Network geometry with flavor: From complexity to quantum geometry

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Network geometry is attracting increasing attention because it has a wide range of applications, ranging from data mining to routing protocols in the Internet. At the same time advances in the understanding of the geometrical properties of networks are essential for further progress in quantum gravity. In network geometry, simplicial complexes describing the interaction between two or more nodes play a special role. In fact these structures can be used to discretize a geometrical d-dimensional space, and for this reason they have already been widely used in quantum gravity. Here we introduce the network geometry with flavor $s = -1,0,1$ (NGF) describing simplicial complexes defined in arbitrary dimension d and evolving by a nonequilibrium dynamics. The NGF can generate discrete geometries of different natures, ranging from chains and higher-dimensional manifolds to scale-free networks with small-world properties, scale-free degree distribution, and nontrivial community structure. The NGF admits as limiting cases both the Bianconi-Barabási models for complex networks, the stochastic Apollonian network, and the recently introduced model for complex quantum network manifolds. The thermodynamic properties of NGF reveal that NGF obeys a generalized area law opening a new scenario for formulating its coarse-grained limit. The structure of NGF is strongly dependent on the dimensionality d. In $d = 1$ NGFs grow complex networks for which the preferential attachment mechanism is necessary in order to obtain a scale-free degree distribution. Instead, for NGF with dimension $d > 1$ it is not necessary to have an explicit preferential attachment rule to generate scale-free topologies. We also show that NGF admits a quantum mechanical description in terms of associated quantum network states. Quantum network states evolve by a Markovian dynamics and a quantum network state at time t encodes all possible NGF evolutions up to time t. Interestingly the NGF remains fully classical but its statistical properties reveal the relation to its quantum mechanical description. In fact the δ-dimensional faces of the NGF have generalized degrees that follow either the Fermi-Dirac, Boltzmann, or Bose-Einstein statistics depending on the flavor s and the dimensions d and δ.

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I. INTRODUCTION

Recently, network geometry [1] is gaining increasing interest. Progress in this field is expected to have relevance for a number of applications, including routing protocols [2–4], data mining [5–9], and advances in the theoretical foundations of network clustering [10]. In this context, several theoretical questions have been recently approached including the formulation of models for emergent geometry [11–13], the characterization of hyperbolic networks [14–17], the modeling of complex networks embedded in the plane or in surfaces [18–23], and finally the development of a geometric information theory of networks [24].

It is also believed that network geometry [25–30] could provide a theoretical framework for establishing cross fertilization between the field of network theory and quantum gravity. In fact most quantum gravity approaches rely on a discretization of space-time that takes a networklike structure. These approaches include causal sets [31,32], causal dynamical triangulations [33–36], group field theory [37,38], loop quantum gravity [39–41], energetic causal sets [42,43], quantum gravity as an information network [44], and quantum graphity [45–47]. Already several works explore the frontier territory between complex networks and quantum gravity. The relation between complex hyperbolic networks and causal sets has been exploited by building a "network cosmology" [48]. Moreover, causal sets have been used to analyze citation networks and measuring their effective dimension [49]. Recently, complex quantum network manifolds (CQNMs) [13] have been introduced as models of discrete manifolds that show the relation between quantum statistics and emergent network geometry.

When faced with the problem of describing a network geometry, simplicial complexes of dimension d become very useful. These are discrete structures formed by the simplices of dimension δ, with $0 \leqslant \delta \leqslant d$, i.e., nodes ($\delta = 0$), links ($\delta = 1$), triangles ($\delta = 2$), tetrahedra ($\delta = 3$), and so on. Simplicial complexes are widely used in the quantum gravity literature. For example, in the context of causal dynamical triangulations [33–36] and group field theory [37,38] space-time is described using these discrete structures. In network theory, much attention [50–52] has been devoted to complex networks described as sets of nodes and links, i.e., forming simplicial complexes of dimension $d = 1$. Only recently additional attention has been addressed to simplicial complexes of higher dimension also called hypergraphs in the network science community. These structures are important to capture relations existing between more than two nodes, such as the one existing in collaboration networks (where each paper might result from a collaboration of more than two individuals, or a movie might have a large cast of actors), protein interaction networks (where proteins form complexes consisting in general of more than two types of proteins), or on Twitter (where one tweet might include several hashtags). Therefore equilibrium and nonequilibrium models of random simplicial complexes and hypergraphs have been recently proposed by physicists and mathematicians [11–13,53–58].

Modeling complex networks has been the subject of intense research in network theory over the years. In particular,

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attention has been focusing on the minimal models able to generate network structure with the universal properties observed in real complex network data sets: the small-world property [59], the scale-free degree distribution [60], and a nontrivial community structure [61]. In this context, nonequilibrium growing network models generating scale-free networks [50–52,61–66] have been widely studied. Scale-free networks have highly inhomogeneous degree distribution $P(k)$ decaying as a power law for large value of k, i.e., $P(k) \simeq k^{-\gamma}$, with the power-law exponent $\gamma \leqslant 3$. The scale-free network distribution affects the properties of dynamical processes defined on networks [1,67,68] such as the Ising model, percolation, epidemic spreading, and quantum phase transitions. In growing network models formed by nodes and links, the so-called preferential attachment mechanism has been identified as a key element for obtaining scale-free networks as shown in the framework of the famous Barabási-Albert model [61]. The preferential attachment rule determines that the probability that a node acquires new links is proportional to its degree. Additional heterogeneity of the nodes, capturing intrinsic characteristics of the nodes that are different from the node degree, have been modeled by associating an energy ϵ to the nodes of the network. The energy ϵ of a node determines its fitness $\eta = e^{-\beta\epsilon}$, measuring the ability of the node to attract new links compared to the ability of other nodes with the same degree. The first growing scale-free network model introducing this heterogeneity of the nodes is the Bianconi-Barabási model [62–64] that has been used to model the Internet and the World Wide Web. This model captures the competition existing between nodes to attract new links. In fact, nodes acquire new links with a generalized preferential attachment rule, which assigns to high degree and high fitness nodes higher probability to acquire new links than to lower degree or lower fitness nodes.

The characterization of the Bianconi-Barabási model has unveiled an important relation between complex networks and quantum statistics. In fact, the Bianconi-Barabási model [62–64] can be mapped to a quantum Bose gas and, under the same circumstances in which the Bose gas undergoes a Bose-Einstein condensation, a structural phase transition is observed in the network structure in which one node grabs a finite fraction of all the links [63,64]. Interestingly, the Fermi-Dirac statistics characterizes growing Cayley trees with energy of the nodes [69], and these results have been extended in different directions [65,70,71], including weighted networks and multiplex networks. It is noteworthy that not only growing network models but also equilibrium network models have been shown to be related to quantum statistics [72].

Recently, the new results obtained in Ref. [13] for CQNMs show that also growing network manifolds describing a complex network geometry are related to quantum statistics. In fact, in complex quantum network manifolds the Fermi-Dirac, the Boltzmann, and the Bose-Einstein statistics coexist in the same network geometry describing the statistical properties of the δ-dimensional faces of the CQNM.

Here our goal is to introduce network geometry with flavor (NGF) $s = -1,0,1$ showing the strong effect of dimensionality d on the geometry emergent from these models and the relation between NGF and quantum statistics. The NGFs describe growing simplicial complexes with energies associated to all their simplices (i.e., to their nodes, links, triangular faces, etc.) and evolving with (case $s = 1$) or without (cases $s = -1,0$) explicit preferential attachment, forming either manifolds (case $s = -1$) or more general simplicial complexes (cases $s = 0,1$). The NGF generalizes the CQNM introduced in Ref. [13], which constitutes the NGF with flavor $s = -1$. For $s = -1$, $d = 3$, and $\beta = 0$ the model reduces to the random Apollonian network [19–22]. Moreover, the NGF with flavor $s = 1$ and dimension $d = 1$ reduces to the Bianconi-Barabási model.

We will focus specifically on the thermodynamic properties of NGF, on the relation of NGF to complexity theory, and on the relation between these geometrical network structures and their quantum mechanical description. In particular we will characterize the thermodynamic relations satisfied by the NGF evolving by a nonequilibrium dynamics and obeying a generalized area law; we will identify in which dimension d and for which flavor s NGFs are scale-free networks; and finally we will provide a quantum mechanical description of NGFs, constructing quantum network states characterizing the evolution of these models, and showing how quantum statistics emerges from the statistical properties of these networks.

In order to determine the thermodynamics of NGF, we define its total energy E, total entropy S and area A. The thermodynamic properties of the NGFs reveal that these structures follow a generalized area law. Since in quantum gravity the celebrated Jacobson [73–75] result relates the area law to the Einstein equations as equation of state, this result could play a crucial role in determining the dynamics of NGFs at the macroscopic, coarse-grained level.

Our results highlight the strong effect of the dimensionality d on the structure of the NGF. For NGF in $d = 1$, as in the Barabási-Albert model, preferential attachment is a necessary element for obtaining scale-free networks. Here we show that for NGFs formed by simplicial complexes of dimension $d > 1$ an explicit preferential attachment is not necessary to obtain scale-free networks, as an effective preferential attachment can emerge in simplicial complexes of dimension $d > 1$ by dynamical rules that do not include an explicit preferential attachment. Therefore, in dimension $d = 2$ also network geometry with flavor $s = 0$ that is not driven by an explicit preferential attachment generates scale-free networks. In dimension $d \geqslant 3$ all the NGFs are scale free, independently of their flavor s.

The NGF can be mapped to quantum network states evolving by a Markovian dynamics. The relation between the NGFs and their quantum mechanical description is also emerging from their statistical properties. In fact, NGFs in dimension d have the generalized degree of their δ faces, which as a function of the flavor s and the dimensions d,δ follows Fermi-Dirac, Boltzmann, or Bose Einstein statistics. The dimension $d = 3$ again plays a special role because it is the lowest dimension for observing the coexistence of the Fermi-Dirac, Boltzmann, and Bose-Einstein statistics describing the statistical properties of the faces of the NGF of dimensions d.

II. NETWORK GEOMETRY WITH FLAVOR s

A. Network geometry with flavor s and simplicial complexes

Here we define NGFs in a constructive way by characterizing their nonequilibrium dynamical evolution.

By d-dimensional simplex here we indicate a fully connected graph (a clique) of $(d+1)$ nodes. Its δ faces are all the δ-dimensional simplices that can be built by a subset of $(\delta+1)$ of its nodes. In general, a simplicial complex of dimension d is formed by a set of simplices of dimension $d' \leqslant d$.

A NGF of dimension $d \geqslant 1$ is a simplicial complex formed by d-dimensional simplices glued along their $(d-1)$-dimensional faces also called $(d-1)$ faces. For example, a NGF of $d=1$ is formed by links glued at their end nodes, a NGF of $d=2$ is formed by triangles glued along their links, and a NGF of $d=3$ is formed by tetrahedra glued along their triangular faces. The set of all possible δ-dimensional faces (or δ faces) belonging to the d-dimensional NGF with N nodes is here indicated by $\mathcal{Q}_{d,\delta}(N)$. The set of all δ-dimensional faces belonging to the d-dimensional NGF with $\delta < d$ is indicated by $\mathcal{S}_{d,\delta}$.

