Spontaneous symmetry breaking and phase coexistence in two-color networks

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(Received 8 August 2015; revised manuscript received 19 November 2015; published 6 January 2016)

We consider an equilibrium ensemble of large Erdős-Renyi topological random networks with fixed vertex degree and two types of vertices, black and white, prepared randomly with the bond connection probability p. The network energy is a sum of all unicolor triples (either black or white), weighted with chemical potential of triples μ. Minimizing the system energy, we see for some positive μ the formation of two predominantly unicolor clusters, linked by a string of N_{bw} black-white bonds. We have demonstrated that the system exhibits critical behavior manifested in the emergence of a wide plateau on the $N_{bw}(\mu)$ curve, which is relevant to a spinodal decomposition in first-order phase transitions. In terms of a string theory, the plateau formation can be interpreted as an entanglement between baby universes in two-dimensional gravity. We conjecture that the observed classical phenomenon can be considered as a toy model for the chiral condensate formation in quantum chromodynamics.

DOI: 10.1103/PhysRevE.93.012302

I. INTRODUCTION

During past two decades it has been recognized that Landau classification of phase transitions does not cover all patterns of symmetry breaking. The fractional quantum Hall effect and the Levin-Wen string networks [1] provide well known examples of topological order for so-called gapped states. Corresponding quantum phase transitions exist at $T = 0$ and do not have any local order parameters. Some symmetry aspects of two-dimensional (2D) topological phases with $\mathbb{Z}_2$ symmetry were discussed in [2]. Quantities describing topological phases are (i) the degeneracy of the ground state (which depends on the topology of the system) and (ii) the holonomy of the non-Abelian Berry connection. Another way to identify topologically ordered states is to use the topological entanglement entropy [3]. In any cases, the topological order is manifested in the emergence of long-range correlations in the system. The interplay between some local rules and global constraints influences the behavior of the whole system and is responsible for the long-range order.

Colored random networks have become a ubiquitous paradigm for a wide range of physical and social phenomena in distributed systems, spread from producer-consumer relations to string theory and from budding in lipid membranes to the creation of baby universes in cosmology. Here we study random black-white vertex networks with the Levin-Wen type of the Hamiltonian [1], evolving by bond reconnections and tending to increase the number of unicolor triples under vertex degree conservation. The local rule is the conservation of the vertex degree in all nodes of the network, while the global constraint is an attempt to maximize the number of triples of the same color. As the energy of the triples increases, the network breaks into two predominantly unicolor (black and white) clusters, connected by a bunch of links between black and white vertices. Unexpectedly, this bunch is stable in a wide interval of triple energy. We conjecture that this statistical phenomenon has an origin relevant to quantum entanglement.

Ensembles of random Erdős-Renyi topological graphs (networks) provide an efficient laboratory for testing various collective phenomena in statistical physics of complex systems, being also tightly linked to random matrix theory. Besides investigating typical statistical properties of networks, such as vertex degree distribution, clustering coefficients, small world structure, and spectra of adjacency matrices, the past two decades have been marked by rapidly growing interest in more refined graph characteristics, for example, the distribution of triadic motifs in oriented networks (small subgraphs involving different triads of vertices).

Triadic interactions, being the simplest interactions beyond the free-field theory, play a crucial role in network statistics. Apparently, just the presence of triadic interactions is responsible for the emergence of phase transitions in complex distributed systems. The first signature of a phase transition in a random network, known as the Strauss clustering model [4], was treated by random matrix theory in [5] and identified with the first-order phase transition in the framework of a mean-field cavitylike approach in [6]. Another example of phase transition is connected with the triadic motif pattern formation, known as motif superfamilies, in real evolutionary networks [7]. This problem was theoretically analyzed in [8], where it was conjectured that stable motif profiles constituting superfamilies [7] may correspond to stability islands associated with localized states in the space of motifs. The localization of states occurs as a first-order transition if the chemical potential associated with triadic motif energy is large enough. The critical behavior of triadic motif concentration, as a function of the chemical potential, has been studied in detail in [8].

