(276x^{10}-1077x^8+1448x^6-822x^4+189x^2-2)u^{12}+2(504x^{12}-2379x^{10}+3782x^8-2486x^6+608x^4\\
&+23x^2-2)u^{11}+(960x^{14}-6216x^{12}+12\,232x^{10}-9979x^8+3859x^6-653x^4+144x^2+12)u^{10}-2(384x^{16}\\
&-1392x^{14}+4732x^{12}-13\,211x^{10}+19\,709x^8-15\,737x^6+5958x^4-831x^2+35)u^9+(1920x^{16}-12\,928x^{14}\\
&+36\,424x^{12}-43\,266x^{10}+12\,444x^8+14\,203x^6-7449x^4-555x^2-30)u^8+2(960x^{16}-5424x^{14}+19\,196x^{12}\\
&-35\,027x^{10}+27\,991x^8-8894x^6+3596x^4-2362x^2+100)u^7-2(960x^{16}-4048x^{14}+3788x^{12}+6533x^{10}\\
&-9068x^8+1887x^6-355x^4+540x^2-20)u^6-2(576x^{16}-3392x^{14}+10\,932x^{12}-14\,129x^{10}+10\,891x^8\\
&-6265x^6+3798x^4-2140x^2+115)u^5+(3872x^{14}-18\,728x^{12}+24\,086x^{10}-7679x^8-7550x^6+3120x^4\\
&+2295x^2-30)u^4+2(640x^{14}-1732x^{12}-1438x^{10}+7839x^8-10\,052x^6+5060x^4-525x^2+62)u^3\\
&+(704x^{12}-3302x^{10}+6571x^8-8133x^6+5047x^4-984x^2+12)u^2-2(16x^{10}-192x^8+531x^6-450x^4\\
&+89x^2+13)u+16x^8-53x^6+47x^4-9x^2-2]-4,
\end{aligned}
$$

$$
\begin{aligned}
B_{03}={}&2(x^2-1)\{d^2(u-1)[2(x^2-1)^2u^{10}+x^2(23x^4-52x^2+29)u^9+(84x^8-188x^6+59x^4+59x^2-10)u^8\\
&+2x^2(40x^8-20x^6-209x^4+227x^2-22)u^7+(-64x^{12}+512x^{10}-1096x^8+345x^6+524x^4-140x^2+20)u^6\\
&-x^2(128x^{10}-704x^8+1880x^6-1515x^4+2x^2+42)u^5+2(168x^{10}-738x^8+819x^6-213x^4+53x^2-10)u^4\\
&+2x^2(80x^8-304x^6+416x^4-199x^2+50)u^3+(-72x^8+277x^6-182x^4-8x^2+10)u^2-x^2(32x^6-80x^4\\
&+2x^2+43)u-8x^6+23x^4-13x^2-2](u+1)^2+2d[2(x^2-1)^2(2x^2-1)u^{13}+x^2(46x^6-161x^4+178x^2\\
&-63)u^{12}+(168x^{10}-851x^8+1202x^6-504x^4-17x^2+4)u^{11}+x^2(160x^{10}-1588x^8+3106x^6-1724x^4\\
&+102x^2-31)u^{10}+(-128x^{14}-272x^{12}+12x^{10}+4391x^8-6335x^6+2825x^4-399x^2+10)u^9+x^2(832x^{12}\\
&-4800x^{10}+11\,884x^8-10\,219x^6+902x^4+1265x^2+288)u^8+(448x^{14}-2192x^{12}+8428x^{10}-12\,119x^8\\
&+5180x^6-1020x^4+1174x^2-40)u^7+x^2(-832x^{12}+3408x^{10}-5148x^8+2525x^6-2498x^4+1720x^2-46)u^6\\
&+(-320x^{14}+2208x^{12}-7708x^{10}+7901x^8-4098x^6+1572x^4-1078x^2+50)u^5+x^2(1232x^{10}-4508x^8\\
&+1961x^6+2681x^4-2392x^2-335)u^4+(256x^{12}-132x^{10}-2370x^8+3914x^6-2660x^4+267x^2-28)u^3\\
&+x^2(128x^8-491x^6+798x^4-870x^2+189)u^2+(-24x^8+133x^6-203x^4+45x^2+6)u+x^2(2x^4-3x^2-2)]\\
&-2[2(x^2-1)^2u^{13}+(23x^6-28x^4-7x^2+12)u^{12}+(84x^8+116x^6-473x^4+269x^2-20)u^{11}+(80x^{10}\\
&+1290x^8-2959x^6+1250x^4+165x^2-64)u^{10}+(-64x^{12}+2648x^{10}-5024x^8-795x^6+3131x^4-861x^2\\
&+70)u^9+(224x^{12}+1744x^{10}-10\,572x^8+7879x^6-384x^4-632x^2+140)u^8+(-896x^{14}+3840x^{12}-7960x^{10}\\
&+56x^8+6645x^6-3538x^4+898x^2-120)u^7+(512x^{14}-256x^{12}-3904x^{10}+5184x^8+1041x^6-2372x^4\\
&+954x^2-160)u^6+(384x^{14}-1344x^{12}+1464x^{10}+3352x^8-537x^6-480x^4-158x^2+110)u^5+(-992x^{12}\\
&+1600x^{10}+2876x^8-1268x^6+1004x^4-615x^2+100)u^4+(-384x^{12}+8x^{10}+2052x^8-1333x^6+1403x^4\\
&-255x^2-52)u^3+(-288x^{10}+710x^8-630x^6+562x^4+113x^2-32)u^2+(-8x^8-45x^4+111x^2+10)u\\
&+10x^6-32x^4+22x^2+4]\},
\end{aligned}
$$

$$
\begin{aligned}
B_{04}={}&-4x^2(x^2-1)(-2u^2+ux^2-x^2+2)[u^2+u(4x^2-2)+1]^3\{d(u+1)^3(x^2-1)+2[u^3(3x^4-4x^2+1)\\
&+u^2(x^4-8x^2+3)+u(3-5x^2)-x^2+1]\},
\end{aligned}
$$

$$
\begin{aligned}
B_{10}={}&-(u+1)(x^2-1)(u^2+u(4x^2-2)+1)\{-d^2x^2[u^3(16x^2-1)+u^2(64x^4-5)+u(144x^4-52x^2+5)+52x^2\\
&+1]+d[u^4(2x^4-3x^2+1)+2u^3(4x^6-6x^4+x^2+1)+2u^2(8x^6-19x^4+12x^2-2)-2u(8x^4-7x^2+1)\\
&-16x^6+16x^4-5x^2+3]+4x^2[u^3(12x^2-1)+u^2(48x^4-x^2-3)+u(108x^4-38x^2+3)+39x^2+1]\},
\end{aligned}
$$

