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Scale invariance implies conformal invariance for the three-dimensional Ising model

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Using the Wilson renormalization group, we show that if no integrated vector operator of scaling dimension -1 exists, then scale invariance implies conformal invariance. By using the Lebowitz inequalities, we prove that this necessary condition is fulfilled in all dimensions for the Ising universality class. This shows, in particular, that scale invariance implies conformal invariance for the three-dimensional Ising model.

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I. INTRODUCTION

Conformal symmetry plays a considerable role in both high-energy and condensed-matter physics. There has been a renewed interest in recent years, particularly because of the anti–de Sitter–conformal field theory (AdS-CFT) conjecture [1] and the successful use of conformal methods in three-dimensional critical physics [2–7]. The groundbreaking papers of the 1970s and 1980s [8–13] solved two fundamental issues in two dimensions: First, scale invariance implies conformal invariance under mild assumptions [12,13], and second, conformal symmetry enables us to solve most of the scale-invariant problems, that is, to determine critical exponents and correlation functions [14].

An important ingredient for the exact solution of two-dimensional conformal models is the existence of an infinite number of generators of the conformal group. In higher dimensions the number of generators is finite, and we could naively conclude that symmetry arguments alone are not sufficient to solve a model in the critical regime. However, it is well known that scale-invariant theories are in a one-to-one correspondence with the fixed points of the Wilson renormalization group (RG) [15], and that the fixed point of a theory completely determines all the correlation functions of a critical model for sufficiently small wave numbers. Therefore, at the level of principles, scale (and *a fortiori* conformal) invariance is sufficient to determine all the universal critical properties of a model. Of course, in practice, the computation of these critical properties requires us to solve the functional Wilson RG equations. This is a formidable task that we do not know how to perform without approximations. Any supplementary information, even if redundant, is therefore welcome, and this is what conformal invariance could provide. A breakthrough in this direction was achieved in recent years with the conformal bootstrap program [2–4], which led to the *exact* (although numerical) computation of the critical exponents of the Ising model in three dimensions assuming, among other things, conformal invariance.

In parallel, a large amount of activity has been devoted to understanding the relation between scale and conformal invariance in—or close to—four dimensions. It has been proven to all orders of perturbation theory that scale invariance implies conformal invariance [16] in four-dimensional unitary and Poincaré invariant theories. Moreover, there are strong indications that a nonperturbative proof could be within reach in this dimension [17–19].

Despite decades of effort, it is still an open question whether a typical statistical model is conformally invariant at criticality in three dimensions. The aim of this article is twofold. First, we derive a sufficient condition that, when fulfilled, ensures that scale invariance implies conformal invariance. In the second part of the paper, we prove that this condition is fulfilled in any dimension for the Euclidean $\mathbb{Z}_2$ model.

The rest of the paper is organized as follows. In Sec. II, we present a brief review of the nonperturbative renormalization group. We then recall in Sec. III the relation between scale invariance and the nonperturbative renormalization group (NPRG). By using the same methods, we generalize these considerations to the case of conformal invariance in Sec. IV. In Sec. V, we finally derive a sufficient condition for the validity of conformal invariance in scale-invariant models. We use general arguments to show that this condition is expected to be fulfilled in O(N) models (and in generalizations thereof). In Sec. VI, these considerations are made rigourous for the Ising universality class. We give our conclusions in Sec. VII.

II. NONPERTURBATIVE RENORMALIZATION-GROUP FORMALISM

The proof of conformal invariance in all dimensions presented below is intimately related to the deep structure of the Wilson RG[1] and scale invariance. We therefore start by recalling, in the case of the $\mathbb{Z}_2$ model, the formalism of the modern formulations (sometimes called the nonperturbative RG, or the functional RG) of the Wilson RG [22–26]. The coarse-graining procedure at some RG scale k is implemented by smoothly decoupling the long-wavelength modes $\varphi(|q| < k)$ of the system, also called the slow modes, by giving them a large mass, while keeping unchanged the dynamics of the short-wavelength/rapid ones $\varphi(|q| > k)$. This decoupling is conveniently implemented by modifying the action or the Hamiltonian of the model: $S[\varphi] \to S[\varphi] + \Delta S_k[\varphi]$, where $\Delta S_k[\varphi]$ is quadratic in the field and reads, in Fourier space, $\Delta S_k[\varphi] = 1/2 \int_q R_k(q^2)\varphi(q)\varphi(-q)$. The precise shape of $R_k(q^2)$ does not matter for what follows as long as it can be written as

$$R_k(q^2) = Z_k k^2 r(q^2/k^2), \tag{1}$$

[1]The history of the relation between conformal invariance and Wilson RG is long; see, for example, [20,21].

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where Z_k is the field renormalization factor and r is a function that (i) falls off rapidly to 0 for $q^2 > k^2$—the rapid modes $\varphi(|q| > k)$ are not affected by ΔS_k—and (ii) goes to a constant for $q^2 = 0$—the slow modes $\varphi(|q| < k)$ acquire a mass of order k and thus smoothly decouple. The partition function, which now depends on the RG parameter k, reads

$$\mathcal{Z}_k[J] = \int \mathcal{D}\varphi \, \exp\left(-S[\varphi] - \Delta S_k[\varphi] + \int_x J\varphi\right), \tag{2}$$

where the field J is a source term that corresponds to the magnetic field in the Ising model and where the ultraviolet (UV) regime of the functional integral is assumed to be regularized at a momentum scale Λ; see, for instance, [27] for a lattice regularization in this formalism. It is convenient to define the free energy $\mathcal{W}_k[J] = \ln \mathcal{Z}_k[J]$ and its (slightly modified) Legendre transform by

$$\Gamma_k[\phi] + \mathcal{W}_k[J] = \int_x J\phi - \frac{1}{2}\int_{xy} R_k(|x-y|)\,\phi(x)\phi(y), \tag{3}$$

with $\phi(x) = \delta\mathcal{W}_k/\delta J(x)$, $R_k(|x-y|)$ is the inverse Fourier transform of $R_k(q^2)$, and where the last term has been added for the following reason. When k is close to Λ, all modes are completely frozen by the ΔS_k term because, for all q, $R_\Lambda(q^2)$ is very large. Thus, $\mathcal{Z}_{k\to\Lambda}$ can be computed by the saddle-point method, and it is then straightforward to show that the presence of the last term in Eq. (3) leads to $\Gamma_\Lambda[\phi] \simeq S[\varphi = \phi]$. On the contrary, when $k = 0$, the definition of R_k implies that $R_{k=0}(q^2) \equiv 0$ and the original model is recovered: $\mathcal{Z}_{k=0}[J] = \mathcal{Z}[J]$ and $\Gamma_{k=0}[\phi] = \Gamma[\phi]$, with $\Gamma[\phi]$ the usual Gibbs free energy or generating functional of one-particle-irreducible correlation functions.