Specifically, in this work we investigate the dependence of an equilibrium number of links N_{bw} connecting black and white vertices in networks with fixed vertex degree p as a function of the chemical potential μ controlling the number of unicolor vertex triples (all three vertices are black or all three are white). The attempt to minimize the energy of unicolor triples leads to almost immediate color separation, when at any infinitesimal positive value of μ two clusters with opposite colors ($\mathbb{Z}_2$ charges) are formed. A more striking phenomenon is observed for large enough μ: The $N_{bw}(\mu)$ curve has a wide plateau separating two Arrhenius-type dependences at low and high μ. Such a plateau is never seen (for any p) if the energy of the system is the sum of unicolor vertex pairs (but not triples) or if the vertex degree is not conserved. We suggest two parallel interpretations of the phenomenon observed in our simulations: classical (in terms of classical first-order transitions) and quantum (in terms of spins interacting with 2D topological gravity). Describing the system in terms of classical statistical physics, we interpret the plateau emergence as a spinodal decomposition similar to the vapor-liquid first-order phase transition.

On the other hand, the quantum gravity language implies the realization of the network as the triangulation of a 2D fluctuating surface. Since the network does not involve a metric, one deals with a purely topological version of 2D gravity where the chemical potential of unicolor triples μ can be understood as a 2D cosmological constant. The spin (i.e., color) separation phenomenon is interpreted as anomalous transport on the string worldsheet due to an axial anomaly in the gravitational field. In terms of 2D gravity, it is conjectured that the plateau formation occurs as the specific entanglement of two baby universes. We propose a synchronization mechanism of phase transitions in each universe and suggest that their entanglement results in a sort of a phase coexistence phenomenon. To describe the phase coexistence properly, one could try to find gravity counterparts of thermodynamic variables. This is a subtle issue and we rely on recent interpretation of (P,V) variables in cosmological terms [9] dealing with van der Waals thermodynamics of charged or rotating black holes [10].

We also mention a useful analogy with the chiral symmetry breaking in the instanton–anti-instanton ensemble in quantum chromodynamics (QCD) described by the random matrix model. In this case, the system at small μ and low temperature is in a phase with the spontaneously broken chiral symmetry characterized by the nonvanishing chiral condensate $\langle\bar{\Psi}\Psi\rangle$. Staying at $T=0$ and increasing the chemical potential μ, we force the system to undergo a quantum phase transition at some μ_c. The density of the chiral condensate is proportional to the number of instanton–anti-instanton connections and the chiral symmetry restoration signifies the emergence of another condensate $\langle\Psi\Psi\rangle$. We conjecture that our two-color network could be a toy model for the formation of such a condensate.

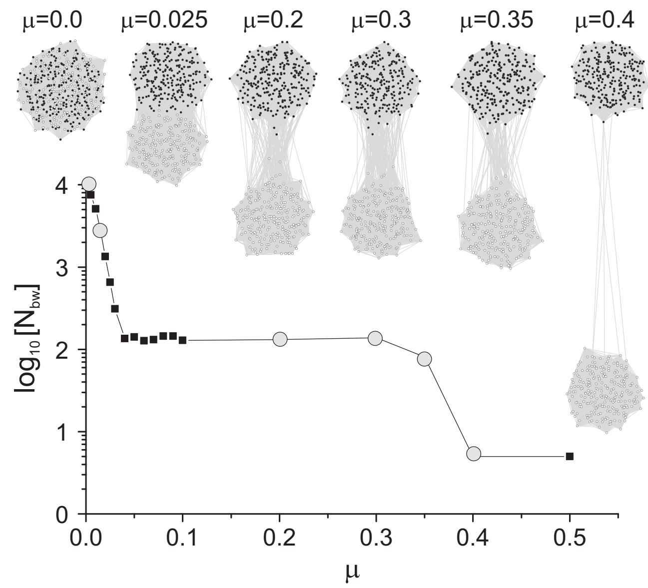

FIG. 1. Dependence of the equilibrium number N_{bw} of black-white bonds on the chemical potential μ per each unicolor triple for a $N=500$ vertex random network of probability $p=0.15$ with equal composition of black and white nodes. The typical network topologies for several values of μ along a $N_{bw}(\mu)$ curve are shown above the $N_{bw}(\mu)$ curve.