$$
\begin{aligned}
B_{11}={}&-(x^2-1)\{2d^2x^2[u^2+u(4x^2-2)+1]^2[u^2+u(4x^2+2)+4x^2+1]-d[2u^7x^2(x^2-1)+u^6(16x^6-6x^4-7x^2\\
&+6)+2u^5(16x^8+40x^6-53x^4+35x^2-6)+u^4(288x^8-176x^6+114x^4-3x^2-6)+2u^3(336x^8-312x^6\\
&+223x^4-55x^2+12)+u^2(288x^8+80x^6+78x^4-13x^2-6)-2u(32x^8-208x^6+91x^4-37x^2+6)+16x^6\\
&+38x^4-9x^2+6]+4x^2[u^6(8x^2-2)+u^5(64x^4-26x^2-1)+u^4(128x^6+16x^4-39x^2+10)+2u^3(176x^6\\
&-180x^4+92x^2-5)-2u^2(56x^6-152x^4+53x^2-1)+u(-64x^6+40x^4+10x^2+3)-16x^4+x^2-2]\},
\end{aligned}
$$

$$
\begin{aligned}
B_{12}={}&(x^2-1)\{d^2x^2[u^6(4x^2-1)+2u^5(16x^4-7x^2+4)+u^4(64x^6+30x^2-9)+4u^3(40x^6-20x^4+17x^2-2)\\
&+u^2(-32x^6+224x^4-80x^2+13)-2ux^2(64x^4-120x^2+27)-32x^4+46x^2-3]+d[u^7(2x^4-3x^2+1)
\end{aligned}
$$

$$+u^6(16x^6-22x^4+19x^2-5)+u^5(32x^8+56x^4-43x^2+5)+u^4(160x^8+64x^6-96x^4-13x^2+7)$$
$$+u^3(448x^8-240x^6-94x^4+47x^2-13)+u^2(384x^8-288x^6-54x^4+49x^2+1)+u(64x^8-144x^6+164x^4$$
$$-65x^2+7)-48x^6+44x^4+9x^2-3]+2x^2[4u^6(3x^2+1)+u^5(96x^4+54x^2-29)+u^4(192x^6+432x^4$$
$$-240x^2+43)+2u^3(560x^6-360x^4+154x^2-1)+2u^2(192x^6+144x^4-62x^2-11)+u(128x^6-32x^4$$
$$+38x^2-1)+32x^4-48x^2+7]\},$$

$$B_{13}=-x^2(x^2-1)\{2d^2[u^6(2x^2-1)+u^5(16x^4-12x^2+3)+2u^4(16x^6-8x^4+x^2+1)+2u^3(32x^6-40x^4$$
$$+28x^2-5)+u^2(-32x^6+80x^4-26x^2+3)+u(-64x^6+112x^4-44x^2+7)-16x^4+22x^2-4]$$
$$+d[2u^7(x^2-1)+u^6(16x^4-18x^2+5)+2u^5(16x^6-8x^4+7x^2-4)+u^4(96x^6+16x^4-42x^2-5)$$
$$+2u^3(144x^6-104x^4+7x^2+3)+u^2(224x^6-176x^4+10x^2+27)+2u(32x^6-64x^4+65x^2-14)-16x^4$$
$$+18x^2+5]-4[u^6(6x^2-4)+u^5(48x^4-54x^2+11)+u^4(96x^6-144x^4+47x^2-6)+2u^3(16x^6-52x^4$$
$$+28x^2-5)-8u^2(10x^6-8x^4+x^2-1)+u(-128x^6+184x^4-58x^2+7)-32x^4+43x^2-6]\},$$

$$B_{14}=2x^2(x^2-1)[u^2+u(4x^2-2)+1]^2[d^2(u+1)(x^2-1)+d(u+1)+u-6x^2+1],$$

$$B_{20}=u(u^2-1)(x^2-1)[d^3x^2+d^2(-2x^4+5x^2+4)-3dx^2+3(2x^4-5x^2-4)],$$

$$B_{21}=(u+1)(x^2-1)\{d^2(u^2+2u-3)x^2+d[u^2(-2x^4+3x^2+4)+2ux^2+2x^4-7x^2-4]$$
$$+2[u^2(2x^4-5x^2-4)+u(-2x^4+x^2+4)+2x^4-x^2-4]\},$$

$$B_{22}=(x^2-1)\{d^2(u^2-1)(2x^4-9x^2-4)+d[3u^3x^2+u^2(-2x^4+21x^2+4)+u(-4x^4+11x^2+8)-2x^4$$
$$+5x^2+4]+u^3(-6x^4+15x^2+12)+u^2(-12x^4+22x^2+24)+3u(2x^4-5x^2-4)+4x^4-18x^2-8\},$$

$$B_{23}=-2(x^2-1)\{d^2(u^2-1)(x^4-3x^2-2)+d[u^2(-2x^4+13x^2+4)+u(-2x^4+6x^2+4)-x^2]$$
$$+u^2(-6x^4+7x^2+12)+u(2x^4-5x^2-4)+4x^4-10x^2-8\},$$

$$B_{24}=0,$$

$$B_{30}=(u-1)(u+1)^2[u^2(x^2-1)-2u-1]\{d^3ux^2[u^3+7u^2+u(4x^2+11)-4x^2+5]+d^2[-u^5(x^2-1)$$
$$+u^4(-6x^4+x^2+5)-u^3(4x^6+2x^4+3x^2-10)+u^2(4x^6-4x^4-11x^2+10)+u(2x^4-4x^2+5)-6x^4$$
$$+2x^2+1]+d[2u^5(x^2-1)^2(2x^2-1)+u^4(4x^6-18x^4+19x^2-8)+u^3(-2x^6-8x^4+x^2-12)$$
$$+u^2(2x^6-16x^4-23x^2-8)+u(2x^6+10x^4-13x^2-2)-2x^4(x^2-1)]+3u^5(x^2-1)+u^4(16x^4-x^2-15)$$
$$+u^3(12x^6+4x^4+11x^2-30)-3u^2(4x^6-4x^4-11x^2+10)+u(-8x^4+14x^2-15)+16x^4-4x^2-3\},$$

$$B_{31}=(u+1)\{d^2(u-1)x^2[u^7(6x^4-11x^2+5)+u^6(8x^6-6x^4-25x^2+21)-u^5(8x^6+8x^4+12x^2-29)$$
$$+u^4(12x^4+4x^2+5)+u^3(10x^4-3x^2-25)+u^2(2x^4-9x^2-25)-3u(2x^2+3)-2x^2-1]$$
$$+d[-u^9(x^2-1)^2(12x^4-15x^2+4)+u^8(x^2-1)^2(8x^4+47x^2-26)+2u^7(8x^8-18x^6-49x^4+98x^2-38)$$
$$-2u^6(22x^8-26x^6+64x^4-141x^2+66)+u^5(-12x^8+189x^6-294x^4+354x^2-148)+u^4(24x^8+39x^6$$
$$-298x^4+344x^2-104)-2u^3(4x^8+9x^6+53x^4-114x^2+18)+2u^2(-2x^8+x^6-16x^4+51x^2+2)$$
$$+u(18x^6-32x^4+31x^2+8)+4x^6-6x^4+5x^2+2]+2[-2u^9(x^2-1)^3+u^8(-4x^6+20x^4-27x^2+11)$$
$$+u^7(-4x^8+40x^4-62x^2+26)+u^6(14x^8-4x^6+35x^4-86x^2+38)-2u^5(5x^8+13x^6-8x^4+30x^2-23)$$
$$+2u^4(x^8+23x^6-15x^4-5x^2+26)+2u^3(-x^8+x^6-40x^4+x^2+23)+u^2(2x^6-33x^4-6x^2+26)$$
$$+2u(x^6+5x^4-x^2+4)+x^2+1]\},$$

