PHYSICAL REVIEW E **93**, 020201(R) (2016)

Onset of time dependence in ensembles of excitable elements with global repulsive coupling

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(Received 5 October 2015; published 10 February 2016)

We consider the effect of global repulsive coupling on an ensemble of identical excitable elements. An increase of the coupling strength destabilizes the synchronous equilibrium and replaces it with many attracting oscillatory states, created in the transcritical heteroclinic bifurcation. The period of oscillations is inversely proportional to the distance from the critical parameter value. If the elements interact with the global field via the first Fourier harmonics of their phases, the stable equilibrium is in one step replaced by the attracting continuum of periodic motions.

DOI: 10.1103/PhysRevE.93.020201

I. INTRODUCTION

Classification of interactions among elements of various ensembles is often defined by the way in which coupling affects the difference between the close states of units. An interaction that tends to decrease the difference is usually called attractive; an interaction that sets the elements further apart is named repulsive. In the context of synchronization, the focus has been understandably put on the attractive coupling: It brings the elements closer, facilitating their synchrony. Repulsive interactions have received less attention, although they are widely present both in nature and in technical applications. Indeed, the famous observation of synchronization by Huygens apparently dealt with repulsive (in terms of the phase) interaction: His phrase "Oscillations of the pendulums, when they are in agreement, are not parallel to each other; on the contrary, they approach each other and depart" [1] portrays pendulums of two clocks, oscillating in counterphase. Inhibitory interactions, widespread in biophysical systems, e.g., neuronal ensembles, turn in models into repulsive coupling. Recently, a rich dynamics was reported for phase oscillators with purely repulsive [2,3] and mixed [4,5] coupling.

In neuronal dynamics, excitability of separate cells is often combined with repulsive interaction between them. Below we consider bifurcations in the ensemble of identical excitable elements. Each unit alone possesses a stable state of equilibrium. If mutual repulsion is introduced into such an ensemble, its intensity should be sufficiently large to overcome the individual stability of each element and to destabilize the collective equilibrium. For global coupling, we derive the scenario of the onset of oscillations with finite amplitude and large (asymptotically infinite) period.

We start with the highly degenerate transcritical bifurcation that occurs when a synchronous state of equilibrium loses stability. None of the $\sim 2^{N-1}$ steady states participating in the bifurcation is stable beyond it. New attractors are periodic orbits born in the global event: the transcritical heteroclinic bifurcation that accompanies destabilization of the synchronous equilibrium. Remarkably, the period of oscillations is inversely proportional to the distance from the bifurcational parameter value. Generally, these orbits are isolated limit cycles, but in the situation of *sinusoidal* coupling when each element interacts with the mean field via the first Fourier harmonics of its phase, the Watanabe-Strogatz phenomenon takes place: The dynamics preserves $N-3$ quantities [6], hence periodic orbits form in the phase space $(N-3)$-dimensional attracting manifolds. In this way, the bifurcation scenario leads in just one step from the equilibrium to continua of periodic solutions.

II. STEADY STATES AND THEIR STABILITY

A paradigmatic simplification for ensembles of excitable elements is the set of globally coupled active rotators

$$\frac{d\varphi_i}{dt} = \omega - \sin\varphi_i + \kappa \sum_j \sin(\varphi_j - \varphi_i), \tag{1}$$

introduced by Shinomoto and Kuramoto [7]. Here the on-site dynamics is described by the Adler equation. At $\omega < 1$ each uncoupled unit is excitable: Sufficiently large perturbations evolve to the stable equilibrium $\varphi = \arcsin\omega$ nonmonotonically, imitating a spike. The coupling term contains equal contributions from all elements. Studies of Eq. (1), mostly with added noisy terms, concerned the case of positive coupling strength κ [7–10].

Global coupling is a mean field created by the elements of the ensemble and acting upon all of them. Not restricting it to an arithmetic mean over units, we view it as a function or set of functions that depend on the instantaneous state of every element.