II. MODEL AND KEY OBSERVATIONS

The setup of the model is as follows. Take N vertices and label them by integers $1,\dots,N$. Vertices $i=1,\dots,cN$ $(0<c<1)$ are black, i.e., are associated with Ising spins $\sigma_i=-1$, while vertices $i=cN+1,\dots,N$ are white and carry $\sigma_i=1$. The initial realization of the Erdős-Renyi network is prepared by connecting any randomly taken pair of vertices with the probability p ($p>\frac{1}{N}$, i.e., p is above the percolation threshold). Then one randomly chooses two arbitrary links, say, between vertices A and B (A-B) and between C and D (C-D) and reconnects them, getting new links A-C and B-D (the former connections A-B and C-D are destroyed). Such reconnection conserves the vertex degree [11]. Now one applies the standard Metropolis algorithm with the following rules: (i) If after the reconnection the number of unicolor vertex triples is increased, a move is accepted; (ii) if the number of unicolor vertex triples is decreased by $\Delta N_{\rm tripl}$ or remains unchanged, a move is accepted with the probability $e^{-\mu\Delta N_{\rm tripl}}$. We have run the Metropolis algorithm repeatedly for a large set of randomly chosen pairs of links, until it converges. For one-color networks it was proven [12] that such a Metropolis algorithm converges to the Gibbs measure $e^{\mu N_{\rm tripl}}$ in the equilibrium ensemble of random undirected Erdős-Renyi networks with fixed vertex degree. For two-color networks the convergence has not been proven rigorously, however extensive numerical tests do not show any pathologies.

The obtained results are as follows. The dependence of the number of links connecting black and white vertices $N_{bw}(\mu)$ demonstrates for small and large μ the Arrhenius activation kinetics $N_{bw}(\mu)\sim e^{-\beta(p)\mu}$, where $\beta(p)$ is some connectivity-dependent constant. However, the function $N_{bw}(\mu)$ develops a plateau in a wide region of μ, manifesting a kind of a critical behavior. The corresponding plot for the network of $N=500$ vertices is shown in Fig. 1 for $p=0.15$. Analyzing the clustering structure of the network for different μ, one can see that at small μ ($\mu\sim\frac{1}{N}$) the system splits for any $p\gg\frac{1}{N}$ into two clusters of predominantly black and white vertices, meaning spontaneous $\mathbb{Z}_2$ symmetry breaking (the clusters are allocated by a preferential number of links entering it, rather than going outside). So the plateau is developed in the $\mathbb{Z}_2$ broken phase. These results are reproducible for various values of c (where c is the fraction of black vertices). The same behavior is seen for random regular networks, which have a fixed vertex degree d in all network nodes.

Recall that we consider very dense networks $p = O(1)$, far above the percolation threshold $p = \frac{1}{N}$. Certainly, even in this regime the entire network contains some number of disjoint clusters, exponentially suppressed in their sizes. In colored network, weights of different rewirings are color dependent and could affect the distribution in cluster sizes. We believe that for dense networks (considered here) the effect of a color-dependent redistribution in cluster sizes is negligible against the background of the exponentially suppressed overall number of disjoint components. However, near the percolation threshold this effect might be visible and definitely deserves special attention. The following two key results are the subjects of our discussion and interpretation in terms of lattice gases and string theory: immediate spontaneous formation of black and white clusters (breaking $\mathbb{Z}_2$ symmetry) at $\mu \sim \frac{1}{N}$ and the emergence of a wide plateau in the $N_{bw}(\mu)$ dependence.

III. SYMMETRY BREAKING

A. Statistical interpretation

First of all, it is worth pointing out that the $\mathbb{Z}_2$ symmetry breaking and the plateau formation are different physical effects. Separation of clusters is typical phenomenon that can be understood using naive mean-field arguments. Let N_b and N_w be the number of black and white nodes, respectively ($N_b + N_w = N$). In a mixed system the interaction energy of one b node (w node) with m randomly chosen other nodes is $u_b = \frac{m}{N}(N_b\phi_{bb} + N_w\phi_{bw})$ [$u_b = \frac{m}{N}(N_b\phi_{bb} + N_2\phi_{wb})$], where ϕ_{bb} and ϕ_{bw} (ϕ_{wb}) are the b-b and b-w (w-b) interaction energies. The free energy of the system reads $F = \frac{1}{2}(N_b u_b + N_w u_w) - kT(N_b \ln \frac{N_b}{N} + N_w \ln \frac{N_w}{N})$. Defining $\phi = \phi_{bb} + \phi_{bw}$ and $\Delta = \phi_{bb} - \phi_{bw}$, passing to concentrations $c_{b,w} = \frac{N_{b,w}}{N}$ ($c_b + c_w = 1$), and introducing the asymmetry of the network composition $\eta = c_b - c_w$, we can write the network free energy $f = \frac{F}{N}$ as follows:

$$f(\eta) = f_0 + \frac{m\Delta}{4}\eta^2 - \frac{kT}{2}[(1-\eta)\ln(1-\eta) + (1+\eta)\ln(1+\eta)], \qquad (1)$$

where $f_0 = \frac{m}{4}\phi + kT\ln 2$.