$$B_{32}=d^2(u^2-1)[u^8(x^2-1)^3-2u^7(x^2-1)^2(2x^4+3)-2u^6(7x^8-22x^6+15x^4-7x^2+7)+2u^5(3x^8+33x^6$$
$$-42x^4-x^2-7)-u^4x^2(2x^6-25x^4+127x^2+10)-2u^3(x^8+2x^6+43x^4-4x^2-7)+2u^2(x^6-6x^4+11x^2$$
$$+7)+2u(x^6+6x^4+7x^2+3)+4x^4+3x^2+1]+d(u+1)[3u^9(x^2-1)^3(4x^2-1)+u^8(-32x^8+33x^6+48x^4$$
$$-64x^2+15)+u^7(6x^8+54x^6+3x^4-103x^2+36)+u^6(72x^8-204x^6+262x^4-251x^2+68)+u^5(16x^8$$
$$-327x^6+651x^4-533x^2+114)+u^4(-24x^8-99x^6+568x^4-623x^2+138)+u^3(14x^8-8x^6+229x^4-397x^2$$
$$+100)+u^2(-2x^6+86x^4-145x^2+36)+u(-16x^6+32x^4-34x^2+3)-4x^4-5x^2-1]+u^{10}(x^2-1)^2(2x^2$$
$$-3)+4u^9(x^2-1)^2(6x^4-6x^2-5)+u^8(28x^8-184x^6+227x^4-6x^2-65)-2u^7(97x^6-186x^4+13x^2+72)$$

$$+ u^6(-80x^8 + 90x^6 + 275x^4 - 26x^2 - 238) + 2u^5(24x^8 + 90x^6 - 5x^4 - 29x^2 - 140) - u^4(12x^8 + 4x^6 + 119x^4 + 54x^2 + 210) - 2u^3(4x^8 - 7x^6 + 36x^4 - 17x^2 + 40) + u^2(32x^6 - 84x^4 + 74x^2 + 1) + 2u(4x^6 - 27x^4 + 17x^2 + 6) - 4x^4 + 4x^2 + 3,$$

$$B_{33} = -d^2(u-1)(u+1)^2[u^7(x^2-1)^3 - u^6(x^2-1)^2(4x^4 - 3x^2 + 5) + u^5(-8x^8 + 37x^6 - 46x^4 + 26x^2 - 9) + u^4(4x^8 + 33x^6 - 74x^4 + 32x^2 - 5) + u^3(-2x^6 - 53x^4 + 23x^2 + 5) + u^2(-8x^6 - 7x^4 + 5x^2 + 9) + u(10x^4 - 4x^2 + 5) + 4x^4 - 2x^2 + 1] - 2d(u+1)[u^9(x^2-1)^3(2x^2-1) - u^8(x^2-1)^2(10x^4 + x^2 - 4) + 4u^7(2x^8 + 2x^6 - 6x^4 + x^2 + 1) + u^6(10x^8 - 62x^6 + 59x^4 - 6x^2 - 4) + 2u^5(3x^8 - 30x^6 + 77x^4 - 34x^2 - 5) + u^4(2x^8 - 33x^6 + 118x^4 - 98x^2 - 4) + u^3(-18x^6 + 50x^4 - 56x^2 + 4) + u^2(-2x^8 + 2x^6 + 21x^4 - 14x^2 + 4) + u(5x^6 + 3x^4 - 3x^2 + 1) + x^2(2x^4 - 2x^2 - 1)] + 2[u^{10}(x^2-1)^3 - u^9(x^2-1)^2(12x^4 - 14x^2 + 7) + u^8(-16x^8 + 101x^6 - 164x^4 + 97x^2 - 18) + u^7(-34x^8 + 152x^6 - 279x^4 + 175x^2 - 16) + u^6(84x^8 - 15x^6 - 218x^4 + 183x^2 + 14) + u^5(-20x^8 - 208x^6 + 57x^4 + 103x^2 + 42) + u^4(-4x^8 - 73x^6 + 242x^4 + 3x^2 + 28) + u^3(2x^8 + 36x^6 + 151x^4 - 43x^2 - 8) + u^2(2x^6 + 17x^4 - 30x^2 - 21) - u(2x^6 + 10x^4 + 7x^2 + 11) - 2(x^4 + 1)],$$

$$B_{34} = -2(u-1)^2x^2\{d^2(u+1)^3[u^3(x^2-1)^3 + u^2(-2x^4 + 5x^2 - 3) - u(x^4 - 2x^2 + 3) - 1] + d(u+1)^2(x^2-1)[u^4(x^2-1) + u^3(3x^4 - 2x^2 - 4) + u^2(x^4 - 8x^2 - 6) - 2u(3x^2 + 2) - x^2 - 1] + u^6(x^2-1)^2 + u^5(-24x^6 + 41x^4 - 23x^2 + 6) + u^4(-12x^6 + 91x^4 - 72x^2 + 15) + u^3(-4x^6 + 67x^4 - 98x^2 + 20) + u^2(20x^4 - 62x^2 + 15) + u(4x^4 - 15x^2 + 6) + 1\},$$

$$B_{40} = (1-u)(u+1)^2x(2u^3 + u^2(7x^2 - 2) - 2u(2x^4 - 5x^2 + 1) - x^2 + 2)\{d^3[u^2 + 4u(x^2+1) + 3][u^2 + u(4x^2 - 2) + 1]^2 - d^2[2u^7 + u^6(2x^2 + 9) + u^5(94x^2 - 72x^4) + u^4(-160x^6 + 152x^4 + 72x^2 - 49) + 2u^3(64x^8 - 160x^6 + 140x^4 - 70x^2 + 21) + u^2(160x^6 - 248x^4 + 74x^2 - 9) + 2u(40x^4 - 57x^2 + 10) + 3(4x^2 - 5)] + d[u^6(1 - 4x^2) + 12u^5(4x^4 - 10x^2 + 3) + u^4(-144x^6 + 176x^4 - 168x^2 + 13) + 16u^3(8x^8 - 40x^6 + 31x^4 - 7x^2 - 4) + u^2(-64x^8 + 304x^6 - 720x^4 + 396x^2 - 69) - 4u(8x^6 - 32x^4 + 62x^2 - 23) - 9] + 6u^7 + 3u^6(2x^2 + 9) - 2u^5(108x^4 - 148x^2 + 7) + u^4(-480x^6 + 496x^4 + 150x^2 - 121) + 2u^3(192x^8 - 512x^6 + 472x^4 - 232x^2 + 65) + u^2(480x^6 - 816x^4 + 322x^2 - 55) + u(232x^4 - 344x^2 + 70) + 34x^2 - 43\},$$

