PHYSICAL REVIEW E **93**, 020103(R) (2016)

Universal structure of two- and three-dimensional self-gravitating systems in the quasiequilibrium state

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(Received 17 May 2015; revised manuscript received 3 July 2015; published 16 February 2016)

We study a universal structure of two- and three-dimensional self-gravitating systems in the quasiequilibrium state. It is shown numerically that the two-dimensional self-gravitating system in the quasiequilibrium state has the same kind of density profile as the three-dimensional one, especially when null virial conditions are fulfilled. It is unveiled why the conditions are necessary for the universal structure by the envelope equation. We develop a phenomenological model to describe this universal structure by using a special Langevin equation with a distinctive random noise to self-gravitating systems. We find that the density profile derived theoretically is very consistent with results of observations and simulations.

DOI: 10.1103/PhysRevE.93.020103

Introduction. Systems with long-range forces exhibit various specific properties which systems with short-range forces do not have. One of the prime examples is the presence of another stable state differing from the thermal equilibrium state. In this paper, we call it *quasiequilibrium state* (QES). It has been found numerically that the distribution of QES depends not only on the energy or other thermodynamical quantities but also on how particles distribute at initial time [1]. The three-dimensional self-gravitating system (3DSGS) satisfying initial conditions that the velocities are isotropic and the virial ratio $V^3(\equiv 2K^3(0)/|\Omega^3(0)|)$ is 1 is trapped in QES, whose number density can be obtained by coarse graining the distribution function of particles under the mean field potential in phase space [2], where we denote respectively the total kinetic and gravitational energy at a time t by $K^3(t)$ and $\Omega^3(t)$ and the superscript means the three-dimensional space. On the other hand, we have obtained numerically the result that the number density $\mathcal{N}$ of 3DSGS around the center of the system can be approximated by the following representation especially when $V^3 = 0$:

$$\mathcal{N}(r) \simeq \frac{\mathcal{N}(0)}{(1 + r^2/a^2)^{\kappa}} \tag{1}$$

with $\kappa \sim 3/2$, where r means the distance from the center of the system [3,4]. The number density of globular clusters which are the best example of 3DSGS also can be depicted by Eq. (1) with $\kappa \sim 3/2$ [5–7]: This density profile is universal for 3DSGS.

The recent observations have made it clear that the universality is not just limited to 3DSGS. As discussed later, the cylindrically symmetric filamentary structure of molecular clouds can be treated as a two-dimensional SGS (2DSGS). The Herschel Space Observatory revealed the 27 filamentary structures in IC 5146, which is a reflection nebula in the Cygnus, and the number density of molecular clouds have a cylindrical symmetry around the axis of the filament [8]. All the number densities of molecular clouds around the axis can be fitted by Eq. (1) with κ from 0.75 to 1.25 [8], where r means the distance from the axis. On the other hand, Eq. (1) with $\kappa = 1$ was utilized in order to describe the number density of a filament in the Taurus [9]. Therefore, 2DSGS has a similar universality depicted by Eq. (1) with $\kappa \sim 1$. As explained later, the interaction potentials of 2D and 3DSGS differ, with one being bounded and the other unbounded. However, the universality is beyond this difference.

Obviously, the universal number density cannot be derived by assuming the system is in the isothermal equilibrium: 2DSGS has a global maximum of the Massieu function if the temperature is larger than a critical value [10,11]. The equilibrium state for 2DSGS has an exact solution for number density represented by Eq. (1) with $\kappa = 2$ [11,12]. On the other hand, 3DSGS has a local maximum of the Massieu function if the temperature is larger than a critical value [13,14]. The number density of 3DSGS in this state decays with r^{-2} at large radius r [7,14]. Therefore, a new theory explaining the physical mechanism behind the universality is necessary.

Here, let us explain that there are two kinds of QES in systems with long-range forces. One is a collisionless state described by the Vlasov equation [1,7]. During the two-body relaxation time, the system is in this state. Generally, this relaxation time is dependent on the number of particles N in the system, e.g., $0.1N/\ln N t_{\rm cross}$ for 3DSGS where $t_{\rm cross}$ is the *crossing time* [7]. Therefore, in the limit $N \to \infty$, the system is trapped by the state permanently. The other one is a collisional state. The method with a mean field approximation to describe this state is by utilizing the Fokker-Planck or the Boltzmann equation [7,14].

In this paper, we shall derive the universal density Eq. (1) from a phenomenological model particular for gravity by utilizing a special Langevin equation for QES and the corresponding Fokker-Planck equation in the μ-dimensional space where μ is 2 or 3. We shall treat them in the overdamped limit. Indeed, the *normal* Langevin equation in the limit and the corresponding Fokker-Planck equation (i.e., Smoluchowski equation) is appropriate for describing 3DSGS enclosed in a box near the thermal equilibrium [14], since the Maxwell-Boltzmann distribution is stable for 3DSGS in the thermal equilibrium state with the total mass and energy fixed if the radius of system is less than a critical value [13–16]. However, it is well known empirically that SGSs without boundary are trapped by another stable state, that is, QES, so that we must modify the Langevin and the Fokker-Planck equation in order to describe the state.