B. Energies and generalized degrees of NGF

To each node i of the NGF we assign an energy of the node ϵ_i from a distribution $g(\epsilon)$. The energy of the node is quenched and does not change during the evolution of the network. This parameter describes the intrinsic and heterogeneous properties of the nodes. To every δ face $\alpha \in \mathcal{S}_{d,\delta}$ we associate an energy ϵ_α given by the sum of the energy of the nodes that belong to the face α,

$$\epsilon_\alpha = \sum_{i\in\alpha} \epsilon_i. \tag{1}$$

Therefore, each link will be associated to an energy of the link given by the sum of energies of the two nodes incident to it, and each triangular face will be associated to the sum of the energy of the three nodes incident to it, and so on. The energy $\epsilon_{(i,j)}$ of the links $\alpha=(i,j)$ belonging to any given triangle of the NGF formed by the nodes i,j, and r satisfy the triangular inequality

$$|\epsilon_{(i,r)} - \epsilon_{(j,r)}| \leqslant \epsilon_{(i,j)} \leqslant \epsilon_{(i,r)} + \epsilon_{(j,r)}. \tag{2}$$

This result remains valid for any permutation of the order of the nodes i,j, and r belonging to the triangle. The energy of the links can therefore be interpreted as length of the links and related to the use of spins in spin networks and loop quantum gravity [41].

Although most of the derivations shown in this paper can be performed similarly for either continuous or discrete energy of the nodes and of the higher-dimensional δ faces, here we consider the case in which the energies of the nodes $\{\epsilon_i\}$ and consequently the energy of the δ faces $\{\epsilon_\alpha\}$ are integers.

The generalized degrees $k_{d,\delta}(\alpha)$ of the δ face α (i.e., $\alpha \in \mathcal{S}_{d,\delta}$) in a d-dimensional NGF is defined as the number of d-dimensional simplices incident to it. Let us define the adjacency indicator function $\mathbf{a}$ of elements $a_{\alpha'}$ with $\alpha' \in \mathcal{Q}_{d,d-1}(N)$ taking value $a_{\alpha'}=1$ if the d-dimensional complex α' is part of the NGF and otherwise taking value zero, $a_{\alpha'}=0$. Using the adjacency indicator function, we can define the generalized degree $k_{d,\delta}$ of a δ face α as

$$k_{d,\delta}(\alpha) = \sum_{\alpha'|\alpha\subset\alpha'} a_{\alpha'}. \tag{3}$$

Therefore, in a NGF of dimension $d=1$ the generalized degree $k_{1,0}(\alpha)$ is the number of links incident to a node α, i.e., its degree. In $d=2$, the generalized degree $k_{2,1}(\alpha)$ is the number of triangles incident to a link α while the generalized degree $k_{2,0}(\alpha)$ indicates the number of triangles incident to a node α. Similarly, in a NGF of dimension $d=3$, the generalized degrees $k_{3,2}, k_{3,1}$, and $k_{3,0}$ indicate the number of tetrahedra incident, respectively, to a triangular face, a link, or a node.

C. NGF evolution

The NGF comes in three flavors indicated by the variable $s=-1,0,1$. In order to define the nonequilibrium dynamics of NGF we associate to each $(d-1)$ face α the number n_α given by the sum of the d-dimensional simplices incident to α minus one, i.e.,

$$n_\alpha = k_{d,d-1}(\alpha) - 1. \tag{4}$$

If the variable n_α can only take values 0,1 the NGF is a manifold also called CQNM. If instead the variable n_α can also take values greater than two we have a NGF that is not a manifold. As we will see in the following, NGFs with flavor $s=-1$ describe manifolds, the CQNMs, while NGFs with flavor $s=0,1$ do not generate manifolds.

The NGFs in dimension d are evolving according to a nonequilibrium dynamics enforcing that at each time the NGF is growing by the addition of a new d-dimensional simplex. Here we describe the NGF evolution for NGF with every type of flavor $s=-1,0,1$ (see Supplemental Material [76] for the MATLAB code generating NGF in dimensions $d=1,2,3$).

At time $t=1$ the NGF is formed by a single d-dimensional simplex. At each time $t>1$ we add a simplex of dimension d to a $(d-1)$ face $\alpha \in \mathcal{S}_{d,d-1}$, which is chosen with probability $\Pi_\alpha^{[s]}$ given by

$$\Pi_\alpha^{[s]} = \frac{1}{Z^{[s]}(t)} e^{-\beta\epsilon_\alpha}(1+sn_\alpha), \tag{5}$$

where $\beta \geqslant 0$ is a parameter of the model called inverse temperature, $s=-1,0,1$ and $Z^{[s]}(t)$ is a normalization sum given by

$$Z^{[s]}(t) = \sum_{\alpha\in\mathcal{S}_{d,d-1}} e^{-\beta\epsilon_\alpha}(1+sn_\alpha). \tag{6}$$

Having chosen the $(d-1)$ face α, we glue to it a new d-dimensional simplex containing all the nodes of the $(d-1)$ face α plus the new node i. It follows that the new node i of the new simplex is linked to each node j belonging to α. Finally, we note here that the number of nodes N at time t is given by $N=t+d$. In fact for $t=1$ the NGF is formed by a single d-dimensional simplex, and has $N=d+1$ nodes. At each time $t>1$, a new d-dimensional simplex is added to the NGF. This simplex has a single new node. Therefore the number of nodes grows at each time step by one, and is given by $N=t+d$.

In Fig. 1 we show the first few steps of the NGF evolution for the cases $d=1,2$ and $s=-1,0,1$. In Fig. 2 we show a visualization of NGF with $s=-1,0,1, d=1,2,3$, and $\beta=0.1$. These NGFs for $d=1$ are trees, for $d>1$ they have at the same time large clustering and small average distance between

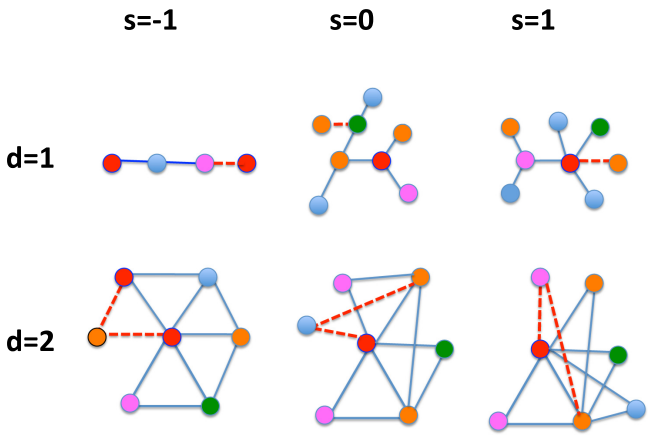


FIG. 1. The figure schematically illustrates the temporal evolution of the NGF of flavor s in dimension $d = 1,2$. In dimension $d = 1$ at each time step a new node is added to the network and is connected to the existing network by a single link. In dimension $d = 2$, at each time a new node is added to the network. This node is connected to the existing network by a triangle, i.e., it is linked to two adjacent nodes of the network. When NGF has flavor $s = -1$, each $(d-1)$ simplex can be connected at most to two d-dimensional simplices. This implies that in $d = 1$ each node can have at most degree two and in $d = 2$ each link can be adjacent to at most two triangles. Therefore, NGFs with flavor $s = -1$ are manifolds, and are also called CQNM [13]. On the contrary, NGF with flavor $s = 0,1$ does not have this constraint and each $(d-1)$-dimensional simplex can be connected to an arbitrarily large number of d-dimensional simplices. Therefore, in $d = 1$ each node can have an arbitrarily large degree and in $d = 2$ each link can be incident to an arbitrarily large number of triangles. For $s = 1$ the NGF evolution includes an explicit preferential attachment rule implying that each new d-dimensional simplex is linked to a $(d-1)$ face α with a probability that increases linearly with its generalized degree $k_{d,d-1}(\alpha)$. Therefore, the NGF with $d = 1, s = 1$ for $\beta = 0$ reduces to the Barabási-Albert model [61] and for $\beta > 0$ it reduces to the Bianconi-Barabási model [62,63].

the nodes, i.e., they are small world and they have a nontrivial community structure.

D. NGFs of different flavor s have significantly different structure and dynamics

The NGFs of different flavor s have significantly different geometry and statistical properties. In fact, depending on the flavor s either manifolds ($s = -1$) or more general simplicial complexes are generated. The dynamical properties of NGFs of different flavor s are also very different, with NGFs of flavor $s = 1$ including an explicit preferential attachment while NGFs with flavor s are driven by an homogeneous attachment dynamics. In the following we will discuss the properties of NGFs as a function of their flavor s and their dimension d. Moreover, we will relate specific limiting cases of NGFs with existing models of complex networks.

The dynamical rules of the NGF imply that only for the case $s = -1$ NGFs are actually manifolds, also called CQNMs [13]. In fact, for $s = -1$ the probability $\Pi_\alpha^{[-1]}$ defined in Eq. (5) is zero, (i.e., $\Pi_\alpha^{[-1]} = 0$) for every $(d-1)$ face α with $n_\alpha = 1$. If a $(d-1)$ face has $n_\alpha = 1$ it is already incident to two d-dimensional simplices, as its generalized degree is $k_{d,d-1}(\alpha) = n_\alpha + 1 = 2$. Such a face α cannot be glued to any additional d-dimensional simplex because the probability that we glue an additional d-simplex to this face is $\Pi_\alpha^{[-1]} = 0$. In particular the NGF of $d = 1$ and flavor $s = -1$ is a chain.

For $s = 1$, we observe that the probability to attach a new simplex to the $(d-1)$ face $\alpha, \Pi_\alpha^{[1]}$, is proportional to its generalized degree $k_{d,d-1}(\alpha) = 1 + n_\alpha$ providing a generalization of the so-called preferential attachment mechanism, known to be necessary for generating scale-free networks in simplicial complexes of dimension $d = 1$.

The evolution of NGF is related to existing complex network models with fitness of the nodes [61–65,69–71]. In particular the NGF with $d = 1, \beta = 0$, and $s = 1$ is the Barabási-Albert model [61] (with the number of initial links of each node given by one), while for $d = 1, \beta > 0$, and $s = 1$ it is the Bianconi-Barabási model [62,63] (always with the number of initial links given by one). Moreover, the NGF of $d = 2$ with flavor $s = 0$ and $\beta = 0$ has been first proposed as a scale-free network model in Ref. [66]. The NGF in $d = 2$ is related to models proposed in the recent literature on emergent network geometry [11,12].