The system described by the thermodynamic potential (1) experiences the standard phase transition with the stationary states η_c determined by the equations $\frac{df(\eta)}{d\eta} = 0$ and $\frac{df(\eta)}{d\eta}|_{\eta_c} > 0$. Supposing that $|\eta| \ll 1$, we can expand $f(\eta)$ up to the fourth term and get the equation

$$\left(\frac{m\Delta}{2} - kT\right)\eta - \frac{kT}{3}\eta^3 = 0 \qquad (2)$$

having a single solution $\eta_c = 0$ at $m \leqslant \frac{kT}{\Delta}$ and exhibiting spontaneous symmetry breaking into two phases $\eta_c^{a,b} = \pm\sqrt{3(\frac{m\Delta}{kT} - 1)}$ at $m > \frac{kT}{\Delta}$.

The mean-field arguments accounting for a plateau are provided in the next section in terms of lattice gas statistics, random matrices, and string theories. Being applied to networks, the random matrix approach means the discretization of a two-dimensional string worldsheet embedded in some D-dimensional target space. Since the colored (black and white) vertices can be described by Ising spin variables, our system is some specific model of matter (Ising spins) coupled to gravity (a fluctuating surface over an ensemble of allowed reconnections in the random network). In the matrix model framework, which parallels the mean-field analysis, such systems have been considered in [13], however, with an important difference: In our system Ising spins are quenched, while in matrix approaches they are typically annealed.

B. Stringy interpretation

Nowadays string theory plays two different roles in fundamental physics. First, it is a model of quantum gravity since the spectrum of closed strings involves a massless graviton. Second, the string can be regarded as a probe and the physics of a 2D string worldsheet should reflect all phenomena happening in the entire target space in which this worldsheet is embedded. This is the key idea behind the gauge-string duality. Having in mind these roles of string theory and considering a network as a discretized string worldsheet, we rephrase in stringy terms the phenomenon observed in the black-white network.

The spontaneous symmetry breaking and the black-white cluster separation can be interpreted as occurrences of anomalous transport on the string worldsheet. Anomalous transport emerges due to the quantum nonconservation of classically conserved currents in the external gauge or gravitational fields. The spin separation phenomenon in our problem means that a nonvanishing axial current takes place on the worldsheet when the spins in subensembles $\mathcal{S} = \{\sigma = +1\}$ and $\mathcal{S} = \{\sigma = -1\}$ move in opposite directions. Since spins are quenched and do not fluctuate, the current is induced purely by fluctuations of the 2D surface. The network is a topological object, hence it is natural to describe it as the topological 2D gravity of Jackiw-Teitelboim type which involves the cosmological constant Λ,

$$L = \int d^2x\sqrt{g}\Phi(R + \Lambda), \qquad (3)$$

where Φ is the scalar dilation field and R is the Ricci curvature of the two-dimensional metric g. This Lagrangian can be written as the topological 2D Yang-Mills theory with an SL$(2,R)$ gauge group (see [14] for a review).

It is known that in the pure 2D gravity the total number of triangles measures the total area, hence the corresponding chemical potential μ of triples can be interpreted as an effective 2D cosmological constant. For the equation of motion we have $R = -\Lambda$. There is an axial current due to the chiral anomaly in an external gravitational field

$$\partial_\nu J_\nu^5 = R, \qquad (4)$$

which for the equation of motion equals the cosmological constant μ. Thus, for the axial current itself in terms of the spin connection ω_ν we have

$$J_\alpha^5 = \epsilon_{\alpha\nu}\omega_\nu \propto \mu. \qquad (5)$$

Equation (5) demonstrates the emergence of spin current for any nonzero chemical potential of triads μ, which is the microscopic mechanism of black and white node separation.

We describe the $\mathbb{Z}_2$ spontaneous symmetry breaking in a colored dynamic network using mean-field statistical arguments and from a more involved viewpoint in terms of fluctuating surfaces. In what follows we briefly discuss these two different approaches.