Two-dimensional gravity. The two-dimensional gravitational potential per unit mass ϕ^2 generated by mass source ρ^2 satisfies the following Poisson equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial\phi^2(r,\theta)}{\partial r}\right\}+\frac{1}{r^2}\frac{\partial^2\phi^2(r,\theta)}{\partial\theta^2}=4\pi G'\rho^2(r,\theta), \quad (2)$$

where the superscript means the two-dimensional space and we denote the two-dimensional gravitational constant by G' in order to distinguish it from the *ordinary* gravitational constant G. If the distribution of mass is circularly symmetric, the above equation becomes

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d\phi^2(r)}{dr}\right\}=4\pi G'\rho^2(r)\,. \quad (3)$$

When the mass source is a mass point with a mass m existing in the origin, $\rho^2(r)=m\delta(r)/\pi r$, where $\delta(r)$ means the Dirac delta function. Then, one can see that a single-particle potential per unit mass $\phi^2_{\rm sp}$ satisfies the following Poisson equation:

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d\phi^2_{\rm sp}(r)}{dr}\right\}=\frac{4mG'\delta(r)}{r}. \quad (4)$$

Therefore, the potential can be solved like

$$\phi^2_{\rm sp}(r)=2G'm\ln r+\text{const.} \quad (5)$$

Thus, the interaction potential of 2DSGS is bounded, whereas one of 3DSGS is proportional to $-1/r$.

Finally, we shall provide a brief explanation for the reason that cylindrically symmetric filaments of molecular clouds can be regarded as 2DSGS: By using the mass density of molecular clouds $\rho^3_{\rm mc}$ and the gravitational potential per unit mass $\phi^3_{\rm mc}$ where the superscript also means the three-dimensional space, because of the symmetry, the Poisson equation becomes

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d\phi^3_{\rm mc}(r)}{dr}\right\}=4\pi G\rho^3_{\rm mc}(r), \quad (6)$$

which is mathematically equivalent to the Poisson equation for 2DSGS, Eq. (3).

N-body simulations. Because the equivalence is merely formal, we followed the time evolution of 10^4-body system in the two-dimensional space interacting by Eq. (5) numerically in order to investigate QES of 2DSGS.

A polytrope solution with a polytrope index n was adopted as the initial condition. The solutions with $n=0$ and $n=\infty$ respectively correspond to the uniform distribution and the thermal equilibrium state: The finite n represents the deviation from equilibrium. Generally, the solution in the three-dimensional space is well known [7]. Here, we have extended it to the two-dimensional space. The one-particle distribution function (DF) $f^2(r,v)$ in phase space can be shown as

$$f^2(r,v)\propto\begin{cases}\mathcal{E}^{n-1} & (\mathcal{E}<0)\\ 0 & (\mathcal{E}\geqslant 0),\end{cases} \quad (7)$$

where $\mathcal{E}\equiv\phi^2(\mathcal{R}^2)-\{\frac{1}{2}v^2+\phi^2(r)\}$ and $\mathcal{R}^2$ means a radius of the system. Then, we run the N-body simulation by varying an initial virial ratio V^2 which can be represented by $V^2\equiv 2K^2(0)/G'M^2$ (see Refs. [17] and [18]) where we have utilized $N-1\simeq N$ and denote the total kinetic energy of 2DSGS at t and the total mass by $K^2(t)$ and M, respectively. Note that the system is not enclosed in a circle.

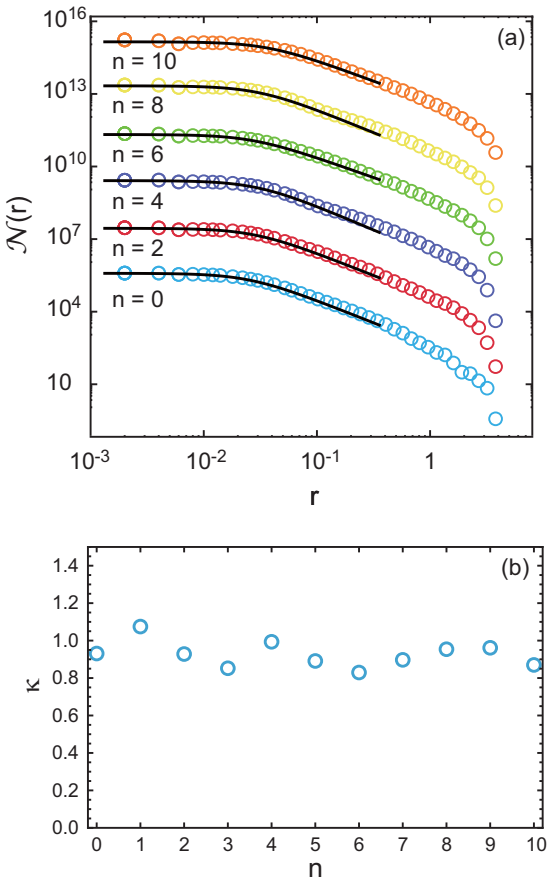

FIG. 1. (a) Number densities at QES for several initial polytrope indices are plotted by open circles. The curve passing the circles is a fitting function, Eq. (1). So as to be seen easily, each density is shifted to the two digits. (b) The optimum values of κ in Eq. (1) for fitting the number densities with respect to the initial polytrope index n.

