Hopf bifurcation analysis for a dissipative system with asymmetric interaction: Analytical explanation of a specific property of highway traffic

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A dissipative system with asymmetric interaction, the optimal velocity model, shows a Hopf bifurcation concerned with the transition from a homogeneous motion to the formation of a moving cluster, such as the emergence of a traffic jam. We investigate the properties of Hopf bifurcation depending on the particle density, using the dynamical system for the traveling cluster solution of the continuum system derived from the original discrete system of particles. The Hopf bifurcation is revealed as a subcritical one, and the property explains well the specific phenomena in highway traffic: the metastability of jamming transition and the hysteresis effect in the relation of car density and flow rate.

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I. INTRODUCTION

Over the last few decades, physicists have shown growing interest in complex systems as many-particle systems of simple components. The society formed by people, the collective biomotion formed by organisms, the granular media formed by particles, and the traffic flow formed by vehicles all are examples of complex systems. One of the most interesting subjects is the pattern formation caused by effects of collective motions in many-particle systems [1]. We focus on the formation of a traffic jam of a vehicular flow as a typical example and phenomena related to the cluster formation of a particle flow [2].

As a mathematical model for describing such phenomena, we investigate the optimal velocity model (the OV model), which was first introduced as a model for traffic flow in 1994 [3,4]. The model reproduces well actual data of highway traffic [5]. From the physical point of view, the model is a nonequilibrium dissipative system describing a one-dimensional chain of interacting particles formulated by nonlinear ordinary differential equations. The interaction between particles in the OV model is asymmetric, meaning that a particle interacts with the particle in front in the direction of motion, not with the particle behind. This interaction breaks the action-reaction principle, and the momentum conservation law is not preserved. The several interesting dynamical properties are originated in the asymmetry of interactions [6]. The model has two kinds of solutions: a homogeneous flow solution and a moving-cluster solution. If a control parameter exceeds a certain critical value, a homogeneous flow solution becomes unstable and a stable moving-cluster solution appears. As a model for traffic flow, a jam cluster emerges beyond the critical vehicle density.

The change in the stability from a homogeneous flow to a cluster flow is caused by the collective effect in many-particle systems. The phenomenon is called a dynamical phase transition. Another property of the transition is a bifurcation in dynamical system. This property is an important characteristic in nonequilibrium dissipative systems of OV-type models. In many numerical simulations, we observe the profile of a jam flow solution as a kind of limit cycle in the phase space of headway and velocity [4]. The appearance of the profile indicates that the transition in the OV model is a Hopf bifurcation [7]. However, there have been few analytical studies of this issue [8–11].

In our previous paper [12], we analytically show that the transition between the solutions is a Hopf bifurcation and that it originates from asymmetric interactions. For this purpose, we derive the continuum system written in a partial differential equation from the original OV model and investigate by a linear stability analysis. We generalize the dissipative particle system with asymmetric interaction and show that the transition of forming a traveling cluster in the flow of such particles is a Hopf bifurcation, as well as the OV model.

In this paper, we further investigate the property of the Hopf bifurcation of the OV model using the continuum system, introduced in the previous work. This paper is organized as follows. We first review the OV model briefly in Sec. II. Next, we derive the continuum system formulated as the partial differential equation and construct the dynamical system for a moving cluster corresponding to a jam flow solution in Sec. III. In Sec. IV we provide the procedure for determine the dynamical system of a cluster for analyzing the property of Hopf bifurcation using the consistency of phase transition and Hopf bifurcation. In Sec. V we determine the velocity of a cluster and define the dynamical system for investigations by systematic calculations. In Sec. VI we investigate the linear term and derive the eigenvalue. In Sec. VII we calculate the normal form of the dynamical system and investigate the property of Hopf bifurcation. In Sec. VIII we provide the explanation of the metastability in jamming transition in highway traffic, using our result for Hopf bifurcation in an OV model. Section IX is devoted to the summary and discussion.

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II. REVIEW OF CONTINUUM SYSTEM FOR A MANY-PARTICLE SYSTEM IN THE OV MODEL

We briefly review the basic features of the OV model [3,4]. The model describes a one-dimensional particle-following system, where N particles move on a circuit with the length L. We express the equation of motion for the nth particle $(n = 1,2,\dots,N)$ as

$$\ddot{x}_n = a\{V(\Delta x_n) - \dot{x}_n\}, \tag{1}$$

where x_n denotes the position of the nth particle and Δx_n is the headway distance defined by $\Delta x_n = x_{n+1} - x_n$. The dot represents the time derivative. The parameter a is called the sensitivity constant $(a > 0)$, whose dimension is the inverse of time and controls the response of motion of a vehicle in the interaction. The function $V(\Delta x_n)$ is the so-called optimal velocity function, which monotonically increases and has an upper bound for $\Delta x_n \to \infty$.

Equation (1) has a homogeneous flow solution expressed as

$$x_n(t) = b\,n + V(b)\,t + \text{const}, \tag{2}$$

where b is an average distance expressed as $b = L/N$. In the solution, all particles move uniformly at the same velocity $V(b)$ with the same headway b. By linear stability analysis, the solution is unstable under the condition that there exists a mode $\theta = n\pi/N$ satisfying the following inequality [3]:

$$\cos^2\frac{\theta}{2} \geqslant \frac{a}{2V'(b)}, \tag{3}$$

where $V'(b)$ is the derivative of V at b.

The equality of Eq. (3) with $\theta \to 0$, which gives the limit of existence of unstable mode, provides the critical condition $a = 2V'(b)$ for the stability of the homogeneous flow solution. The condition predicts the critical vehicle density for given a. The change in the stability is a phase transition in many-particle systems. In the case $a < 2V'(b)$, there exists the long-wavelength-mode, which make the solution of the homogeneous flow unstable and decaying. Instead, the moving cluster solution emerges and becomes stable as in Fig. 1.

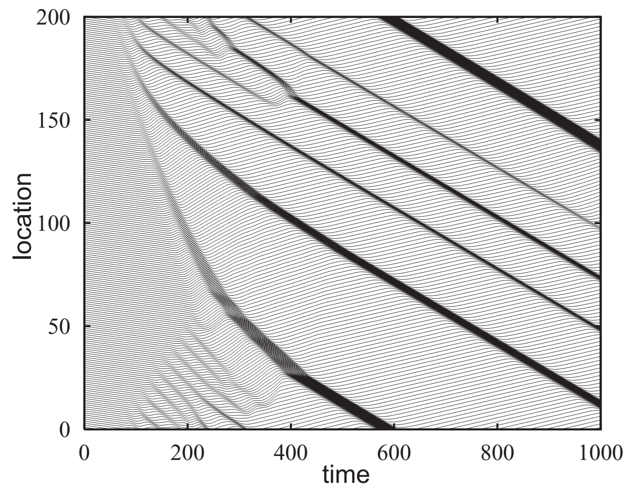

FIG. 1. Space-time plot of cluster formation of the OV model with OV function as $V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2$ in $N = 100$ on the periodic boundary condition on a circuit. The vertical axis is location on the circuit. The horizontal axis is time evolution.

The profile of cluster flow solution is shown as the trajectory of particles in the phase space of headway and velocity $(\Delta x_n, \dot{x}_n)$ in Fig. 2. We recognize the closed curve as a limit cycle [4,6]. The position of the point labeled "Fast" is denoted by $(\Delta x_F, V(\Delta x_F))$, which presents the smooth movement of particles, and the position of the point labeled "Jam" is denoted by $(\Delta x_J, V(\Delta x_J))$, which presents the particles in a jam cluster.

We can observe various limit cycles like Fig. 2 corresponding to each value of the parameter a. This indicates the trajectories for jam flow solutions can be understood as limit cycles concerned with Hopf bifurcation with respect to the parameter a. Actually, we have verified that the transition is a Hopf bifurcation in our previous work [12].

III. MODEL OF TRAVELING CLUSTERS

In this paper, we investigate the property of the Hopf bifurcation in the OV model. For the analysis of this model, we derive the ordinary differential equation of the traveling cluster associated with the jam flow solution in the OV model.