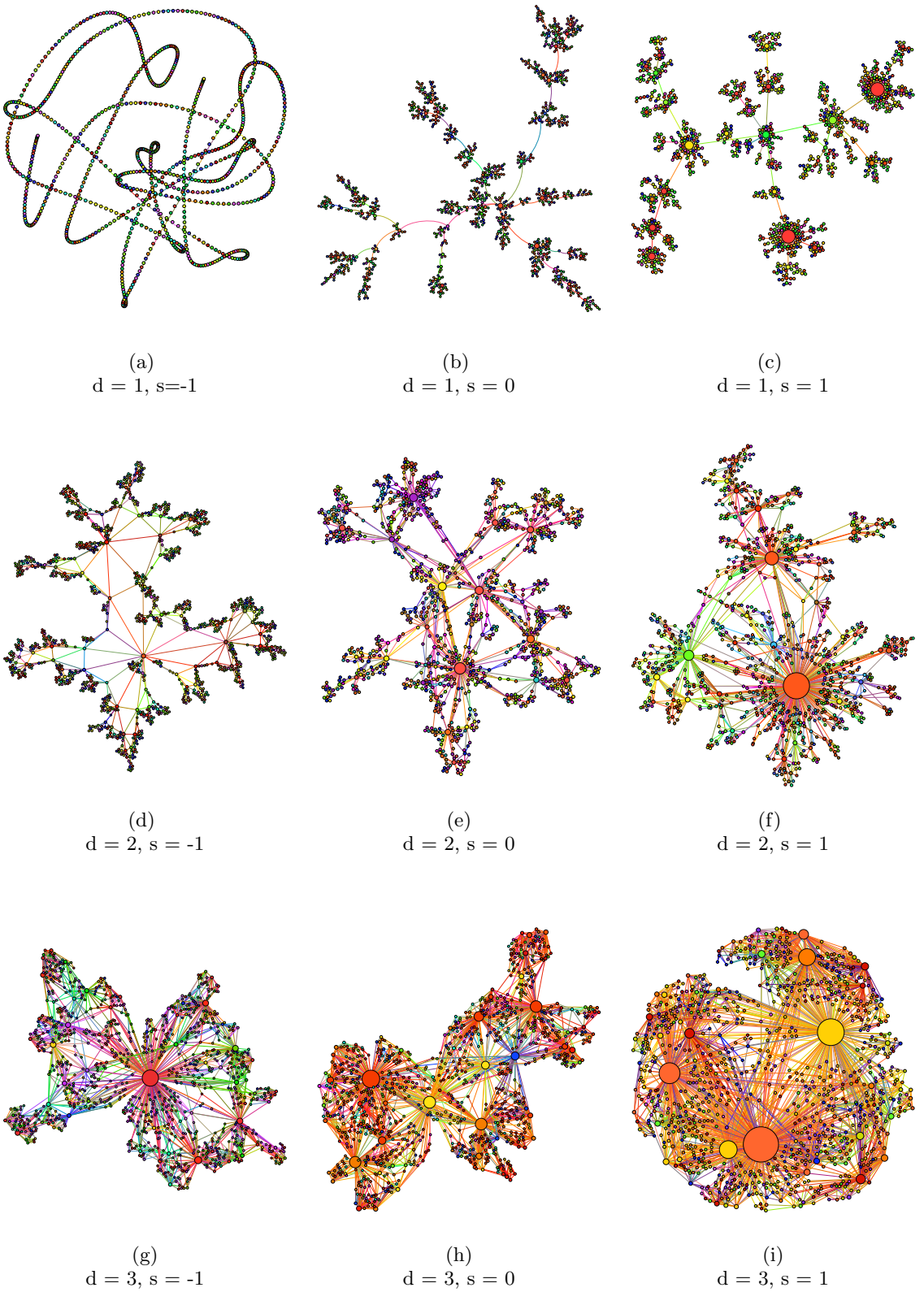
Finally, the NGF for $s = -1, d = 3$, and $\beta = 0$ is a stacked polytope model and as such reduces to the stochastic Apollonian network [19–22]. We note here that it is possible to define NGFs allowing also for a $\Pi_\alpha^{[s]}$ given by Eq. (5) with real values of s, as long as $s > 0$. These models will include energy of the δ faces and preferential attachment with an initial additive constant [77]. These models will qualitatively behave like the NGF with $s = 1$. Also it is possible to consider negative values $s \neq -1$. Nevertheless, to avoid having negative probabilities $\Pi_\alpha^{[s]}$ given by Eq. (5), we should impose that s takes negative rational values $s = -1/m$ with $m \geqslant 1$. This model allows the generalized degree of $(d-1)$ faces to be at most m and therefore $n_\alpha \leqslant m$. These models are related to the ones recently proposed in Ref. [11] for simplicial complexes in $d = 2$. For simplicity here we restrict our study only to NGFs with flavor $s = -1,0,1$ that display a significant change in their structural properties.

E. Area and volume of NGFs

The boundary of the NGF is defined as the set of $(d-1)$ faces with $n_\alpha = 0$, i.e., incident to exactly one d-dimensional simplex. We will call the area A of the NGF the number of $(d-1)$ faces in the boundary, i.e.,

$$A = \sum_{\alpha \in \mathcal{S}_{d,d-1}} \delta(n_\alpha, 0). \tag{7}$$

At each time step of the NGF dynamical evolution, a $(d-1)$ face is chosen and a new simplex is attached to it. If this face is initially at the boundary of the NGF, after the addition of the simplex it will leave the boundary, contributing to a negative change of A of one. At the same time the new simplex adds d new $(d-1)$ faces to the boundary, contributing to an increase of A by d. For NGF with flavor $s = -1$ (i.e., for CQNMs), the new d-dimensional complex is attached exclusively to a $(d-1)$ face at the boundary. Moreover, at time $t = 1$ the area is the area of a single d-dimensional simplex, and is given by



(a) d = 1, s=-1 (b) d = 1, s = 0 (c) d = 1, s = 1

(d) d = 2, s = -1 (e) d = 2, s = 0 (f) d = 2, s = 1

(g) d = 3, s = -1 (h) d = 3, s = 0 (i) d = 3, s = 1

FIG. 2. Network geometry with flavor $s = -1,0,1$ and dimension $d = 1,2,3$. The NGFs have $N = 10^3$ nodes, $\beta = 0.1$ and uniform distribution of the energy of the nodes $g(\epsilon) = 1/10$ for $0 \leqslant \epsilon < 10$. The color of the nodes indicates their energy; the color code keeps the same order of the frequency of light (in order of increasing energy we have red, orange, yellow, green, blue, violet); the size of the nodes is proportional to their degree.

$A = d + 1$. Therefore, we have

$$A = (d-1)t + 2. \tag{8}$$

In general for NGF with every flavor $s = -1,0,1$, and sufficiently low values of β, we have

$$A \simeq \lambda t \tag{9}$$

for $t \gg 1$ with $\lambda \in [d-1,d)$. The volume V of the NGF is given by the total number of d-dimensional simplices that form the NGF. The volume V of the NGF at time t is equal to the time, i.e.,

$$V = t, \tag{10}$$

since at each time step one d-dimensional simplex is added to the NGF. Therefore, in NGF the area A is proportional to the volume V, i.e., $A \propto V$. This property of the NGF is crucial to determine the NGF small-world diameter, i.e., a diameter at most increasing like the logarithm of time t, for sufficiently low values of the inverse temperature β.

F. Dual of the NGFs

The NGF have a particularly simple dual network structure. The dual network is formed by considering nodes indicating the d-dimensional simplices and links connecting d-dimensional simplices that share a $(d-1)$ face. For NGFs with flavor $s = -1$, i.e., for the CQNMs, the dual is a tree with degree bounded by $d+1$. In fact each $(d-1)$ face connects at most two d-dimensional simplices and each d-dimensional simplex has exactly $d+1$ $(d-1)$-dimensional faces. Interestingly, as it is possible to see in Fig. 2, the CQNMs, also if they have very homogeneous dual networks, can display very complex structure, and as we will see in the next section they are scale free for $d \geqslant 3$. This shows a clear example in which the relation between simplicial complexes and their dual networks might not preserve the same complexity properties. For network geometry with flavor $s = 0,1$ the dual network remains a tree but the degree of its nodes is no longer bounded. The treelike nature of the dual network of the NGF allows for relevant simplifications in the analytical calculations.

III. THERMODYNAMICS OF NGFs

A. Probability of a given NGF evolution and total energy of a given NGF

Given the evolutionary dynamics of the NGFs, the evolution of the NGF up to time t is fully determined by the sequence $\{\alpha_{t'}\}_{t' \leqslant t}$, where $\alpha_{t'}$ indicates the $(d-1)$ face to which the new d-dimensional simplex is added at time $t' > 1$. Moreover, the NGF is associated with the sequence of the energies of its $N = t + d$ nodes $\{\epsilon(t')\}_{t' \leqslant t+d}$. Of those, only the energy of the nodes arrived in the NGF before time t, i.e., the sequence $\{\epsilon(t')\}_{t' < t+d}$ determines the probabilities of choosing a particular sequence of $\{\alpha_{t'}\}_{t' \leqslant \ldots t}$. Finally, it is possible to evaluate the probability $P(\{\alpha(t')\}_{t' \leqslant t}|\{\epsilon(t')\}_{t' < t+d},s)$ that the temporal evolution until time t of the NGF with flavor s is described by the subsequent addition of d simplices to the $(d-1)$ faces $\{\alpha(t')\}_{t' \leqslant t}$ given that the energies of the nodes until time $t-1$ are $\{\epsilon(t')\}_{t' < t+d}$. In fact $P(\{\alpha(t')\}_{t' \leqslant t}|\{\epsilon(t')\}_{t' < t+d},s)$ is given by the product of the probability of each subsequent addition of the new simplex to the $\alpha(t)$ face, i.e.,

$$P(\{\alpha(t')\}_{t' \leqslant t}|\{\epsilon(t')\}_{t' < t+d},s) = \prod_{t' \leqslant t} \Pi^{[s]}_{\alpha(t')}, \tag{11}$$

where $\Pi_{\alpha(t')}$ is given by Eq. (5). Inserting the explicit expression of $\Pi_{\alpha(t')}$ in Eq. (11), we obtain

$$P(\{\alpha(t')\}_{t' \leqslant t}|\{\epsilon(t')\}_{t' < t+d},s) = \frac{1}{\mathcal{Z}^{[s]}(t)} e^{-\beta E} \prod_{\alpha \in \mathcal{S}_{d,d-1}(t)} [1 + s n_\alpha(t)]!. \tag{12}$$

Here we have indicated by E the total energy of the NGF, given by

$$E(t) = \sum_{\alpha \in \mathcal{S}_{d,d-1}} \epsilon_\alpha n_\alpha(t), \tag{13}$$

and with $\mathcal{Z}^{[s]}(t)$ the normalization constant,

$$\mathcal{Z}^{[s]}(t) = \sum_{\{\alpha(t')\}_{t' \leqslant t}} e^{-\beta E} \prod_{\alpha \in \mathcal{S}_{d,d-1}(t)} [1 + s n_\alpha(t)]! = \prod_{t' \leqslant t} Z^{[s]}(t'). \tag{14}$$

Moreover, $\mathcal{S}_{d,d-1}(t)$ is the set of $(d-1)$ faces in the NGF formed by the subsequent addition of d-dimensional simplices to the faces $\{\alpha(t')\}_{t' \leqslant t}$.

