

Ballistic annihilation with superimposed diffusion in one dimensionSoham Biswas,^{*} Hernán Larralde,[†] and Francois Leyvraz[‡]*Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Cuernavaca 62210, Mexico*

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We consider a one-dimensional system with particles having either positive or negative velocity, and these particles annihilate on contact. Diffusion is superimposed on the ballistic motion of the particle. The annihilation may represent a reaction in which the two particles yield an inert species. This model has been the subject of previous work, in which it was shown that the particle concentration decays faster than either the purely ballistic or the purely diffusive case. We report on previously unnoticed behavior for large times when only one of the two species remains, and we also unravel the underlying fractal structure present in the system. We also consider in detail the case in which the initial concentration of right-going particles is $1/2 + \varepsilon$, with $\varepsilon \neq 0$. It is shown that remarkably rich behavior arises, in which two crossover times are observed as $\varepsilon \rightarrow 0$.

DOI: [10.1103/PhysRevE.93.022136](https://doi.org/10.1103/PhysRevE.93.022136)**I. INTRODUCTION**

Deviations from mean-field theory caused by fluctuations have been the subject of a considerable amount of research [1]. Thus, when reactants diffuse and react on contact, they create spatial correlations that invalidate the usual concept of mean-field theory, and indeed, invalidate the applicability of rate equations, which are normally viewed as fundamental in chemical kinetics. A very simple example of this is one-species annihilation (or aggregation) in which one species reacts with itself via a bimolecular reaction,



where B represents an inert species, which we disregard in the following. The rate equation for such a process is given by

$$\dot{c}_A = -Kc_A^2, \quad (2)$$

and the decay for large times is therefore $1/t$. However, as is well known (see, for example, [2]), if we consider a model of point random walkers on the line, which annihilate whenever two walkers alight on the same site, then the decay is in fact given by $(Dt)^{-1/2}$. At a deeper level, the amplitude of the $1/t$ decay determined by (2) depends on the initial concentration, whereas the amplitude of the $1/\sqrt{t}$ decay does not.

These and many similar results have been discussed extensively. For a review, see for example the book [3] by Redner and Krapivsky. In the following, we shall consider a variation on the model described by (1), in which the transport has both a ballistic component, in which the particles can have one of two velocities v and $-v$, and a diffusive component. The purely ballistic model was initially discussed by Elskens and Frisch in [4]. Similar to the diffusive model discussed in [2], the decay of concentration goes as $t^{-1/2}$, but for quite different reasons. As was then pointed out in [5,6], combining both diffusion and ballistic motion leads, in one dimension, to a decay that is more rapid than the one brought about by either mechanism.

The kinetics of particle annihilation, $A + A \rightarrow \emptyset$, has attracted considerable attention from the scientific commu-

nity from another completely different point of view. The dynamics of particle annihilation with $A + A \rightarrow \emptyset$ have a direct one-to-one correspondence with the kinetic Ising model in one dimension. For example, the zero-temperature Glauber dynamics in a one-dimensional Ising spin system with nearest-neighbor interaction can be mapped to the $A + A \rightarrow \emptyset$ system, in which the A are pure random walkers [7]. On the other hand, the binary opinion dynamic models in one dimension (which can be directly mapped to a one-dimensional Ising spin system) have a much more complicated walker picture, where the A walkers are not purely random. There are also binary opinion dynamic models, in which A walkers perform a complicated ballistic motion. For example, in the dynamics introduced in [8], the boundaries of the domains (which could be construed as walkers) move ballistically unless they meet the boundary of some other domain. Once two boundaries meet, the annihilation of the boundaries is more involved than the simple $A + A \rightarrow \emptyset$. In another example of a binary opinion dynamics model introduced in [9], the walkers in the corresponding walker picture have ballistic motion; in this case, the A walkers always move ballistically in the direction of their nearest walker and annihilate upon meeting, following $A + A \rightarrow \emptyset$. This leads to complex changes in the direction of motion of the walkers, somewhat akin to a random walk [9]. Understanding the dynamics of the simple ballistic annihilation process with superimposed diffusion can thus also be of help in understanding these complex dynamics.

The simple model of ballistic annihilation discussed in this paper has quite a rich structure. Introducing a corresponding binary opinion dynamics model could be a motivation and direction for future work.

Here we study the model of ballistic annihilation with superimposed diffusion in greater detail than was previously done, showing in particular a remarkably rich structure when the number of left- and right-going particles is allowed to be different. In particular, we study the nature of the crossovers appearing as these numbers come close to each other.

II. THE MODEL**A. Description of the model**

Let us first describe in some detail the model we study in this paper. The system we consider consists of point particles on

^{*}soham@fis.unam.mx[†]hernan@fis.unam.mx[‡]leyvraz@fis.unam.mx

a one-dimensional lattice. Each particle moves, always, either to the left or to the right, that is, each particle has positive or negative velocity. Initially there is a fraction $(1/2 + \varepsilon)$ of particles with positive velocity and a fraction $(1/2 - \varepsilon)$ of particles with negative velocity (with $-1/2 \leq \varepsilon \leq 1/2$) randomly distributed on the lattice. Whenever a particle lands on a site at which another particle is found, both are removed from the system. This is therefore a model of the so-called one-species annihilation type ($A + A \rightarrow \emptyset$). It is of central importance that the particles are moved asynchronously, that is, at each time step one particle is chosen at random and moved in the direction corresponding to the velocity of that particle.

Note that the asynchronous updating rule leads to the possibility that particles having the same velocity can react, since the random choice of the particle leads to an effective diffusive motion of the particles with respect to their neighboring particles having the same speed, whereas in a perfectly synchronous update, these particles' distances would remain fixed. It is largely the consequences of this effect that we will explore in this paper. This model is equivalent to a model in which a fraction $1/2 + \varepsilon$ of particles performs a random walk with bias v , whereas the others have a bias $-v$. We shall generally work with a bias equal to v in order to keep the dimensions explicit.

For $\varepsilon = 0$, the number of positive and negative velocity particles is equal, on average, at the beginning. Taking $\varepsilon \neq 0$ introduces inequality in the numbers of positive and negative velocity particles, and $\varepsilon = \pm 1/2$ means there is only one kind of particle.

The synchronous update version of this system has been studied by Elskens and Frisch [4], who showed that for $\varepsilon = 0$, the concentration of particles decays as $t^{-1/2}$. Here we study the following variant: instead of letting the particles move ballistically in continuous time, we discretize time and choose, at each time step, a particle at random and move it to the right if it has positive velocity, and to the left otherwise. In other words, we use an asynchronous updating rather than a synchronous one. This introduces diffusion, and thereby the possibility of two particles with the same velocity annihilating each other. This apparently minor change affects the system profoundly, as already noted in [5,6] for the case $\varepsilon = 0$. Here we extend the study to the general case, and we also display some nontrivial large time behavior in the case $\varepsilon = 0$ that had previously gone unreported. Finally, we propose a mechanism, first studied by Alemany [10], for modifying the decay exponent of diffusive annihilation kinetics in one dimension, and we show that it is indeed operative in this particular system.

Let us briefly summarize our results for the general case $\varepsilon > 0$. We find three different regimes, of which the easiest is certainly the last: at very large times, all particles with negative velocities have disappeared. Additionally, all the spatial correlations that their presence might have induced—we will see that such correlations can in fact arise—have also disappeared. We thus have a system consisting solely of right-moving particles moving ballistically, with superimposed diffusion. This is equivalent to pure diffusive dynamics, so that the asymptotic decay goes as $(Dt)^{-1/2}$. This is the third stage of the system's evolution. If $\varepsilon \ll 1$, there are two other stages: the first one is the one in which we may neglect

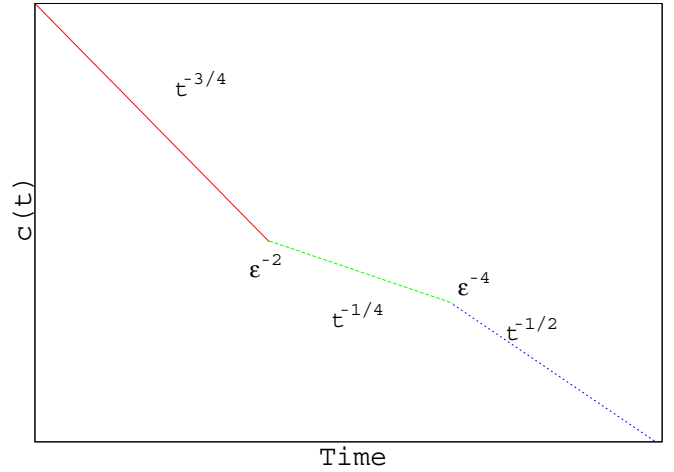


FIG. 1. A schematic figure with two crossovers for the decay of $c(t)$, which is the concentration of the total number of particles at time t .

the difference in concentration between left- and right-going particles. In this regime, the usual decay exponent reported and analyzed by ben-Naim, Redner, and Krapivsky [6] applies, and the concentration decays as $t^{-3/4}$. This stage ends when all left-going particles have disappeared. This happens, as we shall see, at a time $t_1(\varepsilon)$ of order ε^{-2} (Fig. 1). At this point, a second stage sets in, characterized by a decay with a $t^{-1/4}$ leading term (Fig. 1). This is due to the fact that the surviving right-going particles are far from being uniformly distributed, which leads to an anomalous decay, as pointed out by Alemany [10]. The third stage, when these correlations finally disappear, sets in at a time $t_2(\varepsilon)$ that scales as ε^{-4} (Fig. 1).