Many ensembles (e.g., those of neurons) admit a dichotomy: Each element either activates or inhibits the rest. A corresponding generalization of (1) is a set of N globally coupled identical one-dimensional elements, each one with its own coupling coefficient γ_j that defines its action upon every other unit: It attracts all elements if γ_j is positive and repels them if γ_j is negative. The equations read

$$\frac{d\varphi_i}{dt} = F(\varphi_i) + \sum_j \gamma_j \sin(\varphi_j - \varphi_i), \quad i = 1,\ldots,N, \tag{2}$$

with a 2π-periodic continuous function $F(\varphi)$ that has two zeros in the interval $[0,2\pi)$. Of these zeros, one corresponds to the stable fixed point $\varphi_{\rm FP}$ in dynamics of the isolated element.

We do not demand that all γ_j are negative; as we will see, the sign of $\Gamma = \sum_j \gamma_j$ is important. Since φ_i have the

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meaning of angles, we represent the instantaneous state of the system by plotting φ_i on the same circle which we call the state space. Several units that incidentally share the value of φ stay together forever, forming in the state space a cluster. In the N-dimensional phase space, solutions with M clusters form M-dimensional invariant manifolds. Common for systems of identical units, this property is crucial for the bifurcation scenario considered.

The second-order Taylor expansion of (2) near the synchronous equilibrium $\varphi_i = \varphi_{\rm FP}$ reads

$$\frac{dx_i}{dt} = ax_i + bx_i^2 + \sum_j^N \gamma_j(x_j - x_i), \quad i = 1,\ldots,N, \tag{3}$$

where $x_i = \varphi_i - \varphi_{\rm FP}$, $a = F'(\varphi_{\rm FP})$, and $b = 2F''(\varphi_{\rm FP})$. Stability of equilibrium for the decoupled unit implies $a < 0$. At the synchronous equilibrium all x_i vanish; the Jacobian matrix has the simple eigenvalue a and the eigenvalue $a - \Gamma$ with multiplicity $N-1$. At $\Gamma > 0$ when attractive interactions dominate, the largest eigenvalue is a, hence the equilibrium stays stable. To observe the synchrony breakup, we concentrate on the case of prevailing repulsion $\Gamma < 0$.

In terms of $\varepsilon = a - \Gamma$, the synchronous state is stable at negative ε and unstable at $\varepsilon > 0$. Below we treat ε as a bifurcation parameter. Other steady states of (3) fulfill

$$\varepsilon x_i + bx_i^2 + S = 0, \quad i = 1,\ldots,N, \tag{4}$$

with $S = \sum_j \gamma_j x_j$. For each steady state, every coordinate x_i can, regardless of i, attain either of two values

$$2bx_i = -\varepsilon \pm \sqrt{\varepsilon^2 - 4bS}. \tag{5}$$

Therefore, each steady solution is a two-cluster state, uniquely characterized by the set $\{\sigma_1,\ldots,\sigma_N\}$, where σ_i equals 1 if the square root in (5) for the ith coordinate is chosen with $+$ and -1 if it is taken with $-$. Summation of $\gamma_i x_i$ yields the self-consistent equation for S:

$$2bS = -\varepsilon\Gamma + Q\sqrt{\varepsilon^2 - 4bS}, \tag{6}$$

where $Q = \sum_j \gamma_j\sigma_j$ is the weighted sum of signs. In contrast to the overall coupling Γ shared by all states, the value of Q depends on configuration: on the choices of signs in (5) for all coordinates. Formally, Q assumes 2^N different values, but for the bifurcation at $\varepsilon = 0$ only the half is relevant. Smallness of solutions near $\varepsilon = 0$ implies

$$S = \frac{Q^2 - \Gamma^2}{4bQ^2}\varepsilon^2 + O(\varepsilon^3). \tag{7}$$

This turns (5) into

$$2bx_i = -\varepsilon + \sigma_i\frac{\Gamma}{Q}\varepsilon + O(\varepsilon^2), \quad i = 1,\ldots,N. \tag{8}$$

Solvability of (6) at $Q \neq 0$ and small $|\varepsilon|$ requires $\varepsilon\Gamma Q \geqslant 0$. Negative Γ turns it into $\varepsilon\, Q \leqslant 0$. Hence, the number of two-cluster steady states that at $\varepsilon = 0$ and $Q \neq 0$ collide with the synchronous equilibrium equals 2^{N-1}.