For sufficiently low values of β we have that for large times, i.e., for $t \gg 1$, the ratio $Z^{[s]}/t$ is a self-averaging quantity and $\lim_{t\to\infty} Z^{[s]}/t = e^{-\beta \mu^{[s]}_{d,d-1}}$, with $\mu^{[s]}_{d,d-1}$ indicating the chemical potential associated to the $(d-1)$ faces in NGF of flavor s. Therefore, we can approximate $\mathcal{Z}^{[s]}$ as

$$\mathcal{Z}^{[s]}(t) \simeq e^{-\beta \mu^{[s]}_{d,d-1}} t! \simeq e^{-\beta \mu^{[s]}_{d,d-1}} N! \tag{15}$$

for large times $t \gg 1$ and $t \simeq N$. Finally, $P(\{\alpha(t')\}_{t' \leqslant t}|\{\epsilon(t')\}_{t' < t+d},s)$ can be expressed as

$$P(\{\alpha(t')\}_{t' \leqslant t}|\{\epsilon(t')\}_{t' < t+d},s) = \frac{1}{N!} e^{-\beta(E - \mu^{[s]}_{d,d-1} N - F)}, \tag{16}$$

where F is given by

$$e^{\beta F} = \prod_{\alpha \in \mathcal{S}_{d,d-1}(t)} [1 + s n_\alpha(t)]!. \tag{17}$$

B. Entropy of the NGF and the generalized area law

We note that different histories of the NGF up to time t can give rise to the same network structure. This network structure is indicated by G_N where $N = N(t)$ is the number of nodes of the network and $\{\epsilon_i\}_{i \leqslant N}$ are the energies of the nodes. All the possible temporal evolutions $\{\alpha(t')\}_{t' \leqslant t}$ of the NGF corresponding to the same network G_N have the same probability $P(\{\alpha(t')\}_{t' \leqslant t}|\{\epsilon_i\}_{i \leqslant N},s) = P(\{\alpha(t')\}_{t' \leqslant t}|\{\epsilon(t')\}_{t' < t+d},s)$, and they can be obtained from a given history by considering all causal relabelings of the nodes. We define the probability $P(G_N|\{\epsilon_i\}_{i \leqslant N},s)$ that the NGF of flavor s at time t results in a given network structure G_N, independently of its temporal evolution, given the energy of the nodes $\{\epsilon_i\}_{i \leqslant N}$. Using the fact that the dual of the NGF is a tree, this probability can be calculated with methods already developed in Refs. [78,79]

by evaluating the number of possible causal relabelings of the dual tree. Specifically, we have

$$P(G_N|\{\epsilon_i\}_{i\leqslant N},s) = e^{-\beta(E-\mu^{[s]}_{d,d-1}N-F)}z^{[s]}_N, \tag{18}$$

where

$$z^{[s]}_N = \frac{1}{N!}L^{[s]}(T) \tag{19}$$

and where $L^{[s]}(T)$ indicates the number of different NGF temporal evolutions giving rise to the same network G_N. It can easily be realized that $L^{[s]}(T)$ indicates also the number of different labelings of the tree T that is the dual network of the NGF. The introduced quantity $z^{[s]}_N$ can be calculated by following the derivation given in Ref. [78] as long as the NGF is in a stationary state and the degree distribution of the tree describing the dual network of the NGF is known. In fact, it is possible to evaluate the scaling of $z^{[s]}_N$ by writing a recursive equation for $L^{[s]}(T)$ for a tree T given by a root node connected to p subtrees $T_1,T_2,\ldots,T_p$ of $N_1,N_2,\ldots,N_p$ nodes, respectively. The recursive equation is given by

$$L^{[s]}(T) = \frac{(N-1)!}{N_1!N_2!\ldots N_p!}\delta_{\sum_i N_i+1,N}\prod_{i=1}^{p}L^{[s]}(T_i). \tag{20}$$

Here, differently from the case analyzed in Ref. [78], the different branches $T_1,T_2,\ldots,T_p$ of the tree T are not exchangeable since the tree T is a dual of a labeled NGF where the labels indicate the different energies of the nodes. Using Eq. (20), it is found (see Supplemental Material for details) that $z^{[s]}_N$ scales with the number of nodes as

$$z^{[s]}_N = C^{[s]}e^{\beta\nu^{[s]}N} \tag{21}$$

as long as the NGF is not a chain (it is different from the case $s=-1,d=1$), and the NGF reaches a stationary state (low enough values of β). In fact, the prefactor $1/N!$ in Eq. (19) is compensated by the number of terms in the summand. Therefore, in Eq. (21), $C^{[s]}$ is a subleading factor, and $\nu^{[s]}$ depends on the degree distribution of the dual of the NGF, and therefore depends on its flavor s.

Finally, the probability $P(G_N,s)$ scales exponentially with the number of nodes and can be written for large networks $N \gg 1$ as

$$P(G_N|\{\epsilon_i\}_{i\leqslant N},s) = C^{[s]}e^{-\beta(E-\mu^{s}_{d,d-1}N-\nu^{[s]}N-F)}. \tag{22}$$

The entropy $S(N)$ of the NGF has the natural definition

$$S(N) = -\sum_{G_N} P(G_N|\{\epsilon_i\}_{i\leqslant N},s)\ln P(G_N|\{\epsilon_i\}_{i\leqslant N},s).$$

The total energy E and the entropy S of NGF satisfy thermodynamics relations. In order to derive them, let us evaluate the variation in entropy of the network ΔS given by

$$\Delta S(N) = S(N) - S(N-1). \tag{23}$$

It can be easily shown, using the definition of the total energy E in Eq. (13) and the rules determining the NGF evolution, that

$$\langle\epsilon_\alpha\rangle_{\Pi^{[s]}} = \langle\Delta E\rangle_{\Pi^{[s]}}. \tag{24}$$

Finally, since the dynamics of the NGF reaches stationarity for sufficiently low values of β, both $\langle\epsilon_\alpha\rangle_\Pi$ and $\langle\ln(1+sn_\alpha)\rangle_\Pi$ are independent of time for sufficiently large times $t \gg 1$. Therefore, the relation between ΔS and $\langle\Delta E\rangle$ calculated over the interval $\Delta t = 1$ can be found using Eqs. (22), (24) and is given by

$$\Delta S = \left\{\beta\left(\langle\epsilon_\alpha\rangle_{\Pi^{[s]}} - \mu^{[s]}_{d,d-1} - \nu^{[s]}\right) - \langle\ln(1+sn_\alpha)\rangle_{\Pi^{[s]}}\right\}\Delta t.$$

Using the scaling of the area A with time given by Eq. (9), it follows that the change in entropy ΔS can then be expressed as

$$\Delta S = \left\{\beta\left[\langle\epsilon_\alpha\rangle_{\Pi^{[s]}} - \mu^{[s]}_{d,d-1} - \nu^{[s]}\right] - \langle\ln(1+sn_\alpha)\rangle_{\Pi^{[s]}}\right\}\frac{\Delta A}{\lambda}.$$

This relation provides a special type of area law because for NGF the area A scales like the volume $V = N$, i.e., $A \propto V$. Nevertheless, we believe that this result opens new avenues for formulating the macroscopic description of NGF at the coarse-grained level, in the light of the results obtained in Refs. [73–75].

C. Relation between the Regge curvature and the total energy E of NGF with flavor $s = -1$

We note here that the NGF with flavor $s=-1$ are manifolds, specifically, they are the CQNM. For these manifolds, one may wish to characterize their geometry using Regge's definition of curvature [41,80,81]. The Regge curvature is localized on $(d-2)$ faces and is given by the excess angle formed by the d-dimensional simplices incident to a given $(d-2)$ face. Therefore, in the case in which the d-dimensional simplices are assumed all equilateral the curvature R_α associated to the $(d-2)$ face α is uniquely determined by the generalized degree $k_{d,d-2}(\alpha)$, i.e.,

$$R_\alpha = a_\alpha\pi - \theta_d\, k_{d,d-2}(\alpha), \tag{25}$$

where $\theta_d > 0$ indicates the angle between any two $(d-1)$ faces of the d-dimensional simplex and where $a_\alpha = 1$ [81] for all $\alpha \in S_{d,d-2}(N)$ because for the NGF all $(d-2)$ faces are at the boundary.

The total energy E of the NGF with flavor s is defined in Eq. (13) as

$$E = \sum_{\alpha\in\mathcal{S}_{d,d-1}}\epsilon_\alpha n_\alpha, \tag{26}$$

where n_α is related to the generalized degree of the $(d-1)$ face α by $n_\alpha = k_{d,d-1}(\alpha) - 1$ [Eq. (4)], and where the energy of the face α is given by the sum of the energy of the nodes belonging to that face [Eq. (1)]. We note now that it is possible to show (see Supplemental Material for details), using simple combinatorial calculations, that

$$\sum_{\alpha\in\mathcal{S}_{d,d-1}}\epsilon_\alpha k_{d,d-1}(\alpha) = B_d\sum_{\alpha'\in\mathcal{S}_{d,d-2}}\epsilon_{\alpha'}k_{d,d-2}(\alpha'), \tag{27}$$

with $B_d = 2/(d-1)$. Using this expression we can express the total energy E and the total energy of the boundary $\hat{E}$ of the NGF in terms of the Regge curvature R_α of the $(d-2)$ faces. The total energy E of the NGF can then be written as

$$E = \frac{B_d}{\theta_d}\left(\Lambda - \sum_{\alpha'\in\mathcal{S}_{d,d-2}}\epsilon_{\alpha'}R_{\alpha'}\right) \tag{28}$$

with Λ being independent of the curvature and it can be shown to be given by

$$\Lambda = \left(\pi - \frac{\theta_d}{2}\right) \sum_{\alpha' \in \mathcal{S}_{d,d-2}} \epsilon_{\alpha'}. \tag{29}$$

We note that the expression for E in Eq. (28) differs from the Regge action [41,80,81] by an overall sign, and by the fact that Eq. (32) contains the energy of the $(d-2)$ faces while in the Regge action their role is played by the volume of the $(d-2)$ faces. Additionally, it is possible to define the total energy of the boundary $\hat{E}$ of the NGF as given by the sum of the energies of the $(d-1)$ faces at the boundary, i.e.,

$$\hat{E} = \sum_{\alpha \in \mathcal{S}_{d,d-1}} \epsilon_\alpha (1 - n_\alpha), \tag{30}$$

with $\hat{E}$ and E being related by

$$E + \hat{E} = \sum_{\alpha \in \mathcal{S}_{d,d-1}} \epsilon_\alpha. \tag{31}$$

The total energy $\hat{E}$ of the the boundary can be written as

$$\hat{E} = \frac{B_d}{\theta_d} \left(\sum_{\alpha' \in \mathcal{S}_{d,d-2}} \epsilon_{\alpha'} R_{\alpha'} - \hat{\Lambda} \right) \tag{32}$$

with $\hat{\Lambda}$ being independent of the curvature and given by

$$\hat{\Lambda} = (\pi - \theta_d) \sum_{\alpha' \in \mathcal{S}_{d,d-2}} \epsilon_{\alpha'}. \tag{33}$$

We note that the expression for $\hat{E}$ in Eq. (32) differs from the Regge action [41,80,81] by the fact that Eq. (32) contains the energy of the $(d-2)$ faces while in the Regge action their role is played by the volume of the $(d-2)$ faces.