$$B_{41} = -x\{d^2(u-1)(u+1)^2[14u^5 + (58 - 55x^2)u^4 + 2(92x^4 - 123x^2 + 26)u^3 - 4(20x^6 - 62x^4 + 21x^2 + 17)u^2 - 2(24x^4 - 67x^2 + 33)u - 5(x^2 - 2)](u^2 + (4x^2 - 2)u + 1)^2 + 2[10u^{13} + (2x^2 + 61)u^{12} + (-638x^4 + 883x^2 - 83)u^{11} + (-2336x^6 + 2468x^4 + 269x^2 - 195)u^{10} + (-448x^8 - 5328x^6 + 9644x^4 - 4291x^2 + 525)u^9 + (4096x^{10} - 16\,640x^8 + 22\,248x^6 - 12\,626x^4 + 3537x^2 - 672)u^8 - 2(768x^{12} - 6400x^{10} + 16\,448x^8 - 19\,852x^6 + 11\,678x^4 - 2787x^2 + 199)u^7 - 2(512x^{12} - 1792x^{10} - 1216x^8 + 8748x^6 - 10\,580x^4 + 5403x^2 - 1045)u^6 + 2(256x^{12} - 3520x^{10} + 12\,352x^8 - 17\,092x^6 + 9036x^4 - 671x^2 - 382)u^5 + (512x^{12} - 2176x^{10} + 640x^8 + 9080x^6 - 16\,500x^4 + 10\,076x^2 - 1715)u^4 + (-512x^{12} + 3456x^{10} - 8704x^8 + 9736x^6 - 3638x^4 - 1417x^2 + 977)u^3 + (-384x^{10} + 2304x^8 - 5352x^6 + 6020x^4 - 3111x^2 + 457)u^2 + (-64x^8 + 312x^6 - 596x^4 + 593x^2 - 267)u - 10x^4 + 33x^2 - 26] - d(u+1)[4u^{12} + (92 - 54x^2)u^{11} + (822x^4 - 915x^2 + 350)u^{10} + (-2608x^6 + 8242x^4 - 4748x^2 + 132)u^9 + (-2160x^8 + 1736x^6 + 7016x^4 - 3319x^2 - 1018)u^8 + 4(2352x^{10} - 9696x^8 + 15\,802x^6 - 11\,806x^4 + 4809x^2 - 758)u^7 - 2(1792x^{12} - 8480x^{10} + 14\,728x^8 - 12\,884x^6 + 4166x^4 + 1655x^2 - 1902)u^6 + 4(576x^{12} - 4480x^{10} + 12\,248x^8 - 15\,478x^6 + 10\,463x^4 - 4066x^2 + 854)u^5 + 2(256x^{12} - 768x^{10} - 1864x^8 + 8876x^6 - 10\,996x^4 + 6441x^2 - 1936)u^4 + (-256x^{12} + 2368x^{10} - 7744x^8 + 9912x^6 - 3928x^4 + 610x^2 - 772)u^3 + (-64x^{10} + 1040x^8 - 4296x^6 + 6598x^4 - 3743x^2 + 678)u^2 + 2(48x^8 - 204x^6 + 273x^4 - 158x^2 + 82)u + 16x^4 - 59x^2 + 54]\},$$

$$B_{42} = x\{24u^{13} + 2(11x^2 + 70)u^{12} + (-1414x^4 + 1883x^2 - 134)u^{11} + (-5504x^6 + 4430x^4 + 1471x^2 - 662)u^{10} + (-1472x^8 - 9536x^6 + 11\,138x^4 - 2787x^2 + 22)u^9 + (9728x^{10} - 19\,072x^8 - 1040x^6 + 11\,030x^4 - 8353x^2 + 1914)u^8 - 2(1792x^{12} - 10\,240x^{10} + 6976x^8 + 7016x^6 - 2722x^4 + 425x^2 + 110)u^7 + (-2560x^{12} + 7424x^{10}$$

$$
\begin{aligned}
&+960x^8+9456x^6-32\,724x^4+15\,750x^2-2524)u^6+(-1536x^{12}+512x^{10}+2752x^8+8496x^6-11\,228x^4\\
&-710x^2+868)u^5+4(640x^{12}-4224x^{10}+8448x^8-8044x^6+5451x^4-2403x^2+312)u^4+(1024x^{12}\\
&-3584x^{10}+256x^8+7440x^6-6782x^4+2887x^2-638)u^3+(768x^{10}-3392x^8+4688x^6-2490x^4+715x^2\\
&-110)u^2+(128x^8-560x^6+794x^4-423x^2+78)u-2x^4+7x^2+2d^2[4u^{12}-4(5x^2-6)u^{11}\\
&+(4x^2-87x^4)u^{10}+4(76x^6-204x^4+69x^2-26)u^9+(256x^8+544x^6-1717x^4+468x^2-108)u^8-8(24x^{10}\\
&-164x^8+144x^6-28x^4+97x^2-30)u^7+(-960x^{10}+3456x^8-4512x^6+3562x^4-1624x^2+288)u^6\\
&+8(32x^{12}-176x^{10}+212x^8-60x^6-32x^4+133x^2-42)u^5+2(576x^{10}-2432x^8+3184x^6-1901x^4\\
&+916x^2-138)u^4-4(64x^{12}-400x^{10}+744x^8-288x^6-296x^4+201x^2-62)u^3+(-192x^{10}+1152x^8\\
&-2400x^6+2045x^4-684x^2+96)u^2-4(8x^8-44x^6+84x^4-65x^2+18)u-(x^2-2)^2]+d[8(20x^2-17)u^{12}\\
&+(-2212x^4+3399x^2-1150)u^{11}+(832x^6-10\,972x^4+10\,809x^2-1250)u^{10}+(592x^8-6528x^6-5116x^4\\
&+4461x^2+3646)u^9+(-6208x^{10}+21\,840x^8-52\,800x^6+46\,476x^4-22\,581x^2+6954)u^8+2(1280x^{12}\\
&-4928x^{10}+8088x^8-21\,728x^6+21\,644x^4-5917x^2-2038)u^7-2(640x^{12}-8608x^{10}+20\,952x^8-15\,168x^6\\
&+7676x^4-9117x^2+5610)u^6-2(1408x^{12}-5888x^{10}+5016x^8+2944x^6-3716x^4+755x^2-542)u^5\\
&+2(640x^{12}-4960x^{10}+13\,112x^8-14\,752x^6+9180x^4-6117x^2+3622)u^4+(256x^{12}-896x^{10}-368x^8\\
&+5440x^6-7988x^4+3171x^2+1226)u^3-(64x^{10}+16x^8-1984x^6+5740x^4-5597x^2+1578)u^2\\
&+(-224x^8+1280x^6-2636x^4+2313x^2-730)u-4x^4+15x^2-14]-6\},
\end{aligned}
$$