First, we transform Eq. (1) to the equation for the headway Δx_n by subtracting x_{n+1} from x_n, and rewrite the equation using a deviation from the average headway distance b as a dynamical variable, $r_n = \Delta x_n - b$:

$$\ddot{r}_n = a\{V(r_{n+1} + b) - V(r_n + b) - \dot{r}_n\}. \tag{4}$$

In this formula, the homogeneous flow solution (2) is translated to $(r_n, \dot{r}_n) = (0,0)$. We use the shift operator $(\exp\frac{\partial}{\partial n})f(n) = f(n+1)$ by treating the index of a particle number n as the continuous variable. We replace $r_n(t)$ by $r(x,t)$, where x is the continuous variable defined by $x = bn$ by taking the continuum limit as $N \to \infty$, $L \to \infty$, at fixed $b = L/N$.

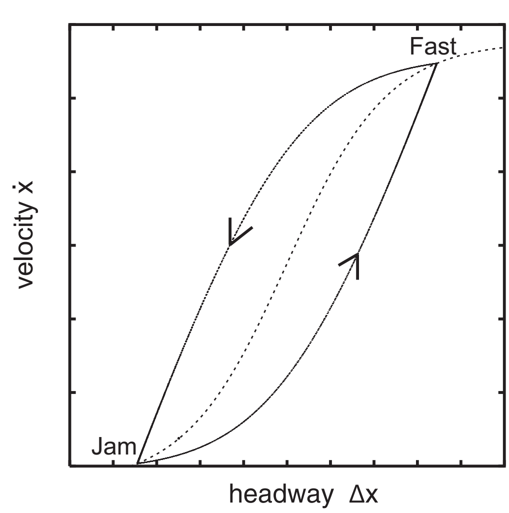

FIG. 2. The profile of a cluster flow solution. "Fast" and "Jam" denote smoothly moving regions and jam clusters, respectively. All vehicles move along the closed loop in the direction of counter clockwise. A dotted curve represents OV function.

Then Eq. (4) is rewritten as

$$\frac{\partial^2 r(x,t)}{\partial t^2} = a\left\{\left(\exp b\frac{\partial}{\partial x} - 1\right)V[r(x,t)+b] - \frac{\partial r(x,t)}{\partial t}\right\}. \tag{5}$$

We have derived the continuum system expressed by the partial differential equation [8] for the original OV model formulated by the set of ordinary differential equations for many particles [Eq. (1)]. Using this formula, we have proved that the transition to a jam flow solution is a Hopf bifurcation in the previous paper [12].

In the next step, we derive the ordinary differential equation of the traveling cluster, as follows. We suppose the hypothesis of the traveling wave solution in Eq. (5) to be $r(x,t) = u(\xi)$ with $\xi = x - ct$, where c is the velocity of a cluster. Then we expand the differential operator $\exp b\frac{\partial}{\partial x}$ up to the third order as

$$c^2\frac{d^2 r}{d\xi^2} - ca\frac{dr}{d\xi} - a\left\{b\frac{d}{d\xi} + \frac{b^2}{2!}\frac{d^2}{d\xi^2} + \frac{b^3}{3!}\frac{d^3}{d\xi^3}\right\}V(r+b) = 0. \tag{6}$$

We can integrate out with respect to ξ once and choose the constant of integration satisfying $r = 0$ as the trivial solution of the following equation at an arbitrary value of b:

$$c^2\frac{dr}{d\xi} - ca\,r - a\left\{b + \frac{b^2}{2!}\frac{d}{d\xi} + \frac{b^3}{3!}\frac{d^2}{d\xi^2}\right\}V(r+b) = 0, \tag{7}$$

where $V(b) = 0$ in Appendix A.

We rewrite Eq. (7) as the formula in autonomous dynamical system (u,v) defining by $u = r, v = du/d\xi$ as

$$\frac{du}{d\xi} = v$$
$$\frac{dv}{d\xi} = \frac{6}{a\,b^3 V'(u+b)}[c^2 v - c\,a\,u - abV(u+b)] - \frac{3v}{b} - \frac{V''(u+b)}{V'(u+b)}v^2. \tag{8}$$

Hereafter, we analyze the two-dimensional dynamical system Eq. (8) to investigate the property of Hopf bifurcation associated with the transition from the trivial solution $(u,v) = (0,0)$ as a homogeneous flow [Eq. (2)] to the limit cycle solution as a jam flow.

Equation (8) can be well defined after the parameter c is verified. We provide the procedure to determine it in the next section.

IV. PROCEDURE OF DETERMINING THE VELOCITY OF A TRAVELING CLUSTER AND DYNAMICAL SYSTEM FOR THE OV MODEL

A. Critical point and Hopf bifurcation point

We have proved that the critical point, which is denoted by a_c for a given $b = b_c$, as

$$a_c = 2V'(b_c), \tag{9}$$

is a Hopf bifurcation point in the meaning that there exists a mode in which eigenvalues are pure imaginary conjugates at each a ($\leqslant a_c$), but not in the region $a > a_c$, for the OV model as a many-particle system [Eq. (1) or (5)] [12]. The critical curve for Eq. (9), which is drawn by the solid curve in Fig. 3, shows the Hopf bifurcation point a_c for a given $b = b_c$.

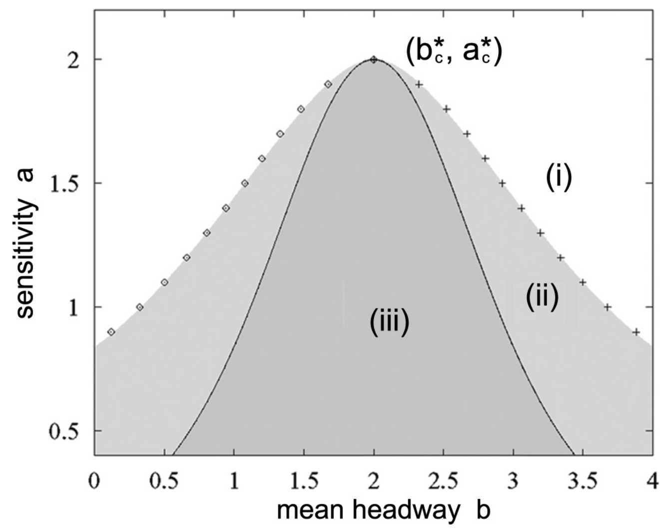

FIG. 3. The phase diagram of OV model. (b_c^*,a_c^*) is the inflection point of the OV function. In phase (i), a homogeneous flow is stable, while a jam flow is unstable. In phase (ii), both a homogeneous flow and a jam flow are stable. In phase (iii), a jam flow is stable, while a homogeneous flow is unstable. The figure is drawn for an OV function as $V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2$ and the inflection point is $(b_c^*,a_c^*) = (2,2)$, for example.

In the dynamical system for traveling clusters [Eq. (8)] a_c is regarded as a usual Hopf bifurcation point in a two-dimensional dynamical system. The condition of Hopf bifurcation should be satisfied at a_c. We should notice that a Hopf bifurcation in the OV model appears in the phase boundary between the bistable phase of a homogeneous flow solution and a jam flow solution (ii) and the stable phase of a jam flow only (iii), as shown in Fig. 3.

We note that the velocity of a cluster c introduced in Eq. (7) is an unknown value at this stage. We have to provide c for defining the system well to perform further calculations. Then we use the following procedure to determine c.

First, we obtain the condition of Hopf bifurcation at a_c in Eq. (8). This condition leads to the formula for c in terms of the parameters defined in the OV model. Next, we should check whether c is correctly expressed as the velocity of a cluster (jam cluster) in the original many-particle system of the OV model. Then the procedure provides the faithful interpretation between the model of the original many -particle system and the dynamical system in the OV model. Details are given in the following subsections and Sec. V.