IV. GENERALIZED DEGREE DISTRIBUTIONS AT $\beta = 0$

A. Dependence of the generalized degree distribution on dimensions d, δ and flavor s

The NGFs display a number of critical dimensions marking changes in the structure of these networks as their dimension d changes. These structural changes are revealed by the statistical properties associated with the distribution of the generalized degree $k_{d,\delta}$ of their δ faces with $0 \leqslant \delta < d$. To show this, here we focus on the effect of the dimensions d and δ on the distribution $P^{[s]}_{k,\delta}(k)$ of the generalized degrees $k_{d,\delta}$ of NGF of flavor s. For simplicity, our study will focus first on the simpler case $\beta = 0$, where the energies of the nodes play no role in the NGF dynamics. Using the master equation approach [50–52] we show that depending on the dimensions d and δ, and on the flavor s, the generalized degrees $k_{d,\delta}$ can follow either binomial or exponential or power-law distributions. The power-law distributions are characterized by the asymptotic behavior for large generalized degree $k_{d,\delta} = k \gg 1$ given by

$$P^{[s]}_{k,\delta}(k) \simeq k^{-\gamma^{[s]}_{d,\delta}}. \tag{34}$$

Our results on the generalized degree distribution of NGF of different flavor s, dimension d, and $\beta = 0$ are summarized in Table I.

TABLE I. Distribution of generalized degrees of faces of dimension δ in a d-dimensional NGF of flavor s at $\beta = 0$. For $d \geqslant d_c^{[\delta,s]} = 2(\delta+1) - s$ the power-law distributions are scale-free, i.e., the second moment of the distribution diverges.

flavor	$s=-1$	$s=0$	$s=1$
$\delta = d-1$	Binomial	Exponential	Power law
$\delta = d-2$	Exponential	Power law	Power law
$\delta \leqslant d-3$	Power law	Power law	Power law

Additionally, power-law distributions can be characterized either by a power-law exponent $\gamma^{[s]}_{d,\delta} > 3$ or $\gamma^{[s]}_{d,\delta} \leqslant 3$ indicating, in the second case, a divergent second moment $\langle k^2_{d,\delta} \rangle$ of the generalized degree distribution $P^{[s]}_{d,\delta}(k)$. The critical dimension $d_c^{[\delta,s]}$ is the smallest dimension d of the NGF of flavor s for which the generalized degree distribution $P^{[s]}_{d,\delta}(k)$ is scale free. For obtaining the exact asymptotic expression for the generalized degree distribution $P^{[s]}_{d,\delta}(k)$ of generalized degree $k_{d,\delta} = k$ in NGF of flavor s with $s = -1,0,1$ we use the master equation approach [50–52]. Here we discuss in detail the results in the cases $s = -1,0,1$. For details of the calculation we refer the reader to the Supplemental Material [76].

B. Generalized degree distribution $P^{[-1]}_{d,\delta}(k)$ for $s = -1, \beta = 0$

In the case $s = -1$ NGF generates manifolds also called CQNM [13]. At $\beta = 0$ the generalized degree follows a binomial distribution for faces of dimension $\delta = d-1$, an exponential distribution for faces of dimension $\delta = d-2$, and a power-law distribution for faces of dimension $\delta \leqslant d-3$ (see Table I). In particular, the distributions $P^{[-1]}_{d,\delta}(k)$ of generalized degrees $k_{d,\delta}$ are given by

$$P^{[-1]}_{d,d-1}(k) = \begin{cases} \frac{d-1}{d} & \text{for} \quad k = 1 \\ \frac{1}{d} & \text{for} \quad k = 2 \end{cases},$$

$$P^{[-1]}_{d,d-2}(k) = \left(\frac{2}{d+1}\right)^k \frac{d-1}{2},$$

$$P^{[-1]}_{d,\delta}(k) = \frac{d-1}{d-\delta-2} \frac{\Gamma[1+(d+1)/(d-\delta-2)]}{\Gamma[1+2/(d-\delta-2)]} \times \frac{\Gamma[k+2/(d-\delta-2)]}{\Gamma[k+1+(d+1)/(d-\delta-2)]},$$

$$\text{for} \quad \delta \leqslant d-3. \tag{35}$$

These distributions perfectly match the simulation results as shown in Fig. 3. For $\delta \leqslant d-3$ and for large values of k, the distribution $P^{[-1]}_{d,\delta}(k)$ can be fitted by a power law given by Eq. (34) with power-law exponent $\gamma^{[-1]}_{d,\delta}$ given by

$$\gamma^{[-1]}_{d,\delta} = 1 + \frac{d-1}{d-\delta-2}. \tag{36}$$

This exponent is lower than 3, i.e., $\gamma^{[-1]}_{d,\delta} \leqslant 3$ indicating a scale-free distribution of generalized degrees above the critical dimension, i.e., for $d \geqslant d_c^{[\delta,-1]}$ where

$$d_c^{[\delta,-1]} = 2\delta + 3. \tag{37}$$

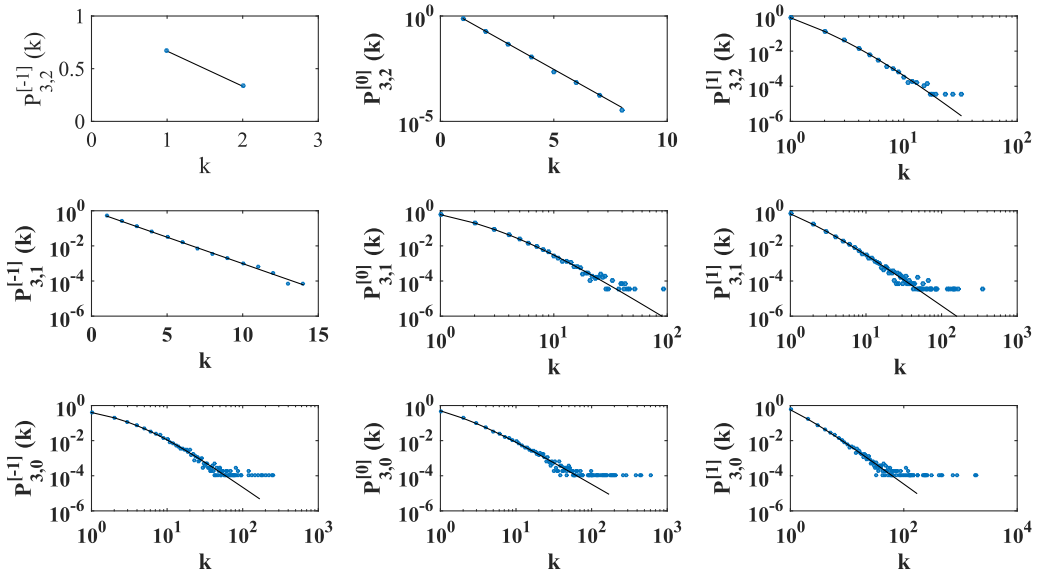


FIG. 3. The distribution $P^{[s]}_{d,\delta}(k)$ of generalized degrees $k_{d,\delta} = k$ in NGF of dimension $d = 3$ with value of flavor $s = -1, 0, 1$ and $\beta = 0$. The simulation results indicated with blue circles are shown for networks of $N = 10^4$ nodes. These results perfectly match the theoretical predictions of Eqs. (35), (38), and (41) indicated here with solid black lines.

Therefore for NGF with flavor $s = -1$ and $\beta = 0$ the generalized degree of faces of dimension $\delta = d - 2$ follows an exponential distribution. This result implies that in this case the Regge curvature R given by Eq. (25) is following an exponential distribution too.

C. Generalized degree distribution $P^{[0]}_{d,\delta}(k)$ for $s = 0, \beta = 0$

In the case $s = 0$, the generalized degree of $(d-1)$ faces follows an exponential distribution, while the generalized degree of faces of dimension $\delta \leqslant d-2$ follows a power-law distribution (see Table I). Specifically, the distribution $P^{[0]}_{d,\delta}(k)$ of generalized degree $k_{d,\delta}$ is given by

$$P^{[0]}_{d,d-1}(k) = \left(\frac{1}{d+1}\right)^k d$$

$$P^{[0]}_{d,\delta}(k) = \frac{d}{d-\delta-1}\frac{\Gamma[1+(d+1)/(d-\delta-1)]}{\Gamma[1+1/(d-\delta-1)]} \times \frac{\Gamma[k+1/(d-\delta-1)]}{\Gamma[k+1+(d+1)/(d-\delta-1)]},$$

$$\text{for} \quad \delta \leqslant d-2. \tag{38}$$

These distributions perfectly match the simulation results as shown in Fig. 3. For $\delta \leqslant d-2$ and for large values of k the distribution $P^{[0]}_{d,\delta}(k)$ can be fitted by a power law given by Eq. (34) with power-law exponent $\gamma^{[0]}_{d,\delta}$ given by

$$\gamma^{[0]}_{d,\delta} = 1 + \frac{d}{d-\delta-1}. \tag{39}$$

This exponent is lower than 3, i.e., $\gamma^{[0]}_{d,\delta} \leqslant 3$ indicating a scale-free distribution of generalized degrees above the critical dimension, i.e., for $d \geqslant d_c^{[\delta,0]}$ where

$$d_c^{[\delta,0]} = 2\delta + 2. \tag{40}$$

D. Generalized degree distribution $P^{[1]}_{d,\delta}(k)$ for $s = 1, \beta = 0$

In the case $s = 1$ the generalized degree distribution $P^{[-1]}_{d,\delta}$ is power-law (see Table I) for any dimension $\delta \leqslant d-1$ and is given by

$$P^{[1]}_{d,\delta}(k) = \frac{d+1}{d-\delta}\Gamma[1+(d+1)/(d-\delta)] \times \frac{\Gamma[k]}{\Gamma[k+1+(d+1)/(d-\delta)]}. \tag{41}$$