It should be pointed out that the spin separation phenomenon is known in other systems as well. At low temperature the 2D Ising model exhibits spontaneous magnetization, developing the coherent spin domains separated by the domain walls. The $\mathbb{Z}_2$ symmetry is spontaneously broken in this phase. In the case of fluctuating Ising spins coupled to 2D gravity, the spontaneous creation of baby universes takes place and it is known that the spin direction in the daughter universe is always opposite to the spin directions in the parent one (see [15] for a review). In our case we see the formation of clusters with opposite spin signs due to dynamic rearrangement of bonds only (since the spins are quenched). A similar spin separation phenomenon occurs in 2D matter with nonvanishing chemical potential [16] and in 4D matter in an external magnetic field (see [17] for a review). The chirality separation emerges in some lattice QCD systems when sheets of opposite chiralities become connected by stringy skeletons [18].

IV. PLATEAU FORMATION

The most intriguing question concerns the interpretation of the plateau formation, since it seems to be a phenomenon with the signature of a quantum phase transition. In contrast to the symmetry breaking, the plateau formation is an essentially collective effect that disappears for networks with a nonconserved vertex degree.

The relevant image that highlights the basic features of this phenomenon is the famous game of fifteen puzzle. The analogy goes as follows. When the concentration of graph links is small, it is always possible to minimize the system energy (the sum of unicolor triples) almost locally. This resembles the initial stages of ordering in the puzzle, when one takes care of single-tile placements only. However, our network gets highly frustrated due to the vertex conservation condition: In order to find the way to place some new good link that minimizes the unicolor triple energy, it might be necessary to pass over a high potential barrier and remove other good links that have already been placed. Thus, the energy minimization involves unfavorable sequences of link permutations until all necessary constraints are satisfied. This is exactly what happens at late stages of ordering in the game of fifteen.

A. Statistical interpretation

To proceed with the combinatorial interpretation, we represent the graph by the symmetric adjacency matrix J_{ij} and split it into four sectors with the corresponding numbers of particles (links) $N_{bb}, N_{bw}, N_{wb}, N_{ww}$ (see Table I). The matrix J_{ij} is symmetric, namely, $N_{bw} = N_{wb}$. The unicolor triple is any pair of particles in one row or in one column in the sectors [●●] or [○○]. The vertex degree conservation implies the condition $\sum_{i=1}^{N} J_{ij} = J_j^{(0)}$, where $J_j^{(0)}$ $(j = 1, \ldots, N)$ are the initial vertex degrees, fixed by the network preparation conditions. Hence the simplest rearrangements of graph links should involve pairs of particles (links) as shown in Fig. 2. As we shall see, this figure helps explain the appearance of the plateau on the $N_{bw}(\mu)$ curve depicted in Fig. 1.

TABLE I. Number of particles in each sector for the symmetric adjacency matrix J_{ij}.

No. of particles	Sector	Symmetric adjacency matrix
N_{bb}	[●●]	$[1 \leqslant \{i,j\} \leqslant Nc]$
N_{ww}	[○○]	$[Nc+1 \leqslant \{i,j\} \leqslant N]$
N_{bw}	[●○]	$[Nc+1 \leqslant i \leqslant N, 1 \leqslant j \leqslant Nc]$
N_{wb}	[○●]	$[1 \leqslant i \leqslant Nc, Nc+1 \leqslant j \leqslant N]$

The physics behind the $N_{bw}(\mu)$ dependence is as follows. When the chemical potential μ of unicolor triples is increasing, the formation of aligned pairs of particles (in rows or in columns) in the sectors [●●] and [○○] becomes favorable. There are two mechanisms that increase the number of aligned pairs in these sectors; they are schematically shown in Figs. 2(a) and 2(b). In the first mechanism I the exchange of particles [shown by arrows in Fig. 2(a)] $1 \leftrightarrows 2$ (and of their symmetric counterparts $1' \leftrightarrows 2'$) pushes particles $1,1'$ to [●●] and particles $2,2'$ to [○○]. In the second mechanism II [see Fig. 2(b)] the exchange $1 \leftrightarrows 2$ (and $1' \leftrightarrows 2'$) preserves the number of particles in all sectors.