B. Condition of Hopf bifurcation

Now we investigate the condition of Hopf bifurcation in Eq. (8). The trivial fixed point for Eq. (8) is $(u,v) = (0,0)$, which corresponds to a homogeneous flow solution. A Jacobian matrix of the linearized equation in the vicinity of the trivial fixed point is written as

$$\begin{pmatrix} 0 & 1 \\ -\frac{6c}{b^3V'(b)} - \frac{6}{b^2} & -\frac{3}{b} + \frac{6c^2}{ab^3V'(b)} \end{pmatrix}. \quad (10)$$

The eigenvalue λ of the matrix (10) is given by the solution for the following equation:

$$\lambda^2 - \left[-\frac{3}{b} + \frac{6c^2}{ab^3V'(b)}\right]\lambda + \left[\frac{6c}{b^3V'(b)} + \frac{6}{b^2}\right] = 0. \quad (11)$$

If a_c is a Hopf bifurcation point, the real part of λ is 0. So the following equation should be satisfied at $a = a_c$, as

$$\lambda + \bar{\lambda} = -\frac{3}{b} + \frac{6c^2}{a_c b^3 V'(b)} = \frac{3[-2c^2 + a_c b^2 V'(b)]}{a_c b^3 V'(b)} = 0. \quad (12)$$

Equation (12) leads the condition for c and a_c for a given $b = b_c$ as

$$c^2 = \tfrac{1}{2} a_c b_c^2 V'(b_c). \quad (13)$$

Here we consider the velocity of a cluster at a_c, which is the critical point as well as the Hopf bifurcation point defined here. Together with the condition of a critical point Eq. (9), then the velocity of a cluster c should be

$$c = -b_c V'(b_c). \quad (14)$$

Here we choose the negative value for the velocity of a cluster, because a cluster moves in the opposite direction of the motion of particles.

C. Consistency of the condition of Hopf bifurcation and the velocity of a cluster

For the purpose to check that the condition Eq. (14) obtained in the previous subsection is justified as the velocity of a jam cluster in the OV model, we review the property of the velocity in the original system for many particles.

In the case that the system consists of a large number of particles, the velocity of a jam cluster is written by using the values of a limit cycle denoted in Fig. 2 as

$$V_{jam}(a) = -\frac{\Delta x_J\, V(\Delta x_F) - \Delta x_F\, V(\Delta x_J)}{\Delta x_F - \Delta x_J}. \quad (15)$$

The velocity is completely determined by a limit cycle. That is given by the value of the intercept on the axis of velocity with the line drawn between $(\Delta x_F, V(\Delta x_F))$ and $(\Delta x_J, V(\Delta x_J))$ in the phase space [5].

We should remark that a limit cycle depends only on a sensitivity parameter a but not on an average distance (a particle density) b. This property is justified for a large number of particles N by numerical simulations. Moreover, if we choose the OV function as the Heaviside step function, as the limiting case of $\tanh(x)$, the exact solution of a cluster flow for arbitrary N proves the above fact [13].

The changing of the parameter a deforms the size of limit cycle. As the value of a becomes larger, the limit cycle becomes smaller to shrink to the inflection point of $V(\Delta x)$, which is denoted by (b_c^*, a_c^*), as $a \to a_c^*$, as shown in Fig. 4. The two points $(\Delta x_F, V(\Delta x_F))$ and $(\Delta x_J, V(\Delta x_J))$ of a limit cycle are always on the curve of OV function $V(\Delta x)$, and the limit cycle is symmetric with respect to the inflection point.

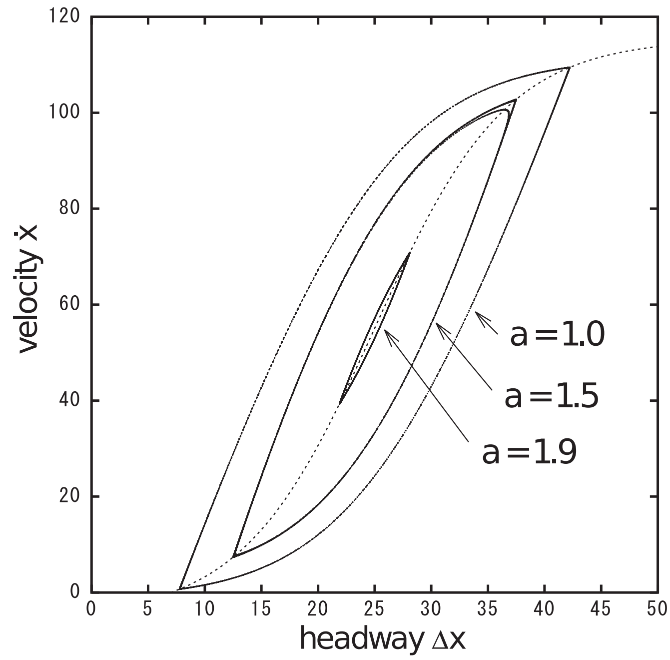

FIG. 4. Profile of the jam flow solution for various values of the sensitivity $a = 1.0, 1.5, 1.9$ with $b = b_c^* = 2$ in the headway-velocity space in the OV model. The OV function is chosen as $V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2$. In this case, the Hopf bifurcation point is $a_c^* = 2$.

We derive the velocity of a jam cluster at the inflection point denoted by V_{jam}^*, as the limit of the formula (15) as $a \to a_c^*$.

We define $\Delta x_J(\epsilon) = b_c^* - \epsilon$, and $\Delta x_F(\epsilon) = b_c^* + \epsilon$, and then calculate the limit as $\epsilon \to +0$:

$$\begin{aligned} V_{jam}^* &= \lim_{a \to a_c^*} V_{jam}(a) = \lim_{\epsilon \to +0} V_{jam}(\epsilon) \\ &= \lim_{\epsilon \to +0} -\frac{\Delta x_J(\epsilon)\, V[\Delta x_F(\epsilon)] - \Delta x_F(\epsilon)\, V[\Delta x_J(\epsilon)]}{\Delta x_F(\epsilon) - \Delta x_J(\epsilon)} \\ &= -b_c^* V'(b_c^*) + V(b_c^*). \end{aligned} \quad (16)$$

We note that the result is obtained by the expansion with respect to ϵ up to the second order.

The derived velocity of a jam cluster is the value in the real configuration space in the circuit. On the contrary, the velocity c is defined in the continuous space denoted by x corresponding to the index n for $r_n = \Delta x_n - b$. The velocity of the cluster at a_c^* denoted by $c^* = c(a_c^*)$, is obtained from V_{jam}^* by Euler-Lagrange transformation, as

$$c^* = V_{jam}^* - V(b_c^*) = -b_c^* V'(b_c^*). \quad (17)$$

This result (17) is consistent with the velocity of a traveling cluster [Eq. (14)] derived from the condition of Hopf bifurcation in the previous subsection. The consistency of the determination of the velocity c by using the condition of Hopf bifurcation has been checked, at least, at the special point (b_c^*, a_c^*). So we are convinced that the velocity of a traveling cluster at Hopf bifurcation point (b_c, a_c) is provided as Eqs. (14) and (9).

Next, we determine the velocity of a cluster $c(a)$ in the vicinity of a_c^*. At the Hopf bifurcation point (b_c^*, a_c^*) the limit cycle shrinks to the point. In the vicinity $a \sim a_c^*$, the size of the limit cycle, denoted by $O(\epsilon)$, is estimated as large as $O(a_c^* - a)$ shown in Fig. 4. We determine the velocity $c(a)$ by calculating the relation between $O(\epsilon)$ and $O(a_c^* - a)$. For this purpose, we use the formulas $V_{jam}(\epsilon)$ in Eq. (16), as the expansion in terms of ϵ. The concrete calculations are individually given for the cases $b_c = b_c^*$ and $b_c \neq b_c^*$, in the next section.

V. DETERMINATION OF THE VELOCITY OF A CLUSTER, $c(a)$

In general properties for the velocity of a cluster, the inequality $0 > c(a) > c^* = c(a_c^*)$ holds as $a < a_c^*$. This means that a cluster moves backward against the movement of particles with the larger speed as the larger value of the parameter a. We assume $V(\Delta x)$ as a sigmoidal function. This fact is easily seen in Fig. 4.

We remark again that the velocity of a cluster c is determined only by a not by b. We use this important property in the procedure of calculation hereafter.

A. The case $a < a_c^*, b_c = b_c^*$

In our previous paper [12], we have proved that the critical point a_c with b_c in Eq. (9) is a Hopf bifurcation point. Of course, a_c^* with b_c^* is also. The phase diagram in Fig. 3 shows that no cluster flow solution exists in the region $a > a_c^*$. On the other hand, a limit cycle should appear in the region $a < a_c^*$. In order to investigate the dynamical system [Eq. (8)] in this region, we determine the velocity of the cluster $c(a)$ as follows.