These distributions perfectly match the simulation results as shown in Fig. 3. For any $\delta \leqslant d-1$ for large values of k the distribution $P^{[1]}_{d,\delta}(k)$ can be fitted by a power law given by Eq. (34) with power-law exponent $\gamma^{[1]}_{d,\delta}$ given by

$$\gamma^{[1]}_{d,\delta} = 1 + \frac{d+1}{d-\delta}. \tag{42}$$

This exponent is lower than 3, i.e., $\gamma^{[1]}_{d,\delta} \leqslant 3$ indicating a scale-free distribution of generalized degrees above the critical dimension, i.e., for $d \geqslant d_c^{[\delta,1]}$ where

$$d_c^{[\delta,1]} = 2\delta + 1. \tag{43}$$

E. Critical dimensions $d_c^{[\delta,s]}$

Summarizing the results of the previous paragraphs, NGFs of flavor s follow a regular pattern, with the flavor s having the effect of shifting the statistical properties of generalized degree $k_{d,\delta}$ as indicated in Table I. The critical dimension for having a scale-free distribution of generalized degree for faces of dimension δ in NGF of dimension d at $\beta = 0$ is given by

$$d_c^{[\delta,s]} = 2(\delta+1) - s, \tag{44}$$

which is a simple expression, which summarizes the Eqs. (37), (40), (43). Therefore, the generalized degree distribution $P_{d,\delta}(k)$ of NGF of flavor s is scale free for every dimension d

of the NGF satisfying

$$d \geqslant d_c^{[\delta,s]} = 2(\delta+1) - s. \tag{45}$$

Since in NGF the generalized degree of node α,$k_{d,0}(\alpha)$, is related to its degree $K(\alpha)$ by the simple relation

$$K(\alpha) = k_{d,0}(\alpha) + d - 1, \tag{46}$$

the critical dimension $d_c^{[0,s]}$ indicates also the smallest dimension d of the NGF for which the NGF has a scale-free degree distribution. Therefore, the NGFs at $\beta = 0$ are scale free networks as long as the dimension d is greater than the critical dimension $d_c^{[0,s]}$, i.e.,

$$d \geqslant d_c^{[0,s]} = 2 - s. \tag{47}$$

Therefore, for $s = -1$ NGF at $\beta = 0$ are scale free for $d \geqslant d_c^{[0,-1]} = 3$, while for $s = 0$ they are scale free for any $d \geqslant d_c^{[0,0]} = 2$, and for $s = 1$ they are scale free for any dimension $d \geqslant d_c^{[0,-1]} = 1$.

This interesting result implies that an explicit preferential attachment rule is not necessary for generating scale-free NGFs in dimension $d > 1$. In fact both NGFs with flavor $s = 0$ and $s = -1$ do not have an explicit preferential attachment rule, but they can generate scale-free networks respectively for $d \geqslant 2$ and $d \geqslant 3$. This apparent contradiction with the results obtained by the seminal Barabási-Albert model [61] is solved by observing that NGFs of dimension $d > 1$ and flavor $s \neq -1$ that are scale free, although they do not evolve according to an explicit preferential attachment rule, follow an effective preferential attachment rule emergent from their dynamics (see Supplemental Material [76] for details).

V. QUANTUM NETWORK STATES

To each NGF of flavor s, evolved up to time t, we can associate a quantum network state $|\psi^{[s]}(t)\rangle$ belonging to the Hilbert space $\mathcal{H}_{\rm tot}^{[s]}$ by following a similar procedure as the one used in precedent works [12,13,45–47]. A Hilbert space $\mathcal{H}_{\rm tot}^{[s]}$ is associated to a simplicial complex of N nodes formed by gluing together d-dimensional simplices along $(d-1)$ faces. The Hilbert space $\mathcal{H}_{\rm tot}^{[s]}$ is the tensorial product of the Hilbert spaces $\mathcal{H}_{\rm node}$ associated to the nodes of the NGF and of two Hilbert spaces $\mathcal{H}_{d,d-1}$ and $\tilde{\mathcal{H}}_{d,d-1}^{[s]}$ associated to each of the possible $(d-1)$ faces of the NGF, i.e.,

$$\mathcal{H}_{\rm tot}^{[s]} = \bigotimes^{N} \mathcal{H}_{\rm node} \bigotimes^{P} \mathcal{H}_{d,d-1} \bigotimes^{P} \tilde{\mathcal{H}}_{d,d-1}^{[s]}, \tag{48}$$

with $P = \binom{N}{d}$ indicating the maximum number of $(d-1)$ faces in a network of N nodes. The Hilbert space $\mathcal{H}_{\rm node}$ is the one of a fermionic oscillator of energy ϵ_i, with basis $\{|o_i,\epsilon\rangle\}$, with $o_i = 0,1$. We indicate with $b_i^\dagger(\epsilon)$,$b_i(\epsilon)$, respectively, the fermionic creation and annihilation operators acting on this space. The Hilbert space $\mathcal{H}_{d,d-1}$ associated to a $(d-1)$ face α is the Hilbert space of a fermionic oscillator with basis $\{|a_\alpha\rangle\}$, with $a_\alpha = 0,1$. We indicate with $c_\alpha^\dagger$,c_α respectively the fermionic creation and annihilation operators acting on this space. Finally, the Hilbert space $\tilde{\mathcal{H}}_{d,d-1}^{[s]}$ associated to a $(d-1)$ face α has a different definition depending on the flavor s of the NGF. For $s = -1$,$\tilde{\mathcal{H}}_{d,d-1}^{[-1]}$ is the Hilbert space of a fermionic oscillator with basis $\{|n_\alpha\rangle\}$, with $n_\alpha = 0,1$. For $s = 1$,$\tilde{\mathcal{H}}_{d,d-1}^{[1]}$ is the Hilbert space of a bosonic oscillator with basis $\{|n_\alpha\rangle\}$, with $n_\alpha = 0,1,2,3,\ldots$. For $s = 0$,$\tilde{\mathcal{H}}_{d,d-1}^{[0]}$ is the Hilbert space with basis $\{|n_\alpha\rangle\}$, with $n_\alpha = 0,1,2,3,\ldots$. For $s = 1$, and $s = -1$ we indicate with $h_\alpha^{\dagger,[s]}$,$h_\alpha^{[s]}$ the fermionic and bosonic creation and annihilation operators acting respectively on the space $\tilde{\mathcal{H}}_{d,d-1}^{[-1]}$ and $\tilde{\mathcal{H}}_{d,d-1}^{[1]}$. For $s = 0$ we indicate with $h_\alpha^{\dagger,[0]}$,$h_\alpha^{[0]}$ the operators with commutation relations

$$\left[h_\alpha^{\dagger,[0]}, h_\alpha^{[0]}\right] = b \tag{49}$$

with the operator b having elements

$$b_{mn} = \langle m|b|n\rangle = \delta_{m,n}\delta_{m,0}, \tag{50}$$

such that

$$h^{\dagger,[0]}|n\rangle = |n+1\rangle \tag{51}$$

and

$$h^{[0]}|n\rangle = |n-1\rangle \quad \text{for} \quad n > 0$$
$$h^{[0]}|0\rangle = 0.$$

Having introduced the Hilbert space $\mathcal{H}_{\rm tot}^{[s]}$, we can decompose any quantum network state $|\phi\rangle \in \mathcal{H}_{\rm tot}^{[s]}$ as

$$|\phi\rangle = \sum_{\{o_i,\epsilon_i,a_\alpha,n_\alpha\}} C(\{o_i,\epsilon_i,a_\alpha,n_\alpha\}) \bigotimes_{i=1}^{N} |o_i,\epsilon_i\rangle \times \bigotimes_{\alpha\in\mathcal{Q}_{d,d-1}(N)} (|a_\alpha\rangle \otimes |n_\alpha\rangle), \tag{52}$$

where with $\mathcal{Q}_{d,d-1}(N)$ we indicate all the possible $(d-1)$ faces of a network of N nodes.

The node states $|o_i,\epsilon\rangle$ are mapped respectively to the presence $(|o_i = 1,\epsilon\rangle)$ or the absence $(|o_i = 0,\epsilon\rangle)$ of a node i of energy $\epsilon_i = \epsilon$ in the simplicial complex. The state $|a_\alpha = 1\rangle$ is mapped to the presence of the $(d-1)$ face $\alpha \in \mathcal{S}_{d,d-1}$ in the network while the quantum state $|a_\alpha = 0\rangle$ is mapped to the absence of such a face. Moreover, when $a_\alpha = 1$, the quantum number n_α is mapped to the generalized degree of the face α minus one $k_{d,d-1}(\alpha) - 1$. Note that for $s = -1$ the Hilbert space $\tilde{\mathcal{H}}_{d,d-1}^{[s]}$ is the one of a fermionic oscillator therefore allowing only $n_\alpha = 0,1$ corresponding to generalized degrees $k_{d,d-1}(\alpha) = 1,2$.

As already proposed in the literature [12,13,45], here we assume that the quantum network state follows a Markovian evolution. In particular we assume that at time $t = 1$ the state is given by

$$|\psi^{[s]}(1)\rangle = \frac{1}{\sqrt{\hat{\mathcal{Z}}^{[s]}(1)}} \sum_{\{\epsilon_i\}_{i=1,..d+1}} \prod_{i=1}^{d+1} \sqrt{g(\epsilon_i)} b_i^\dagger(\epsilon_i) \times \prod_{\alpha\in\mathcal{Q}_{d,d-1}(d+1)} c_\alpha^\dagger |0\rangle, \tag{53}$$

where $\mathcal{Z}^{[s]}(1)$ is fixed by the normalization condition $\langle\psi^{[s]}(1)|\psi^{[s]}(1)\rangle = 1$. The quantum network state is updated at each time $t > 1$ according to the transition matrix $T_t^{[s]}$,

i.e.,

$$|\psi^{[s]}(t)\rangle = T_t^{[s]}|\psi(t-1)\rangle \tag{54}$$

with $T_t^{[s]}$ given by

$$T_t^{[s]} = \sqrt{\frac{\hat{\mathcal{Z}}^{[s]}(t-1)}{\hat{\mathcal{Z}}^{[s]}(t)}} \sum_{\epsilon_{t+d}} \sqrt{g(\epsilon_{t+d})} b_{t+d}^{\dagger}(\epsilon_{t+d}) \times \sum_{\alpha \in \mathcal{Q}_{d,d-1}(t+d-1)} e^{-\beta\epsilon_\alpha/2} \left[\prod_{\alpha' \in \mathcal{F}(t+d,\alpha)} c_{\alpha'}^{\dagger} \right] h_{\alpha}^{\dagger,[s]} c_{\alpha}^{\dagger} c_{\alpha},$$

where $\mathcal{F}(i,\alpha)$ indicates the set of all the $(d-1)$ faces α' formed by the node i and a subset of the nodes in $\alpha \in \mathcal{Q}_{d,d-1}(N)$, $\mathcal{Z}^{[s]}(t)$ is fixed by the normalization condition

$$\langle \psi^{[s]}(t)|\psi^{[s]}(t)\rangle = 1. \tag{55}$$

The quantity $\hat{\mathcal{Z}}^{[s]}(t)$ is a path integral over NGF evolutions determined by the sequences $\{\epsilon_i\}_{i\leqslant t+d}, \{\alpha_{t'}\}_{t'\leqslant t}$. In fact, using the normalization condition in Eq. (55) and the evolution of the quantum network state given by Eqs. (54), (55) we get

$$\hat{\mathcal{Z}}^{[s]} = \mathcal{Z}^{[s]}, \tag{56}$$

where $\mathcal{Z}^{[s]}$ defined in Eq. (14) describes the temporal evolution of NGF, and therefore

$$\hat{\mathcal{Z}}^{[s]} = \sum_{\{\alpha(t')\}_{t'\leqslant t}} e^{-\beta E} \prod_{\alpha \in \mathcal{Q}_{d,d-1}(t)} [1 + s n_\alpha(t)]!. \tag{57}$$

This implies that the set of all classical evolutions of the CQNM fully determines the properties of the quantum network state evolving through the Markovian dynamics given by Eq. (54).