Pitchfork bifurcations occur in the nongeneric situation of vanishing Q; at negative Γ pitchforks are subcritical and their branches lie in the region $\varepsilon < 0$. At $Q \neq 0$, all branches are transcritical: While approaching $\varepsilon = 0$, every coordinate of every steady state varies linearly and changes sign after passing

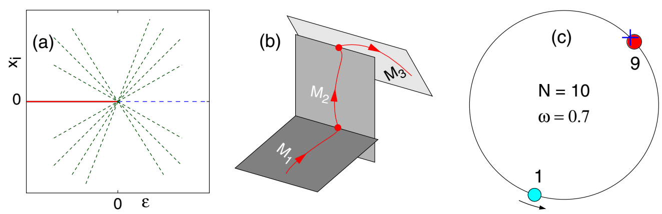

FIG. 1. (a) Transcritical bifurcation: collision of 2^{N-1} steady states with synchronous equilibrium at $\varepsilon = 0$, with $N = 4$. The solid line shows the stable state and the dashed lines show the unstable states. (b) Sketch of heteroclinic trajectory in the phase space. Here M_1, M_2, and M_3 are the invariant manifolds built by two-cluster states. Circles are replicas of equilibrium. (c) Snapshot of state space at $\kappa = \kappa_{\rm cr}$. Circles are the positions of clusters; 1 and 9 show the populations of clusters, and the plus indicates the location of synchronous equilibrium.

through $\varepsilon = 0$. Accordingly, along each branch the value of $|Q|$ stays constant, but Q has a different sign for $\varepsilon > 0$ and $\varepsilon < 0$. A typical bifurcation diagram is sketched in Fig. 1(a).

Concerning stability, eigenvalues λ of the steady state fulfill

$$\left(1 + Q\sum_{j=1}^N \frac{\gamma_j}{\Gamma\varepsilon\sigma_j - Q\lambda}\right)\prod_{j=1}^N(\Gamma\varepsilon\sigma_j - Q\lambda) = 0. \tag{9}$$

If the number of positive entries in the set $\{\sigma_i\}$ is p, then (9) has $p-1$ equal positive roots $\Gamma\varepsilon/Q$ and $N-p-1$ negative roots $-\Gamma\varepsilon/Q$. Two remaining eigenvalues solve

$$\lambda^2 - \lambda\Gamma - \Gamma\varepsilon - (\Gamma\varepsilon/Q)^2 = 0 \tag{10}$$

and at small $|\varepsilon|$ are $\lambda \approx \Gamma$ and $\lambda \approx -\varepsilon$. The latter ensures that at negative ε, when the synchronous equilibrium is stable, all other steady states are unstable. Summarizing, while crossing $\varepsilon = 0$ along transcritical branches, for each steady state all coordinates and all eigenvalues, except for one negative, change their signs. After the bifurcation, at positive ε, all nontrivial steady states with $p > 1$ are unstable. Stable states with $p = 1$ acquire relevance only under strongly unbalanced coupling, which in systems like (1) never occurs: One element repels more strongly than the whole rest of the ensemble.[1] Then events around $\varepsilon = 0$ remind us of the classical transcritical bifurcation: a complete exchange of stability between two steady states, one of them stable, the other one with an $(N-1)$-dimensional unstable manifold. Below we treat the case where none of the $|\gamma_j|$ exceeds the sum of all other coupling coefficients: There, all nontrivial steady states at either sign of ε are unstable.

III. TRANSCRITICAL HETEROCLINIC BIFURCATION

After destabilization of the synchronous equilibrium, its vicinity contains neither a stable steady state (see above) nor a small limit cycle: Eq. (2), like many similar systems, obeys

[1] At $p = 1$, the plus sign in (5) is present only once; let it correspond to the coordinate x_1. Then $Q = \gamma_1 - \sum_{j=2}^N \gamma_j = 2\gamma_1 - \Gamma$. Hence, inequalities $\Gamma < 0$ and $2\gamma_1 - \Gamma < 0$ hold simultaneously; this implies that γ_1 is negative, whereas $|\gamma_1|$ exceeds the absolute value of the sum of all other γ_j.

gradient dynamics. Instead, a big limit cycle appears: All oscillators, one after another, rotate their phases by 2π. This limit cycle is born at $\varepsilon = 0$ in the transcritical heteroclinic bifurcation [11] from the contour which, in the phase space, connects the neutral point of equilibrium to its N replicas shifted by 2π along one of the coordinates.[2]

Components of heteroclinic connections lie on the invariant manifolds of two-clustered states. The system evolves along one of such manifolds, switches to the other one when it reaches their crossing, proceeds to the third one, etc. [Fig. 1(b)]. In the simplest case when all γ_j are negative, the ensemble breaks into a cluster of $N-1$ units and a solitary element.