The exact RG equation for Γ_k reads [23–25]

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\int_{xy} \partial_t R_k(|x-y|) G_{k,xy}[\phi], \tag{4}$$

where $t = \ln(k/\Lambda)$, and $G_{k,xy}[\phi]$ is the field-dependent propagator:

$$G_k = \left(\Gamma_k^{(2)} + R_k\right)^{-1}, \quad \Gamma^{(2)}_{k,xy}[\phi] = \frac{\delta^2\Gamma_k[\phi]}{\delta\phi(x)\,\delta\phi(y)}. \tag{5}$$

III. SCALE INVARIANCE

We now discuss how scale invariance emerges in the NPRG formalism. We first consider a scale-invariant model described by an action $S_{\rm scal}$. As we discuss in the following, the existence of such a model is not necessary for our proof, but it is convenient to imagine that it exists to motivate the form of the expected Ward identity (WI) of scale invariance. If such a model existed, this WI would be obtained by performing the infinitesimal change of variables $\varphi(x) \to \varphi(x) + \epsilon(D_\phi + x_\mu\partial_\mu)\varphi(x)$ in the functional integral, with D_ϕ the scaling dimension of the field, usually written in terms of the anomalous dimension η as $D_\phi = (d-2+\eta)/2$. Actually, the analysis of this model and of its WI faces both UV and infrared (IR) problems. In the IR regime, nonanalyticities are present, and in the UV regime it is difficult to control mathematically the continuum limit: $\Lambda \to \infty$. Let us first discuss the IR aspect. Since $\Delta S_k[\phi]$ acts as an IR regulator, the Wilson RG offers a solution to the IR problem: the regularized model, Eq. (2), is not scale-invariant even if the original model associated with $S_{\rm scal}$ was, and thus all $\Gamma^{(n)}_{k>0}(\{p_i\})$ are regular contrary to $\Gamma^{(n)}_{k=0}(\{p_i\})$. The price to pay for regularity is the breaking of scale invariance that manifests itself through a modification of the WI (see [28,29] for situations in which R_k breaks symmetries). By enlarging the space of cutoff functions R_k to arbitrary functions $R_k(x,y)$ that are neither constrained to satisfy (1) nor to be invariant under rotations and translations (as also done in [30]), this modified WI for scale invariance, obtained from Eqs. (2) and (3), reads

$$\int_{xy}(D^x + D^y + D_R)R_k(x,y)\frac{\delta\Gamma_k}{\delta R_k(x,y)} + \int_x (D^x + D_\phi)\phi(x)\frac{\delta\Gamma_k}{\delta\phi(x)} = 0, \tag{6}$$

where $D^x = x_\mu\partial^x_\mu$, and $D_R = 2d - 2D_\phi$ is the scaling dimension of R_k, which implies that the field renormalization in Eq. (1) behaves as $Z_k \propto k^{-\eta}$. By constraining $R_k(q)$ to be of the form (1), Eq. (6) can be conveniently rewritten (following [29]) as

$$\partial_t\Gamma_k[\phi] = -\int_x (D^x + D_\phi)\phi(x)\,\frac{\delta\Gamma_k[\phi]}{\delta\phi(x)}. \tag{7}$$

Introducing dimensionless and renormalized quantities (denoted with a tilde)

$$x = k^{-1}\tilde{x}, \tag{8}$$

$$\phi(x) = k^{D_\phi}\tilde{\phi}(\tilde{x}), \tag{9}$$

$$\tilde{\Gamma}_k[\tilde{\phi}] = \Gamma_k[\phi], \tag{10}$$

Eq. (7) is rewritten as

$$\partial_t\tilde{\Gamma}_k[\tilde{\phi}] = 0. \tag{11}$$

Equation (11) means that a hypothetical scale-invariant action $S_{\rm scal}$ would lead, in its regularized and *not* scale-invariant version $S_{\rm scal} + \Delta S_k$, to a RG flow where $\tilde{\Gamma}_k[\tilde{\phi}]$ would be at a fixed point $\tilde{\Gamma}^*[\tilde{\phi}]$ for all values of t: $\partial_t\tilde{\Gamma}^*[\tilde{\phi}] = 0$.

The very structure of the Wilson RG (or NPRG) also solves the UV problem. Actual models have a natural UV cutoff Λ (e.g., a lattice spacing) at which is defined their microscopic action S. The momentum integrals are therefore cut off at Λ and are thus UV-finite. At scales of order Λ (i.e., if we consider correlation functions in the regime where at least one external momentum or k or the magnetic field in appropriate units is of the order of Λ), the model is not scale-invariant. In fact, scale invariance is an emergent property that appears in the IR when some parameter (such as the temperature) has been fine-tuned. We call $S_{\rm crit}$ the corresponding action. In the RG formalism, scale invariance emerges in the IR regime when integrating the RG flow starting at $\Gamma_\Lambda[\phi] = S_{\rm crit}[\varphi = \phi]$ because $\tilde{\Gamma}_k$ gets close to a fixed point for large negative t, that is, $k \ll \Lambda$. As discussed above, the fixed-point condition coincides with the WI for scale invariance in the presence of a regulator; see Eqs. (7) and (11). As a consequence, if the RG flow is attracted toward an IR fixed point, then scale invariance emerges in the universal, long-distance regime.[2] We stress that this discussion does not rely on the actual existence of a well-defined continuum limit associated with a scale-invariant action $S_{\rm scal}$, which is, *per se*, an interesting issue, but one that we do not need to address in the present work.

When the microscopic action is slightly different from the critical one (e.g., choosing a temperature slightly away from the critical one), the RG trajectory approaches the fixed point and then stays close to it for a long RG "time" before departing. In this situation, the correlation length ξ is finite but large, and the WI is almost fulfilled for momenta $\Lambda \gg p \gg \xi^{-1}$. This defines the critical regime of the theory. The closer the microscopic action is tuned to the critical one, the larger the correlation length ξ, and the better the WI is fulfilled.