Especially when $V^2=0$, it is found that the number density in QES has a universality and the density around the center of the system can be fitted well by Eq. (1). The QES is so stable that the characteristic of the number density does not change during the simulations. The results are shown in Fig. 1. It can be seen in Fig. 1(a) that the number density from $n=0$ to 10 has the same profile. In addition, Fig. 1(b) denotes that κ which is the index in Eq. (1) ranges from 0.8 to 1.1. These results have a good consistency with the observations of molecular clouds [8,9].

Null virial conditions. We have made sure that the number density of 2DSGS without boundary in QES can be represented well by Eq. (1) with $\kappa\sim 1$, especially when $V^2=0$. On the other hand, the number density of most globular clusters is well known to be fitted by Eq. (1) with $\kappa\sim 3/2$ [5–7]. Furthermore, we have reported that the same density profile is obtained through N-body simulations of 3DSGS, especially when $V^3=0$ (see Refs. [3] and [4]). Therefore, we can conclude that SGSs without boundary have the universal density profile depicted by Eq. (1) in QES, especially if the *null virial conditions* are fulfilled.

In order to find out why the conditions are necessary, we shall utilize the following *envelope equation* (EEq) of the μ-dimensional SGS with the total mass M where μ is 2 or 3 [1,18]:

$$\ddot{r}_{\rm e}^{\mu}(t) = \begin{cases} \dfrac{1}{(G'M)^2}\left(\dfrac{\varepsilon(t)^2}{4r_{\rm e}^2(t)^3} - \dfrac{G'M\left\langle \boldsymbol{r}^2\cdot\frac{\partial\phi^2}{\partial\boldsymbol{r}^2}\right\rangle_t}{r_{\rm e}^2(t)}\right) & (\mu=2) \\ M^2\left(\dfrac{\varepsilon(t)^2}{4r_{\rm e}^3(t)^3} - \dfrac{1}{M}\dfrac{\left\langle \boldsymbol{r}^3\cdot\frac{\partial\phi^3}{\partial\boldsymbol{r}^3}\right\rangle_t}{r_{\rm e}^3(t)}\right) & (\mu=3) \end{cases}, \tag{8}$$

with $\varepsilon(t)^2 = 4(\langle \boldsymbol{r}^\mu\cdot\boldsymbol{r}^\mu\rangle_t\langle \boldsymbol{v}^\mu\cdot\boldsymbol{v}^\mu\rangle_t - \langle \boldsymbol{r}^\mu\cdot\boldsymbol{v}^\mu\rangle_t{}^2)$, in which $\boldsymbol{r}^\mu, \boldsymbol{v}^\mu$, and ϕ^μ are the position, the velocity vector, and the mean gravitational potential per unit mass in the μ-dimensional space, respectively. The average $\langle\bullet\rangle_t$ of EEq can be represented as follows by using DF $f^\mu(\boldsymbol{r}^\mu,\boldsymbol{v}^\mu,t)$ satisfying the Vlasov equation: $\langle\bullet\rangle_t \equiv \int d^\mu\boldsymbol{r}^\mu\int d^\mu\boldsymbol{v}^\mu\,\bullet\, f^\mu$. Therefore, EEq is valid for collisionless systems and $\varepsilon(t)^2$ cannot be less than 0 because of the Cauchy-Schwarz inequality

$$\langle \boldsymbol{r}^\mu\cdot\boldsymbol{r}^\mu\rangle_t\langle \boldsymbol{v}^\mu\cdot\boldsymbol{v}^\mu\rangle_t \geqslant \langle \boldsymbol{r}^\mu\cdot\boldsymbol{v}^\mu\rangle_t{}^2 . \tag{9}$$

Moreover, the envelope $r_{\rm e}^\mu(t)$ is defined as $r_{\rm e}^2(t) = \sqrt{\langle \boldsymbol{r}^2\cdot\boldsymbol{r}^2\rangle_t/G'M}$ or $r_{\rm e}^3(t) = \sqrt{M\langle \boldsymbol{r}^3\cdot\boldsymbol{r}^3\rangle_t}$. For simplicity, we shall utilize the following notations: $r \equiv |\boldsymbol{r}^\mu|$ and $v \equiv |\boldsymbol{v}^\mu|$. Then, the total kinetic energy at t can be represented by using the average as $M\langle v^2\rangle_t/2$. Therefore, the initial virial ratios are $V^2 = \langle v^2\rangle_0/G'M$ and $V^3 = M\langle v^2\rangle_0/|\Omega^3(0)|$.