Heteroclinic dynamics looks as follows. At $\varepsilon = 0$ the synchronous state is neutrally stable. Since the bifurcation is transcritical, in the state space only one direction is unstable. If the equilibrium is slightly disturbed, the element farthest from φ_{FP} departs from the rest of the cluster. The latter sways close to φ_{FP}, whereas the renegade element performs a full rotation around φ [Fig. 1(c)].

The loss of one repelling element consolidates the remainder: The cluster loses a portion of repulsion and the remaining amount of repulsion is insufficient for further fragmentation. Let the renegade have number r. In (2) the instantaneous growth rate for the perturbations in the cluster of size $N-1$ is $\lambda_e(t) \approx \varepsilon + \gamma_r\{1 - \cos[\varphi_r(t) - \varphi_{\mathrm{FP}}]\}$; at $\varepsilon = 0$ and $\gamma_r < 0$, $\lambda_e(t)$ is negative for all φ_r outside φ_{FP}. When the turn around the circle is accomplished, the renegade joins the cluster, which thereby again becomes fragile; the next element breaks away to perform a full rotation and so on. Since identical one-dimensional units cannot overtake each other, all of them, one by one, make the excursion around the circle. When the last one returns to the cluster, the contour gets completed. Permutations of elements produce $(N-1)!$ contours that differ by the succession of individual rotations. For the permutationally invariant model (1), all contours belong to the same orbit of the symmetry group, but in the generic case of different γ_j none of them is equivalent to another one.

As soon as ε becomes positive, heteroclinic contours break into stable periodic orbits. In general (see Sec. IV for the remarkable exception), such orbits are isolated in the phase space. Scaling argument shows that the passage duration near the neutral equilibrium is inversely proportional to the initial distance from its stable manifold. Therefore, when ε grows, the period of oscillations decreases as ε^{-1}. At small ε the dynamics reminds us of the heteroclinic situation: Elements, one by one, depart from the cluster (which, in the state space, swells to finite size), perform a rotation, and return from the other side (Fig. 2). Every rotation of one unit results in a short bump of the mean field. At larger ε clusters get fuzzy, but the basic pattern persists: The cycle includes one rotation around the circle for each element.

If some units are attracting, the picture is less uniform: After departure of an attracting element from the neutral cluster,

[2]Heteroclinic terminology refers to the Euclidean phase space. If the phase space is viewed as a torus $[0:2\pi)^N$, the replicas coincide with the equilibrium and heteroclinic orbits turn into homoclinic ones.

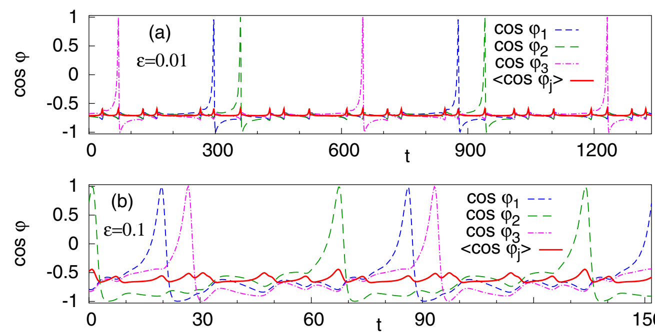

FIG. 2. Periodic states in Eq. (1) at $N = 10$ and $\omega = 0.7$ for (a) $\kappa = -0.724\,143$ ($\varepsilon = 0.01$), period 580.657, and (b) $\kappa = -0.814\,143$ ($\varepsilon = 0.1$), period 66.532. Solid curves show the mean field and dashed curves show the separate elements.

repulsion prevails in the remainder, hence further splittings occur before the return of the renegade [12]. Since dynamics preserves the ordering on the circle, oscillations retain the basic pattern, whereas the details of grouping depend on the arrangement of repelling and attracting units.