Let us now make two comments. First, when $k \to 0$, $\partial_t \Gamma_k[\phi]$ becomes negligible compared to any k-independent finite scale and $\Gamma_k \to \Gamma$. In this limit, Eqs. (7) and (11) become the usual WI of scale invariance, as expected. Second, the above analysis shows that among the continuous infinity of solutions of the fixed-point equation $\partial_t \tilde\Gamma_k[\tilde\phi] = 0$, only those that are regular for all fields are acceptable since they must be the limit when $k \to 0$ of the smooth evolution of $\tilde\Gamma_k[\tilde\phi]$ from $S_{\rm crit}[\tilde\phi]$. There is generically only a discrete, often finite, number of such physical fixed points.[3]

A characteristic feature of the physical fixed points is that the linearized flow around them has a discrete spectrum of eigenvalues from which some critical exponents can be straightforwardly obtained [15]. The discrete character of the eigenperturbations around a fixed point has been studied intensively by perturbative means. As for the Wilson RG, it has been studied in detail in the particular case of the O(N) models in [34–36]. Although obtained within the derivative expansion of the exact RG flow (4), its discrete character certainly remains true beyond this approximation. The eigenvalues λ are obtained from the flow by substituting $\tilde\Gamma_k[\tilde\phi] \to \tilde\Gamma^*[\tilde\phi] + \epsilon\, \exp(\lambda t)\tilde\gamma[\tilde\phi]$ and retaining only the $O(\epsilon)$ terms. (With our definition of t, a relevant operator has a negative eigenvalue.) This leads to the eigenvalue problem

$$\lambda\, \tilde\gamma[\tilde\phi] = \int_{\tilde x} (D^{\tilde x} + D_\phi)\tilde\phi(\tilde x)\, \frac{\delta \tilde\gamma}{\delta \tilde\phi(\tilde x)} - \frac{1}{2}\int_{\tilde x_i} [(D^{\tilde x} + D_R) r(\tilde x - \tilde y)]\, \tilde G^*_{\tilde x \tilde z}\, \tilde\gamma^{(2)}_{\tilde z \tilde w}\, \tilde G^*_{\tilde w \tilde y}, \quad (12)$$

where $\tilde x_i = \{\tilde x, \tilde y, \tilde z, \tilde w\}$, $\tilde G^*[\tilde\phi]$ is the dimensionless renormalized propagator at the fixed point: $\tilde G^* = (\tilde\Gamma^{*(2)} + r)^{-1}$, and $r(\tilde x)$ is the dimensionless inverse Fourier transform of $r(q^2/k^2)$ defined in Eq. (1).

We conclude from the above discussion that regularity selects among all the fixed-point functionals $\tilde\Gamma^*[\tilde\phi]$ those that are physical, that is, those that can be reached by an RG flow from a physical action $S_{\rm crit}$ and that have a discrete spectrum of eigenperturbations.

[2] A running anomalous dimension can be defined by $\eta_k = -\partial_t \log Z_k$. It is only around the fixed point that η_k approaches a fixed-point value, which is simply η.

[3] Two well-known exceptions to this rule are the line of fixed points of the $O(2)$ model in $d = 2$ [31] and the discrete infinity of (multicritical) Z_2-invariant fixed points in $d = 2$ [32,33].

IV. SPECIAL CONFORMAL TRANSFORMATIONS

Let us now study conformal invariance by following the same method as above. As in the case of scale invariance, to motivate the form of the WI, we imagine that a conformally invariant model, associated with an action $S_{\rm conf}[\varphi]$ written in terms of a primary field φ, exists. We put aside for now the problem of the existence of this model since, as discussed below, we do not need it to actually exist for our proof. If such a model existed, the modified WI would follow from performing the infinitesimal change of variables $\varphi(x) \to \varphi(x) + \epsilon_\mu(x^2\partial_\mu - 2x_\mu x_\nu \partial_\nu + 2\alpha x_\mu)\varphi(x)$ in Eq. (2). By considering general cutoff functions as in Eq. (6), we find that it reads

$$\int_{xy} \left(K^x_\mu - D_R\, x_\mu + K^y_\mu - D_R\, y_\mu\right) R_k(x,y) \frac{\delta \Gamma_k}{\delta R_k(x,y)} + \int_x \left(K^x_\mu - 2D_\phi x_\mu\right)\phi(x)\frac{\delta \Gamma_k}{\delta \phi(x)} = 0, \quad (13)$$

with $K^x_\mu = x^2\partial^x_\mu - 2x_\mu x_\nu \partial^x_\nu$. By specializing R_k to functions of the form Eq. (1) and requiring again that $Z_k \propto k^{-\eta}$, Eq. (13) can be rewritten as

$$0 = \Sigma^\mu_k[\phi] \equiv \int_x \left(K^x_\mu - 2D_\phi x_\mu\right)\phi(x)\frac{\delta \Gamma_k}{\delta \phi(x)} - \frac{1}{2}\int_{xy} \partial_t R_k(|x-y|)\,(x_\mu + y_\mu)\, G_{k,xy}. \quad (14)$$

Again, this identity boils down to the usual WI of conformal invariance in the limit $k \to 0$ where $R_k \to 0$.

At any fixed point, the scaling dimension of $\Sigma^\mu_k[\phi]$ is fixed by Eq. (14) to be -1. We thus define $\tilde\Sigma^\mu_k[\tilde\phi] = k\, \Sigma^\mu_k[\phi]$. Its flow equation reads

$$\partial_t \tilde\Sigma^\mu_k[\tilde\phi] - \tilde\Sigma^\mu_k[\tilde\phi] = \int_{\tilde x} (D^{\tilde x} + D_\phi)\tilde\phi(\tilde x)\, \frac{\delta \tilde\Sigma^\mu_k}{\delta \tilde\phi(\tilde x)} - \frac{1}{2}\int_{\tilde x_i} [(D^{\tilde x} + D_R) r(\tilde x - \tilde y)] \times \tilde G_{k,\tilde x \tilde z}\, \tilde\Sigma^{\mu\,(2)}_{k,\tilde z \tilde w}\, \tilde G_{k,\tilde w \tilde y}, \quad (15)$$

where $\tilde\Sigma^{\mu\,(2)}_k[\tilde\phi]$ is the second functional derivative of $\tilde\Sigma^\mu_k$.