If we start with N particles, then one finds that at $t_1(\varepsilon)$ there remain $N\varepsilon^{3/2}$ particles, whereas at $t_2(\varepsilon)$ there only remain $N\varepsilon^{5/2}$. This brings out some numerically challenging features of the model: if we wish to have a clean separation of time scales, we need at least $\varepsilon \sim 10^{-2}$, and if we wish to have, say, 100 particles at $t_2(\varepsilon)$ (in order to be able to observe the final exponent reliably), we would need to start out with $N \sim 10^7$, which is altogether unrealistic. We will therefore rely on various simulations with different values of the parameters to bring out the various features of the system.

B. Quantities calculated

We have studied the following quantities in the present work:

(i) The decay of the concentration of the total number particles, $c(t)$, with time, starting from concentration $c(0) = 1$, which means that all sites are occupied initially.

(ii) The decay of the concentration of an excess number of particles, $c_{\text{ex}}(t)$, with time, which is defined as the absolute difference between positive and negative velocity particles. For $\varepsilon = 0$, $c_{\text{ex}}(0) \simeq 1/\sqrt{N}$ [up to an $O(1)$ factor], where N is the initial number of particles.

(iii) The change of the domain size of positive and negative velocity particles with time.

(iv) The distribution of interparticle distances between two neighboring particles (independent of their velocities) at late time. This distribution is also studied for particles with the same velocity and those with different velocities.

(v) The persistence probability $P(t)$: This is the probability that a site remains unvisited by any of the walkers A up to time $t > 0$. When the walkers perform a pure random walk, the persistence probability $P(t)$ shows a power-law decay given by $P(t) \sim t^{-\theta}$, where θ is the persistence exponent and is unrelated to any other known static or dynamic exponents [11,12].

We have studied the dynamics starting from initially N particles, with $10^4 \leq N \leq 8 \times 10^4$. The results are averaged over 2000 to 2500 configurations. Periodic boundary conditions have been used.

III. THEORY

For $\varepsilon = \pm 1/2$, only one kind of ballistic particle (either $+v$ or $-v$ velocity particles) exists in the system. Due to the random update rule for the simulation, by which diffusion is incorporated, the relative motion of the particles is actually diffusive in this case. The relative diffusion constant $D_{\text{eff}} = 1$ is the same as if the particles performed symmetric random walks (see Appendix A for details). Naturally, in this situation, the concentration $c(t)$ will decay as $t^{-1/2}$.

If $\varepsilon \neq \pm 1/2$, for any typical configuration, the number of $+v$ and $-v$ velocity particles will not be equal (this is clear for $\varepsilon \neq 0$, but also when $\varepsilon = 0$, due to statistical fluctuations). Due to this inequality in numbers, the system will go from a regime of ballistic annihilation with superimposed diffusion to a long-time regime of pure diffusion when only one kind of particle is present. In the following, we start by considering the regime in which no significant difference in the numbers of left- and right-going particles exists, $\varepsilon = 0$. In this case, as has been shown in [5,6], the particle number decays as $t^{-3/4}$. In the following subsection, we shall rederive this result for the sake of keeping the paper self-contained. Afterward, we will proceed to analyze the case of finite systems, in which the particles eventually all have one given velocity due to the effect of statistical fluctuations. Finally, we analyze the case of $\varepsilon \neq 0$.

A. The infinite system with $\varepsilon = 0$: Dimensional analysis

In the following, we analyze the system for $\varepsilon = 0$ using dimensional considerations. The various parameters involved are $c(0)$, D , t , and v . These can be combined in two dimensionless parameters:

$$x = \frac{v}{Dc(0)}, \quad (3a)$$

$$\tau = \frac{v^2 t}{D}. \quad (3b)$$

The former is the ratio of the time required for nearest neighbors to cross the average distance between them if they move ballistically, to the time required to cross the same distance with diffusive dynamics. It thus states whether diffusion or drift dominates the short-time dynamics. τ is a dimensionless

time, which separates the regime in which diffusion dominates, $\tau \ll 1$, from that in which drift dominates, $\tau \gg 1$.

The concentration of particles at time t can thus be written in terms of the adimensional quantities x and τ as follows:

$$c(t) \simeq c(0)\Phi(x, \tau). \quad (4)$$

If $x \ll 1$, then $c(t)$ should not depend on v for initial times, since the process is (in a first approximation) purely diffusive, leading to a $(Dt)^{-1/2}$ behavior. Therefore, for $x \ll 1$, using the expression of x and τ given by Eqs. (3a) and (3b), we get

$$\Phi(x, \tau) = \frac{x}{\sqrt{\tau}}. \quad (5)$$

This is only valid up to $\tau \sim 1$, since, as we have seen above, this is the crossover time between drift and diffusion.

On the other hand, for $x \gg 1$, the behavior of $c(t)$ is purely ballistic for moderate τ . It is thus independent of D and given by $\sqrt{c(0)/vt}$, as shown in [4]. Therefore, for $x \gg 1$, again using Eqs. (3a) and (3b), we find

$$\Phi(x, \tau) = \sqrt{\frac{x}{\tau}}. \quad (6)$$

Let us now determine the time τ up to which this is valid. The qualitatively new feature introduced by diffusion is the possibility for particles of the same velocity to annihilate via diffusion. But this happens on a time scale $1/[Dc(0)^2]$, which, in terms of the adimensional quantities, is $\tau \sim x^2$. We see, therefore, that the approximation (6) should hold up to $\tau \sim x^2$.

The previous Eqs. (5) and (6) only hold for short times. For the former, this is because the influence of drift is eventually felt, whereas for the latter, it is due to the diffusive annihilation of particles with like velocity. If we now describe the large τ behavior for the full dynamics by

$$\Phi(x, \tau) \simeq x^\alpha \tau^\beta, \quad (7)$$

then we obtain two conditions on α and β as follows: when $\tau \sim 1$ and $x \ll 1$, we may apply both (5) and (7). This leads to

$$x \sim x^\alpha \quad (8)$$

implying $\alpha = 1$. Similarly, if we consider the case in which $x \gg 1$ and $\tau \sim x^2$, we may apply (6) as well as (7), and we are thereby led to

$$\sqrt{\frac{1}{x}} \sim x^{\alpha+2\beta} = x^{1+2\beta}, \quad (9)$$

from which follows that

$$\beta = -\frac{3}{4}. \quad (10)$$

We thus obtain

$$\Phi(x, \tau) \simeq x\tau^{-3/4}. \quad (11)$$

This finally yields, for the concentration $c(t)$ at large times,

$$c(t) \simeq v^{-1/2} D^{-1/4} t^{-3/4}. \quad (12)$$

This dimensional analysis parallels that made in [5,6] and is presented for the sake of keeping the paper self-contained.

Note that the above derivation contains a weak point: it is assumed in (7) that the x dependence of the prefactor of τ^β is

always the same power α , both in the $x \ll 1$ and in the $x \gg 1$ regime. This is, in itself, not obvious, and we will see in later sections a derivation of the same result that is free from this objectionable feature.

B. The case of finite systems

Let us first describe the behavior of the set of surviving particles in ballistic annihilation with synchronous updating. Here we follow [4].