VI. QUANTUM STATISTICS IN NETWORK GEOMETRY WITH FLAVOR s

A. Fermi-Dirac, Boltzmann, and Bose-Einstein statistics describe the properties of the generalized degree of δ faces

For $\beta > 0$, as long as β is sufficiently low, we can define self-consistently the chemical potentials $\mu_{d,\delta}^{[s]}$ and express the distributions $P_{k,\delta}^{[s]}(k)$ of the generalized degrees $k_{d,\delta}$ as convolution of binomial, exponential, or power-law distributions corresponding to the generalized degrees of δ faces of energy ϵ. These distributions depend on the chemical potentials $\mu_{d,\delta}^{[s]}$. When we average the generalized degrees of δ faces of energy ϵ and subtract one, i.e., we evaluate $\langle k_{d,\delta} - 1|\epsilon,s\rangle$, we observe that these quantities obey either the Fermi-Dirac, the Boltzmann, or the Bose-Einstein statistics, depending on the dimensions d and δ and on the flavor s of the NGF, where the Fermi-Dirac $n_F(\epsilon,\mu)$, the Boltzmann $n_B(\epsilon,\mu)$, and the Bose-Einstein statistics are given [82] by the expressions

$$n_F(\epsilon,\mu) = \frac{1}{e^{\beta(\epsilon-\mu)}+1}, \quad n_Z(\epsilon,\mu) = e^{-\beta(\epsilon-\mu)}, \quad n_B(\epsilon,\mu) = \frac{1}{e^{\beta(\epsilon-\mu)}-1}. \tag{58}$$

TABLE II. The average $\langle k_{d,\delta} - 1|\epsilon,s\rangle$ of the generalized degrees $k_{d,\delta}$ of δ faces with energy ϵ minus one in a d-dimensional NGF of flavor s follows either the Fermi-Dirac, the Boltzmann, or the Bose-Einstein statistics depending on the values of the dimensions d and δ.

flavor	$s=-1$	$s=0$	$s=1$
$\delta = d-1$	Fermi-Dirac	Boltzmann	Bose-Einstein
$\delta = d-2$	Boltzmann	Bose-Einstein	Bose-Einstein
$\delta \leqslant d-3$	Bose-Einstein	Bose-Einstein	Bose-Einstein

The results are summarized in Table II and simulation results are compared with the theoretical expectations in Fig. 4.

We note here that the average of $k_{d,d-1}(\alpha) - 1 = n_\alpha$ obeys the Fermi-Dirac statistics for $s=-1$, the Boltzmann statistics for $s=0$, and the Bose-Einstein statistics for $s=1$. This is particularly surprising because it shows that the statistical properties of NGF are intertwined with the properties of quantum network states in which n_α is mapped to a quantum number, which is fermionic in the case $s=-1$ and bosonic in the case $s=1$. Therefore, statistically, on the NGF n_α follows the Fermi-Dirac statistics for $s=-1$ and the Bose-Einstein statistics for $s=1$ even if the NGF does not follow quantum equilibrium statistical mechanics. In order to show this result, let us give the results of the master equation approach for the generalized degree distribution $P_{d,\delta}^{[s]}(k)$ for $\beta > 0$ (for the details of the derivation see the Supplemental Material [76]). We will distinguish the cases in which the flavor s takes value $s=-1,0,1$.

B. Generalized degree distribution $P_{d,\delta}^{[-1]}(k)$ for $s=-1, \beta > 0$

As long as the NGF is not a chain, i.e., $d > 1$, and as long as we consider sufficiently low values of the inverse temperature β, we can define a set of self-consistent quantities that we call the chemical potentials $\mu_{d,\delta}^{[-1]}$. The generalized degrees $k_{d,\delta} = k$ of NGF with $d > 1$ follow the distribution $P_{d,\delta}^{[-1]}(k)$ that depends on the chemical potential $\mu_{d,\delta}^{[-1]}$, and is given by a binomial distribution defined only for $k = 1,2$ (for $\delta = d-1$), by a convolution of exponentials (for $\delta = d-2$), or by a convolution of power-law distributions (for $\delta \leqslant d-3$) [13]. In fact the exact asymptotic expression of the distribution $P_{d,\delta}^{[-1]}(k)$ of the generalized degree $k_{d,\delta} = k$ obtained with the master equation approach is given by

$$P_{d,d-1}^{[-1]}(1) = \sum_{\epsilon} \rho_{d,d-1}^{[-1]}(\epsilon) \left(1 - \frac{1}{\exp\left[\beta\left(\epsilon - \mu_{d,d-1}^{[-1]}\right)\right]+1}\right),$$

$$P_{d,d-1}^{[-1]}(2) = \sum_{\epsilon} \rho_{d,d-1}^{[-1]}(\epsilon) \frac{1}{\exp\left[\beta\left(\epsilon - \mu_{d,d-1}^{[-1]}\right)\right]+1},$$

$$P^{[-1]}_{d,d-2}(k) = \sum_{\epsilon} \rho^{[-1]}_{d,d-2}(\epsilon) \frac{\exp\left[\beta\left(\epsilon - \mu^{[-1]}_{d,d-2}\right)\right]}{\left\{\exp\left[\beta\left(\epsilon - \mu^{[-1]}_{d,d-2}\right)\right] + 1\right\}^{k}},$$

$$P^{[-1]}_{d,\delta}(k) = \sum_{\epsilon} \rho^{[-1]}_{d,\delta}(\epsilon) \frac{\exp\left[\beta\left(\epsilon - \mu^{[-1]}_{d,\delta}\right)\right]\Gamma[k + 2/(d-\delta-2)]}{\Gamma\left[k + 1 + 2/(d-\delta-2) + \exp\left[\beta\left(\epsilon - \mu^{[-1]}_{d,\delta}\right)\right]\right]} \times \frac{\Gamma\left[1 + 2/(d-\delta-2) + \exp\left[\beta\left(\epsilon - \mu^{[-1]}_{d,\delta}\right)\right]\right]}{\Gamma[1 + 2/(d-\delta-2)]},$$

where $\rho^{[-1]}_{d,\delta}(\epsilon)$ indicates the probability that a δ face has energy ϵ, the dimension d is greater than one, i.e., $d > 1$, and the last expression is valid for values of δ satisfying $0 \leqslant \delta \leqslant d-3$. The average of the generalized degree minus one, performed over δ faces of energy ϵ in dimension $d > 1$, is given by the Fermi-Dirac statistics for $\delta = d-1$, the Boltzmann statistics for $\delta = d-2$ and the Bose-Einstein statistics for $\delta \leqslant d-3$ [13]

$$\begin{aligned} \langle k_{d,d-1} - 1 | \epsilon, s = -1 \rangle &= n_F\left(\epsilon, \mu^{[-1]}_{d,d-1}\right), \\ \langle k_{d,d-2} - 1 | \epsilon, s = -1 \rangle &= n_Z\left(\epsilon, \mu^{[-1]}_{d,d-2}\right), \\ \langle k_{d,\delta} - 1 | \epsilon, s = -1 \rangle &= A^{[-1]}_{d,\delta} n_B\left(\epsilon, \mu^{[-1]}_{d,\delta}\right), \end{aligned} \tag{59}$$

where the last expression is valid for $\delta \leqslant d-3$, and where $n_F(\epsilon,\mu)$,$n_Z(\epsilon,\mu)$, and $n_B(\epsilon,\mu)$ are given by Eqs. (58), while $A^{[-1]}_{d,\delta}$ is given by

$$A^{[-1]}_{d,\delta} = \frac{(d-\delta)}{(d-\delta-2)}. \tag{60}$$

These relations perfectly match the simulation results for sufficiently low value of the inverse temperature β (see Fig. 4). The self-consistent value of the chemical potential can be found by imposing the following geometrical relations satisfied by the generalized degrees of the NGF of every flavor s,

$$\lim_{t\to\infty} \frac{\sum_{\alpha \in \mathcal{S}_{d,\delta}(t)} k_{d,\delta}(\alpha)}{N_{d,\delta}(t)} = \frac{d+1}{\delta+1}. \tag{61}$$

Imposing such condition is equivalent to fixing the normalization conditions for $n_F(\epsilon,\mu^{[-1]}_{d,d-1})$,$n_Z(\epsilon,\mu^{[-1]}_{d,d-2})$, and $n_B(\epsilon,\mu^{[-1]}_{d,\delta})$. These conditions are given by

$$\begin{aligned} \sum_{\epsilon} \rho_{d,d-1}(\epsilon) n_F\left(\epsilon, \mu^{[-1]}_{d,d-1}\right) &= \frac{1}{d}, \\ \sum_{\epsilon} \rho_{d,d-2}(\epsilon) n_Z\left(\epsilon, \mu^{[-1]}_{d,d-2}\right) &= \frac{2}{d-1}, \\ \sum_{\epsilon} \rho_{d,\delta}(\epsilon) n_B\left(\epsilon, \mu^{[-1]}_{d,\delta}\right) &= \frac{d-\delta-2}{\delta+1}. \end{aligned} \tag{62}$$

The case $d = 1$ is an exception because it is the only case in which the area A of the NGF is not growing in time, in fact we have $A = 2$ for every value of t. This property of the NGF of flavor $s = -1$ in dimension $d = 1$ makes this case significantly different from the other cases, but fortunately this NGF has a much simpler dynamics, since it is a chain.