Here, we shall consider the following situation which is consistent with the numerical simulations: The system is in the dynamical equilibrium state until $t=0$, which means that the virial ratio is 1. Then, when $t=0$, we shall alter the velocity of each particle by multiplying a constant $\sqrt{V^\mu}$ in order to set the virial ratio V^μ. Therefore, the system will be in the collisionless state until the *collapse* which means the contraction of the system as will be explained later occurs and the system becomes collisional.

Let us calculate $\langle \boldsymbol{r}^\mu\cdot\frac{\partial\phi^\mu}{\partial\boldsymbol{r}^\mu}\rangle_t$. From here, we shall postulate the system is circularly ($\mu=2$) or spherically ($\mu=3$) symmetric and isotropic in velocity. Thus, we obtain

$$\left\langle \boldsymbol{r}^\mu\cdot\frac{\partial\phi^\mu}{\partial\boldsymbol{r}^\mu}\right\rangle_t = \left\langle r\frac{\partial\phi^\mu}{\partial r}\right\rangle_t = \frac{\mathcal{S}^\mu}{M}\int_0^{\mathcal{R}^\mu} dr r^{\mu-1} r\frac{\partial\phi^\mu}{\partial r}\rho^\mu \tag{10}$$

where $\mathcal{R}^\mu$ is a radius of the system and $\mathcal{S}^\mu$ means the surface area of a unit sphere in the μ-dimensional space: $\mathcal{S}^\mu = \mu\pi^{\mu/2}/\Gamma(\mu/2+1)$, where Γ is the gamma function. Additionally, we have utilized the relation between DF and the density ρ^μ: $\rho^\mu = M\int dv\mathcal{S}^\mu v^{\mu-1}f^\mu$.

By using the Poisson equation

$$\frac{1}{r^{\mu-1}}\frac{\partial}{\partial r}r^{\mu-1}\frac{\partial\phi^\mu}{\partial r} = 4\pi\mathcal{G}^\mu\rho^\mu, \tag{11}$$

where $\mathcal{G}^\mu$ relates quantities that appeared previously as $\mathcal{G}^2 = G'$ and $\mathcal{G}^3 = G$, the representation (10) can be calculated as

$$\left\langle r\frac{\partial\phi^\mu}{\partial r}\right\rangle_t = \frac{\mathcal{S}^\mu}{4\pi\mathcal{G}^\mu M}\int_0^{\mathcal{R}^\mu} dr r\frac{\partial\phi^\mu}{\partial r}\frac{\partial}{\partial r}r^{\mu-1}\frac{\partial\phi^\mu}{\partial r} . \tag{12}$$

When $\mu=2$, Eq. (12) can be simplified as [1,18]

$$\left\langle r\frac{\partial\phi^2}{\partial r}\right\rangle_t = \frac{1}{4G'M}\left(r\frac{\partial\phi^2}{\partial r}\right)^2\Big|_{r=\mathcal{R}^2} . \tag{13}$$

Here, because of $M = \int dr 2\pi r\rho^2 = \frac{1}{2G'}\int dr\frac{\partial}{\partial r}r\frac{\partial\phi^2}{\partial r} = \frac{1}{2G'}r\frac{\partial\phi^2}{\partial r}\Big|_{r=\mathcal{R}^2}$, Eq. (13) becomes

$$\left\langle r\frac{\partial\phi^2}{\partial r}\right\rangle_t = G'M . \tag{14}$$

When $\mu=3$, by using Eq. (11), the total gravitational potential $\Omega^3 (= -\int dr 4\pi r^2\rho^3\frac{G}{r}\int dr' 4\pi r'^2\rho^3)$ can be calculated as

$$\Omega^3(t) = -\frac{1}{G}\int_0^{\mathcal{R}^3} dr r\frac{\partial\phi^3}{\partial r}\frac{\partial}{\partial r}r^2\frac{\partial\phi^3}{\partial r} . \tag{15}$$

Therefore,

$$\left\langle r\frac{\partial\phi^3}{\partial r}\right\rangle_t = \frac{|\Omega^3(t)|}{M} . \tag{16}$$

Eventually, EEq becomes

$$\ddot{r}_{\rm e}^{\mu}(t) = \begin{cases} \dfrac{1}{4(G'M)^2}\dfrac{\varepsilon(t)^2}{r_{\rm e}^2(t)^3} - \dfrac{1}{r_{\rm e}^2(t)} & (\mu=2) \\ \dfrac{M^2}{4}\dfrac{\varepsilon(t)^2}{r_{\rm e}^3(t)^3} - \dfrac{|\Omega^3(t)|}{r_{\rm e}^3(t)} & (\mu=3) \end{cases} . \tag{17}$$