We set $\Delta x_J = b_c^* - \epsilon$ and $\Delta x_F = b_c^* + \epsilon$, where $\epsilon > 0$, at $a(<a_c^*), b = b_c^*$. Here we consider ϵ as the size of a limit cycle. Then, $O(\epsilon)$ and $O(a_c^* - a)$ are related by the dependency of the size of a limit cycle on a.

From Eq. (15), expanding with respect to ϵ, we can calculate $c(a)$ as

$$c = -b_c^* \left[V'(b_c^*) + \frac{V'''(b_c^*)}{3!} \epsilon^2 \right]. \tag{18}$$

The first order term with respect to $O(\epsilon)$ does not exist for an arbitrary b. Here we use $V''(b_c^*) = 0$ as the inflection point at b_c^*, and we note $V'''(b_c^*) < 0$, as $V(\Delta x)$ is assumed as a sigmoidal function.

We should calculate the relation between $O(a_c^* - a)$ and $O(\epsilon)$. We note that the velocity of a cluster does not depend on b. So we can determine the velocity at an arbitrary a on the critical curve. Then we expand $a = 2V'(b)$ beyond $a_c^* = 2V'(b_c^*)$ as

$$\begin{aligned} a &= 2V'(b) \\ &= 2\left\{ V'(b_c^*) - V''(b_c^*)(b_c^* - b) + \frac{1}{2!} V'''(b_c^*)(b_c^* - b)^2 \right\}. \end{aligned} \tag{19}$$

We can estimate the size of limit cycle ϵ by $b_c^* - b$. Thus, using $V''(b_c^*) = 0$, Eq. (19) provides the relation between $O(a_c^* - a)$ and $O(\epsilon)$:

$$a_c^* - a = -V'''(b_c^*)\epsilon^2. \tag{20}$$

Finally, we determine $c(a)$ from Eqs. (18) and (20), in $b = b_c^*, a < a_c^*$, as

$$\begin{aligned} c(a) &= -b_c^* V'(b_c^*) + \frac{b_c^*}{3!}(a_c^* - a) = c^* + \frac{b_c^*}{3!}(a_c^* - a) \\ &= c^* \left(1 - \frac{a_c^* - a}{3a_c^*} \right). \end{aligned} \tag{21}$$

Here we use the relation $c^* = -\frac{1}{2} a_c^* b_c^*$, which is derived from the condition (14) for a_c^*, b_c^*, and c^* together with $a_c^* = 2V'(b_c^*)$.

B. The case $a > a_c, b_c \neq b_c^*$

We denote b_c, which satisfies the critical condition $a_c = 2V'(b_c)$ where $a_c < a_c^*$ and $b_c \neq b_c^*$. It is enough to consider the case $b_c < b_c^*$ for the symmetry of b_c with respect to b_c^*, shown as Fig. 3.

We remind the bistable (or coexistence) phase (ii) for a homogeneous flow and a cluster flow solutions in the phase diagram in Fig. 3. The bistability of two solutions indicates the existence of an unstable limit cycle in this region. Then we investigate the limit cycle appearing at the Hopf bifurcation point a_c with $b_c(\neq b_c^*)$. For this purpose, we should determine the velocity of the cluster $c(a)$ in the region $a > a_c$ in the vicinity of this Hopf bifurcation point.

In the similar way as the previous case, we set $\Delta x_J = b_c - \epsilon$, and $\Delta x_F = b_c + \epsilon$, where $\epsilon > 0$, at $a(> a_c)$, for a given $b_c \neq b_c^*$. We first calculate $c(a)$ as the expansion with respect to ϵ, which is the size of the limit cycle in the vicinity of (a_c, b_c). Using Eq. (15), we derive the formula of c as

$$c = -b_c V'(b_c) + \left\{ \frac{V''(b_c)}{2!} - \frac{V'''(b_c) b_c}{3!} \right\} \epsilon^2. \tag{22}$$

We note the first order of ϵ is vanishing again as the previous case. In this case, we remark that $V''(b_c) \neq 0$. We should calculate the relation between $O(\epsilon)$ and $O(a - a_c)$ by the dependency on a of the limit cycle from this formula.

Next, in the same way as the previous case, the dependency on a for $c(a)$ is determined along the critical relation $a = 2V'(b)$ and $a_c = 2V'(b_c)$ by expanding $O(b - b_c)$, where $a > a_c$, $b > b_c$ in the case $b_c < b_c^*$:

$$\begin{aligned} a &= 2V'(b) \\ &= 2\left\{ V'(b_c) + V''(b_c)(b - b_c) + \frac{1}{2!} V'''(b_c)(b - b_c)^2 \right\}. \end{aligned} \tag{23}$$

Owing to $V''(b_c) \neq 0$ the first order term is dominant, and $a - a_c = 2V''(b_c)(b - b_c)$, where $V''(b_c) > 0$ in the case $b_c < b_c^*$. The size of the limit cycle is estimated as ϵ by $b_c^* - b$, in the same way as in case A.

Then we obtain the relation as

$$a - a_c = 2V''(b_c)\epsilon. \tag{24}$$

From Eqs. (22) and (24), we obtain

$$c(a) = c(a_c) + \left\{ \frac{V''(b_c)}{2!} - \frac{V'''(b_c) b_c}{3!} \right\} \left[\frac{a - a_c}{2V''(b_c)} \right]^2. \tag{25}$$

The obtained correction term in $c(a)$ beyond $c(a_c)$ is second order with respect to $O(a-a_c)$.

The coefficient is calculated by using the formula in Appendix B to rewrite $V''(b_c), V'''(b_c)$, which depends on a_c^* as well as a_c. For the condition that $c(a) > c(a_c)$ should be satisfied, the calculation is performed with respect to $O(a_c^*-a_c)$ as well as $O(a-a_c)$. After the calculation, we obtain the final form of $c(a)$ up to the order $O(a-a_c)^2$ as

$$c(a) = c(a_c)\left\{1 - \frac{a_c^*(a-a_c)^2}{12a_c^2(a_c^*-a_c)}\right\}. \tag{26}$$

Here we use the relation $c(a_c) = -\frac{1}{2}a_c b_c$.

We note that the cluster whose velocity is given as Eq. (26) corresponds to the unstable limit cycle appearing in the bistable phase. In this case, the meaning of the velocity of a cluster is different from the velocity in case A, where the cluster is a real jam cluster observed in a circuit. However, we can investigate the property of the Hopf bifurcation at (b_c,a_c) using the dynamical system Eq. (8) with the velocity $c(a)$ in Eq. (26).

We should remark that $a_c < a < a_c^*$, which is easily seen by the phase diagram Fig. 3. In the case of changing the parameter a along $a > a_c$, at a given $b_c \neq b_c^*$, the inequality of these parameters leads the inequality of orders as

$$O(a-a_c)^2 < O\left[\frac{(a-a_c)^2}{a_c^*-a_c}\right] < O(a-a_c). \tag{27}$$

Thus, we should make double expansions with respect to $O(a_c^*-a_c)$ as well as $O(a-a_c)$ carefully to perform calculations in the case $b_c \neq b_c^*$.

VI. ANALYSIS OF DYNAMICAL SYSTEM FOR THE OV MODEL: EIGENVALUE

We have prepared the analysis of the dynamical system well defined by determining the velocity $c(a)$. The analysis is carried out using the dynamical system [Eq. (8)] by expanding with respect to dynamical variables (u,v). Up to the first order the equations are written as

$$\frac{du}{d\xi} = v$$
$$\frac{dv}{d\xi} = \frac{-6}{b^2}\left[1 + \frac{c}{b\,V'(b)}\right]u + \frac{3}{b}\left[-1 + \frac{2\,c^2}{a\,b^2\,V'(b)}\right]v. \tag{28}$$

We should note that the above dynamical system for the first order of (u,v) is not the linearized system of original OV model but the nonlinear system describing the cluster flow, which is formulated by the nonlinear effect in the interaction of many particles. The analysis is individually given for each case in $b_c = b_c^*$ and $b_c \neq b_c^*$.