An important property of Σ^μ_k is that, at the fixed point, it is the integral of a density with no explicit dependence on x_μ. Indeed, observe that the left-hand sides of Eqs. (6) and (13) can be interpreted as the action of the generators of dilatations $\mathcal{D}$ and conformal transformations $\mathcal{K}_\mu$ on Γ_k,

$$\mathcal{D}\Gamma_k = \int_x (D^x + D_\phi)\phi(x)\frac{\delta \Gamma_k}{\delta \phi(x)} + \int_{xy} (D^x + D^y + D_R) R_k(x,y) \frac{\delta \Gamma_k}{\delta R_k(x,y)}, \quad (16)$$

$$\mathcal{K}_\mu \Gamma_k = \int_x \left(K^x_\mu - 2D_\phi x_\mu\right)\phi(x)\frac{\delta \Gamma_k}{\delta \phi(x)} + \int_{xy} \left(K^x_\mu - D_R x_\mu + K^y_\mu - D_R y_\mu\right) \times R_k(x,y)\frac{\delta \Gamma_k}{\delta R_k(x,y)}, \quad (17)$$

that is, $\Sigma_\mu = \mathcal{K}_\mu \Gamma_k$. Similar expressions can be obtained for the generators of translations $\mathcal{P}_\mu$ and rotations $\mathcal{J}_{\mu\nu}$:

$$\mathcal{P}_\mu \Gamma_k = \int_x \partial_\mu \phi(x) \frac{\delta \Gamma_k}{\delta \phi(x)} + \int_{xy} \left(\partial_\mu^x + \partial_\mu^y\right) R_k(x,y) \frac{\delta \Gamma_k}{\delta R_k(x,y)}, \tag{18}$$

$$\mathcal{J}_{\mu\nu} \Gamma_k = \left[\int_x x_\mu \partial_\nu \phi(x) \frac{\delta \Gamma_k}{\delta \phi(x)} + \int_{xy} \left(x_\mu \partial_\nu^x + y_\mu \partial_\nu^y\right) R_k(x,y) \frac{\delta \Gamma_k}{\delta R_k(x,y)} \right] - [\mu \leftrightarrow \nu]. \tag{19}$$

It can easily be checked that the generators satisfy the algebra of the conformal group. In particular, applying $[\mathcal{P}_\mu, \mathcal{K}_\nu] = 2\delta_{\mu\nu}\mathcal{D} + 2\mathcal{J}_{\mu\nu}$ to a translation, rotation, and dilatation invariant Γ_k yields

$$\mathcal{P}_\mu \Sigma_k^\nu = 0. \tag{20}$$

Thus, at the fixed point, Σ_k^μ is the integral of a density that does not have an explicit dependence on x. This density only depends on the field and its derivatives. This proof generalizes trivially to other scalar models.

V. A SUFFICIENT CONDITION FOR CONFORMAL INVARIANCE

Let us now consider a physical model at criticality. At the scale Λ, $\Gamma_\Lambda = S = S_{\text{crit}}$ and the model is neither conformally invariant nor scale-invariant. However, when $k \ll \Lambda$, the regularized model gets close to a fixed point and thus $\tilde{G}_k \simeq \tilde{G}^*$ and $\tilde{\Sigma}_k^\mu[\tilde{\phi}] \simeq \tilde{\Sigma}^{\mu *}[\tilde{\phi}]$. The key point of our proof is that at the fixed point, Eq. (15) is identical to (12) with $\tilde{\gamma}[\tilde{\phi}] \to \tilde{\Sigma}^{\mu *}[\tilde{\phi}]$ and $\lambda = -1$, although these two equations have different physical meanings. A *sufficient* condition for proving conformal invariance is therefore to show that there is no integrated vector eigenperturbation $V_\mu = \int_x \mathcal{V}_\mu$ of $\tilde{\Gamma}^*[\tilde{\phi}]$ of scaling dimension $D_V = -1$. If no such eigenperturbation exists, the conformal WI is satisfied, which means that the system is conformally invariant at criticality in the long-distance regime. Moreover, the form of the conformal WI (14) fixes the transformation law fulfilled by φ, which is the one of a primary field.

To understand how conformal invariance is related with the scaling dimension of the vector eigenperturbations, it proves useful to consider three simple examples. The first one is the Ising model in $d = 4$. The fixed point being Gaussian, the eigenvalues are trivially given by the canonical dimensions. By using the fact that $\tilde{\Sigma}_k^\mu$ is $\mathbb{Z}_2$- and translation-invariant, see Eqs. (14) and (20), we find by inspection that the vector operator with lowest dimension reads $\int_x \phi\, \partial_\mu \phi (\partial\phi)^2$. It has therefore dimension +3. Note that there exists local vector operators with lower scaling dimensions [$\phi\partial_\mu\phi$, $\phi^3\partial_\mu\phi$, and $\phi\partial_\mu\phi(\partial_\nu\phi)^2$]. However, these are total derivatives and are not associated with integrated vector operators. In the absence of a vector operator of dimension -1, we retrieve the well-known property that this model is conformally invariant at criticality in the long-distance regime [13,16]. Using standard methods, we can compute the corrections to the scaling dimension of the vector operator $\int_x \phi\, \partial_\mu \phi (\partial\phi)^2$ in a systematic expansion in $\epsilon = 4 - d$. We performed the calculation at one loop and found that the correction vanishes. The scaling dimension is therefore $3 + O(\epsilon^2)$.

This analysis can be extended to the O(N) models. In $d = 4$, there exists now two independent integrated vector operators, $\int_x \phi^a (\partial_\mu \phi^a)(\partial_\nu \phi^b)^2$ and $\int_x \phi^a (\partial_\nu \phi^a)(\partial_\mu \phi^b)(\partial_\nu \phi^b)$, with the lowest scaling dimension 3. As in the Ising case, there exist local operators with lower scaling dimensions that are, however, total derivatives. Our sufficient condition is again fulfilled, and we recover the well-known fact that these models are conformally invariant in the critical regime for $d = 4$. At one loop, the degeneracy of the scaling dimensions is lifted and we obtain $3 + O(\epsilon^2)$ and $3 - \frac{6\epsilon}{N+8} + O(\epsilon^2)$.