Let us discuss initial conditions for Eq. (17). Obviously, $r_{\rm e}^2(0) = \sqrt{\langle r^2\rangle_0/G'M}, r_{\rm e}^3(0) = \sqrt{M\langle r^2\rangle_0}$, and $\dot{r}_{\rm e}^\mu(0) \propto \langle \boldsymbol{r}^\mu\cdot\boldsymbol{v}^\mu\rangle_0$. Because the initial DF is circularly or spherically symmetric and isotropic in velocity, $\langle \boldsymbol{r}^\mu\cdot\boldsymbol{v}^\mu\rangle_0 = 0$: The DF does not have any information about the angle between the position and the velocity. Indeed, the polytrope solution is such a kind of distribution. Therefore, $\dot{r}_{\rm e}^\mu(0) = 0$.

The potential energy in Eq. (17) has a global minimum when $\varepsilon(t)^2 > 0$ which holds if $V^\mu > 0$ as explained later. The $r_{\rm e}^{\mu *}$ minimizing this potential at $t=0$ is as follows: $r_{\rm e}^{2*}(0) = \sqrt{\langle r^2\rangle_0\langle v^2\rangle_0}/G'M = \sqrt{V^2}r_{\rm e}^2(0)$ or $r_{\rm e}^{3*}(0) = M\sqrt{\langle r^2\rangle_0\langle v^2\rangle_0/|\Omega^3(0)|} = \sqrt{V^3}r_{\rm e}^3(0)$, which reflect how the particles initially distributed. Therefore, if $V^\mu \simeq 1$, the systems are stable [1,18].

If and only if $V^\mu = 0$, the potential has no global minimum, i.e., $\varepsilon(0)^2 = 0$. Therefore, only if the null virial conditions are fulfilled, the envelope is attracted to the origin independently of the initial distribution at the moment.

Fokker-Planck model. This behavior of the envelope under the null virial conditions means that all the particles are attracted to the origin, which is the *collapse*.[1] Then, because of the finite N, the inner particles arrive before the outer ones,

[1]Note that the emittance generally becomes nonzero while the system contracts. Therefore, the collapse does not mean that all the particles concentrate at the origin. When the initial distribution is homogeneous, the position is proportional to the velocity, so that the equality holds in Eq. (9), resulting in $\varepsilon(t)^2 = 0$. Because of the finite N, however, it is impossible that all the particles reach the origin at the same time [19].

so that the density around the center becomes quite high. The high density causes a transition from the collisionless to the collisional state.[2] After this transition, EEq is not appropriate for describing this system because this equation is valid for collisionless systems as mentioned before. Therefore, we shall derive the density profile uniformly by a special Fokker-Planck equation in the μ-dimensional space. The reason why we adopt the Fokker-Planck equation approach rather than the Boltzmann equation is as follows: Because of the high density around the center of the system, it is natural to consider that each collision there cannot be distinguished. Thus, the Boltzmann equation premising that the collision is distinguishable is not appropriate.

Before constructing the Fokker-Planck equation, we shall model forces influencing an particle of the system. In other words, we shall begin by constructing a Langevin equation.

We assume that the frictional force $-m\gamma\dot{\boldsymbol{r}}^\mu(t)$ and the random noise with constant intensity $\sqrt{2D}\boldsymbol{\xi}^\mu(t)$ which are essential for a many-body system to reach the thermal equilibrium state act on the particle [14], where m is a mass of the particle and $\boldsymbol{\xi}^\mu(t)$ represents Gaussian-white noises. The superscript means that the vector is in the μ-dimensional space. Postulating the system to be circularly or spherically symmetric,[3] one can see that the particle is also influenced by a mean gravitational force $-F^\mu(r)$ along the radial direction of the system, which is derived by differentiating $m\phi^\mu(r)$: $-F^\mu(r) = -m\partial_r\phi^\mu(r)$. However, this is just a mean gravity. It is natural to consider that the particle is actually influenced by a fluctuating gravity around the mean value: The number density producing $\phi^\mu(r)$ through the Poisson equation is the mean value, and the actual distribution of particles must fluctuate around the value. This means that another noise which prevents the system from reaching the thermal equilibrium state is added to the normal Langevin equation, so that this system goes to another stable state, i.e., QES. Therefore, we can consider the noise distinctive to SGS. Note that the fluctuations appear only because of the finiteness of the number of particles. As mentioned in the introduction, if the number goes to infinite the Vlasov equation becomes appropriate to describe the system [1], which means that each particle is influenced only by the mean potential.