A. The case $a < a_c^*, b_c = b_c^*$

At the critical point $a = a_c^*$, $b_c = b_c^*$, the first order terms of (u,v) in the r.h.s. of the second equation in Eq. (28) vanish due to the relation of the velocity of a cluster, $c^* = -a_c^* b_c^*/2 = -b_c^* V'(b_c^*)$.

Using Eq. (21) for $c(a)$ we expand Eq. (28) in terms with $O(a_c^*-a)$. The result is

$$\frac{du}{d\xi} = v$$
$$\frac{dv}{d\xi} = -\frac{a_c^*}{2c^{*2}}(a_c^*-a)u - \frac{1}{2c^*}(a_c^*-a)v. \tag{29}$$

From the above dynamical system formulated up to first order $O(a_c^*-a)$ [Eq. (29)], the eigenvalues are

$$\lambda_\pm = -\frac{a_c^*-a}{4c^*} \mp i\,\frac{\sqrt{(a_c^*-a)(a+7a_c^*)}}{4c*}. \tag{30}$$

The eigenvalues are complex conjugate, and the real part of the eigenvalues vanishes at the bifurcation point $a = a_c^*$, faster than the imaginary part. This result is consistent with that the transition shows Hopf bifurcation, which is proved in our previous paper [12].

The real part of the eigenvalues is positive for $a < a_c^*$, because of $c^* < 0$. The flow in the phase space (u,v) for $a < a_c^*$, $b_c = b_c^*$ diverges as an unstable spiral, which means that the trivial solution corresponding to a homogeneous particle flow is unstable. This result is consistent with the analysis in Ref. [12].

B. The case $a > a_c, b_c \neq b_c^*$

At the critical point $a = a_c$, $b_c \neq b_c^*$, the r.h.s. of the second equation in Eq. (28), which are the first order terms for u,v, also vanish due to the relation $c(a_c) = -a_c b_c/2 = -b_c V'(b_c)$.

Using the formula of $c(a)$ in Eq. (26) we expand Eq. (28) in terms with $O(a-a_c)$ for $a > a_c$. The result is

$$\frac{dv}{d\xi} = -\frac{a_c^*}{8c^2(a_c)}\frac{(a-a_c)^2}{(a_c^*-a_c)}u + \left\{\frac{3(a-a_c)}{2c(a_c)} + \frac{(5a_c^*-6a_c)}{4c(a_c)\,a_c}\frac{(a-a_c)^2}{(a_c^*-a_c)}\right\}v. \tag{31}$$

The contribution of the first order $O(a-a_c)$ vanishes in the term u. We should calculate by taking into account the next to leading contribution as $O(\frac{(a-a_c)^2}{a_c^*-a_c})$, considering the relation Eq. (27).

Thus, the linearized dynamical system for (u,v) is obtained as

$$\frac{du}{d\xi} = v$$
$$\frac{dv}{d\xi} = -\frac{a_c^*}{8c^2(a_c)}\frac{(a-a_c)^2}{(a_c^*-a_c)}u + \frac{3(a-a_c)}{2c(a_c)}v. \tag{32}$$

The eigenvalues are obtained by carefully taking into account a contribution $O(\frac{(a-a_c)^2}{a_c^*-a_c})$ up to the order $O(a-a_c)$ as

$$\lambda_\pm = \frac{3(a-a_c)}{4c(a_c)} \pm i\,\frac{(a-a_c)}{4c(a_c)}\sqrt{\frac{2a_c}{a_c^*-a_c}}. \tag{33}$$

This result also shows that the real part of the eigenvalues vanishes faster than the imaginary part at the bifurcation point $a = a_c$, which is understood by the comparison of the order of each part as Eq. (27). Then the transition is a Hopf bifurcation at $a = a_c$, as well as the previous case at $a = a_c^*$

In contrast to the previous case, the real part of the eigenvalues is negative for $a > a_c$, because of $c(a_c) < 0$. The flow in the phase space (u,v) for $a > a_c$, $b_c \neq b_c^*$ converges as a stable spiral, which means that the trivial solution corresponding to a homogeneous flow is stable. This indicates that the Hopf bifurcation at $a = a_c$ is subcritical, because a uniform flow is unstable for $a < a_c$ by the analysis in Ref. [12].

VII. THE FEATURE OF HOPF BIFURCATION IN THE OV MODEL: ANALYSIS INCLUDING HIGHER ORDER

The dynamical system for OV model [Eq. (8)] is formulated by expansion with respect to (u,v). The second equation in Eq. (8) is denoted formally up to the third order term as follows:

$$\frac{dv}{d\xi} = f_1(u,v) + f_2(u,v) + f_3(u,v), \tag{34}$$

where $f_1(u,v)$ is the first order term in Eq. (28) and $f_2(u,v), f_3(u,v)$ are higher order terms of (u,v), as

$$f_2(u,v) = \left\{\frac{6\,c}{b^3}\frac{V''(b)}{V'(b)^2} + \frac{3}{b^2}\frac{V''(b)}{V'(b)}\right\}u^2 - \frac{6\,c^2}{a\,b^3}\frac{V''(b)}{V'(b)^2}\,u\,v - \frac{V''(b)}{V'(b)}\,v^2 \tag{35}$$

and

$$\begin{aligned} f_3(u,v) &= \left\{-\frac{6c}{b^3}\frac{V''(b)^2}{V'(b)^3} + \frac{3cV'''(b)}{b^3V'(b)^2} - \frac{3V''(b)^2}{b^2V'(b)^2} + \frac{2V'''(b)}{b^2V'(b)}\right\}u^3 \\ &+ \left\{\frac{6\,c^2}{a\,b^3}\frac{V''(b)^2}{V'(b)^3} - \frac{3\,c^2}{a\,b^3}\frac{V'''(b)}{V'(b)^2}\right\}u^2\,v \\ &+ \left\{\frac{V''(b)^2}{V'(b)^2} - \frac{V'''(b)}{V'(b)}\right\}u\,v^2, \end{aligned} \tag{36}$$

up to the third order.

In the case $b_c = b_c^*$ even order terms of (u,v) vanish for such as $V''(b_c^*) = 0$, $V^{(4)}(b_c^*) = 0, \ldots,$ while in the case $b_c \neq b_c^*$ even order terms exist as well as odd order terms. We make calculations individually in each case.

A. The case $a < a_c^*, b_c = b_c^*$

In this case, $f_2(u,v) = 0$. As for the third order term $f_3(u,v)$, using the relations $2V'(b_c^*) = a_c^*, V''(b_c^*) = 0$, and $V'''(b_c^*) = -m^2 a_c^*$ in Appendix A; moreover, $c^* = -a_c^* b_c^*/2 = -b_c^* V'(b_c^*)$, the leading term is the 0th order with respect to $O(a_c^* - a)$, as

$$f_3 = m^2\left(\frac{2}{b_c^{*2}}\,u^3 + \frac{3}{b_c^*}u^2\,v + 2u\,v^2\right), \tag{37}$$

where we use $a \sim a_c^*$ and $c \sim c^*$.

We note that m is a parameter appeared in the general explicit formulation of the OV function, which has the dimension of the inverse of length. The parameter can be formally removed by the scale transformation defined as $mb \to b$, $m\xi \to \xi$, $mu \to u$, $v \to v$, and $mc \to c$. In the transformation, the equations of dynamical system Eq. (8) are invariant.

Next, we rewrite the equations of dynamical system Eq. (34) with Eq. (37) in a two-dimensional system of (u,v) to the equation of the complex conjugate pair with $(z,\bar{z})$. After the calculation, which is shown in Appendix C, the equation is expressed as

$$\frac{dz}{d\xi} = \lambda z + F_3(z,\bar{z}), \tag{38}$$

where $(\lambda,\bar{\lambda}) = (\lambda_+,\lambda_-)$, with $\lambda_\pm = \alpha \pm i\beta$ (α is the real part of the eigenvalues, and β is the imaginary part of them) in Eq. (30), and

$$F_3(z,\bar{z}) = \frac{1}{2\beta}(1 - i)f_3(u,v(x(z,\bar{z}),y(z,\bar{z})). \tag{39}$$

$F_3(z,\bar{z})$ is calculated using the formulas in Appendix D.