Let the initial condition of a system undergoing ballistic annihilation be given by the numbers σ_k , where k runs from 0 to L , where L is the length of the system, and σ_k can take three possible values: $\sigma_k = \pm 1$ means that site k is initially occupied by a particle having velocity σ_k , whereas $\sigma_k = 0$ means that site k is initially unoccupied. Assume that $\sigma_k = 1$ represents the $+v$ velocity particles and $\sigma_k = -1$ represents the $-v$ velocity particles. Under these circumstances, once the initial condition is set, something we will assume to have been done at random, then the fate of each particle is uniquely determined. Indeed, each particle has a unique annihilation partner, or else it survives indefinitely. If $\sigma_k = 1$, the unique annihilation partner is initially at position $\pi_+(k)$, defined by the following condition: let \mathcal{A}_k be defined as the following set:

$$\mathcal{A}_k := \left\{ m \in \mathbb{N} : \sum_{r=k}^{k+m} \sigma_r = 0 \right\}. \quad (13)$$

Then $\pi_+(k)$ is the smallest element of \mathcal{A}_k . If $\mathcal{A}_k = \emptyset$, then the particle initially at k survives indefinitely and $\pi_+(k) = \infty$ [a wholly similar definition works if $\sigma_k = -1$, in which case we would say the partner is at initial position $\pi_-(k)$].

Given the initial condition, each particle survives until it encounters its reaction partner. The collision time is thus

$$\tau(k) = \frac{\pi_+(k)}{2}, \quad (14)$$

where we have considered the positive velocity particles.

We now determine the structure of the set Σ_t defined as

$$\Sigma_t = \{k : \tau_+(k) > t\}. \quad (15)$$

Let us consider a pair of particles with positive velocity separated by a distance k . Without loss of generality, we may assume that we have two particles, one at 0 and one at $k > 0$. The particle at k we call the *leader*, while the one at 0 is the *follower*. For both to belong to Σ_t , the following conditions are necessary:

- (i) \mathcal{A}_0 should not have any element $r \leq t$.
- (ii) \mathcal{A}_k should not have any element $s \leq t$.

If $k > t$, then the two intervals $[0, t]$ and $[k, k + t]$ are disjoint. The probability of both sites belonging to Σ_t is thus simply the product of either site belonging to Σ_t , and no dependence on k exists. Note that this is so regardless of whether $\varepsilon = 0$ in the initial condition.

Let us now consider the opposite case. In this case, the leader “clears the way” for the follower: since \mathcal{A}_k has no element $s \leq t$, it follows that \mathcal{A}_0 cannot contain any elements in the interval $[k, k + t]$. Thus, for the follower to belong to Σ_t , it is sufficient that there be no elements in \mathcal{A}_0 belonging to the interval $[0, k]$. The probabilities that the leader and the

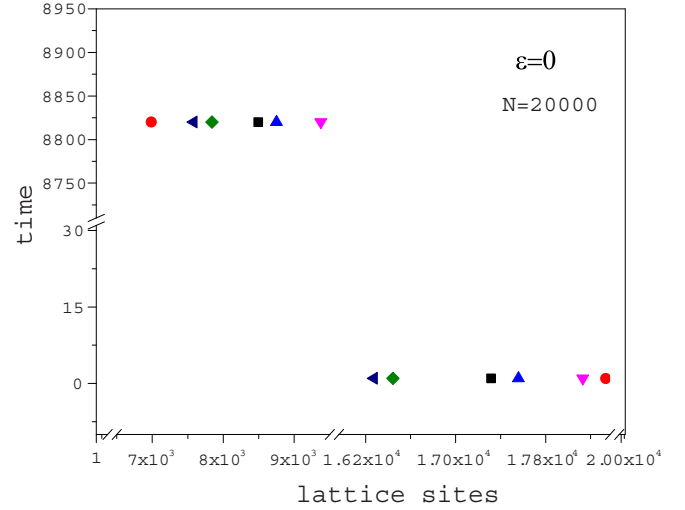


FIG. 2. Initial and very late time position of the particles that have survived for a long enough time. The picture suggests that the long-surviving particles are fractally distributed over the lattice from the beginning. This plot is generated starting initially from $N = 2 \times 10^4$ particles. Note that the number of final particles is quite small—namely six—in order to claim that these lie on a fractal. Nevertheless, we may view this as evidence of the fact that, in the limit of infinitely many initial particles, the surviving particles would indeed lie on a fractal.

follower both belong to Σ_t are thus again a product, but this time of the probability that for all $r < k$ we have

$$\sum_{m=0}^r \sigma_m > 0. \quad (16)$$

In other words, this is the probability $p_0(k)$ that a random walk, which starts at the origin and takes a step to the right, does not return to the origin before time k . The probability for this, as is well known [13], scales as $k^{-1/2}$ for $k \gg 1$, if the walk is symmetric, which corresponds, in our case, to an initial condition with $\varepsilon = 0$. Thus, if $\varepsilon = 0$, the probability of having two particles separated by a distance $k < t$ both surviving a time t is of the order of $k^{-1/2}$. In other words, this description is compatible with the set Σ_t forming a fractal set—below the cutoff value t —with fractal dimension $1/2$. The correlation function for $k \gg 1$ and $t \gg 1$, but $k < t$ is of order $(kt)^{-1/2}$.

The initial and very late time positions of the surviving particles, as shown in the schematic in Fig. 2, form a number of clusters (of two or more particles), and the clusters are well separated from each other. This indicates that the long-surviving particles are on a fractal from the beginning.

In the case with no diffusion, we therefore see that Σ_t is a fractal with a lower cutoff at length 1 (lattice spacing) and an upper cutoff t with fractal dimension $1/2$. This was the case in which we have synchronous updating. If instead we have asynchronous dynamics, the annihilation of similar particles eliminates all particles with a distance less than \sqrt{t} . The corresponding set of surviving particles then becomes a fractal of dimension $1/2$ with lower cutoff \sqrt{t} and upper cutoff t , leading to the fact that the set has $t^{1/4}$ elements in each domain of size t , thereby leading to a concentration $c(t)$

of $t^{-3/4}$. We therefore see that the particles surviving at time t_1 , at which only one species survives, also lie on a fractal of dimension $d_f = 1/2$.

The fact that Σ_{t_1} is a fractal further implies, as shown by Alemany [10], that the decay of a purely diffusive reaction starting from such an initial condition is not given by $t^{-1/2}$ but rather by $t^{-d_f/2}$, where d_f is the fractal dimension of the initial condition. In the particular case with which we are concerned here, since $d_f = 1/2$, we have a decay law of $t^{-1/4}$. This is seen in qualitative terms as follows: the probability that a given particle survives for a time t is negligible if this particle is both followed and preceded by a particle significantly closer than \sqrt{Dt} . We therefore require, for the particle to survive, that one of either neighbors be further from the central particle than \sqrt{Dt} , which has the probability $(Dt)^{-d_f/2}$. In our case, this means that the final decay, after the annihilation of all minority particles, goes as $t^{-1/4}$.

Let us look at this decay law in greater detail. When the particles are distributed on a fractal of dimension d_f ($0 < d_f < 1$), the probability distribution function for initial interparticle distances for two nearest particles will be $P(x)$,

$$P(x) = \frac{1}{\zeta(\lambda)} \sum_{l=1}^{\infty} l^{-\lambda} \delta(x-l), \quad (17)$$

where $\lambda = d_f + 1$. The mean distance between nearest-neighbor particles diverges for $0 < d_f < 1$ or for $1 < \lambda < 2$. We have obtained the decay law considering this discrete distribution given by Eq. (17) following the formalism developed by Alemany [10]. The number of particles $n(t)$ at time t [normalized by the initial number of particles $n(0)$] will be

$$n(t) = -\frac{\Gamma(1-\lambda)}{2\zeta(\lambda)\Gamma(\frac{3-\lambda}{2})} \tau^{-\frac{\lambda-1}{2}} + \frac{\Gamma(1-\lambda)}{4(1-\lambda)[\zeta(\lambda)]^2} \tau^{-(\lambda-1)} + \frac{\zeta(\lambda-1)}{2\zeta(\lambda)\sqrt{\pi}} \tau^{-1/2} + O(\tau^{-\lambda/2}), \quad (18)$$

where $\tau = D_{\text{eff}}t$. See Appendix B for details. For the fractal dimension $d_f = 1/2$, that is $\lambda = 1.5$, the leading term will be $t^{-1/4}$ and the coefficient of this leading term is positive, as $\Gamma(1-\lambda)$ is *negative* for all $\lambda > 1$. The first correction to scaling will involve both the second and third terms of (18), thereby leading to

$$n(t) = \frac{\sqrt{\pi}}{\zeta(3/2)\Gamma(3/4)} \tau^{-1/4} + \left(\frac{\sqrt{\pi}}{\zeta(3/2)^2} + \frac{\zeta(1/2)}{2\sqrt{\pi}\zeta(3/2)} \right) \tau^{-1/2} + O(\tau^{-3/4}) \approx 0.5537 \tau^{-1/4} + 0.102 \tau^{-1/2} + O(\tau^{-3/4}). \quad (19)$$

In the following, we shall always take the leading correction to scaling into account, since it modifies the behavior considerably.