If we assume the intensity of the noise to be constant, we obtain the following Langevin equation in the overdamped limit:

$$m\gamma\dot{\boldsymbol{r}}^\mu(t) = -F^\mu(r)\{1+\sqrt{2\epsilon}\eta(t)\}\boldsymbol{e}_r^\mu - \frac{\partial}{\partial r}\frac{\epsilon}{2m\gamma}F^\mu(r)^2\boldsymbol{e}_r^\mu + \sqrt{2D}\boldsymbol{\xi}^\mu(t), \tag{18}$$

where $\boldsymbol{e}_r^\mu$ is a unit vector along the radial direction in the μ-dimensional space and the second term on the right-hand side of the above equation is a correction term in order to regard products as the Stratonovich product [21].

The corresponding Fokker-Planck equation is given by

$$\begin{aligned}\frac{\partial}{\partial t}P^\mu(r,t) = {}& \frac{D}{(m\gamma)^2}\left\{\frac{\partial^2}{\partial r^2}+\frac{\mu-1}{r}\frac{\partial}{\partial r}\right\}P^\mu(r,t)\\ &+\frac{1}{m\gamma}\frac{1}{r^{\mu-1}}\frac{\partial}{\partial r}r^{\mu-1}F^\mu(r)P^\mu(r,t)\\ &+\frac{\epsilon}{(m\gamma)^2}\left\{\frac{\partial^2}{\partial r^2}+\frac{\mu-1}{r}\frac{\partial}{\partial r}+\frac{\mu-1}{r^{\mu-1}}\frac{\partial}{\partial r}r^{\mu-2}\right\}\\ &\times F^\mu(r)^2P^\mu(r,t).\end{aligned} \tag{19}$$

We are treating the system as a circularly or a spherically symmetric one including N particles. Hence, the probability distribution function (PDF) P^μ is a function of the distance from the origin r. Note that, in this theory, the relation among P^μ, the number density $\mathcal{N}^\mu$, and the mass density ρ^μ is as follows: $P^\mu = \mathcal{N}^\mu/N = \rho^\mu/(mN)$.

Let us use $\mathcal{P}^\mu = J^\mu P^\mu$ instead of P^μ, where J^μ means Jacobian determinant: $J^\mu = \mathcal{S}^\mu r^{\mu-1}$. In doing so, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{P}^\mu(r,t) = {}& \frac{D}{(m\gamma)^2}\left\{\frac{\partial^2}{\partial r^2}-\frac{\partial}{\partial r}\frac{\mu-1}{r}\right\}\mathcal{P}^\mu(r,t)\\ &+\frac{1}{m\gamma}\frac{\partial}{\partial r}F^\mu(r)\mathcal{P}^\mu(r,t)\\ &+\frac{\epsilon}{(m\gamma)^2}\frac{\partial^2}{\partial r^2}F^\mu(r)^2\mathcal{P}^\mu(r,t).\end{aligned} \tag{20}$$

When the system reaches QES, $\partial_t\mathcal{P}^\mu_{\rm qe} = 0$. Here, we integrate the Fokker-Planck equation over r. Owing to the use of $\mathcal{P}^\mu_{\rm qe}$, the integration becomes easier:

$$\begin{aligned}&-\left\{1+\frac{\epsilon F^\mu(r)^2}{D}\right\}\mathcal{P}^{\mu\,\prime}_{\rm qe}(r)\\ &\quad+\left[\frac{\mu-1}{r}-\frac{2\epsilon F^\mu(r)F^{\mu\prime}(r)}{D}-\frac{m\gamma F^\mu(r)}{D}\right]\mathcal{P}^\mu_{\rm qe}(r)\\ &\quad=\text{const.}\end{aligned} \tag{21}$$

Let us determine the constant of the right-hand side of the above equation by using the boundary condition at $r = 0$. Because of the symmetry, the mean field force can be represented as $F^\mu(r) \propto \int_0^r dr' J^\mu N P^\mu_{\rm qe}/r^{\mu-1}$. Then,

$$\begin{aligned}F^\mu(0) &\propto \lim_{r\to0}\frac{\int_0^r dr' J^\mu N P^\mu_{\rm qe}(r')}{r^{\mu-1}}\\ &= \lim_{r\to0}\frac{\frac{\mu\pi^{\mu/2}r^{\mu-1}}{\Gamma(\mu/2+1)}N P^\mu_{\rm qe}(r)}{(\mu-1)r^{\mu-2}} = 0,\end{aligned} \tag{22}$$

where we utilized the fact that $P^\mu_{\rm qe}(0)$ is bounded. Since $P^{\mu\,\prime}_{\rm qe}(0)$ is also bounded, $\mathcal{P}^\mu_{\rm qe}(0) = 0$ and $\mathcal{P}^{\mu\,\prime}_{\rm qe}(0) = \mu\pi^{\mu/2}(\mu-1)\delta_{\mu,2}P^\mu_{\rm qe}(0)/\Gamma(\mu/2+1)$, where $\delta_{\mu,2}$ is the Kronecker delta. Thus, by taking the limit $r\to0$, the second term of the

[2] In the case of 3DSGS, it is well known that the *local two-body relaxation time* by which the system becomes collisional is inversely proportional to the number density [20].