In order to obtain the normal form, we perform the nonlinear transformation for $z,\bar{z}$ to eliminate the terms except $z^2\bar{z}$. The normal form of Eq. (38) is obtained as

$$\frac{dz}{d\xi} = \lambda z + \lambda_3\,z^2\bar{z}, \tag{40}$$

where

$$\lambda_3 = \frac{3}{4b_c^*} - \left(\frac{3}{2b_c^{*2}\beta} + \frac{9\alpha}{4b_c^*\beta}\right)i. \tag{41}$$

The explicit equation of the normal form expressed in the polar coordinates $z = r\exp(i\theta)$ up to the leading order in the expansion with respect to $O(a_c^* - a)$ is

$$\begin{aligned} \dot{r} &= \alpha r + \frac{3}{4b_c^*}r^3 \\ \dot{\theta} &= \beta - \frac{1}{\beta}\left(\frac{3}{2b_c^{*2}} + \frac{9\alpha}{4b_c^*}\right)r^2. \end{aligned} \tag{42}$$

To verify that Eq. (40) with Eq. (41) shows Hopf bifurcation at $a = a_c^*$, we rewrite Eq. (40) using the invariance in the property of normal form by a spatiotemporal rescaling: $z \to z' = z/\beta$, $\xi \to \xi' = \xi\beta$. The normal form is rewritten as

$$\frac{dz'}{d\xi'} = \left(\frac{\alpha}{\beta} + i\right)z' + (\lambda_3\,z'^2\bar{z}' + \beta^2\lambda_5\,z'^3\bar{z}'^2), \tag{43}$$

where $\alpha/\beta = O(a_c^* - a)^{\frac{1}{2}}$ and higher orders follow with respect to $O(\beta^2) = O(a_c^* - a)$. At the Hopf bifurcation point, the real part of the eigenvalues vanishes, whereas the imaginary part remains.

The result in the normal form Eq. (42) shows that the coefficients of linear and the third order terms of r are both positive. So the attractive limit cycle is not obtained up to this order of expansion. However, we convince the existence of a stable limit cycle for the following reason. First, we note that the trivial fixed point $r = 0$ corresponding $(u,v) = (0,0)$ exists, and it is stable in $a > a_c^*$, by the analysis in our previous paper [12]. The stability of the trivial fixed point $r = 0$ is changed at the Hopf bifurcation point a_c^* from $a > a_c^*$ to $a < a_c^*$, from the stable fixed point to the unstable one, and the flow diverges as spiral, which is analytically verified by Eq. (30). We remark that the fixed point is only

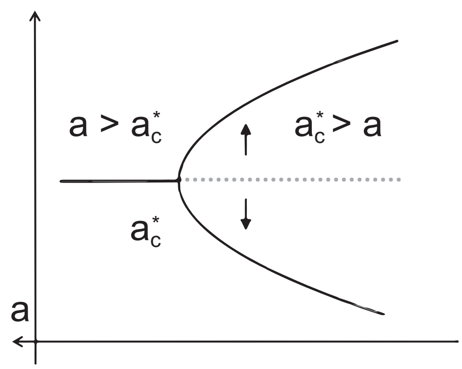

FIG. 5. Hopf bifurcation of the OV model in the case of $b_c = b_c^*$. The horizontal axis is a sensitivity a, the vertical axis is the projection of two-dimensional phase space (u,v). The center line shows the trivial solution (a homogeneous flow). The solid curve shows the limit cycle (a jam flow solution). The solid line and curve denote stable solutions, and the dotted line means an unstable homogeneous flow solution. Arrows are examples of flows from an unstable solution to a stable one in the phase space, which shows a supercritical Hopf bifurcation.

$(u,v) = (0,0)$ in Eq. (8). The OV model has no solution of infinity $r = \infty$, and flow in the phase space (u,v) moves inward. It is because the model is a dissipative system with asymmetric interaction, which has a relaxation state. By the Poincaré-Bendixson theorem [14], the stable limit cycle should exist in the intermediate region. Thus, we can illustrate the Hopf bifurcation in the case at $b = b_c^*$ in Fig. 5 and conclude the transition is a supercritical Hopf bifurcation.

B. The case $a > a_c, b_c \neq b_c^*$

In this case, $f_2(u,v)$ exists. From Eq. (35) by $2V'(b_c) = a_c$:

$$f_2(u,v) = V''(b_c)\left\{\left[\frac{6\,c}{b_c\,c(a_c)^2} - \frac{3}{b_c\,c(a_c)}\right]u^2 - \frac{6\,c^2}{ab_c\,c(a_c)^2}\,uv - \frac{2}{a_c}v^2\right\}. \tag{44}$$

Using the relation in Appendix B,

$$V''(b_c) = \mp m a_c\sqrt{1 - \frac{a_c}{a_c^*}} = \mp m\,O(a_c^* - a_c)^{\frac{1}{2}}, \tag{45}$$

we perform the double expansion with respect to $O(a_c^* - a_c)$ as well as $O(a - a_c)$. Moreover we use the relation $c(a_c) = -\frac{1}{2}a_c b_c$ and after m is rescaled:

$$f_2(u,v) = O(a_c^* - a_c)^{\frac{1}{2}}\left\{\frac{6}{b_c^2}u^2 + \frac{24}{b_c}\,uv + 2v^2 + O(a - a_c)\right\}. \tag{46}$$

As for f_3 in Eq. (36), we note that

$$V''^2(b_c) = m^2 O(a_c^* - a_c) \tag{47}$$

and

$$V'''(b_c) = m^2 a_c\left(2 - 3\,\frac{a_c}{a_c^*}\right) = m^2\{-a_c + O(a_c^* - a_c)\}, \tag{48}$$

from Appendix B. We rewrite it using $2V'(b_c) = a_c$ [Eqs. (47) and (48)] as

$$f_3(u,v) = m^2\{-a_c + O(a_c^* - a_c)\}\left\{\left[\frac{3c}{b_c\,c(a_c)^2} - \frac{2}{b_c\,c(a_c)}\right]u^3 - \frac{3\,c^2}{ab_c\,c(a_c)^2}\,u^2 v + \frac{b_c}{c(a_c)}\,uv^2\right\} + m^2 O(a_c^* - a_c). \tag{49}$$

Using the relation $c(a_c) = -\frac{1}{2}a_c b_c$ and the rescaling with m, leads to

$$f_3(u,v) = \frac{2}{b_c^2}u^3 + \frac{3}{b_c}\,u^2 v + 2uv^2 + O(a_c^* - a_c) + O(a - a_c). \tag{50}$$

The contribution of $f_2(u,v)$ to higher order $O^3(u,v)$ in the normal form is negligible up to the order $O(a_c^* - a_c)$ as well as $O(a - a_c)$, because the overall factor of f_2 is $O(a_c^* - a_c)^{\frac{1}{2}}$, while the order of the leading term in f_3 is the 0th order with respect to $O(a_c^* - a)$.

Thus, the normal form Eq. (40) in case B is obtained by replacing the corresponding eigenvalue λ and λ_3.

The final result of the normal form up to the leading order in the expansion with respect to $O(a - a_c)$ is

$$\dot{r} = \alpha\,r + \frac{3}{4b_c}r^3$$
$$\dot{\theta} = \beta - \frac{1}{\beta}\left(\frac{3}{2{b_c}^2} + \frac{9\alpha}{4b_c}\right)r^2, \tag{51}$$

where α and β are the real and imaginary parts of eigenvalue λ in Eq. (33).

In this case, α is negative valued. Then the unstable limit cycle is derived, as

$$r = \sqrt{-\frac{4\alpha b_c}{3}} = \sqrt{\frac{2(a - a_c)}{a_c}}; \tag{52}$$

here we use $c(a_c) = -a_c b_c/2$. The limit cycle is only dependent on a, not on b. The result is consistent with the property of the OV model. A flow starting the region inside of the unstable limit cycle converges as spiral into the trivial fixed point.

A stable limit cycle cannot be obtained up to the third order term r^3 in the case $b_c \neq b_c^*$, also. We need a calculation of higher order at least the fifth order. However, we can stress the same statement that in the case $b_c = b_c^*$ again. Thus, we can illustrate the Hopf bifurcation at a_c in the case $b_c \neq b_c^*$ in Fig. 6 and conclude the transition is a subcritical Hopf bifurcation.