The third example involves a vector field $A_\mu(x)$ and is described by the (Euclidean) action

$$S = \int_x \left[\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{\alpha}{2} (\partial_\mu A_\mu)^2 \right]. \tag{21}$$

This model is interesting because it is scale-invariant but generically not conformally invariant, except for $\alpha = \alpha_c = (d-4)/d$ [37,38]. This situation can be understood in our context by considering the contraposition of our sufficient condition, which states that, assuming that A_μ is a primary field [38], a necessary condition for having scale invariance without conformal invariance is that there exists an integrated vector operator with scaling dimension -1. It is actually easy to find such an operator (which is unique, up to a normalization): $C \int_x A^\mu (\partial_\nu A_\nu)$. We can understand the particular case $\alpha = \alpha_c$ by an explicit calculation of $\Sigma_k^\mu[A_\nu]$ from Eq. (14). This shows that $C = \alpha d + 4 - d$, which, as expected, vanishes when $\alpha = \alpha_c$.

To conclude this section, we discuss the plausibility of *not* having conformal invariance in $d = 3$ for the O(N) models at criticality. The only possibility would be to have a vector eigenperturbation with eigenvalue -1 in this dimension; see Fig. 1. This would mean that the $d = 3$ model has an integer critical exponent, a property that is highly improbable. Nevertheless, let us suppose that one of these eigenvalues crosses -1 right in $d = 3$, as in curve (c) of Fig. 1. Then, for

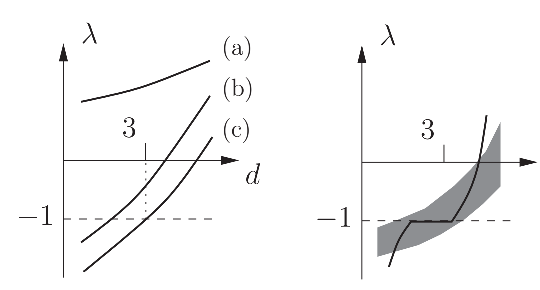

FIG. 1. Possible behavior of the lowest eigenvalue D_V associated with a vector perturbation as a function of dimension. Left panel: (a) and (b) correspond to typical behavior, while (c) corresponds to the exceptional case in which $D_V = -1$ right in $d = 3$. In the three cases, conformal invariance holds. Right panel: the shaded area represents a continuum of eigenvalues, and the curve denotes an eigenvalue D_V having a plateau at -1 around $d = 3$. In these cases, conformal invariance can be broken.

any dimension infinitesimally smaller or larger than 3, there would exist no eigenperturbation of dimension -1. The critical system would exhibit conformal symmetry above and below $d = 3$. Since correlation functions of the critical theory are expected to be continuous functions of d, we conclude that, even in this highly improbable situation, the model would exhibit conformal invariance at criticality in $d = 3$. We are thus led to the more stringent necessary (but not sufficient) condition: for a critical model *not* to be conformally invariant, there must exist a vector perturbation of scaling dimension -1 in a finite interval of dimensions containing $d = 3$. This could happen either because a discrete eigenvalue is independent of the dimension in some range of dimension around 3, or because there exists a continuum of eigenvalues, see Fig. 1. Such a behavior is, to say the least, not standard. To our knowledge, this has never been observed in any interacting model.

The previous arguments are compelling but not mathematically rigorous. In particular, assuming that the theory can be properly defined in noninteger dimensions, which is standard but not obvious, it is hard to control the analytic structure of the critical exponents in d. In the next section, we give a proof in the physically important case of the Ising model in $d = 3$ that does not rely on such arguments.

VI. PROOF IN THE ISING UNIVERSALITY CLASS

We concentrate here on the Ising universality class and show that, in this case, the smallest eigenvalue D_V associated with an integrated vector perturbation is larger than -1 for $d < 4$. Using our necessary condition, this proves that, for the Ising universality class, the critical regime is conformally invariant.

For simplicity, we consider a lattice version of the Ginzburg-Landau model whose dynamics is described by the Hamiltonian (or action)

$$S = -J \sum_{\langle ij \rangle} \varphi(i)\varphi(j) + \sum_i U(\varphi(i)), \tag{22}$$

where the index i labels the lattice sites, the $\varphi(i)$ take values in the real domain, and $U(\varphi)$ is an even function that diverges for $|\varphi| \to \infty$. We choose here a hypercubic lattice with lattice spacing a. The original Ising model can be recovered by considering a potential $U(\varphi)$ strongly peaked around $\varphi = \pm 1$, but the Ginzburg-Landau model, which is in the Ising universality class for a generic potential, is more convenient for what follows.

The use of the Ginzburg-Landau model has another advantage. In the case of a quartic potential,

$$U[\varphi(i)] = \frac{r_0}{2}\varphi^2(i) + \frac{u_0}{4!}\varphi^4(i), \tag{23}$$

and for $d < 4$ the model is super-renormalizable. Its ultraviolet behavior is therefore controlled by a Gaussian fixed point. In this case, the existence of a controlled scaling limit seems to be under control even if, to the best of our knowledge, there is no mathematical proof of its existence [39]. To compare different models in the Ising universality class, we assume below that this scaling limit does exist in the following precise sense. Let us consider two local operators $O_1(i,a)$ and $O_2(i,a)$. Let us also introduce (i) a smooth interpolating field $\phi^{\text{interp}}(x)$ with $x \in \mathbb{R}^d$ that coincides on the lattice points with $\phi(i)$, and (ii) two interpolating operators $O_{1,2}^{\text{interp}}(x,a)$ defined by $O_i^{\text{interp}} = O_i(\phi^{\text{interp}})$. Of course the construction of these interpolating operators is not unique. We now consider the particular case in which O_1 and O_2 are such that $O_1^{\text{interp}} \to O_2^{\text{interp}}$ when $a \to 0$. (We notice that this limit is independent of the choice of interpolation used to define ϕ^{interp}.) Our assumption, which we call the "scaling limit" for short, is that there exists a multiplicative factor $Z_O(a)$ depending on the lattice spacing such that the correlation functions of the operators $O_1(x,a)$ and $Z_O(a)O_2(x,a)$ are the same for distances much larger than a:

$$\begin{aligned}\langle O_1(x,a)O_3(y_3)\cdots O_n(y_n)\rangle \\ \sim Z_O(a)\langle O_2(x,a)O_3(y_3)\cdots O_n(y_n)\rangle,\end{aligned} \tag{24}$$

where $\{O_3,\dots,O_n\}$ are arbitrary local operators and where the equivalence occurs for $a \ll \min\{|y_3 - x|,\dots,|y_n - x|\}$. This hypothesis is a prerequisite of all Monte Carlo simulations, and it is, of course, satisfied to all orders of perturbation theory in any renormalizable theory. We assume here that it is also valid nonperturbatively.