$$
\begin{aligned}
B_{43}={}&-2x\{(8-70x^2)u^{11}+(-640x^4+207x^2-34)u^{10}+2(600x^6-2795x^4+1693x^2-258)u^9+(9440x^8\\
&-24\,472x^6+17\,610x^4-6699x^2+1358)u^8-4(1120x^{10}-7200x^8+12\,302x^6-7028x^4+1641x^2-88)u^7\\
&+2(64x^{10}-2288x^8+12\,100x^6-17\,920x^4+8663x^2-1522)u^6-12(128x^{12}-800x^{10}+2304x^8-3366x^6\\
&+1937x^4-81x^2-94)u^5+2(256x^{12}-2752x^{10}+8400x^8-13\,348x^6+13\,586x^4-7055x^2+1078)u^4\\
&+2(512x^{12}-2816x^{10}+4992x^8-3028x^6-512x^4+1637x^2-644)u^3+(768x^{10}-4256x^8+8536x^6-7792x^4\\
&+3283x^2-442)u^2+2(64x^8-332x^6+617x^4-499x^2+158)u-d^2[2u^{11}+(47x^2-30)u^{10}+2(36x^4\\
&+4x^2-9)u^9+(-208x^6+396x^4-159x^2+94)u^8+4(116x^8-538x^6+604x^4-194x^2+45)u^7-2(96x^{10}\\
&-592x^8+1404x^6-896x^4-135x^2+102)u^6+4(144x^{10}-684x^8+1334x^6-1336x^4+634x^2-113)u^5\\
&-2(128x^{12}-896x^{10}+2256x^8-2772x^6+1236x^4+291x^2-174)u^4+(-768x^{10}+3440x^8-5752x^6+5328x^4\\
&-2776x^2+426)u^3+(256x^{12}-1600x^{10}+3296x^8-2344x^6-96x^4+755x^2-310)u^2+2(96x^{10}-584x^8\\
&+1284x^6-1236x^4+504x^2-69)u+32x^8-184x^6+380x^4-331x^2+102]u+2x^4-7x^2+d[16(3x^2-2)u^{12}\\
&-2(330x^4-572x^2+185)u^{11}-(504x^6+1415x^4-2548x^2+324)u^{10}-2(12x^8+2102x^6-2325x^4+776x^2\\
&-721)u^9+(96x^{10}-2888x^8-2600x^6+7813x^4-4012x^2+1444)u^8+4(976x^{10}-4058x^8+3514x^6-650x^4\\
&+506x^2-457)u^7-2(448x^{12}-4080x^{10}+8964x^8-5028x^6-269x^4-840x^2+1100)u^6-4(160x^{12}-448x^{10}\\
&-794x^8+2972x^6-2545x^4+826x^2-165)u^5+2(192x^{12}-1584x^{10}+4148x^8-3996x^6+1581x^4-944x^2\\
&+764)u^4+2(64x^{12}-288x^{10}+164x^8+876x^6-1386x^4+576x^2+123)u^3+(32x^{10}-280x^8+1040x^6\\
&-1907x^4+1628x^2-420)u^2-2(24x^8-142x^6+303x^4-268x^2+75)u+(x^2-2)^2]+6\},
\end{aligned}
$$

$$B_{44}=0,$$

$$
\begin{aligned}
B_{50}={}&-(u-1)(u+1)^2(u^2+u(4x^2-2)+1)\{d^3(u^2+4ux^2-1)[u^2+u(4x^2-2)+1]^2+d^2[u^6(6x^2-3)\\
&+u^5(40x^4-26x^2+6)+u^4(32x^6-8x^4+28x^2+3)-4u^3(32x^8-48x^6-6x^4-x^2+3)+u^2(-32x^6\\
&+104x^4-10x^2+3)+2u(16x^4-5x^2+3)+8x^2-3]-d[3u^6-8u^5(2x^4-7x^2+2)+u^4(48x^6+32x^4\\
&-36x^2+19)-8u^3(16x^8-56x^6+40x^4-8x^2-1)+u^2(-64x^8+48x^6+128x^4-56x^2-23)-8u(4x^6\\
&-6x^4+3x^2-1)-4x^2+1]+u^6(9-18x^2)-20u^5(6x^4-4x^2+1)-u^4(96x^6-16x^4+82x^2+3)\\
&+16u^3(24x^8-40x^6+x^4-2x^2+2)+u^2(96x^6-336x^4+58x^2-13)-4u(22x^4-4x^2+3)-22x^2+7\},
\end{aligned}
$$

$$
\begin{aligned}
B_{51}={}&-d^2(u-1)(u+1)^2[u^4+u^3(2-8x^2)+u^2(48x^4-40x^2-4)+2u(8x^2-1)+3][u^2+u(4x^2-2)+1]^2\\
&-d(u+1)[u^{10}(6x^2-9)+u^9(-96x^4+22x^2+34)+u^8(432x^6-1432x^4+964x^2-117)-8u^7(88x^8-32x^6
\end{aligned}
$$

$$
\begin{aligned}
&-173x^4+183x^2-36)-2u^6(1792x^{10}-3680x^8+2456x^6-708x^4+54x^2+149)-4u^5(576x^{10}-1344x^8\\
&+1240x^6-610x^4+135x^2-5)+2u^4(1280x^{10}-4224x^8+5416x^6-3620x^4+1080x^2+43)+8u^3(32x^{10}\\
&+184x^8-616x^6+519x^4-159x^2+14)+u^2(64x^8+304x^6-936x^4+566x^2-173)+u(-96x^6+312x^4\\
&-330x^2+58)-4x^2-1]+2[9u^{11}(2x^2-1)+u^{10}(160x^4-124x^2+37)+u^9(320x^6-240x^4+256x^2-55)\\
&+u^8(-512x^8+1536x^6-424x^4+50x^2+23)-2u^7(768x^{10}-2048x^8+1408x^6-916x^4+270x^2-23)\\
&+u^6(-1024x^{10}+3584x^8-2048x^6+728x^4+8x^2-86)+u^5(512x^{10}-2176x^8+5376x^6-3832x^4+864x^2\\
&+26)+u^4(-512x^{10}+640x^8-384x^6+968x^4-556x^2+70)+u^3(512x^{10}-1920x^8+1152x^6+312x^4-70x^2\\
&-53)+u^2(384x^8-1152x^6+616x^4+100x^2-31)+u(64x^6-120x^4-16x^2+45)+10x^2-13],
\end{aligned}
$$