C. The case $\varepsilon > 0$

Let us now analyze the case in which $\varepsilon > 0$. We proceed exactly as in the previous subsection, and we wish to know the structure of the set Σ_t . The probability that a particle at 0 and

another particle at k both belong to Σ_t , in the case $k < t$, is still given by the probability, which we now call $p_\varepsilon(k)$, that a random walk, which starts at the origin and takes a step to the right, does not return to the origin before time k . The important difference is now that the walk is biased, that is, that a step to the right now has probability $1/2 + \varepsilon$, whereas a step to the left has probability $1/2 - \varepsilon$.

The probability $p_\varepsilon(k)$ has the property that it *saturates* to a positive value $p_\varepsilon(\infty)$ as $k \rightarrow \infty$. More specifically, this saturation happens when $k \sim k_c(\varepsilon)$, where $k_c(\varepsilon) \sim \varepsilon^{-2}$ as $\varepsilon \rightarrow 0$ [13,14]. When $\varepsilon \rightarrow 0$ we have $p_\infty(\varepsilon) \simeq \varepsilon$. The set Σ_t is therefore a fractal set with a cutoff that is either at t or at ε^{-2} , depending on which is smaller. The correlation function in this case is $(kt)^{-1/2}$ for $k \ll \varepsilon^{-2}$, whereas it goes as $\varepsilon t^{-1/2}$ for $k \sim \varepsilon^{-2}$. Of course, $k \simeq \varepsilon^{-2}$ and $k < t$ are only compatible if $t > \varepsilon^{-2}$.

If, on the other hand, $\varepsilon < 0$, that is, if we are looking at the number of surviving particles of the *minority* species, then the probability of surviving for k time steps decays exponentially in k as k becomes larger than a characteristic value $k_c(\varepsilon) \simeq \varepsilon^{-2}$. This means that there are essentially no minority particles when $t > k_c(\varepsilon)$, in other words, after a time of order ε^{-2} . We will call this time $t_1(\varepsilon)$. We will again obtain this time scale from the scaling at the crossover point at the end of this section, with the implication that the relatively few particles that survive at such times are very close to each other.

When all minority particles have been annihilated, and only the majority species remains, the surviving particles lie on a fractal of dimension $1/2$ with a lower cutoff $t_1(\varepsilon)^{1/2}$ and an upper cutoff $t_1(\varepsilon)$, where $t_1(\varepsilon)$ is the time at which the minority species disappears. At that time, one has $N(0)^{1/4}$ particles, where $N(0)$ is the initial number of majority particles. These particles will now undergo diffusive annihilation with diffusion constant D_{eff} (Appendix A), starting from a fractal distribution. Actually, not only these remaining majority particles, but all excess particles $c_{\text{ex}}(t)$ will undergo diffusive annihilation on a fractal of dimension $1/2$ from the beginning, except when $\varepsilon = \pm 1/2$, in which case all particles are excess particles.

Let us determine the crossover time $t_1(\varepsilon)$. This is the time at which we cross from the $t^{-3/4}$ initial behavior to the long-time $t^{-1/4}$ behavior. The latter, as follows from that mentioned above, is

$$at_1(\varepsilon)^{-3/4} = b(\varepsilon)t_1(\varepsilon)^{-1/4}, \quad (20)$$

where the coefficient a does not depend on ε as the decay at the beginning does not depend on the initial concentration. On the other hand, $b(\varepsilon) \sim \varepsilon$ as $p_\infty(\varepsilon) \simeq \varepsilon$ for $\varepsilon \rightarrow 0$. Simplifying Eq. (20), we get

$$t_1(\varepsilon) \sim \varepsilon^{-2}, \quad (21)$$

which is compatible with the previous description. For $\varepsilon \rightarrow 0$, this crossover time t_c will diverge and hence will scale with system size L for finite L .

For $\varepsilon = 0$, the $t^{-1/4}$ decay is difficult to see, as there remain very few particles at this late stage. For $\varepsilon \neq 0$, it is easier to detect the $t^{-1/4}$ (which is the leading term) decay of concentration on the fractal, because there remain more particles in this situation. As the dynamics is diffusive in this regime, the fractal structure will eventually fade out to the uniform distribution. Hence there will be yet another crossover

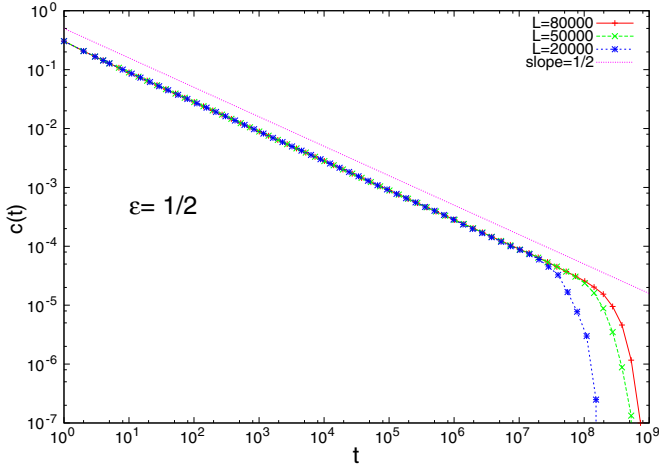


FIG. 3. The concentration $c(t)$ for the total number of particles decaying as $t^{-1/2}$.

and the concentration will decay as $t^{-1/2}$ at very late time. If the crossover time for this second crossover is $t_2(\varepsilon)$, we can write

$$b(\varepsilon)t_2(\varepsilon)^{-1/4} = c t_2(\varepsilon)^{-1/2},$$

where the coefficient c does not depend on the initial concentration and hence on ε . This gives

$$t_2(\varepsilon) \sim \varepsilon^{-4}. \quad (22)$$

It is quite difficult to detect and measure this second crossover time $t_2(\varepsilon)$, as the number of remaining particles is very low and the time very large. To be able to observe it, we need to use comparatively large values of ε , for which other effects, such as the $-3/4$ initial decay, are not so clearly visible.

IV. NUMERICAL RESULTS AND SCALING

A. Results for $\varepsilon = \pm 1/2$

For $\varepsilon = \pm 1/2$, there exists only one kind of particles (either $+v$ or $-v$ velocity particles) and hence the excess number of particles $c_{\text{ex}}(t) = c(0)$. As discussed above, the system is purely diffusive in this case, thus the concentration $c(t)$ decays with time t as $t^{-1/2}$ (Fig. 3).

The domain size of positive or negative velocity particles (depending on $\varepsilon = +1/2$ or $-1/2$) cannot change and is equal to the system size from the beginning. The probability distribution of interparticle distances between two neighboring particles increases linearly with the domain size (due to diffusion) for smaller domains [15] and then drops exponentially, as expected for diffusion. The persistence probability decays exponentially with time as the walkers are ballistic and wipe out the persistence of all the lattice sites.