At small μ [at the beginning of the $N_{bw}(\mu)$ curve] both mechanisms (I and II) of unicolor triple energy increasing are available. However, as the number of particles in the sectors [●●] and [○○] exceeds some critical value, the entropic loss due accumulation of particles in these sectors forces the first-order vapor-liquid-like phase transition in sectors [●●] and [○○]. Note that since the initial concentrations $c_{bb}^{(0)} = N_{bb}^{(0)}/N^2c^2$ and $c_{ww}^{(0)} = N_{ww}^{(0)}/N^2(1-c)^2$ of particles in sectors [●●] and [○○] do not coincide in general (they are fixed by a random network preparation), the difference

$$\left|c_{bb}^{(0)} - c_{ww}^{(0)}\right| \sim (Np)^{-1/2} \tag{6}$$

desynchronizes the vapor-liquid transitions in [●●] and [○○]. So we connect the beginning of the plateau on the $N_{bw}(\mu)$

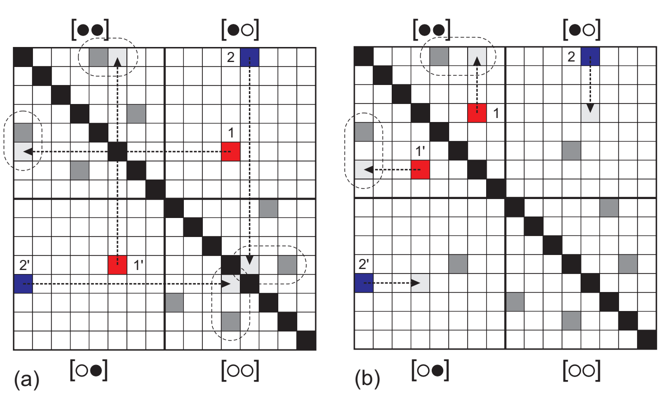

FIG. 2. (a) Pulling particles (bonds) 1 and 2 (and their symmetric counterparts $1'$ and $2'$) from the sectors [●○] and [○●] into the sectors [●●] and [○○]. The rearrangement results in two newborn triples of graph vertices (pairs of particles). (b) Bond rearrangement that conserves the number of particles in all sectors, but increases the number of triples in the sector [●●].

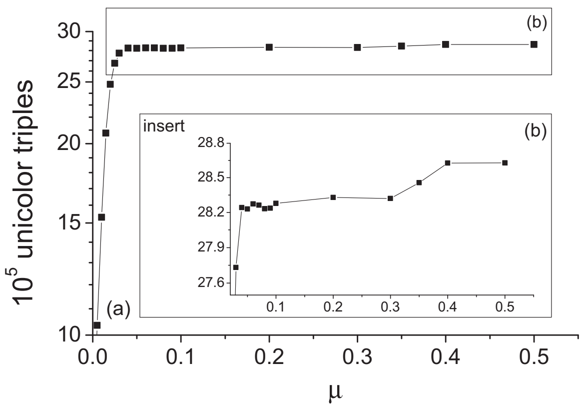

FIG. 3. Number of unicolor triples $N_{\rm tripl}$ as a function of μ in the ensemble of networks with $N = 500$ and $p = 0.15$.

curve with the *first occurrence* of independent vapor-liquid transitions in [••] or in [◦◦]. The plateau itself we identify with the synchronization of these two vapor-liquid transitions in [••] and in [◦◦]. At the plateau mechanism I gets suppressed, because capturing particles by sectors [••] and [◦◦] becomes entropically unfavorable.

The plateau on the $N_{bw}(\mu)$ curve (shown in Fig. 1) occurs simultaneously with the plateau on the $N_{\rm trip}(\mu)$ curve, shown in Fig. 3, where $N_{\rm tripl}$ is the number of unicolor triples. The plateau on the $N_{\rm tripl}(\mu)$ curve indicates the topological quench of each unicolor subnetwork in [••] and in [◦◦]. This behavior is a signature of a spinodal decomposition typical for the vapor-liquid phase transitions. The mechanism that competes with the increase of energy of triples in the plateau region is just mechanism II of particle exchange: Particle pairings, as shown in Fig. 2(b), do not change the numbers of particles in all sectors, but act against the entropy in sectors [•◦] and [◦•] since trapping of particles such as 2 and 2′ is entropically unfavorable.

Altogether, we are led to the following conjecture. The plateau begins at the value $\mu_{\rm beg}$ independently in subnetworks [••] and [◦◦] as the individual vapor-gas phase transitions. At the plateau the interaction between sectors [••] and [◦◦] is ensured by process II, particle trapping in [••] and [◦◦] works against the entropy in [•◦] and [◦•], and we deal with two effectively coupled subsystems [••] and [◦◦], both exhibiting a first-order transition and existing in a joint metastable state. The plateau is finished at the value $\mu_{\rm end}$, at which both sectors [••] and [◦◦] fall down in liquid states and decouple. At $\mu > \mu_{\rm end}$ the entropic contributions in all sectors become negligible and the Arrhenius-type activation kinetics gets restored.