$$
\begin{aligned}
B_{52}={}&(21-42x^2)u^{11}+(-384x^4+290x^2-83)u^{10}+(-832x^6+768x^4-714x^2+123)u^9+(1024x^8-1024x^6\\
&-2512x^4+1282x^2-157)u^8+(3584x^{10}-2560x^8-6400x^6+5552x^4-2340x^2+338)u^7+(2560x^{10}-5376x^8\\
&+5184x^6-6224x^4+2804x^2-446)u^6-2(2304x^{10}-5632x^8+5344x^6-2136x^4+506x^2-43)u^5+2(768x^{10}\\
&-2816x^8+2752x^6-248x^4-702x^2+211)u^4+(1024x^{10}-1536x^8+1408x^6-2160x^4+1870x^2-503)u^3\\
&+(768x^8-1472x^6+1424x^4-918x^2+257)u^2+(128x^6-240x^4+190x^2-65)u-6x^2+2d^2(u^2-1)[-7x^2u^8\\
&-8(2x^4+4x^2-1)u^7+4(16x^6-64x^4+33x^2-8)u^6+8(8x^8-28x^6+18x^4-16x^2+5)u^5+2x^2(32x^6\\
&-128x^4+96x^2-25)u^4+8(32x^{10}-104x^8+120x^6-70x^4+28x^2-5)u^3+4(48x^8-144x^6+144x^4\\
&-51x^2+8)u^2+8(4x^6-10x^4+8x^2-1)u+x^2]-d(u+1)[7u^{10}+2(184x^4-124x^2-7)u^9+(-272x^6\\
&+2224x^4-1648x^2+187)u^8+16(36x^8+192x^6-379x^4+267x^2-50)u^7+2(1280x^{10}-96x^8-3512x^6\\
&+4312x^4-2056x^2+643)u^6+4(960x^{10}-3104x^8+4728x^6-3652x^4+1416x^2-219)u^5-2(1024x^{10}\\
&-3904x^8+6904x^6-6664x^4+2824x^2-147)u^4-16(16x^{10}+60x^8-212x^6+225x^4-97x^2+17)u^3\\
&+(64x^8-400x^6+400x^4-368x^2+275)u^2+(224x^6-672x^4+536x^2-86)u-1]+7,
\end{aligned}
$$

$$
\begin{aligned}
B_{53}={}&2\{u^{10}-2(40x^4-39x^2+6)u^9+(-1120x^6+1656x^4-730x^2+75)u^8-8(400x^8-752x^6+589x^4-220x^2\\
&+27)u^7-2(576x^8+400x^6-1260x^4+680x^2-137)u^6+4(384x^{10}-736x^8-64x^6+438x^4-101x^2-12)u^5\\
&-2(256x^{10}-1216x^8+1040x^6+396x^4-598x^2+129)u^4-8(128x^{10}-320x^8+192x^6-89x^4+102x^2\\
&-35)u^3+(-768x^8+1952x^6-1336x^4+384x^2-99)u^2-2(64x^6-140x^4+53x^2+2)u-2x^2+d^2[u^{10}\\
&-4(8x^4-13x^2+3)u^9+(-112x^6+232x^4-128x^2+35)u^8+8(8x^8-12x^6+11x^4+2x^2-3)u^7+(64x^8\\
&+272x^6-472x^4+224x^2-46)u^6-8(32x^{10}-96x^8+84x^6-43x^4+33x^2-10)u^5-2(128x^8-216x^6\\
&+132x^4-32x^2+9)u^4+8(32x^{10}-104x^8+92x^6-39x^4+18x^2-5)u^3+(192x^8-592x^6+504x^4-160x^2\\
&+29)u^2+4(8x^6-22x^4+13x^2-1)u-1]+d[(112x^4-105x^2+8)u^{10}+(-8x^6+556x^4-470x^2+40)u^9\\
&+(-96x^8+1576x^6-1928x^4+1051x^2-216)u^8+8(264x^8-407x^6+273x^4-76x^2+31)u^7+2(448x^{10}\\
&-624x^8+756x^6-484x^4+211x^2+28)u^6+4(160x^{10}-576x^8+910x^6-520x^4+169x^2-62)u^5-2(192x^{10}\\
&-688x^8+1380x^6-1588x^4+765x^2-60)u^4-8(16x^{10}-24x^8-11x^6+65x^4-38x^2+3)u^3+(-32x^8\\
&+184x^6-392x^4+163x^2+32)u^2+2(24x^6-70x^4+49x^2-8)u-x^2]+7\},
\end{aligned}
$$

$$B_{54}=0,$$

$$
\begin{aligned}
B_{60}={}&(u+1)(u^2+(4x^2-2)u+1)\{6(x^2-1)^2u^8+(88x^6-218x^4+157x^2-36)u^7+(480x^8-1392x^6+1310x^4\\
&-545x^2+84)u^6+(1152x^{10}-3648x^8+4216x^6-2622x^4+797x^2-84)u^5+x^2(1024x^{10}-2816x^8+4128x^6\\
&-4704x^4+2518x^2-465)u^4+(1536x^{12}-1152x^{10}-2656x^8+2744x^6-838x^4-33x^2+84)u^3+(3584x^{10}\\
&-7232x^8+4000x^6-590x^4+133x^2-84)u^2+(512x^{10}-288x^8-648x^6+350x^4-25x^2+36)u+128x^8\\
&-208x^6+84x^4-7x^2-2d^3x^2[u^2+(4x^2-2)u+1]^2[-8x^4+u^3x^2+9x^2+u^2(4x^4-5x^2+4)+u(-12x^4\\
&+19x^2-4)]+d^2[-2(x^2-1)^2u^8+(-30x^6+74x^4-53x^2+12)u^7+(-168x^8+482x^6-450x^4+185x^2\\
&-28)u^6+(-416x^{10}+1288x^8-1470x^6+906x^4-273x^2+28)u^5+x^2(-384x^{10}+992x^8-1408x^6+1626x^4\\
&-886x^2+165)u^4+(-640x^{12}+608x^{10}+784x^8-922x^6+302x^4+x^2-28)u^3+(-1376x^{10}+2776x^8
\end{aligned}
$$