B. Results for $\varepsilon = 0$

When $\varepsilon = 0$, the number of particles of each kind is equal on average. As discussed above, the system will undergo a crossover from ballistic annihilation with superimposed diffusion to purely diffusive annihilation on a fractal. Figure 4 shows the decay of the concentration, $c(t)$, for the total

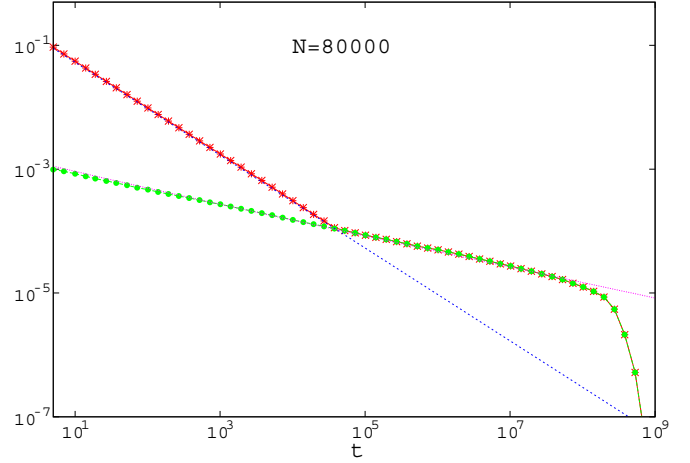


FIG. 4. The concentration $c(t)$ (red points) for the total number of particles decaying as t^{-z_1} , with $z_1 = 3/4$ (blue dotted line) before the crossover and following Eq. (19) after the crossover. The decay of $c_{\text{ex}}(t)$ (green points), which is the concentration for the excess number of particles, is also plotted. $n(t)/\sqrt{N}$, with $\lambda = 1.5$, is plotted as a theoretical curve (pink dotted line), where the expression of $n(t)$ is given by Eq. (18). The decay of the concentrations is plotted starting initially with $N = 80\,000$ particles.

number of particles, and the decay of $c_{\text{ex}}(t)$, the concentration of the excess number of particles. Both concentrations are normalized by the total number of initial particles, denoted by N . The concentration $c(t)$ decays as $t^{-3/4}$ [Eq. (12)] before the crossover and following Eq. (19) after the crossover (Fig. 4).

The initial concentration for the excess particles is $c_{\text{ex}}(0) = 1/\sqrt{N}$. If the excess particles, which decay due to diffusive annihilation, are assumed to lie on a fractal of dimension $1/2$ from the beginning, then they will decay according to Eq. (19). The plot of Eq. (19) with the numerical data for $c_{\text{ex}}(t)$ (Fig. 4) shows excellent agreement, supporting the assumption made above.

After the crossover, the total number of particles is equal to the excess number of particles, as there exists only one kind of particle in this regime. However, the number of particles left after the crossover is quite limited, $O(N^{1/4})$, and a very large number of configurations (more than 2×10^3) are needed to attain the proper statistics. Further, as previously mentioned, one has to consider $n(t)/\sqrt{N}$ to fit the decay of the fraction of excess particles from the beginning. Hence to check the expression of Eq. (19) directly, we have also studied the dynamics starting from an initial configuration, where all the particles have the same velocity and are distributed on a fractal (Fig. 5). The numerics show good agreement with the theoretical plot.

At late times, the distribution of interparticle distances between two neighboring particles was also studied. This distribution for two neighboring particles with the same velocity, which is denoted by $P_s(l)$, goes as $l^{-3/2}$ for large l (see the inset at the bottom of Fig. 6). This is evidence that inside a domain of particles with the same velocity, the particles are distributed on a fractal of dimension $1/2$, while $P_s(l) \sim l$ for small l . This is due to the diffusion-limited annihilation, which follows, for example, from the exact results

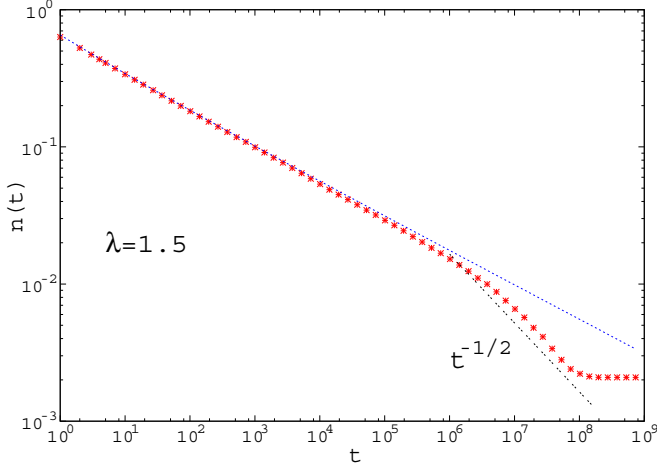


FIG. 5. $n(t)$, normalized by the initial number of particles $n(0)$ for each configurations, are plotted as a function of time t (red dots). The blue line is the theoretical plot, which is the plot of Eq. (18) with $\lambda = 1.5$ [which is the plot of Eq. (19)]. In this plot, the system size $L = 150\,000$ and the average initial number of particles is $\langle n(0) \rangle = O(680)$.

of [16], where it is shown that the interparticle distribution function for diffusion-limited annihilation grows linearly as x for x much less than the average interparticle distance. On the other hand, we have also computed $P_d(l)$, the distribution of interparticle distances between two neighboring particles with different velocities. The distribution is almost flat and then has an exponential decay (see the top-right inset of Fig. 6). At the exponential decay, the value of $P_d(l)$ suddenly increases, indicating that the probability of having some large value of l is very high. This is due to the fact that the two domains or

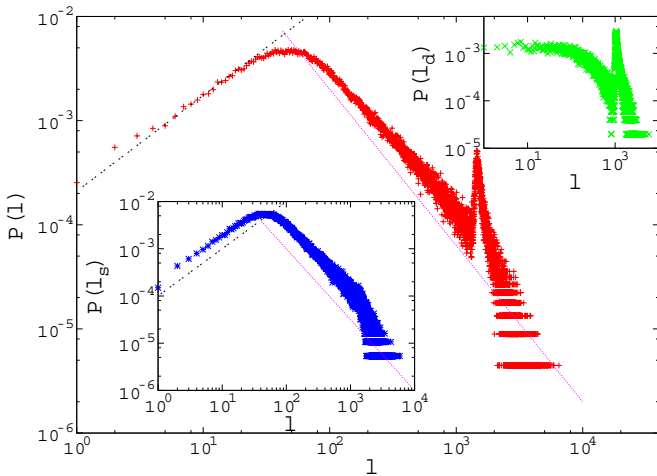


FIG. 6. The distribution $P(l)$ for the interparticle distances between two neighboring particles, independent of their velocity, is plotted in the main plot. The inset at the bottom shows the distribution $P_s(l)$ for the particles with the same velocity, and the top-right inset shows the distribution $P_d(l)$ for two particles with different velocities. The pink line indicates $l^{-3/2}$ decay for both the main plot and the bottom inset. Similarly, the black dashed line shows the linear increase.

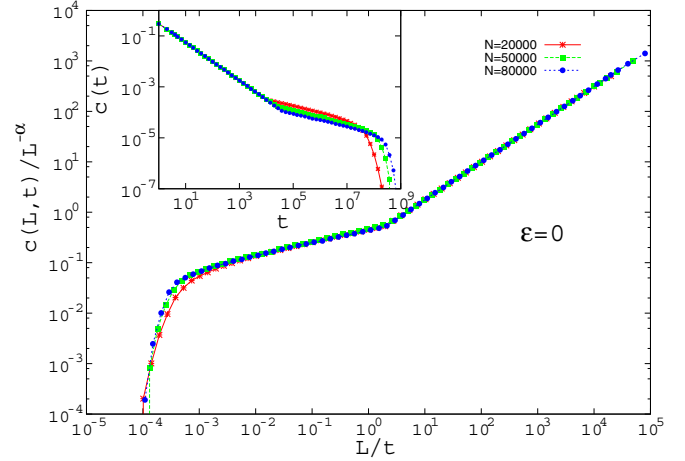


FIG. 7. The collapsed plot of scaled data of the concentration of particles with $\alpha = 0.75$ for a different number of particles for $\varepsilon = 0$. The inset shows the raw data. The collapse is not good for the exponentially decaying part, as the scaling theory applies to the two power-law regions only. The average number of particles is also considerably less than 1 in this exponentially decaying region.

fractals of particles with the same velocity are moving apart from each other, and the distance increases almost linearly with time. Indeed, this sudden increase of probability is not observed in the distribution function $P_s(l)$.

$P(l)$, the general distribution function for interparticle distances between two neighboring particles, where the neighboring particles can have any velocity, is also computed (see the main plot of Fig. 6). As a combined effect of $P_s(l)$ and $P_d(l)$, $P(l) \sim l$ for small l and it also goes as $l^{-3/2}$ for large l (Fig. 6) before the exponential decay. This indicates that the fractal structure is dominating at late times, right up to just before the crossover time, when a small number of minority particles is still left in the lattice. As an effect of $P_d(l)$, the sudden increase of probability for a large value of l is also present in this general distribution function $P(l)$.

The decay of $c(t)$, the concentration of the total number of particles, has a dependence on the initial number of particles (see the inset of Fig. 7). After the crossover, when $c(t, L)$ is equal to $c_{ex}(L, t)$, finite-size scaling analysis can be done using the scaling form

$$c(t, L) \sim L^{-\alpha} f(L/t), \quad (23)$$

where $f(x) \rightarrow x^\alpha$ with $\alpha = 3/4$ for $x \rightarrow \infty$ and $f(x) \rightarrow x^{-1/4}$ when $x \ll 1$. The raw data as well as the scaled data using $\alpha = 3/4$ are shown in Fig. 7.