[3] Note that Pakter *et al.* showed that the symmetries in SGSs with low virial ratio break down [22]. However, the symmetries remain around the centers of SGSs, which is confirmed numerically. Therefore, the assumption of symmetries is relevant as long as we discuss the vicinity of the center.

left-hand side of Eq. (21) goes to

$$\lim_{r\to 0}\frac{\mu-1}{r}\mathcal{P}^{\mu}_{\rm qe}(r)=\lim_{r\to 0}\frac{\mu\pi^{\mu/2}(\mu-1)r^{\mu-2}}{\Gamma(\mu/2+1)}P^{\mu}_{\rm qe}(r)$$
$$=\frac{\mu\pi^{\mu/2}(\mu-1)\delta_{\mu,2}}{\Gamma(\mu/2+1)}P^{\mu}_{\rm qe}(0), \qquad (23)$$

which cancels with the first term. Therefore, the constant of Eq. (21) becomes 0. Additionally, by utilizing the number density in QES $\mathcal{N}^{\mu}_{\rm qe}(=N\mathcal{P}^{\mu}_{\rm qe}/J^{\mu})$, we can obtain

$$\frac{\mathcal{N}^{\mu\,\prime}_{\rm qe}(r)}{\mathcal{N}^{\mu}_{\rm qe}(r)}=-\frac{rF^{\mu}(r)\{m\gamma+2\epsilon F^{\mu\prime}(r)\}+(\mu-1)\epsilon F^{\mu}(r)^2}{r\{D+\epsilon F^{\mu}(r)^2\}}. \qquad (24)$$

By substituting $\phi^{\mu}=\frac{1}{m}\int dr F^{\mu}$ into the Poisson equation $\triangle\phi^{\mu}=4\pi\mathcal{G}^{\mu}m\mathcal{N}^{\mu}_{\rm qe}$, an equation governing F^{μ} can be obtained as follows:

$$F^{\mu\prime}(r)+\frac{\mu-1}{r}F^{\mu}(r)=4\pi\mathcal{G}^{\mu}m^2\mathcal{N}^{\mu}_{\rm qe}(r)\,, \qquad (25)$$

where $\rho^{\mu}_{\rm qe}=m\mathcal{N}^{\mu}_{\rm qe}=mNP^{\mu}_{\rm qe}=mN\mathcal{P}^{\mu}_{\rm qe}/J^{\mu}$.

Here, we nondimensionalize these equations by using the following units of length and force:

$$[\text{length}]=\sqrt{\mu^2(\mu+2)\mathcal{T}^{\mu}/2\pi\mathcal{G}^{\mu}m^3\mathcal{N}^{\mu}_{\rm qe}(0)}, \qquad (26)$$

$$[\text{force}]=\sqrt{8\pi\mu^2(\mu+2)\mathcal{G}^{\mu}m\mathcal{N}^{\mu}_{\rm qe}(0)\mathcal{T}^{\mu}}, \qquad (27)$$

where $\mathcal{T}^{\mu}=D/\{\mu^2\gamma+8\pi(\mu^2+4\mu+2)\epsilon\mathcal{G}^{\mu}m\mathcal{N}^{\mu}_{\rm qe}(0)\}$. Then, Eqs. (24) and (25) are altered to

$$\frac{\bar{\mathcal{N}}^{\mu\,\prime}_{\rm qe}(\bar r)}{\bar{\mathcal{N}}^{\mu}_{\rm qe}(\bar r)}=-2\mu(\mu+2)$$
$$\times\frac{\bar r\bar F^{\mu}(\bar r)\{1+2\mu q\bar F^{\mu\prime}(\bar r)\}+\mu(\mu-1)q\bar F^{\mu}(\bar r)^2}{\bar r\{\mu+2(\mu^2+4\mu+2)q+2\mu^2(\mu+2)q\bar F^{\mu}(\bar r)^2\}} \qquad (28)$$

and

$$\bar F^{\mu\prime}(\bar r)+\frac{\mu-1}{\bar r}\bar F^{\mu}(\bar r)=\bar{\mathcal{N}}^{\mu}_{\rm qe}(\bar r), \qquad (29)$$

where $q\equiv 4\pi\epsilon\mathcal{G}^{\mu}m^2\mathcal{N}^{\mu}_{\rm qe}(0)/(\mu m\gamma)$ and overbars denote dimensionless. We should solve these equations with boundary condition $\bar{\mathcal{N}}^{\mu}_{\rm qe}(0)=1$.