VIII. METASTABILITY OF JAMMING TRANSITION

In the phenomena of the formation of a traffic jam in real highways, the important characteristic feature is the metastability in a jamming transition, as shown in data of the relation between vehicle density and flow rate, the so-called q-ρ diagram, in Fig. 7. The flow rate discontinuously changes from the state of homogeneous flow to the state of jam flow.

From the result of our study in this paper, the property is explained by the subcritical Hopf bifurcation between a trivial solution (homogeneous flow) and a limit cycle solution (jam

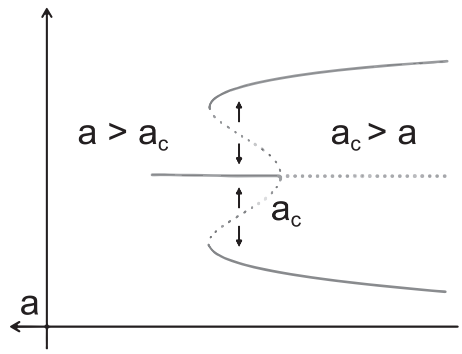

FIG. 6. Hopf bifurcation of the OV model in the case of $b \neq b_c^*$. The horizontal and vertical axes are the same in Fig. 5. The center line and the curve show the trivial and limit cycle solution, respectively. The solid line and curve mean stable, and dotted ones signify unstable. Arrows are presented as flows from an unstable solution to a stable one in the phase space, which shows subcritical Hopf bifurcation in the bistable (coexistence) phase.

flow). The realistic process of the emergence of a jam is as follows.

In the phase diagram as Fig. 3, the sensitivity of vehicles is smaller than the supercritical point a_c^*, as the density of vehicles (the mean headway) becomes larger (smaller), the point on the traffic situation moves from right to left in the phase diagram with fixed sensitivity. The point moves from the phase (i) to (ii). The phase (ii) is the bistable for a free (homogeneous) flow and a jam flow. If a tiny disturbance occurs in the state in this phase, the free traffic flow suddenly changes to the state of a jam flow. This process is mathematically explained as the subcritical Hopf bifurcation obtained in this paper. The similar phenomena to this transition are the first order transition in equilibrium statistical physics in materials, such as the phase transition from water (liquid) to ice (solid).

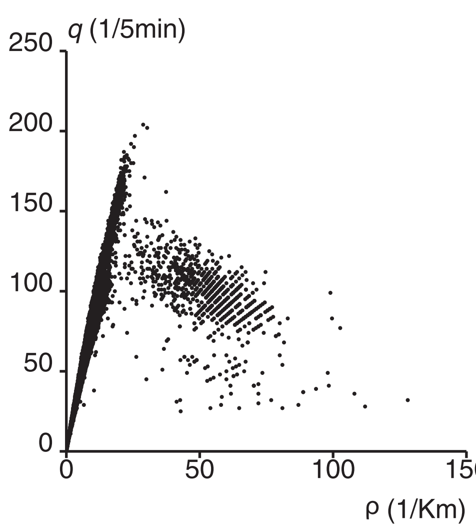

FIG. 7. The typical fundamental diagram (the relation between vehicle density and flow rate) by 1-month data measured at a point on freeway. The critical density is nearly 25 (vehicle/km). The data are measured by Japan Highway Public Cooperation.

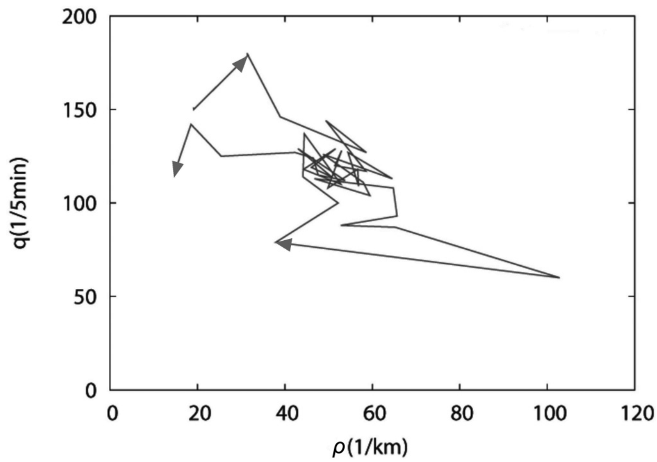

FIG. 8. The time sequence in the q-ρ space from 07:00 to 10:25 on 2 Aug 1996 at the upper stream of Nihonzaka tunnel in the Tomei Expressway [15].

The more explicit reflection of subcritical Hopf bifurcation can be observed in highway traffic as the process of appearance and disappearance in the period of the lifetime of a jam. We observe the whole process of the emergence of a jam from a homogeneous flow and the disappearance of a jam to a homogeneous flow in a day. According to the density increasing and decreasing, the time sequence of changing in state forms the hysteresis loop in q-ρ space. The examples of such an empirical data are shown in Refs. [15] and [16].

In Fig. 8 the time sequence of changing state of flow in the q-ρ space is shown. From a free flow with growing density, the emergence of a jam is seen. After the maximal density it begins to decrease, and the state shows the disappearance of a jam and back to a free flow. The hysteresis loop process in q-ρ space is clearly observed, which has the peak of the maximal flow near the critical density and the minimum flow at the largest density.

IX. SUMMARY AND DISCUSSION

Let us summarize the results in this paper. We investigate the property of transition from a homogeneous flow solution to a moving-cluster (jam flow) solution in the OV model. In the OV model the moving-cluster solution is identified as a limit cycle in the headway-velocity space. Using the continuum system derived from the original discrete system of particles, we show the transition is Hopf bifurcation, as in the previous paper.

In this paper, we further investigate the property of Hopf bifurcation in the dynamical system of an OV model. For this purpose, we introduce a dynamical system of a traveling wave, $r(x,t) \equiv r(x - ct)$, in a continuum system (5), where c is the velocity of a moving cluster corresponding to the limit cycle depending on a.

Then the equation of motion in the continuum system expressed in a partial differential equation (PDE) (5) is rewritten as the ordinary differential equation (ODE) for a traveling wave. In order to define the derived ODE system, we should determine the velocity $c(a)$ of a moving cluster, which can be evaluated by the property of limit cycles in the original discrete system of particles in OV model.

First, we recognize that the Hopf bifurcation point of the ODE system derived from the PDE system is just the critical point a_c of the original many-particle system of the OV model, because the velocity of a cluster in the ODE is just the same velocity in the original OV model.

We can determine the velocity of the cluster $c(a_c)$ at the Hopf bifurcation point a_c. Then we calculate it by expansion in terms of $O(a - a_c)$ for given $b = b_c$ in order to obtain $c(a)$ for the purpose of constructing the dynamical system to investigate the property of Hopf bifurcation.

We investigate each case of $b_c = b_c^*$ and $b_c \neq b_c^*$ individually, considering each with the characteristic property of the phase diagram, that is, the existence of a bistable phase for homogeneous flow and jam flow, as shown in Fig. 3.

We calculate the normal form of the dynamical system in each case and investigate the property of Hopf bifurcation, up to the third order in terms of dynamical variables (u,v). As the result, in the case $a < a_c^*$ for $b_c = b_c^*$ the trivial fixed point corresponding to a homogeneous flow solution is unstable. Then the Hopf bifurcation at a_c^* is supercritical, while in the case $a > a_c, b_c \neq b_c^*$, an unstable limit cycle exists. Then the Hopf bifurcation at a_c is subcritical.

In the realistic situation on highway traffic, the sensitivity of the OV model is almost common with usual vehicles. The reflection of the subcritical Hopf bifurcation for the transition in the dynamical system is observed in the phenomena changing car density in highway. Those are the metastability of jamming transition of flow rate q near the critical density ρ_c, and the hysteresis phenomenon in q-ρ space. These specific phenomena in real traffic flow are well described analytically using the subcritical Hopf bifurcation in the dynamical system investigated in this paper, which is derived from the OV model.