The domain size for particles with the same velocity is defined as the number of consecutive particles with the same velocity (either $+v$ or $-v$). $S_d(t)$ is the average over all the domains of particles with the same velocity at the time t . On the other hand, lattice domain size is defined by the number of lattice sites occupied by these domains of particles with the same velocity and $S_{ld}(t)$ is the average over all these lattice domains at the time t . Both of the average domain sizes $S_d(t)$ and $S_{ld}(t)$ are normalized by the size of the lattice. The average value of the lattice domain size cannot go beyond 0.5 as the entire lattice can be occupied by either the positive or the

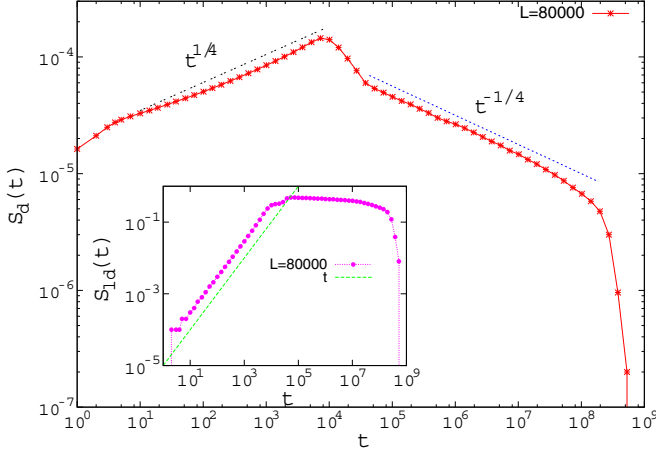


FIG. 8. The plot for the change of $S_d(t)$, which is the average domain size (normalized by the size of the lattice) for particles with the same velocity with time t . The inset shows the change of the lattice domain size $S_{ld}(t)$ (normalized by the size of the lattice) for the particles with the same velocity with time t .

negative velocity particles (with 50% probability) at very late time.

At some late time, but before the crossover, that is, in the region where $c(t) \sim t^{-3/4}$, the average interparticle distance increases as $t^{3/4}$. Now $S_d(t)$, the average lattice domain size, increases linearly in time due to the ballistic nature of the particles (see the inset of Fig. 8). Hence $S_d(t)$, the average domain size for particles with the same velocity, increases with time as $t/t^{3/4} = t^{1/4}$ in this region (Fig. 8). After the crossover, there exists only one domain and hence $S_{ld}(t)$ becomes constant. In this region after the crossover, $S_d(t) \sim t^{-1/4}$ as the concentration of particles decreases as $t^{-1/4}$ (Fig. 8). We have also computed the average interparticle distance between two neighboring particles with different velocities (not shown), which should increase linearly with time. This quantity increases with time with an effective exponent 0.93 in our simulation, which is a little ambiguous but agrees with the observation made in [6].

The persistence probability does not show any finite-size dependence before the crossover, and it fits quite well to the form (Fig. 9)

$$P(t) \simeq a \frac{\log(t)}{t} + \frac{b}{t} \quad (24)$$

with $a = 0.45 \pm 0.01$ and $b = 1.27 \pm 0.05$, obtained by least-squares fitting of the numerical data. Although the fit is certainly quite good and the functional dependence remarkably simple, we would like to mention that we have no rationale at all for this behavior. After the crossover, the persistence probability decays exponentially as the remaining ballistic particles with the same velocity wipe out the persistence of all the remaining sites.

C. Results for $0 < \varepsilon < 1/2$

In this section, we will present the numerical results for the cases in which the number of positive velocity and negative velocity particles are not equal on average. For $\varepsilon \neq 0$, as previ-

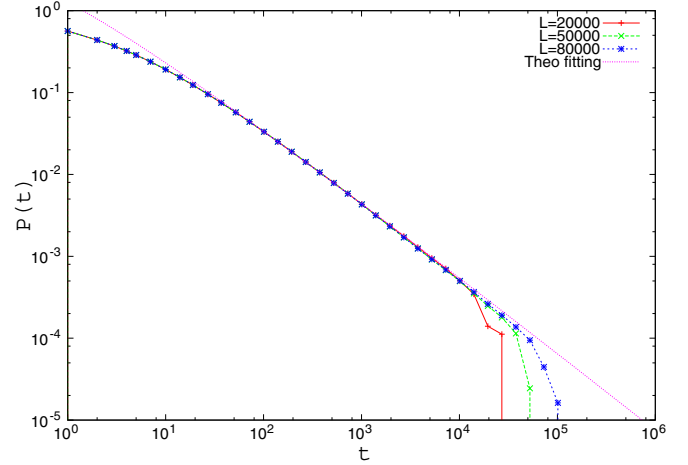


FIG. 9. Decay of the persistence probability $P(t)$ with time t for different system sizes. The theoretical fitting curve is plotted following Eq. (24), where the fitting parameters are $a = 0.45$ and $b = 1.27$.

ously mentioned, there exist three different dynamical regimes where the concentration decays with different exponent values. However, due to the finiteness of the system, the numerics is challenging because the regimes are not that well separated, thus the crossover times are not very clean. This can be seen in Fig. 10, where we plot the behavior of $c(t)$.

To better analyze the behavior of the concentration, we have computed the function $\phi(t)$ defined as

$$\phi(t) = \frac{d}{d[\log(t)]} \{\log[\sqrt{t}c(t)]\}. \quad (25)$$

Thus, when $c(t) \sim t^{-3/4}$, $\phi(t) = -1/4$ and so on. In particular, when $c(t) \approx at^{-3/4} + bt^{-1/4}$, $\phi(t)$ changes from $-1/4$ to $1/4$, whereas when $c(t) \approx bt^{-1/4} + ct^{-1/2}$, $\phi(t)$ changes from $1/4$ to 0 . The inset of Fig. 10 shows the change of $\phi(t)$ with time t for $\varepsilon = 0.08$.

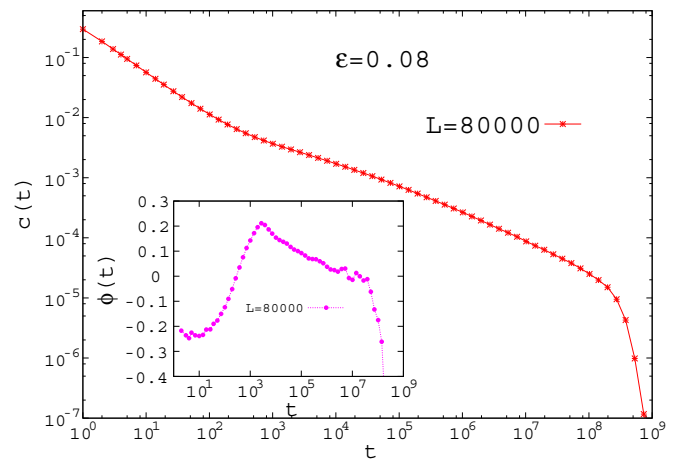


FIG. 10. The decay of $c(t)$, the concentration for the total number of particles, with time t is plotted in the main plot. The inset shows the change of $\phi(t)$ with time t .

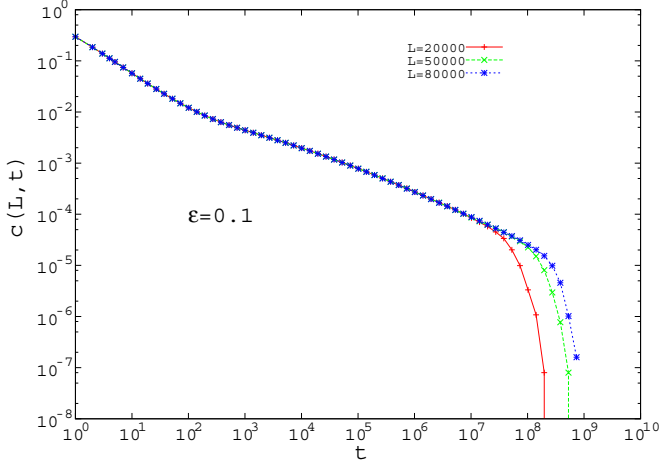


FIG. 11. The decay of $c(t)$ for different finite sizes, starting from $c(0) = 1$. The decay of $c(t)$ is size-independent for any $\varepsilon \neq 0$. The plots here are for $\varepsilon = 0.1$.