Note that just above the percolation threshold, i.e., for $p \gtrsim \frac{1}{N}$, the standard Erdős-Rényi network, besides the giant component, contains many disconnected clusters with the size distribution $c(k) \sim k^{-5/2}$, where $c(k)$ is the concentration of clusters of k vertices. For black-white networks the distribution $c(k)$ should be modified since weights of particular clusters become color dependent. For all p clusters contain only nodes of the same color and all black-white links are external, connecting oppositely colored clusters. The very existence of the plateau is insensitive to the number of disjointed unicolor clusters in the network, however the plateau disappears for quite small p.

B. Two-dimensional gravity or stringy interpretation

In terms of 2D gravity, the emergence of the plateau can be linked with baby universe formation. It was proven in [15,19] that critical phenomena in 2D gravity deal with the formation of a newborn (daughter) bubble connected by the neck to the parent universe. The free energy of the system consists of two competing contributions: the mean value of the total neck diameter and the spontaneous curvature of the newborn (baby) universe of a given size. The same mechanism lies behind the budding of vesicles in liquid membranes [20]. In terms of the black-white network, the color separation can be interpreted as the baby universe formation, whose size depends on the ratio $N_{bb}^{(0)}/N_{ww}^{(0)}$, where $N_{bb}^{(0)}$ (or $N_{ww}^{(0)}$) are the initial numbers of black-black (or white-white) nodes, while the interaction of triples plays the role of a curvature contribution to the free energy (fixed by the cosmological constant μ).

The plateau formation, in terms of 2D gravity, signals that the neck size (or sum of all neck sizes for a multicluster network) connecting the parent and baby universes does not depend in some interval on the bulk cosmological constant μ. The pure 2D gravity undergoes the phase transition at some value μ_c (see [21] for a review), therefore upon the color separation, we have two copies of 2D gravities connected by the neck. Each gravity could be in one of two phases. Since we consider the quantum geometry, there is no reason for initial synchronization of phase transitions in these two universes. The value of the cosmological constant, at which the first transition happens, is presumably the point at which the plateau gets started. Two universes are entangled, which in terms of each 2D gravity can be interpreted as the insertion of the macroscopic loop operator.

So the phase transition in one (say, black) universe happens spontaneously, however, due to the quantum entanglement of two universes ([••] and [◦◦] clusters), the transition in the second (white) universe is induced by the vacuum expectation value of the macroscopic loop operator. The plateau corresponds to the coexistence of two phases in 2D gravity and when the second universe undergoes the induced phase transition, they get separated and the plateau terminates. Note that we consider the Euclidean version of 2D gravity, when the baby universes can be produced.

If we would deal with fluctuating Ising spins coupled to 2D gravity, the results could be borrowed from the literature. The case when spins exist on faces was solved in [13], while the dual Ising model for spins existing on vertices was treated in [22]. In the last case the model was reduced to the $O(1)$ loop model. In our system we have spins on the vertices, hence the dual Ising model [22] is more relevant. It admits the two-matrix representation [22]

$$Z = \int dX\, dY e^{{\rm Tr}\, X^2 + \mu {\rm Tr}\, X^3 + {\rm Tr}\, Y^2 + q\, {\rm Tr}\, XY^2}, \qquad (7)$$

where X is the adjacency matrix for both black-black and white-white links, while Y is the matrix of black-white connections. Besides, we have two additional ingredients compared to [22]: The spins are quenched and the graph vertex degree is conserved. The degree conservation can be introduced into the matrix model by the linear term with the Lagrangian multiplier Tr MX playing the role of an external magnetic field similar to [23]. At $Y = 0$ we are back to the pure matrix model describing 2D gravity with the cosmological constant μ.