Our strategy is to study correlation functions of local vector operators $\mathcal{V}_\mu(x)$ and use their critical behavior to find a bound on the scaling dimension of the *integrated* operator $V_\mu = a^d \sum_i \mathcal{V}_\mu(i)$. We now mention two difficulties.

First, there are local vector operators that are total derivatives, and which are therefore not associated with an integrated one. As discussed before, the vector operator $\partial_\mu(\varphi^2)$ is such an operator. Note that its scaling dimension near $d = 4$ is lower than that of the vector operators that are not total derivatives [such as $\partial_\mu(\varphi)^2(\partial_\nu\varphi)^2$].

Second, operators with the same quantum numbers typically mix together in the calculation of correlation functions. As a consequence, the critical behavior of a two-point function of some vector operator is governed by the lowest scaling dimension of the class of operators with which it mixes. To be more precise, let us call $\mathcal{V}_\mu^{(n)}$ the local vector eigenoperator of scaling transformations with scaling dimension $D_\mathcal{V}^{(n)}$ (ordered such that $D_\mathcal{V}^{(0)} \leqslant D_\mathcal{V}^{(1)} \leqslant \cdots$). The associated two-point correlation function behaves, in the critical regime, as

$$\left\langle \mathcal{V}_\mu^{(n)}(x)\mathcal{V}_\mu^{(n)}(y)\right\rangle_c \sim \frac{1}{|x-y|^{2D_\mathcal{V}^{(n)}}}, \tag{25}$$

where the subscript c indicates connected correlation functions, defined as

$$\langle X(x)Y(y)\rangle_c = \langle X(x)Y(y)\rangle - \langle X(x)\rangle\langle Y(y)\rangle \tag{26}$$

and where an appropriate normalization has been chosen. In $d = 4$, the eigenproblem can be solved and the scaling dimensions are the canonical dimensions. In particular, the eigenoperators with the lowest scaling dimensions are

$$\mathcal{V}_\mu^{(0,d=4)} \propto \partial_\mu\phi^2, \quad D_\mathcal{V}^{(0,d=4)} = 3, \tag{27}$$

$$\mathcal{V}_\mu^{(1,d=4)} \propto \partial_\mu\phi^4, \quad D_\mathcal{V}^{(1,d=4)} = 5, \tag{28}$$

$$\mathcal{V}_\mu^{(2,d=4)} \propto \partial_\mu(\partial_\nu\phi)^2, \quad D_\mathcal{V}^{(2,d=4)} = 5, \tag{29}$$

$$\begin{gathered}\mathcal{V}_\mu^{(3,d=4)} \propto (\partial_\mu\phi^2)(\partial_\nu\phi)^2, \quad D_\mathcal{V}^{(3,d=4)} = 7, \\ \cdots,\end{gathered} \tag{30}$$

where ∂_μ is a lattice discretization of the partial derivative. (Note that the first three operators are total derivatives.) In dimension $d < 4$, although we do not know the explicit form of the eigenoperators, we can, in principle, decompose any vector operator on the basis $\{\mathcal{V}_\mu^{(n)}\}$, and generically there is a nonvanishing overlap with each of the eigenoperators:

$$\mathcal{V}_\mu = \sum_n \alpha_n \mathcal{V}_\mu^{(n)}. \tag{31}$$

As a consequence, the critical regime of the two-point correlation function is dominated by the smallest critical dimension,

$$\langle \mathcal{V}_\mu(x)\mathcal{V}_\mu(y)\rangle_c \sim \frac{\alpha_0^2}{|x-y|^{2D_\mathcal{V}^{(0)}}}. \tag{32}$$

We stress that the list of quantum numbers must include those associated with lattice isometries. For example, we require scalar (vector) operators defined on the lattice to be even (odd) under parity transformations.

The proof is organized as follows. Using Lebowitz inequalities [40,41], we derive a lower bound for $D_\mathcal{V}^{(0)}$ from which follows a lower bound for the scaling dimension of the integrated vector operators. The proof that the scaling dimension D_V of any integrated vector operator is different from -1 for $d \leqslant 4$ is then a direct consequence of this bound.

As a first step, we derive a bound for the correlation function $\langle \varphi^2(x)\varphi^2(y)\rangle_c$. We use here the Lebowitz inequalities [40], which state that, considering two decoupled copies of the ferromagnetic system (that we denote φ and φ'), both described by the action (22), and considering two sets A and B of lattice points,

$$\begin{aligned}&\left\langle \prod_{i\in A, j\in B}[\varphi(i)+\varphi'(i)][\varphi(j)-\varphi'(j)]\right\rangle\\ &\quad\leqslant \left\langle \prod_{i\in A}[\varphi(i)+\varphi'(i)]\right\rangle\left\langle \prod_{j\in B}[\varphi(j)-\varphi'(j)]\right\rangle,\end{aligned} \tag{33}$$

$$\begin{aligned}&\left\langle \prod_{i\in A, j\in B}[\varphi(i)+\varphi'(i)][\varphi(j)+\varphi'(j)]\right\rangle\\ &\quad\geqslant \left\langle \prod_{i\in A}[\varphi(i)+\varphi'(i)]\right\rangle\left\langle \prod_{j\in B}[\varphi(j)+\varphi'(j)]\right\rangle.\end{aligned} \tag{34}$$

In particular, this implies that

$$\begin{aligned}&\langle [\varphi(x)+\varphi'(x)]^2[\varphi(y)-\varphi'(y)]^2\rangle\\ &\quad\leqslant \langle [\varphi(x)+\varphi'(x)]^2\rangle\langle [\varphi(y)-\varphi'(y)]^2\rangle.\end{aligned} \tag{35}$$

Expanding the binomials, we readily obtain the following identity:

$$\langle \varphi^2(x)\varphi^2(y)\rangle_c \leqslant 2G^2(x-y), \tag{36}$$

where we have used the fact that the average of an odd number of fields vanishes for temperatures higher than (or equal to) the critical temperature. This implies that the connected correlation function $\langle \varphi^2(x)\varphi^2(y)\rangle_c$ cannot decrease more slowly than the square of the propagator at long distances. This inequality can be generalized to arbitrary even powers of the fields:

$$0 \leqslant \langle \varphi^m(x)\varphi^n(y)\rangle_c \leqslant C\, G^2(x-y), \tag{37}$$

where C is a positive constant (that depends on m and n), as shown in Appendix A.