IV. WATANABE-STROGATZ DYNAMICS

Two decades ago Watanabe and Strogatz discovered a huge number of conserved quantities in dynamics of identical elements, globally coupled through sinusoidal interactions: For a set of N oscillators, there are $N-3$ constants of motion [6]. This property is shared by every system of identical one-dimensional units, coupled to the mean field through their first Fourier harmonics [13,14]: a system reducible to the form

$$\frac{d\phi_i}{dt} = g + e^{i\phi_i} f(\phi_1,\ldots,\phi_N) + \text{c.c.}, \quad i = 1,\ldots,N, \tag{11}$$

$f(\cdots)$ being a common field. Conserved quantities of the dynamics are cross ratios [13]

$$I_{ijkl} = \frac{\sin\frac{\phi_i-\phi_j}{2}\,\sin\frac{\phi_k-\phi_l}{2}}{\sin\frac{\phi_i-\phi_k}{2}\,\sin\frac{\phi_j-\phi_l}{2}}. \tag{12}$$

Given a set of $N-3$ ratios and values of three phases (ϕ_i,ϕ_j,ϕ_k), solving (12) for ϕ_l reconstructs the remaining $N-3$ phases in a unique way.

Once a set of constants is fixed (through, e.g., the choice of initial conditions), the effective dynamics is three dimensional: Instantaneous states of the system can be transformed into each other by the Möbius transformation on the unit circle. The latter, a tool from complex analysis, is completely defined by three (in this case, time-dependent) real coefficients, whose temporal evolution is mapped onto the evolution of N phases [13]. Different sets of constants for the same equation (11) define different invariant three-dimensional subspaces in its N-dimensional phase space, hence their respective attractors can differ, at least quantitatively.

We immediately notice that Eq. (2) falls in the class (11) whenever $F(\varphi)$ is a constant or the first-order trigonometric polynomial in φ, like the model (1). Indeed, in direct simulations of Eq. (1) at $N > 3$, fixed $\omega < 1$, fixed $\kappa < \kappa_{\mathrm{cr}} = -\sqrt{1-\omega^2}$, and randomly chosen initial states, all trials

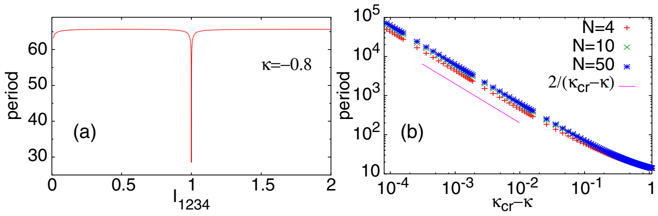

FIG. 3. Period of oscillations in Eq. (1) with $\omega = 0.7$. (a) Dependence on the cross ratio I_{1234} at $\kappa = -0.8$ and $N = 4$. (b) Asymptotics near κ_{cr} at fixed (randomly chosen) I_{ijkl}.

eventually converge to periodic orbits, but practically every new set of initial conditions ends up on a new periodic solution, not transformable into already known ones by permutation of coordinates.

Ambiguity is lifted when $N-3$ cross ratios (and thereby the three-dimensional subspace in the N-dimensional phase space) are fixed. For $N = 4$, each subspace contains only one (up to permutations) limit cycle; these cycles form in the whole phase space the attracting continuum of closed orbits, parametrizable, e.g., by I_{1234}. By varying I under fixed ω and κ we observe a variation of the period along the continuum [Fig. 3(a)]. Fixing I and varying κ confirms that at all N in every three-dimensional subspace periods of attracting orbits diverge as $\sim (\kappa_{cr}-\kappa)^{-1}$ [Fig. 3(b)].

Numerical studies of Eq. (1) show continua of periodic motions for all tested values of $N > 3$ and $\kappa < \kappa_{cr}$. Remarkably, for $N \geqslant 5$ and every fixed (up to permutations) set of $N-3$ cross ratios, there are two attracting periodic orbits with slightly differing period values [12]. In the case of (2) with all coupling weights γ_j negative and different, the three-dimensional subspace with fixed cross ratios contains $N!$ different periodic attractors (except for $N = 4$, with 12 limit cycles).