Results. The numerical solutions for $\mu=2$ and 3 are shown in Fig. 2 for different q. The curves with $q=0$ on both figures correspond to the thermal equilibrium state. For comparison with observations, $(1+\bar r^2)^{-1}$ and $(1+\bar r^2)^{-3/2}$ are also plotted by a dashed curve in Figs. 2(a) and 2(b), respectively. These dashed curves are typical best fit for densities of molecular clouds or stars in a globular cluster. From Fig. 2(b), one notices that the numerical result of our model with $q=0.01$ completely coincide with the typical number density. Figure 2(a) also shows the good agreement of our model with observations and numerical simulations for small radius by setting $q=0.56$, although the deviation between two curves increases as r gets larger. Therefore, we can understand that the best-fit curves are derived from our model by varying q appropriately.

Finally, we examine the range of the index κ. From Eqs. (28) and (29), the number density around the center of the system

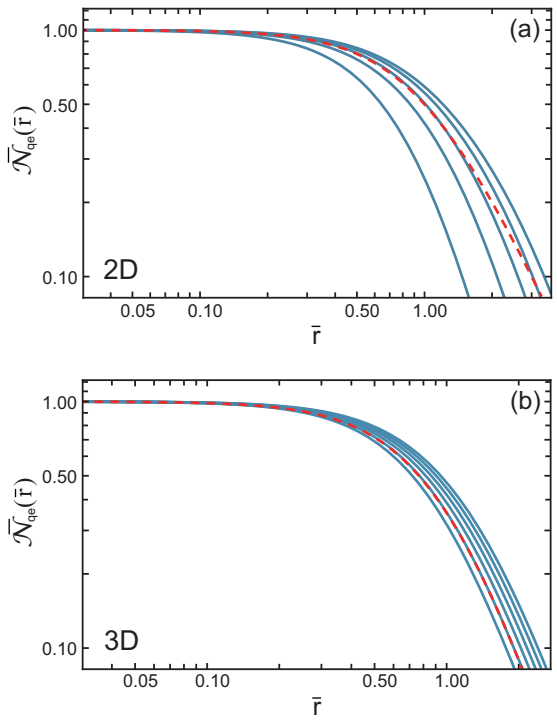

FIG. 2. Numerical solutions of Eqs. (28) and (29) for $\mu=2$ (a) and $\mu=3$ (b). As the curve changes from the left to the right in panel (a), q gets larger from 0 to 0.28 in steps of 0.07. On the other hand, q gets larger from 0 to 0.05 in steps of 0.01 in panel (b). The (red) dashed curve in panels (a) and (b) means $(1+\bar r^2)^{-1}$ and $(1+\bar r^2)^{-3/2}$, respectively.

in QES can be described by $\bar{\mathcal{N}}^{\mu}_{\rm qe}(\bar r)\simeq(1+\bar r^2)^{-\kappa(q)}$, where κ is a function of q:

$$\kappa(q)=\frac{(\mu+2)\{1+(\mu+1)q\}}{\mu+2(\mu^2+4\mu+2)q}\,. \qquad (30)$$

From this equation, as $q\geqslant 0$, the range of κ is

$$\frac{(\mu+1)(\mu+2)}{2(\mu^2+4\mu+2)}<\kappa\leqslant\frac{\mu+2}{\mu}\,. \qquad (31)$$

Therefore, with regard to $\mu=2$ and $\mu=3$, $\frac{3}{7}<\kappa\leqslant 2$ and $\frac{10}{23}<\kappa\leqslant\frac{5}{3}$, respectively. Both ranges include the observed indices. However, the observed ranges of the index are narrower, which means that the value of q is limited. The limitation of q can be regarded as a kind of fluctuation-dissipation relation [23], which is particular for SGS because q includes a ratio of the intensity of *mean gravity fluctuation* ϵ to the friction coefficient γ.

Concluding remarks. In this paper, we made it clear numerically that 2DSGS without boundary goes to QES in which the density profile is depicted by Eq. (1) with $\kappa\sim 1$, especially for the null virial condition. It is well known that QES of 3DSGS without boundary can be described by Eq. (1) with $\kappa\sim 3/2$ through the observations of globular clusters and N-body simulations: It was shown that there is the universal structure in QES between 2D and 3DSGS. The discussion based on EEq could explain why the null virial condition is necessary for the universality, when a DF has a circular

or spherical symmetry. Exploring the universality without these symmetries remains as a future work. Furthermore, we developed the model to derive the universal density from the special Langevin equation including the distinctive noise of SGS. Indeed, the solution of the corresponding Fokker-Planck equation in QES was depicted by Eq. (1). In addition, it was found that κ in Eq. (1) can be represented as a function of the intensity of the particular noise for SGS, the friction coefficient, and others. Therefore, we showed the range of κ for each dimension analytically, which has a good consistency with observations and simulations.

Acknowledgments. The author is grateful to Y. Levin, L. Rodet, M. Morikawa, the referees, and members of astrophysics laboratory at Ochanomizu University for extensive discussions. He also would like to thank T. Tatekawa for N-body simulations. This work was supported by JSPS Grant-in-Aid for Challenging Exploratory Research No. 24654120.

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