$$
\begin{aligned}
&- 1546x^6 + 218x^4 - 37x^2 + 28)u^2 + (-192x^{10} + 104x^8 + 246x^6 - 130x^4 + 5x^2 - 12)u - 48x^8 + 78x^6 - 32x^4 \\
&+ 3x^2 + 2] + 2d[16x^{10} - 50x^8 + 35x^6 - 6x^4 + 11x^2 + u^8(x^2 - 1)^3(2x^2 - 1) + u^7(24x^{10} - 96x^8 + 148x^6 \\
&- 105x^4 + 36x^2 - 4) + u(104x^{10} - 436x^8 + 406x^6 + 28x^4 - 93x^2 + 12) + u^4(64x^{12} - 192x^{10} + 200x^8 + 92x^6 \\
&- 81x^4 + 12x^2 + 10) + u^2(96x^{12} - 352x^{10} - 144x^8 + 1026x^6 - 740x^4 + 193x^2 - 16) + u^6(96x^{12} - 432x^{10} \\
&+ 776x^8 - 666x^6 + 306x^4 - 67x^2 + 8) + u^3(128x^{14} - 256x^{12} - 648x^{10} + 1992x^8 - 1760x^6 + 787x^4 - 142x^2 \\
&+ 4) + u^5(128x^{14} - 640x^{12} + 1320x^{10} - 1316x^8 + 726x^6 - 198x^4 + 55x^2 - 12) - 3] - 6\},
\end{aligned}
$$

$$
\begin{aligned}
B_{61} = {}& 2d^2x^2[(4x^4 - 9x^2 + 4)u^6 + (32x^6 - 80x^4 + 50x^2 - 8)u^5 + (64x^8 - 144x^6 + 68x^4 - 7x^2 + 4)u^4 + 4x^2(32x^6 \\
&- 56x^4 + 6x^2 + 13)u^3 + (64x^8 + 16x^6 - 236x^4 + 145x^2 - 4)u^2 + 2(32x^6 - 52x^4 + 13x^2 + 4)u + 4x^4 - x^2 - 4][u^2 \\
&+ (4x^2 - 2)u + 1]^2 - 2[4(x^2 - 1)^3u^{11} + (64x^8 - 212x^6 + 260x^4 - 135x^2 + 26)u^{10} + (384x^{10} - 1344x^8 + 1892x^6 \\
&- 1306x^4 + 469x^2 - 68)u^9 + 2(512x^{12} - 1728x^{10} + 2560x^8 - 2190x^6 + 1218x^4 - 357x^2 + 39)u^8 + 2(512x^{14} \\
&- 1024x^{12} + 832x^{10} - 1288x^8 + 1936x^6 - 1009x^4 + 163x^2 + 4)u^7 + 2(1536x^{14} - 3584x^{12} + 1248x^{10} + 2136x^8 \\
&- 1188x^6 - 129x^4 + 240x^2 - 70)u^6 + 2(2560x^{14} - 7808x^{12} + 7136x^{10} - 1416x^8 - 468x^6 + 409x^4 - 308x^2 \\
&+ 84)u^5 + 2(2048x^{14} - 5632x^{12} + 2912x^{10} + 3544x^8 - 4056x^6 + 1463x^4 - 127x^2 - 26)u^4 + (512x^{12} + 768x^{10} \\
&- 6288x^8 + 8588x^6 - 4378x^4 + 974x^2 - 68)u^3 + (-1536x^{12} + 6080x^{10} - 7568x^8 + 2316x^6 + 1414x^4 - 761x^2 \\
&+ 82)u^2 - (256x^{12} - 704x^{10} + 272x^8 + 364x^6 + 16x^4 - 243x^2 + 36)u - 64x^{10} + 240x^8 - 292x^6 + 134x^4 - 24x^2 \\
&+ 6] - d[2(x^2 - 1)^2(10x^4 - 13x^2 + 4)u^{11} + (320x^{10} - 1212x^8 + 1770x^6 - 1238x^4 + 401x^2 - 48)u^{10} + 2(960x^{12} \\
&- 4096x^{10} + 6978x^8 - 6087x^6 + 2815x^4 - 667x^2 + 64)u^9 + (5120x^{14} - 24\,192x^{12} + 46\,912x^{10} - 49\,284x^8 \\
&+ 30\,378x^6 - 11\,334x^4 + 2289x^2 - 168)u^8 + 8(640x^{16} - 3264x^{14} + 7056x^{12} - 9148x^{10} + 7979x^8 - 4656x^6 \\
&+ 1495x^4 - 191x^2 + 2)u^7 + 2(512x^{16} - 768x^{14} - 3008x^{12} + 6000x^{10} - 2148x^8 - 2720x^6 + 2194x^4 - 773x^2 \\
&+ 144)u^6 + 4(1280x^{16} - 3200x^{14} + 1024x^{12} + 4200x^{10} - 7502x^8 + 6638x^6 - 3459x^4 + 800x^2 - 96)u^5 \\
&+ 2(2560x^{16} - 6400x^{14} + 8128x^{12} - 10\,288x^{10} + 7676x^8 - 1224x^6 - 590x^4 - 401x^2 + 56)u^4 + 2(2560x^{14} \\
&- 3008x^{12} - 6512x^{10} + 13\,530x^8 - 9079x^6 + 3028x^4 - 855x^2 + 84)u^3 + (4992x^{12} - 14\,560x^{10} + 14\,436x^8 \\
&- 4058x^6 - 2222x^4 + 1417x^2 - 176)u^2 + 2(320x^{12} - 1136x^{10} + 1866x^8 - 1725x^6 + 815x^4 - 189x^2 + 32)u \\
&- 32x^{10} + 172x^8 - 234x^6 + 66x^4 + 33x^2 - 8],
\end{aligned}
$$