The plateau formation is analogous to the chiral symmetry breaking in QCD with a nonvanishing baryonic chemical potential μ. Recall key QCD properties relevant to our system. At $T = 0$ the QCD is in the confinement phase with the chiral condensate $\langle\bar{\Psi}\Psi\rangle \neq 0$. The popular model of the QCD vacuum is the instanton–anti-instanton $I\bar{I}$ liquid, represented by an ensemble of interacting pointlike objects of two types. The value of the chiral condensate is presumably expressed in terms of the number of connected instanton–anti-instanton pairs [24], which is the number of black-white links in the network language. The number of fermionic zero modes at the instanton is an unchanged topological number, meaning a fixed vertex degree in network terms. In the $I\bar{I}$ ensemble zero modes at individual instantons get collectivized. The phase diagram in QCD is quite rich in the temperature-chemical potential plane. At small μ the system is in the confinement phase with broken chiral symmetry. By increasing μ, one reaches a mixed phase with two coexisting condensates $\langle\bar{\Psi}\Psi\rangle$ and $\langle\Psi\Psi\rangle$. Such a mixed phase exists in some region in μ, in which the string tension does not depend on the density, being analogous to the network plateau. Further growth of μ forces the system to fall down into the so-called color superconducting phase, with chiral symmetry being restored and the QCD string disappearing. In the QCD color superconducting phase, which is believed to occur in neutron stars, condensation of a quark-quark state happens similarly to the Cooper pairing in conventional superconductors. The phenomenon of chiral symmetry restoration at large μ apparently can be described by the effective random matrix model (see [25] for a review). We conjecture that the two-color network picture could be regarded as a toy model for zero-mode collectivization and can be related to the holographic representation of the chiral symmetry breaking proposed in [26].

V. DISCUSSION

In this paper we have described a critical phenomenon in the statistics of a two-color random network: the emergence of a wide plateau in the concentration of bonds linking black and white nodes. The key condition for the plateau development is the graph vertex degree conservation, a very natural condition for any topological ensemble. The physics behind this phenomenon can be thought of as a particular mechanism of bond collectivization in Levin-Wen-type topological Hamiltonians. It has evident parallels with the effect of fermionic zero-mode collectivization, emerging from localized solutions of the Dirac equation in the instanton–anti-instanton QCD ensemble. Recall that this effect results in chiral symmetry breaking at small condensate density and its restoration at large density. It would be highly desirable to find an appropriate matrix model description of the phenomenon, similar to the QCD chiral matrix model describing the spectrum of the Dirac operator in the $I\bar{I}$ ensemble.

One could consider ensembles of topological defects of different nature such as instantons, monopoles, or vertices. The effect of plateau formation due to zero-mode collectivization is expected to be universal for them. Recently, the formation of the condensate in the colorless instanton ensemble without anti-instantons was investigated [27], where it was found that a microscopic description of the condensate involves refined knot invariants. The critical behavior in such colorless topological ensembles inspires the conjecture that knot recognition by the topological invariant is different in different phases. It would be very interesting to understand whether a relation to the knot invariants can be found for colored topological ensembles.

In this work we followed the probe approach for the string, however one could question whether the critical plateau formation would have stringy meaning if considering the colored network as a quantum gravity model. The idea to use a colored network in this way was recently suggested in [28]. However, our model differs from the one proposed in [28] in two crucial respects: (i) We consider a conserved vertex degree and (ii) in our case, contrary to [28], the flow increases the number of triads, since the signs of cosmological constants in our model and in [28] are different.

The criticality found in colored networks seems to be a quite general phenomenon and could have practical applications in real life networks where there are sets of separated communities (black and white nodes). Criticality takes place even in highly asymmetric black-white networks (i.e., for the parameter c lying in the region about [0.1,0.9]). As an immediate example we could mention social networks. In this framework, the phenomenon of plateau formation can be regarded as follows. Consider two social communities and assume that each community desires to increase the number of its own triadic connections (its own simplest cliques) measured with some weight (the chemical potential of triads). We predict in such networks the existence of the stability plateau (i.e., in the presence of unavoidable communication between communities), at which the number of links between different communities is insensitive to the weight of its own connections.

ACKNOWLEDGMENTS

We are grateful to D. Kharzeev, I. Kostov, A. Lokhov, A. Mironov, and M. Tamm for fruitful discussions. S.N. was partially supported by an IRSES DIONICOS grant. O.V. and V.A. acknowledge support of the Higher School of Economics program for Basic Research. The work of A.G. was supported in part by Grant No. RBBR-15-02-02092 and Russian Science Foundation Grant for IITP No. 14-050-00150. A.G. thanks the Simons Center for Geometry and Physics for hospitality and support during the program "Knot Homologies, BPS States and Supersymmetric Gauge Theories."

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