In the critical regime, scale invariance implies that connected two-point correlation functions behave as power laws. In particular,

$$\langle \varphi^m(x)\varphi^n(y)\rangle_c \sim \frac{A}{|x-y|^{\aleph_m+\aleph_n}} \tag{38}$$

with A a positive constant (see, for example, [14]). The inequality (37) implies that $\aleph_n \geqslant d-2+\eta$. We can then deduce the asymptotic behavior of the matrix of second derivatives of this correlation function:

$$\begin{aligned}&\left\langle \partial_\mu^x[\varphi^m(x)]\partial_\nu^y[\varphi^n(y)]\right\rangle_c\\ &\quad\sim \frac{1}{|x-y|^{\aleph_m+\aleph_n+2}}\left(B\delta_{\mu\nu}+C\frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2}\right),\end{aligned} \tag{39}$$

where B and C are some constants.

We now consider two local vector operators that are the product of one power of $\partial_\mu\varphi(x)$ and an odd (finite) number of fields evaluated at points in a finite neighborhood of x:

$$\mathcal{W}_\mu^{(1)}(x) = \frac{1}{2}[\partial_\mu\varphi(x)]\sum_{s=\pm1}\prod_{i=1}^{m-1}\varphi\big(x+s\,e_i^{(1)}\big), \tag{40}$$

$$\mathcal{W}_\mu^{(2)}(x) = \frac{1}{2}[\partial_\mu\varphi(x)]\sum_{s=\pm1}\prod_{i=1}^{n-1}\varphi\big(x+s\,e_i^{(2)}\big), \tag{41}$$

where $e_i^{(1)}$ and $e_i^{(2)}$ are some constant lattice vectors.[4] The operators $W_\mu^{(1)}(x)$ and $W_\mu^{(2)}(x)$ are, up to a multiplicative constant, other discretizations of, respectively, the operator $\partial_\mu^x[\varphi^m(x)]$ and $\partial_\mu^x[\varphi^n(x)]$. According to the assumption of the existence of the scaling limit, Eq. (24), the connected correlation function $\langle \mathcal{W}_\mu^{(1)}(x)\mathcal{W}_\nu^{(2)}(y)\rangle_c$ has the same asymptotic behavior as in Eq. (39) up to a multiplicative factor depending on the lattice spacing. Indeed, when $|x-y|$ is much larger than the lattice spacing a, the vectors $e_i^{(1)}$ and $e_i^{(2)}$ can be neglected in (40) and (41) and the local operators $\mathcal{W}_\mu^{(1)}$ and $\mathcal{W}_\mu^{(2)}$ are then proportional to $\partial_\mu(\varphi^m)$ and $\partial_\nu(\varphi^n)$, respectively, as explained before.

Now, any local vector operator $\mathcal{V}_\mu$ on the lattice is a linear combination of vector operators of the form (40). For instance, a discretization of the operator $\partial_\mu(\phi^2)(\partial_\nu\phi)^2$ is given by

$$\frac{\phi(x)}{16a^3}[\phi(x+\hat\mu)-\phi(x-\hat\mu)]\sum_\nu[\phi(x+\hat\nu)-\phi(x-\hat\nu)]^2, \tag{42}$$

where the sum runs over all the nearest neighbors of x.

Using the triangular inequality, we conclude that

$$|\langle \mathcal{V}_\mu(x)\mathcal{V}_\nu(y)\rangle_c| \leqslant \frac{Z_{\mu\nu}}{|x-y|^{2(d-1+\eta)}}, \tag{43}$$

[4]Since Σ_μ, defined in Eq. (14), is odd under parity, it is important in what follows to consider only vector operators that are also odd. This is the reason why the sum over s is necessary in the definitions (40) and (41).

where $Z_{\mu\nu}$ is a positive constant. Using Eq. (32), this implies that, for all n,

$$D_{\mathcal{V}}^{(n)} \geqslant d - 1 + \eta. \tag{44}$$

We conclude that the scaling dimension $D_V = D_{\mathcal{V}} - d$ of any *integrated* vector operator is not smaller than $-1 + \eta$.[5] Using the unitarity of the Minkowskian ϕ^4 theory, one can prove that $\eta \geqslant 0$ [42] for the Ising universality class. Moreover, an interacting massless theory, such as the critical Ising model for $d < 4$, has a nonzero η [43]. As a consequence, our necessary condition is fulfilled, and we conclude that scale invariance implies conformal invariance for the Ising universality class for all $d \leqslant 4$.

VII. CONCLUSIONS

Let us now point out some directions of research for the future. It is clear that the condition of conformal invariance (14) can be straightforwardly extended to other theories (involving scalar, fermionic, or vector fields), and it would be interesting to come to a conclusion on the fate of conformal invariance in this wider class of models. In these systems, it is much more difficult to find rigorous bounds on correlation functions (that would generalize the Lebowitz inequalities). It would then prove useful to approach the problem by computing the scaling behavior of vector operators by Monte Carlo simulations.

Another promising line of investigation consists in making use of the conformal invariance in the Wilson framework to perform actual calculations of universal quantities. On the one hand, and in the best case, this would lead to closed and numerically tractable equations for the critical exponents. On the other hand, because the approximation schemes currently used for solving the Wilson RG flow equation are incompatible with exact conformal invariance, we can expect that constraining them to be conformally invariant at the fixed point would improve their accuracy.

Note finally that, at first glance, our approach could seem similar to the one based on the energy-momentum tensor and on the analysis of the virial current. This is not the case, although there is perhaps a relationship between the two. Σ_k^μ is a functional of ϕ and not of φ; it is built from Γ_k and not from S. What matters is that its density vanishes up to a surface term and not that it is conserved. Moreover, as we already explained, we only deal with a regularized theory, which enables us to consider only the analytic candidates for $\tilde{\Sigma}^{\mu*}$ contrary to what should be done in a nonregularized theory. In any case, a clarification of the relation between the two approaches would be welcome. In this respect, our proof of the nonexistence of a local vector operator of scaling dimension $d - 1$ (conserved or not) might be of interest also when applied to a hypothetical conserved virial current.