$$
\begin{aligned}
B_{62} = {}& 2(x^2 - 1)^2(2x^2 - 3)u^{11} + (64x^8 - 264x^6 + 392x^4 - 233x^2 + 50)u^{10} + 2(192x^{10} - 928x^8 + 1690x^6 - 1343x^4 \\
&+ 492x^2 - 61)u^9 + (1024x^{12} - 5888x^{10} + 12\,544x^8 - 11\,896x^6 + 5544x^4 - 991x^2 + 14)u^8 + 4(256x^{14} - 1920x^{12} \\
&+ 4256x^{10} - 3840x^8 + 1486x^6 + 301x^4 - 420x^2 + 97)u^7 - 2(1024x^{14} + 2048x^{12} - 9728x^{10} + 10\,864x^8 - 7928x^6 \\
&+ 4328x^4 - 1595x^2 + 294)u^6 - 4(3840x^{14} - 8960x^{12} + 8096x^{10} - 4016x^8 - 246x^6 + 1199x^4 - 256x^2 - 35)u^5 \\
&- 2(12\,544x^{12} - 25\,408x^{10} + 9968x^8 + 8392x^6 - 8424x^4 + 2567x^2 - 206)u^4 + (-6144x^{12} + 6272x^{10} + 2432x^8 \\
&+ 2756x^6 - 7782x^4 + 3456x^2 - 414)u^3 + (-512x^{12} - 512x^{10} + 800x^8 + 1368x^6 - 632x^4 - 445x^2 + 122)u^2 \\
&+ 2(256x^{12} - 1024x^{10} + 1408x^8 - 902x^6 + 381x^4 - 108x^2 + 7)u + 128x^{10} - 416x^8 + 456x^6 - 184x^4 + 29x^2 \\
&+ 2d^2(x^2 - 1)[(x^2 - 1)^2u^{11} + (16x^6 - 41x^4 + 30x^2 - 5)u^{10} + (96x^8 - 304x^6 + 291x^4 - 89x^2 + 11)u^9 + (256x^{10} \\
&- 992x^8 + 1152x^6 - 475x^4 + 124x^2 - 15)u^8 + 2(128x^{12} - 640x^{10} + 816x^8 - 368x^6 + 213x^4 - 47x^2 + 5)u^7 \\
&+ (-256x^{12} + 320x^8 + 400x^6 + 114x^4 - 74x^2 + 14)u^6 - 2(128x^{12} - 192x^{10} + 368x^8 - 1112x^6 + 549x^4 - 154x^2 \\
&+ 21)u^5 + 2(128x^{12} - 640x^{10} + 1184x^8 - 504x^6 + 323x^4 - 123x^2 + 17)u^4 + (-512x^{10} + 1824x^8 - 1184x^6 \\
&+ 421x^4 - 64x^2 + 5)u^3 - (512x^{10} - 1408x^8 + 640x^6 + 201x^4 - 164x^2 + 25)u^2 + (-128x^{10} + 256x^8 - 41x^4 \\
&- 59x^2 + 15)u - 32x^8 + 80x^6 - 43x^4 + 2x^2 - 3] + 2d[2(x^2 - 1)^3(7x^2 - 2)u^{11} + (224x^{10} - 858x^8 + 1241x^6 \\
&- 831x^4 + 245x^2 - 24)u^{10} + 2(672x^{12} - 2960x^{10} + 5119x^8 - 4414x^6 + 1953x^4 - 418x^2 + 34)u^9 + (3584x^{14} \\
&- 18\,112x^{12} + 37\,024x^{10} - 39\,806x^8 + 24\,187x^6 - 8441x^4 + 1543x^2 - 96)u^8 + 2(1792x^{16} - 10\,752x^{14} + 26\,848x^{12} \\
&- 37\,608x^{10} + 32\,634x^8 - 17\,713x^6 + 5243x^4 - 592x^2 + 4)u^7 + (-2560x^{16} + 16\,640x^{14} - 40\,512x^{12} + 47\,152x^{10} \\
&- 26\,924x^8 + 5448x^6 + 864x^4 - 746x^2 + 176)u^6 + 2(2816x^{16} - 8704x^{14} + 11\,840x^{12} - 10\,360x^{10} + 4166x^8 \\
&+ 2421x^6 - 3129x^4 + 806x^2 - 108)u^5 + 2(1792x^{16} - 3712x^{14} + 3360x^{12} - 3560x^{10} + 754x^8 + 3694x^6 - 2558x^4
\end{aligned}
$$

$$
\begin{aligned}
&+25x^2+16)u^4+2(1792x^{14}-2592x^{12}-3432x^{10}+8551x^8-5740x^6+1894x^4-627x^2+58)u^3+(3648x^{12}\\
&-12\,848x^{10}+16\,502x^8-8377x^6+407x^4+693x^2-88)u^2+2(288x^{12}-1256x^{10}+2131x^8-1667x^6+540x^4\\
&-52x^2+10)u+x^2(-48x^8+170x^6-191x^4+61x^2+7)]-10,\\
B_{63}={}&2(1-x^2)\{[(x^2-1)^2u^{11}+(16x^6-38x^4+27x^2-5)u^{10}+(96x^8-256x^6+223x^4-75x^2+11)u^9+(256x^{10}\\
&-704x^8+640x^6-290x^4+103x^2-15)u^8+2(128x^{12}-256x^{10}+48x^8-36x^6+135x^4-45x^2+5)u^7\\
&+2(256x^{12}-640x^{10}+176x^8+116x^6+66x^4-30x^2+7)u^6+2(384x^{12}-896x^{10}-176x^8+1020x^6-535x^4\\
&+154x^2-21)u^5+2(256x^{12}-832x^{10}+560x^8-212x^6+278x^4-130x^2+17)u^4+(-896x^{10}+2016x^8\\
&-1720x^6+593x^4-68x^2+5)u^3-(640x^{10}-1312x^8+536x^6+318x^4-185x^2+25)u^2+(-128x^{10}+192x^8\\
&+8x^6-17x^4-73x^2+15)u-32x^8+72x^6-42x^4+5x^2-3]d^2+[2(x^2-1)^2(3x^2-1)u^{11}+(96x^8-282x^6\\
&+283x^4-109x^2+12)u^{10}+2(288x^{10}-1040x^8+1371x^6-793x^4+192x^2-17)u^9+(1536x^{12}-6848x^{10}\\
&+11\,616x^8-9214x^6+3599x^4-713x^2+48)u^8+2(768x^{14}-4608x^{12}+10\,592x^{10}-11\,800x^8+7302x^6\\
&-2473x^4+282x^2-2)u^7-2(1280x^{14}-6528x^{12}+10\,528x^{10}-8120x^8+2954x^6-192x^4-137x^2+44)u^6\\
&+2(1792x^{14}-3584x^{12}+4992x^{10}-4472x^8+834x^6+991x^4-316x^2+54)u^5+2(768x^{14}+128x^{12}-288x^{10}\\
&-664x^8-986x^6+1404x^4-43x^2-8)u^4+2(768x^{12}+544x^{10}-3352x^8+2719x^6-716x^4+273x^2-29)u^3\\
&+(1984x^{10}-4976x^8+4030x^6-659x^4-261x^2+44)u^2+2(160x^{10}-520x^8+571x^6-202x^4+12x^2-5)u\\
&-x^2(16x^6-34x^4+15x^2+1)]d-2[(x^2-1)^2u^{11}+2(8x^6-17x^4+11x^2-2)u^{10}+(96x^8-192x^6+127x^4\\
&-47x^2+13)u^9+(256x^{10}-320x^8+32x^6-138x^4+187x^2-46)u^8+(256x^{12}+512x^{10}-1184x^8-464x^6\\
&+1222x^4-553x^2+90)u^7+(1536x^{12}-512x^{10}-3968x^8+3808x^6-1504x^4+407x^2-56)u^6+(4864x^{12}\\
&-9088x^{10}+3904x^8+928x^6-1618x^4+641x^2-70)u^5+(1024x^{12}+1152x^{10}-3904x^8+64x^6+2172x^4\\
&-1083x^2+132)u^4+(-640x^{10}+3040x^8-3456x^6+513x^4+303x^2-59)u^3-(896x^{10}-1920x^8+672x^6\\
&+726x^4-263x^2+20)u^2+(-256x^{10}+480x^8-208x^6+75x^4-150x^2+25)u-2(32x^8-72x^6\\
&+45x^4-6x^2+3)]\},\\
B_{64}={}&0.
\end{aligned}
$$

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