ACKNOWLEDGMENTS

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APPENDIX A: GENERAL EXPLICIT FORMULA OF OV FUNCTION

We denote OV function $V(\Delta x)$ as defined by

$$V(\Delta x) := V(\Delta x\,; b) = V(r + b\,; b), \tag{A1}$$

where

$$V(\Delta x\,; b) = v_0\{\tanh m(\Delta x - b_c^*) - \tanh m(b - b_c^*)\}. \tag{A2}$$

Then, for an arbitrary value b,

$$V(b\,; b) = 0. \tag{A3}$$

The above definition provides rather simple calculation, which needs no Euler-Lagrange transformation, without the loss of generality:

$$\begin{aligned}
V'(\Delta x) &= m v_0\{1 - \tanh^2 m(\Delta x - b_c^*)\},\\
V''(\Delta x) &= m^2 v_0\,\{-2\tanh m(\Delta x - b_c^*)\}\{1 - \tanh^2 m(\Delta x - b_c^*)\},\\
V'''(\Delta x) &= m^3 v_0\,2\{-1 + 3\,\tanh^2 m(\Delta x - b_c^*)\}\{1 - \tanh^2 m(\Delta x - b_c^*)\},\\
V^{(4)}(\Delta x) &= m^4 v_0\,\{-8\tanh m(\Delta x - b_c^*)\}\{-2 + 3\,\tanh m(\Delta x - b_c^*)\}\{1 - \tanh^2 m(\Delta x - b_c^*)\};
\end{aligned}$$

(i) $b \neq b_c^*$

$$\begin{aligned}
V'(b) &= m v_0\{1 - \tanh^2 m(b - b_c^*)\},\\
V''(b) &= m^2 v_0\,\{-2\tanh m(b - b_c^*)\}\{1 - \tanh^2 m(b - b_c^*)\},\\
V'''(b) &= m^3 v_0\,2\{-1 + 3\,\tanh^2 m(b - b_c^*)\}\{1 - \tanh^2 m(b - b_c^*)\},\\
V^{(4)}(b) &= m^4 v_0\,\{-8\tanh m(b - b_c^*)\}\{-2 + 3\,\tanh m(b - b_c^*)\{1 - \tanh^2 m(b - b_c^*)\};
\end{aligned}$$

(ii) $b = b_c^*$

$$\begin{aligned}
V'(b_c^*) &= m v_0 = \tfrac{1}{2} a_c^*,\\
V''(b_c^*) &= 0,\\
V'''(b_c^*) &= -2m^3 v_0 = -m^2 a_c^*,\\
V^{(4)}(b_c^*) &= 0,\\
&\dots\\
V^{(2n)}(b_c^*) &= 0.
\end{aligned}$$

APPENDIX B: FORMULA OF $v_0 m$

In the case $b_c \neq b_c^*$, we define $\eta = b_c - b_c^*$. (In the case $b_c = b_c^*$, then $a_c = a_c^*, \eta = 0$.)

From the critical relation $2V'(b_c) = a_c$, the formula below follows:

$$\begin{aligned}
a_c^* &= 2 m v_0,\\
a_c &= 2 m v_0\{1 - \tanh^2 m\eta\}\\
&= a_c^*\{1 - \tanh^2 m\eta\}.
\end{aligned}$$

Then

$$\tanh^2 m\eta = 1 - \frac{a_c}{a_c^*},$$

$$\tanh m\eta = \pm\sqrt{1 - \frac{a_c}{a_c^*}} \quad (\pm\eta > 0, < 0).$$

Using the formula in Appendix A, we derive the relations as follows:

$$V'(b_c) = mv_0\{1 - \tanh^2 m\eta\},$$
$$V''(b_c) = m^2 v_0 \{-2\tanh m\eta\}\{1 - \tanh^2 m\eta\},$$
$$V'''(b_c) = m^3 v_0\, 2\{-1 + 3\tanh^2 m\eta\}\{1 - \tanh^2 m\eta\},$$
$$V^{(4)}(b_c) = m^4 v_0 \{-8\tanh m\eta\}\{-2 + 3\tanh m\eta\}\{1 - \tanh^2 m\eta\}.$$

From the above relations, we derive the following:

$$V'(b_c) = \frac{1}{2}a_c, \tag{B1}$$
$$V''(b_c) = \mp m a_c\sqrt{1 - \frac{a_c}{a_c^*}}, \tag{B2}$$
$$V'''(b_c) = m^2 a_c\left(2 - 3\frac{a_c}{a_c^*}\right) = m^2 a_c\left\{-1 + 3\left(1 - \frac{a_c}{a_c^*}\right)\right\}. \tag{B3}$$

APPENDIX C: COMPLEX CONJUGATE FORM

The linear term of the equation of dynamical system

$$\frac{d}{d\xi}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ A & B \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{C1}$$

can be linearly transformed to a Jordan's normal form, as

$$\frac{d}{d\xi}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{C2}$$

by a matrix P such as

$$\begin{pmatrix} x \\ y \end{pmatrix} = P^{-1}\begin{pmatrix} u \\ v \end{pmatrix}, \quad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = P^{-1}\begin{pmatrix} 0 & 1 \\ A & B \end{pmatrix}P. \tag{C3}$$

Then we can rewrite it by $z = x + iy, \bar{z} = x - iy$ as

$$\frac{d}{d\xi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \tag{C4}$$

where $\lambda = \alpha + i\beta, \bar{\lambda} = \alpha - i\beta$. The relation between A, B and α, β is $2\alpha = B$, $\alpha^2 + \beta^2 = -A$. P is obtained as

$$P = \begin{pmatrix} 1 & 1 \\ \alpha + \beta & \alpha - \beta \end{pmatrix}, \quad P^{-1} = \frac{1}{2\beta}\begin{pmatrix} -\alpha + \beta & 1 \\ \alpha + \beta & -1 \end{pmatrix}. \tag{C5}$$

The nonlinear term is obtained by using P.

APPENDIX D: FORMULA FOR $(u,v) \mapsto (z,\bar{z})$

$$u^2 = -\frac{i}{2}z^2 + z\bar{z} + \frac{i}{2}\bar{z}^2, \tag{D1}$$
$$uv = \frac{\beta - \alpha i}{2}z^2 + \alpha z\bar{z} + \frac{\beta + \alpha i}{2}\bar{z}^2, \tag{D2}$$
$$v^2 = \left\{\alpha\beta - \frac{(\alpha^2 - \beta^2)i}{2}\right\}z^2 + (\alpha^2 + \beta^2)z\bar{z} + \left\{\alpha\beta + \frac{(\alpha^2 - \beta^2)i}{2}\right\}\bar{z}^2, \tag{D3}$$
$$u^3 = -\tfrac{1}{4}(1+i)z^3 + \tfrac{3}{4}(1 - i)z^2\bar{z} + \tfrac{3}{4}(1 + i)z\bar{z}^2 - \tfrac{1}{4}(1 - i)\bar{z}^3, \tag{D4}$$
$$u^2 v = -\tfrac{1}{4}\{-\alpha + \beta - (\alpha + \beta)i\}z^3 + \tfrac{1}{4}\{3\alpha + \beta - (3\alpha - \beta)i\}z^2\bar{z} + \tfrac{1}{4}\{3\alpha + \beta + (3\alpha - \beta)i\}z\bar{z}^2 - \tfrac{1}{4}\{-\alpha + \beta + (\alpha + \beta)i\}\bar{z}^3, \tag{D5}$$
$$uv^2 = \tfrac{1}{4}\{-\alpha^2 + 2\alpha\beta + \beta^2 - (\alpha^2 + 2\alpha\beta - \beta^2)i\}z^3 + \tfrac{1}{4}\{3\alpha^2 + 2\alpha\beta + \beta^2 - (3\alpha^2 - 2\alpha\beta + \beta^2)i\}z^2\bar{z} + \tfrac{1}{4}\{3\alpha^2 + 2\alpha\beta + \beta^2 + (3\alpha^2 - 2\alpha\beta + \beta^2)i\}z\bar{z}^2 + \tfrac{1}{4}\{-\alpha^2 + 2\alpha\beta + \beta^2 + (\alpha^2 + 2\alpha\beta - \beta^2)i\}\bar{z}^3. \tag{D6}$$

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