C. Generalized degree distribution $P^{[0]}_{d,\delta}(k)$ for $s = 0, \beta = 0$

For NGF of flavor $s = 0$, using the master equation approach together with the self-consistent derivation, we can derive the distribution $P^{[0]}_{d,\delta}(k)$ of generalized degrees $k_{d,\delta} = k$. Therefore, we define self-consistently the chemical potentials $\mu^{[0]}_{d,\delta}$, and express the distribution $P^{[0]}_{d,\delta}(k)$ as a convolution of exponentials or a convolution of power-law distributions depending on the dimension d and δ. These distributions are given by

$$P^{[0]}_{d,d-1}(k) = \sum_{\epsilon} \rho^{[0]}_{d,d-1}(\epsilon) \frac{e^{\beta(\epsilon - \mu^{[0]}_{d,d-2})}}{(e^{\beta(\epsilon - \mu^{[0]}_{d,d-2})} + 1)^{k}},$$

$$P^{[0]}_{d,\delta}(k) = \sum_{\epsilon} \rho^{[0]}_{d,\delta}(\epsilon) \frac{\exp\left[\beta(\epsilon - \mu^{[0]}_{d,\delta})\right]\Gamma[k + 1/(d-\delta-1)]}{\Gamma\left[k + 1 + 1/(d-\delta-1) + \exp\left[\beta\left(\epsilon - \mu^{[0]}_{d,\delta}\right)\right]\right]} \times \frac{\Gamma\left[1 + 1/(d-\delta-1) + \exp\left[\beta\left(\epsilon - \mu^{[0]}_{d,\delta}\right)\right]\right]}{\Gamma[1 + 1/(d-\delta-1)]}, \tag{63}$$

where $\rho^{[0]}_{d,\delta}(\epsilon)$ indicates the probability that a δ face has energy ϵ, and where the last equation is valid for values of δ satisfying $0 \leqslant \delta \leqslant d-2$. Therefore, the $(d-1)$ faces have generalized degree distribution $P^{[0]}_{d,d-1}(k)$ that is given by a convolution of exponentials, while the δ faces with $\delta \leqslant d-2$ have a generalized degree distribution $P^{[0]}_{d,\delta}(k)$ that is given by a convolution of power laws. When considering the average $\langle k_{d,\delta} - 1 | \epsilon, s = 0 \rangle$, we observe that for $\delta = d-1$ this quantity is a Boltzmann distribution and for every $\delta \leqslant d-2$ is a Bose-Einstein distribution,

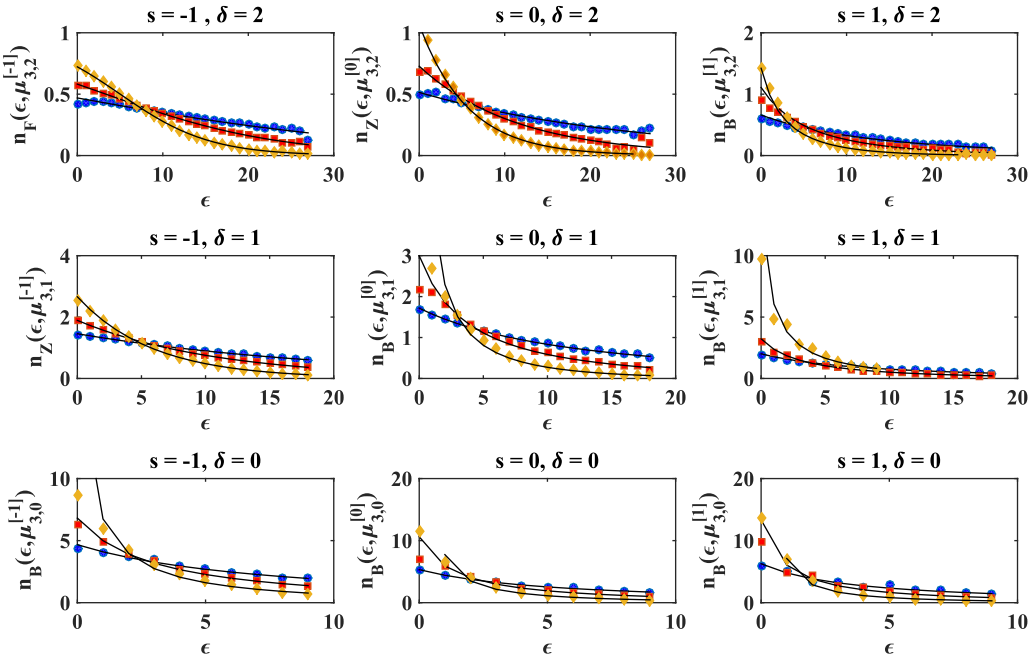


FIG. 4. The average $\langle k_{d,\delta}-1|\epsilon,s\rangle$ for NGF of flavor s in dimension $d=3$ follows either the Fermi-Dirac statistics $n_F(\epsilon,\mu)$, the Boltzmann statistics $n_Z(\epsilon,\mu)$ or the Bose-Einstein statistics $n_B(\epsilon,\mu)$ given by Eqs. (58) depending on the value of δ and s as predicted by Eqs. (60), (65), and (68). Here the simulation results for NGF of dimension $d=3$ formed by $N=10^4$ nodes for $\beta=0.05,0.1,0.2$ (indicated, respectively, by blue circles, red squares, and orange diamonds) averaged 20 times are compared with the theoretical expectations (indicated with solid black lines). The energies of the nodes take integer values and their uniform distribution is given by $g(\epsilon)=1/10$ for $0\leqslant\epsilon<10$.

i.e.,

$$\begin{aligned}\langle k_{d,d-1}-1|\epsilon,s=0\rangle &= n_Z\big(\epsilon,\mu^{[0]}_{d,d-2}\big),\\ \langle k_{d,\delta}-1|\epsilon,s=0\rangle &= A^{[0]}_{d,\delta}n_B\big(\epsilon,\mu^{[0]}_{d,\delta}\big),\quad \text{for}\quad \delta\leqslant d-2\end{aligned} \tag{64}$$

with $n_Z(\epsilon,\mu)$ and $n_B(\epsilon,\mu)$ given by Eqs. (58) and $A^{[0]}_{d,\delta}$ given by

$$A^{[0]}_{d,\delta}=\frac{(d-\delta)}{(d-\delta-1)}. \tag{65}$$

The chemical potential $\mu^{[0]}_{d,\delta}$ can then be found imposing the condition in Eq. (61) that all NGFs must satisfy. Therefore, the self-consistent equations that the chemical potentials must satisfy are

$$\begin{aligned}\sum_\epsilon \rho_{d,d-1}(\epsilon)n_Z\big(\epsilon,\mu^{[-1]}_{d,d-1}\big)&=\frac{1}{d},\\ \sum_\epsilon \rho_{d,\delta}(\epsilon)n_B\big(\epsilon,\mu^{[-1]}_{d,\delta}\big)&=\frac{d-\delta-1}{\delta+1},\ \text{for}\ \delta\leqslant d-2.\end{aligned} \tag{66}$$

D. Generalized degree distribution $P^{[1]}_{d,\delta}(k)$ for $s=1,\beta=0$

The NGF of flavor $s=1$, at sufficiently low inverse temperature β, has the generalized degrees $k_{d,\delta}=k$ with distribution $P^{[1]}_{d,\delta}(k)$ dependent on the chemical potential $\mu^{[1]}_{d,\delta}$. The generalized degree distributions $P^{[1]}_{d,\delta}(k)$ can be found using the master equation approach, and they are given by

$$\begin{aligned}P^{[1]}_{d,\delta}(k)=\sum_\epsilon \rho^{[1]}_{d,\delta}(\epsilon)&\frac{\exp\big[\beta\big(\epsilon-\mu^{[1]}_{d,\delta}\big)\big]\Gamma[k]}{\Gamma\big\{k+1+\exp\big[\beta\big(\epsilon-\mu^{[1]}_{d,\delta}\big)\big]\big\}}\\ &\times\Gamma\big\{1+\exp\big[\beta\big(\epsilon-\mu^{[1]}_{d,\delta}\big)\big]\big\},\end{aligned} \tag{67}$$

where $\rho^{[1]}_{d,\delta}(\epsilon)$ indicates the probability that a δ face has energy ϵ. In this case, if we perform the average $\langle k_{d,\delta}-1|\epsilon,s=1\rangle$ over all δ faces with energy ϵ, we always get the Bose-Einstein distribution, independently of $0\leqslant\delta<d$, i.e., we obtain

$$\langle k_{d,\delta}-1|\epsilon,s=1\rangle=n_B\big(\epsilon,\mu^{[1]}_{d,\delta}\big), \tag{68}$$

with $n_B(\epsilon,\mu)$ given by Eq. (58). The chemical potentials $\mu^{[1]}_{d,\delta}$ must satisfy Eq. (61). Therefore, they can be found self-consistently by solving

$$\sum_\epsilon \rho_{d,\delta}(\epsilon)n_B\big(\epsilon,\mu^{[1]}_{d,\delta}\big)=\frac{d-\delta}{\delta+1}. \tag{69}$$

E. Low-temperature regime

In the regime of low temperatures, i.e., high enough values of β, it is possible to observe a breakdown of the self-consistent hypothesis made for solving the generalized degree distribution and the self-consistent equations might not have a solution. In the NGF of $d=1$ and flavor $s=1$ there is a well-defined phase transition in which one node grabs a finite fraction of all the links. This phase transition is also called Bose-Einstein condensation in complex networks and has been characterized in Ref. [63]. In general NGF of

higher dimensions and also different flavors might show phase transitions modifying the generalized degree distribution of different δ faces as shown for the case $d = 2$ and flavors $s = -1$ and $s = 1$ in Ref. [12]. A full investigation of the nature of the possible phase transitions occurring in NGF is beyond the scope of this paper.

VII. CONCLUSIONS

In conclusion, here we have presented the model of network geometry with flavor s. This is a model for growing simplicial complexes in dimension d. Simplicial complexes are very useful generalizations of networks and can be used to model interactions involving more than just two nodes, as the one occurring, for example, in collaboration networks, or in protein-interaction networks. Moreover, simplicial complexes of dimension d are useful structures to discretize a geometrical d-dimensional space, and for this reason they are widely used in quantum gravity.

Network geometry with flavor s evolves by a nonequilibrium dynamics that enforces an indefinite growth of these geometrical structures. Moreover, these networks are formed by simplices having heterogeneous properties modeled by assigning an energy to them that determines their evolution. The statistical mechanics of the NGF allows us to characterize the thermodynamic properties of these networks and to relate these networks to complexity theory on the one side and to quantum geometry on the other side.

The thermodynamic properties of NGF reveal that these networks obey the area law and the change in their entropy S depends on the change of their area A. From the point of view of network theory we observe that characterizing NGF of dimensionality $d > 1$ allows for a significant generalization of previous results, showing that an explicit preferential attachment is not necessary for obtaining scale-free networks in the case of NGF of $d > 1$. Finally, the significant interplay between the NGF and their quantum mechanical description in terms of quantum network states is revealed by the statistical properties of the generalized degrees of δ faces, whose average follows either the Fermi-Dirac, the Boltzmann, or the Bose-Einstein statistics depending on the dimensions d, δ, and on the flavor s.

Overall we have proposed the theoretical framework of NGF for describing the nonequilibrium dynamics of simplicial complexes. Our framework generates a large variety of network geometries, from chains and higher-dimensional manifolds to scale-free networks with communities and small-world properties. Interestingly, NGFs with flavor $s = -1, 0, 1$ display a strikingly regular pattern in their structural properties. We believe that these results extend our understanding of growing complex networks to simplicial complexes of larger dimensionality and can be used in network theory to model networklike structures where nodes are connected by interactions involving more than two nodes. Finally, we hope that this work, showing the rich interplay between NGFs and their quantum mechanical description, will stimulate the cross fertilization between network theory and quantum gravity.

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