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APPENDIX: BOUND FOR CORRELATION FUNCTIONS $\langle\varphi^m(x)\varphi^n(y)\rangle$

In this appendix, we derive bounds on the correlation functions $\langle\varphi^m(x)\varphi^n(y)\rangle_c$, in the symmetric phase ($T \geqslant T_c$), with m and n arbitrary integers with the same parity (otherwise the correlation function vanishes).

We want to show that

$$\langle\varphi^a(x)\varphi^b(y)\rangle_c \leqslant C_1 G(x-y) \quad \text{for odd } a,b, \tag{A1}$$

$$\langle\varphi^a(x)\varphi^b(y)\rangle_c \leqslant C_2 G^2(x-y) \quad \text{for even } a,b, \tag{A2}$$

where C_1 and C_2 are some strictly positive constants and

$$G(x-y) = \langle\varphi(x)\varphi(y)\rangle. \tag{A3}$$

Note that for odd a and b, the connected and disconnected correlation functions are equal; see Eq. (26).

Property (A1) is obvious for $a = b = 1$. The proof of (A2) for $\{a = 2, b = 2\}$ is presented in the core of the article. For general a and b, the proof is made by induction. Assuming that the inequalities (A1) and (A2) are fulfilled for $\{a \leqslant m, b \leqslant n\} \setminus \{a = m, b = n\}$, we have to prove that these properties are also valid for $a = m$ and $b = n$.

We first consider the case in which m and n are even. Using the Lebowitz inequality [40] [see Eq. (33)],

$$\begin{aligned}&\langle[\varphi(x) + \varphi'(x)]^m[\varphi(y) - \varphi'(y)]^n\rangle \\ &\quad\leqslant \langle[\varphi(x) + \varphi'(x)]^m\rangle\langle[\varphi(y) - \varphi'(y)]^n\rangle,\end{aligned} \tag{A4}$$

as well as translation invariance and the binomial expansion, we obtain

$$\begin{aligned}\sum_{a=0}^{m}\sum_{b=0}^{n}(-1)^b\binom{m}{a}\binom{n}{b}[&\langle\varphi^a(x)\varphi^b(y)\rangle_c\langle\varphi^{m-a}(x)\varphi^{n-b}(y)\rangle_c \\ &+ \langle\varphi^a(0)\rangle\langle\varphi^b(0)\rangle\langle\varphi^{m-a}(x)\varphi^{n-b}(y)\rangle_c \\ &+ \langle\varphi^a(x)\varphi^b(y)\rangle_c\langle\varphi^{m-a}(0)\rangle\langle\varphi^{n-b}(0)\rangle] \leqslant 0.\end{aligned} \tag{A5}$$

Writing explicitly the terms with $a \in \{0,m\}$ and $b \in \{0,n\}$, we get the following bound:

$$\begin{aligned}2\langle\varphi^m(x)\varphi^n(y)\rangle_c \leqslant &\sum_{a\in\{1,3,\ldots,m-1\}}\sum_{b\in\{1,3,\ldots,n-1\}}\binom{m}{a}\binom{n}{b} \\ &\times \langle\varphi^a(x)\varphi^b(y)\rangle_c\langle\varphi^{m-a}(x)\varphi^{n-b}(y)\rangle_c,\end{aligned} \tag{A6}$$

where we have used the fact that connected correlation functions, as well as $\langle\varphi^m(0)\rangle$, are non-negative [44,45] to eliminate the terms with even b. We observe that there appears on the right-hand side only connected correlation functions with at most $m - 1$ powers of $\varphi(x)$ and at most $n - 1$ powers

[5] This bound also applies for correlation functions of more general operators.

of $\varphi(y)$. By hypothesis, properties (A1) and (A2) are thus valid for these correlation functions. Furthermore, we observe that the right-hand side involves a sum of terms that are a product of two correlation functions with odd powers of the fields. In both cases, by using properties (A1) and (A2), this quantity is bounded by some positive constant times $G^2(x-y)$.[6] This concludes the proof of the induction hypothesis in the case of even m and n.

We now turn to the case in which m and n are odd. We now make use of the Lebowitz inequality

$$\langle[\varphi(x)+\varphi'(x)]^{m-1}[\varphi(x)-\varphi'(x)][\varphi(y)-\varphi'(y)]^n\rangle \leqslant \langle[\varphi(x)+\varphi'(x)]^{m-1}\rangle\langle[\varphi(x)-\varphi'(x)][\varphi(y)-\varphi'(y)]^n\rangle, \tag{A7}$$

which is of interest if $m>1$ (we can obviously derive a similar inequality with $\{m,x\}\leftrightarrow\{n,y\}$, which can be applied in the case $m=1$). We again use the binomial expansion and the positivity of (connected and disconnected) correlation functions to obtain the following inequality:

$$\begin{aligned}\langle\varphi^m(x)\varphi^n(y)\rangle_c \leqslant& \sum_{a\in\{0,2,\dots,m-1\}}\sum_{b\in\{1,3,\dots,n-1\}}\binom{m-1}{a}\binom{n}{b}\langle\varphi^a(0)\rangle\langle\varphi^{m-a-1}(0)\rangle\langle\varphi(x)\varphi^b(y)\rangle_c\langle\varphi^{n-b}(0)\rangle\\ &+\sum_{a\in\{1,3,\dots,m-2\}}\sum_{b\in\{0,2,\dots,n\}}\binom{m-1}{a}\binom{n}{b}\langle\varphi^{a+1}(x)\varphi^b(y)\rangle\langle\varphi^{m-1-a}(x)\varphi^{n-b}(y)\rangle.\end{aligned} \tag{A8}$$

The first term involves the product of a correlation function with odd powers of the fields and a positive constant. The second sum involves either a product of two correlation functions, one with even and one with odd powers of the fields, or the product of a correlation function with odd powers of the fields and a positive constant. In all cases, the correlation functions that appear on the right-hand side fulfill the conditions of our hypothesis. We therefore conclude that $\langle\varphi^m(x)\varphi^n(y)\rangle_c$ satisfies property (A1) for m and n odd (see footnote 6). This concludes the proof of the induction hypothesis.

Using the fact that the property (A1) is valid for $a=b=1$, it is easy to check, by applying several times the induction property, that (A1) and (A2) are valid for any a and b.

[6] We use here the fact that, on the lattice, G is bounded by some positive constant $[G(r)<D]$. Consequently, we can further use the bound $G^{2n}(r)\leqslant CG^2(r)$, where n is a positive integer and C is a positive constant.

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