In this case $c_{\text{ex}}(t)$, the concentration for the excess number of particles, decays as $t^{-1/4}$ initially and then as $t^{-1/2}$ at late times.

The concentration of the total number of particles is a function of ε also, so we write it explicitly as $c(L, \varepsilon, t)$. We now turn to the scaling behavior of $c(L, \varepsilon, t)$. As the first crossover time is $t_1(\varepsilon) \sim \varepsilon^{-2}$ [Eq. (21)], the dimensionless quantity controlling the crossover between the first two dynamical regimes will be $\varepsilon^2 t$. Similarly, the dimensionless quantity for the crossover between the second and third regimes will be $\varepsilon^4 t$, as the second crossover time is $t_2(\varepsilon) \sim \varepsilon^{-4}$ [Eq. (22)].

Unlike the case of $\varepsilon = 0$, in this case we are considering infinite systems, since finite-size effects would only be significant when $\varepsilon \ll 1$. Hence the first scaling ansatz is that, except for very long times,

$$c(L, \varepsilon, t) = c(\varepsilon, t) \quad \text{for all } \varepsilon \neq 0. \quad (26)$$

That is, there will be no system size dependence of $c(t)$ for a constant $\varepsilon \neq 0$ (for all ε), which is indeed borne out by the simulations (Fig. 11), except for the long-time exponential decay where this scaling ansatz does not hold.

Now we will discuss the scaling laws involving ε and t , which hold for $|\varepsilon| \ll 1$. The scaling function describing the first two dynamical regimes can be written as

$$c(\varepsilon, t) \sim \varepsilon^{2\delta} f(\varepsilon^2 t) \quad \text{for } \varepsilon \ll 1, \quad (27)$$

where $f(x) \rightarrow x^{-\delta}$ with $\delta = 3/4$ for $x \ll 1$ and $f(x) \rightarrow x^{-1/4}$ when $x \gg 1$. The raw data as well as the scaled data using $\delta = 3/4$ are shown in Fig. 12. The collapse is good for the first two regimes.

The scaling analysis for the second and third dynamical regimes (where $\varepsilon^4 t$ is the relevant dimensionless quantity) can be carried out using the scaling form

$$c(\varepsilon, t) \sim \varepsilon^{4\eta} g(\varepsilon^4 t) \quad \text{for all } \varepsilon \ll 1, \quad (28)$$

where $g(x) \rightarrow x^{-\eta}$ with $\eta = 1/2$ for $x \rightarrow \infty$ and $g(x) \rightarrow x^{-1/4}$ when $x \ll 1$.

The behavior of $c(\varepsilon, t)$ for $\varepsilon^2 t \gg 1$ in Eq. (27) is the same as that of Eq. (28) for $\varepsilon^4 t \ll 1$ [in both cases, one gets that

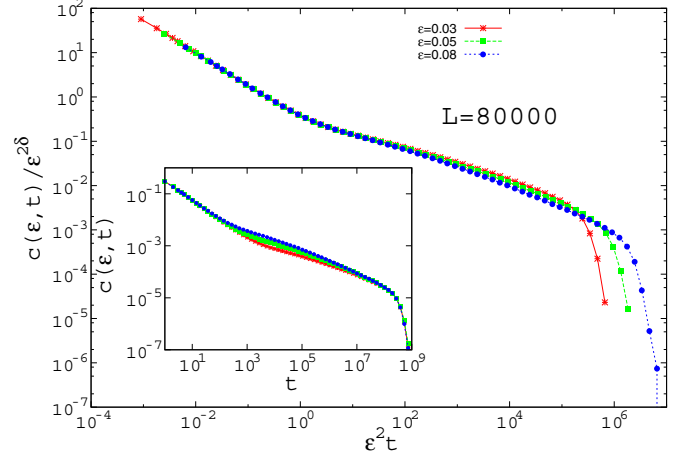


FIG. 12. The collapsed plot of the scaled data of the concentration of particles with $\delta = 0.75$ for different $\varepsilon \neq 0$. The inset shows the raw data. The scaled data show a good collapse for the first two dynamical regions where the exponent values are $3/4$ and $1/4$, respectively.

$c(\varepsilon, t) \sim \varepsilon t^{-1/4}$], so both scaling laws are consistent with each other.

The raw data as well as the scaled data using $\eta = 1/2$ are shown in Fig. 13. The collapse is good for the second and third dynamical regimes.

$S_{\text{id}}(t)$, the average lattice domain size, does not increase linearly in time, rather it increases faster than that (though it does not appear to grow as a well-defined power of time t) and then saturates. $S_d(t)$, the average domain size for particles with the same velocity, again increases with time initially and then follows the concentration $c(\varepsilon, t)$ at late times when minority particles do not exist anymore.

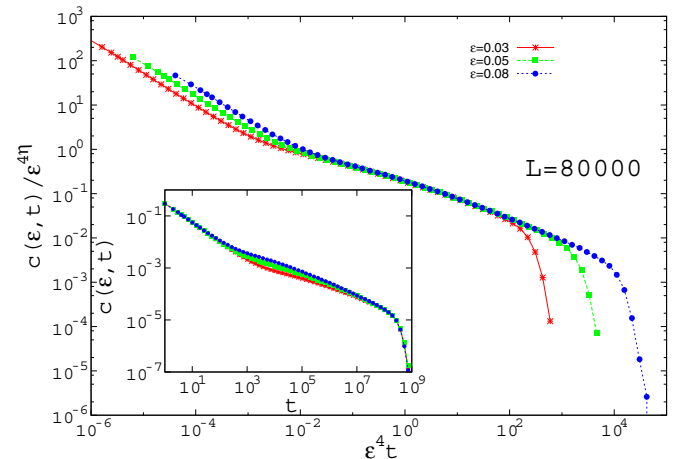


FIG. 13. The collapsed plot of the scaled data for the concentration of particles with $\eta = 1/2$ for different $\varepsilon \neq 0$. The inset shows the raw data. The scaled data show a good collapse for the second and third dynamical regions where the exponent values are $1/4$ and $1/2$, respectively. Of course, data collapse does not occur for the exponentially decaying part, as the scaling theory applies only to the power-law regions. The average number of particles is again considerably less than 1 in this exponentially decaying region.

$P(t)$, the persistence probability, decays exponentially for all values of $\varepsilon \neq 0$. This is due to the presence of more particles with one velocity instead of the other; the majority particles wipe out the persistence of all the sites very quickly.

V. CONCLUSION

To summarize, we have considered the model of annihilating particles moving ballistically with superimposed diffusion. As shown in [5,6], the particle concentration was found to decay as $t^{-3/4}$, that is, faster than either the purely ballistic or the purely diffusive case, both of which decay as $t^{-1/2}$. This result, however, fails once all particles are of the same species, that is, either right- or left-going. If the initial condition has an equal concentration of the two kinds of particles, there will nevertheless remain a number $N^{1/4}$ of particles of one velocity after all the particles of opposite velocity are annihilated. This follows from the fact, which we confirmed numerically, that the number of excess particles decays as $t^{-1/4}$. Since this number starts out at \sqrt{N} , it decays as $\sqrt{N}t^{-1/4}$, which becomes equal to $t^{-3/4}$ when $t \sim N$. From this, it immediately follows that the number of particles remaining at this time scales as $N^{1/4}$.

Thus the number of particles remaining after one kind of particle has been eliminated goes to infinity as N does. It therefore makes sense to ask the following: with what power of t do the remaining particles decay? We have shown that they decay with the exponent $t^{-1/4}$, and we have provided a rationale for this behavior in terms of Alemany's result, namely that annihilating particles that start out distributed on a fractal of dimension d_f decay as $t^{-d_f/2}$. Since it can be argued that the excess particles are constrained to lie on a fractal of dimension $d_f = 1/2$, the result readily follows.

Finally, we have also looked at the case in which the initial concentrations of left- and right-going particles differ, their initial values being given by $1/2 - \varepsilon$ and $1/2 + \varepsilon$, respectively. If $\varepsilon \ll 1$, we have shown that two crossovers arise: one from the usual $t^{-3/4}$ behavior to the $t^{-1/4}$ behavior described in the preceding paragraph. This crossover arises at a time $t(\varepsilon) \sim \varepsilon^{-2}$. A second crossover to an ordinary $t^{-1/2}$ decay, characteristic of ordinary diffusion-limited annihilation in one dimension, is observed at a crossover time $t_2(\varepsilon) \sim \varepsilon^{-4}$. This second crossover is not observed for $\varepsilon = 0$, since the number of remaining particles for that regime turns out to tend to zero as $N \rightarrow \infty$.