A few words about the limit $N \to \infty$ are in order (see [12] for details). There, a description in terms of phase density $\rho(\varphi,t)$ is more appropriate. Its Fourier coefficients $\rho_k = \frac{1}{2\pi}\int_0^{2\pi} \rho e^{k\varphi} d\varphi$ obey the infinite set of equations [7,9]

$$\frac{1}{k}\dot{\rho}_k = \mathrm{i}\omega\rho_k + \frac{\rho_{k-1} - \rho_{k+1} + \kappa(\rho_1\rho_{k-1} - \rho_1^*\rho_{k+1})}{2} \quad (13)$$

complemented by the boundary condition $\rho_0 = 1$. An infinite number of conserved quantities in (1) induces infinite foliation in the space of solutions of (13). The Ott-Antonsen subspace with $\rho_k = \alpha(t)^k$ [15] has an attracting fixed point that for $\kappa > \kappa_{cr}$ corresponds to the δ distribution at $\varphi = \arcsin\omega$ and for $\kappa < \kappa_{cr}$ its coordinate $c_1 = \mathrm{Re}(\alpha)$ solves the equation

$$c_1^3\kappa + c_1^2(1-2\kappa^2) + c_1\kappa(\kappa^2 + 4\omega^2 - 2) + \kappa^2 = 0.$$

The fixed point yields the only stationary distribution of density in (13); all other distributions stay time dependent forever.

V. DISCUSSION

The mechanism presented, in one step, leads the ensemble of rotators from the stable equilibrium either to a rather high ($\sim N!$) number of different periodic attractors or (under sinusoidal coupling) to attracting continua of periodic states. Such systems with a multitude of attractors can be used as simple models of information storage.

The analysis in Sec. II was based on Eq. (3): coupling via the difference of instantaneous states, but its implications stay valid for a broader class of interactions. Any linear coupling can be put into this form by rearrangement of summands, whereas the higher-order coupling terms influence neither the character of bifurcation at $\varepsilon = 0$ nor the number and stability of participating states.

In real systems the elements are not identical and the coupling does not reduce to just one Fourier harmonic. The assumption in (2) that each element acts upon all partners with the same intensity γ_j is also a restriction: In many networks attraction or repulsion is a property of a link and not of a node. To discuss the implications of weak violations of the setup, we start with the attracting high-dimensional Watanabe-Strogatz continuum of periodic orbits. When the units are slightly nonidentical and/or the coupling includes higher Fourier harmonics, the continuum dissolves. What is left depends on the particular kind of disturbances; the general perturbation theory near the Watanabe-Strogatz class is yet to be constructed. Our empirical observations show that in the simplest, but quite common, situation finitely many isolated periodic orbits survive and become ultimate attractors of the system. If the permutation symmetry persists, such orbits are often "ponies on the merry-go-round" [16]: All N oscillators perform the same motion, while with respect to each other they are shifted by $1/N$th of the period. [In the nonperturbed Watanabe-Strogatz situation, equal shifts are incompatible with a generic set of conserved cross ratios (12); this explains uneven distances between the bumps of the mean field in Fig. 2.] Generally, a separation of time scales takes place: $N-3$ local neutral directions in the phase space, corresponding to conservation of cross ratios, turn into the $(N-3)$-dimensional slow manifold and the remaining three directions correspond to fast dynamics. For certain disturbances, the system wanders along the slow manifold. Since in the unperturbed case each point of that manifold corresponds to periodic motion, this wandering looks like a slowly modulated fast oscillation. If, in the phase space, parts of the slow manifold are repelling, trajectories that cross the border of the repelling region can exhibit canardlike dynamics: alternation of slowly modulated oscillations and fast violent bursts.

Transcritical bifurcation of the synchronous equilibrium does not survive introduction of diversity among the units: It is replaced by a succession of saddle-node bifurcations, which, after its completion, leaves no stable steady states. The transcritical heteroclinic bifurcation breaks up as well; depending on the particularities of diversity, limit cycles stem from orbits heteroclinic to saddle points or from heteroclinics to saddle nodes. The onset of time dependence turns into a jump from the equilibrium to one of many available large-amplitude limit cycles.

ACKNOWLEDGMENTS

Fruitful discussions with O. Burylko are gratefully acknowledged. The research of P.T. was conducted within the scope of the IRTG 1740 funded by the DFG; M.A.Z. was supported by the Research Grant No. PI 220/17-1 of the DFG.

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