ACKNOWLEDGMENTS

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APPENDIX A: DIFFUSION CONSTANT

When all the particles are of the same velocity (say $+v$), the only source of diffusion is the random update rule. So whenever a particle is randomly chosen, it will move forward

a single lattice site and one Monte Carlo update will be over. After N such updates, one Monte Carlo time step will be over if there are N particles in the lattice.

We calculate the relative diffusion constant D_{eff} between two neighboring particles. Let $P(l, n)$ be the probability that the relative distance between two chosen neighboring particles be l after the n th Monte Carlo update, where $0 < n < N$. At the $(n + 1)$ th update, any one of the two particles can be chosen with a probability $1/N$, in which case the distance l will be increased or decreased by one lattice site; or neither of these particles will be chosen with a probability $(1 - 2/N)$, in which case the distance between the particles remains unchanged. Thus we can write the following equation:

$$P(l, n + 1) = \frac{1}{N} [P(l - 1, n) + P(l + 1, n)] + \left(1 - \frac{2}{N}\right) P(l, n). \quad (\text{A1})$$

Taking the Fourier transform of (A1), we get

$$\hat{P}(\omega, n + 1) = \left(1 - \frac{2}{N} + \frac{2}{N} \cos \omega\right) \hat{P}(\omega, n), \quad (\text{A2})$$

where $\hat{P}(\omega, n)$ is the Fourier transform of $P(l, n)$.

We have to repeat this update N times to complete one Monte Carlo time step. Initially, for $n = 0$,

$$\hat{P}(\omega, n) = \hat{P}(\omega, 0) = \exp(i\omega\ell),$$

where ℓ is the initial distance between these two particles. After N updates, Eq. (A2) becomes

$$\hat{P}(\omega, N) = \left(1 - \frac{2}{N} + \frac{2}{N} \cos \omega\right)^N \hat{P}(\omega, 0). \quad (\text{A3})$$

After t such time steps, Eq. (A3) becomes

$$\hat{P}(\omega, t) = \left(1 - \frac{2}{N} + \frac{2}{N} \cos \omega\right)^{Nt} \exp(i\omega\ell). \quad (\text{A4})$$

Doing a Taylor series expansion of cosine and exponential functions in (A4),

$$\begin{aligned} \hat{P}(\omega, t) &= \left(1 - \frac{1}{N}\omega^2 + \dots\right)^{Nt} \left(1 + i\omega\ell - \frac{\omega^2\ell^2}{2} + \dots\right) \\ &= 1 + i\omega\ell - \frac{1}{2}(2t + \ell^2)\omega^2 + \dots, \end{aligned} \quad (\text{A5})$$

from which we get

$$\langle l \rangle = \ell \quad \text{and} \quad \langle l^2 \rangle = 2t + \ell^2,$$

which implies that

$$\langle l^2 \rangle - \langle l \rangle^2 = 2D_{\text{eff}}t = 2t. \quad (\text{A6})$$

This in turn gives

$$D_{\text{eff}} = 1. \quad (\text{A7})$$

APPENDIX B: DECAY OF CONCENTRATION STARTING FROM THE INITIAL FRACTAL DISTRIBUTION OF PARTICLES

Let $F(x, t)$ be the probability at time t in which a random walker, starting at $x = 0$, reaches site x for the first time. For

any lattice with translational invariance, the Laplace transform of $F(x,t)$ will be given by $\tilde{F}(x,u)$ [13,14]. It can be written as

$$L_u\{F(x,t)\} = \tilde{F}(x,u) = \tilde{\xi}(u)^{-x},$$

with $\tilde{\xi}(u)^{-x} = 1 + (u/D) + \sqrt{2(u/D) + (u/D)^2}$, (B1)

where $D = D_{\text{eff}}$ is the relative diffusion constant. If $\tilde{P}(v)$ is the Laplace transform of the probability distribution function $P(x)$ for initial interparticle distances for two nearest particles, then the number of particles $n(t)$ at time t [normalized by the initial number of particles $n(0)$] will be [10]

$$n(t) = L_u^{-1} \left\{ u^{-1} \frac{1 - \tilde{P}[v = \ln \tilde{\xi}(u/2)]}{1 + \tilde{P}[v = \ln \tilde{\xi}(u/2)]} \right\}. \quad (\text{B2})$$

If initially the particles are distributed on a fractal with fractal dimension d_f ($0 < d_f < 1$), then we can write

$$P(x) = \frac{1}{\zeta(\lambda)} \sum_{l=1}^{\infty} l^{-\lambda} \delta(x-l), \quad (\text{B3})$$

where $\lambda = d_f + 1$ and hence $1 < \lambda < 2$. The Laplace transform of Eq. (B3) will be

$$\tilde{P}(v) = \int_0^{\infty} e^{-xv} P(x) dx$$

$$n(t) = L_u^{-1} \left\{ u^{-1} \frac{-\Gamma(1-\lambda)(u/D)^{(\lambda-1)/2} + \zeta(\lambda-1)(u/D)^{1/2}}{2\zeta(\lambda) + \Gamma(1-\lambda)(u/D)^{(\lambda-1)/2} - \zeta(\lambda-1)(u/D)^{1/2}} \right\}. \quad (\text{B8})$$

This implies

$$n(t) \approx L_u^{-1} \left\{ -\frac{\Gamma(1-\lambda)}{2\zeta(\lambda)} \frac{u^{(\lambda-3)/2}}{D^{(\lambda-1)/2}} + \frac{\Gamma(1-\lambda)^2}{4\zeta(\lambda)^2} \frac{u^{\lambda-2}}{D^{\lambda-1}} + \frac{\zeta(\lambda-1)}{2\zeta(\lambda)} \frac{u^{-1/2}}{D^{1/2}} + O(D^{-\lambda/2} u^{\lambda/2-1}) \right\},$$

which in turn leads to

$$n(t) = a_1 \tau^{-(\lambda-1)/2} + a_2 \tau^{-(\lambda-1)} + a_3 \tau^{-1/2} + O(\tau^{-\lambda/2}), \quad (\text{B9})$$

where $\tau = Dt$ and

$$a_1 = -\frac{\Gamma(1-\lambda)}{2\zeta(\lambda)\Gamma(\frac{3-\lambda}{2})}, \quad (\text{B10a})$$

$$a_2 = \frac{\Gamma(1-\lambda)}{4(1-\lambda)\zeta(\lambda)^2}, \quad (\text{B10b})$$

$$a_3 = \frac{\zeta(\lambda-1)}{2\zeta(\lambda)\sqrt{\pi}}. \quad (\text{B10c})$$

$$= \frac{1}{\zeta(\lambda)} \sum_{l=1}^{\infty} l^{-\lambda} e^{-lv}$$

$$= \frac{1}{\zeta(\lambda)} \Phi(e^{-v}, \lambda, 1), \quad (\text{B4})$$

where $\Phi(z,s,a)$ is defined in [17], Sec. 1.11. Thus we can write, following Eq. (8) of the same section of [17].

$$\tilde{P}(v) = \frac{1}{\zeta(\lambda)} \left[\Gamma(1-\lambda)v^{\lambda-1} + \sum_{k=0}^{\infty} \frac{\zeta(\lambda-k)}{k!} (-v)^k \right]$$

$$\simeq 1 + \frac{\Gamma(1-\lambda)}{\zeta(\lambda)} v^{\lambda-1} - \frac{\zeta(\lambda-1)}{\zeta(\lambda)} v + \dots \quad (\text{B5})$$

Note that this function has no special name in [17]. It is, however, related to functions denoted as *polylogarithms*, treated, for example, in [18,19]. Now we have the elements to evaluate $n(t)$ given in (B2) [20]. First we write

$$v = \ln \tilde{\xi}(u/2) \approx (u/D)^{1/2} - \frac{3}{8}(u/D)^{3/2} + \dots \quad (\text{B6})$$

and hence, to leading order, we have

$$v^{\lambda-1} = (u/D)^{(\lambda-1)/2} [1 + O(u/D)]. \quad (\text{B7})$$

Then inserting $\tilde{P}(v)$ given by Eq. (B5), along with Eq. (B7), in Eq. (B2), we get

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