Quantum discord length is enhanced while entanglement length is not by introducing disorder in a spin chain

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Classical correlation functions of ground states typically decay exponentially and polynomially, respectively, for gapped and gapless short-range quantum spin systems. In such systems, entanglement decays exponentially even at the quantum critical points. However, quantum discord, an information-theoretic quantum correlation measure, survives long lattice distances. We investigate the effects of quenched disorder on quantum correlation lengths of quenched averaged entanglement and quantum discord, in the anisotropic XY and XYZ spin glass and random field chains. We find that there is virtually neither reduction nor enhancement in entanglement length while quantum discord length increases significantly with the introduction of the quenched disorder.

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I. INTRODUCTION

Correlations are important in physical systems from a fundamental perspective as well as from an application-oriented one [1–3]. Constituent subsystems of a physical system are often interacting, and in such systems the correlations, whether classical or quantum, between the constituent parts are almost invariably decaying with distance between the parts. The efforts to understand the sustainability of correlations in many body systems as a function of lattice distance happens to be crucial, since the low-temperature scaling of correlations is expected to remain universal, irrespective of the microscopic details of various materials governed by the same class of underlying models. In case of gapped systems with short-ranged interactions, using the Lieb-Robinson bounds [4], it can be shown that two-site classical correlation functions, $\langle\mathcal{O}_i\mathcal{O}_j\rangle - \langle\mathcal{O}_i\rangle\langle\mathcal{O}_j\rangle$, decay exponentially with increasing lattice distance, known as clustering of correlations [5,6], except when the systems are critical, in which they decay polynomially, with $\mathcal{O}_i$ and $\mathcal{O}_j$ being the observables at sites i and j, respectively. However, it has been reported that in several physical systems, quantum correlations, as qualified by entanglement, decay much more rapidly in comparison to classical correlators, both near and far from the quantum critical points. For example, in the quantum XY model with transverse magnetic field, entanglement can survive only up to the next nearest neighbor [7], while in the Heisenberg model, it becomes vanishingly small already after the nearest neighbor [8]. In comparison, the information-theoretic measures like quantum discord (QD) [9] and quantum work deficit (QWD) [10] can survive over a longer distance [11,12] in the homogeneous XY spin chain. It is important to note here that such spin models can be prepared in the laboratories using current technologies [13–15].

Establishing finite quantum correlations between distant parties is undoubtedly important to implement several quantum information processing tasks in many body systems. In this paper, we ask the following question: *Is it possible to enhance quantum correlation lengths, namely entanglement and QD lengths, significantly by introducing defects in quantum spin systems?* We find that the answer is in the affirmative. Disorder occurs unavoidably in real materials and can now also be engineered artificially in, e.g., cold atom experiments [16]. Moreover, it was shown that there exists some models, in which disorder plays a constructive role by enhancing physical properties like magnetization, classical correlators, and quantum correlations [17–19]. Such phenomena, known as "order-from-disorder" or "disorder-induced order," run certainly contrary to the naive belief that impurity in the systems can have only a debilitating effect.

In this work, we investigate the trends of quantum correlation lengths for the ground states of 1D quenched disordered anisotropic XY as well as XYZ spin models. Our results show that although the entanglement lengths cannot be improved or reduced by introducing disorder in the system, the lengths of information-theoretic quantum correlation measures can be substantially enhanced (specifically, more than doubled) in the presence of impurities. The features remain unaltered in the XYZ model for low values of the zz interactions. Higher values of the zz interaction, however, interferes destructively with the disordered couplings, to suppress the disorder-induced enhancement in QD length. As a consequence of the fact that QD remains finite between two arbitrary sites, we prove that QD cannot satisfy a large class of monogamy relations for the ground states of the homogeneous and disordered spin models for sufficiently large systems.

The rest of the paper is arranged as follows. Section II introduces the models under study and discusses the methods involved in solving them. Section III reviews various quantum correlation measures used in this work and introduces quantum correlation lengths. The results for the disordered XY and XYZ models are discussed in Secs. IV and V, respectively. Finally, we summarize our results in Sec. VI.

II. MODELS AND METHODOLOGY

In this section, we briefly describe the models that we use in this paper. Among the models described, note that the homogeneous XY spin model is exactly solvable and the analytic technique can be used to handle the corresponding disordered model, while the XYZ spin model with and without disorder cannot be solved analytically. We also present a brief description of quenched averaging.

A. Quenched disorder and averaging

The disorder in the system parameters are taken to be "quenched." That is, we assume that the time scale over which the system dynamics of interest takes place is much smaller compared to the time scale over which there is a change in the particular set of parameters governing the disorder in the system. In order to calculate the quenched averaged value of a physical quantity, we need to perform the averaging over the probability distribution of several realizations, each of which corresponds to a fixed configuration of the system, after calculating the value of the physical quantity for the fixed configurations.

B. Quantum XY spin chain: Homogeneous and disordered models

The general Hamiltonian for the quantum XY spin chain with nearest-neighbor interactions in an external magnetic field is given by

$$H = \kappa\left\{\sum_{i=1}^{N}\frac{J_i}{4}\left[(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y\right] - \sum_{i=1}^{N}\frac{h_i}{2}\sigma_i^z\right\}, \tag{1}$$

where κJ_i are the coupling constants, κh_i is the magnetic field strength at the ith site, and γ is the anisotropy constant. The constant κ has the units of energy, while J_i, h_i, and γ are dimensionless. Here σ^j, for $j = x,y,z$, correspond to the Pauli spin matrices. Moreover, we assume the periodic boundary condition, i.e., $\vec{\sigma}_{N+1} = \vec{\sigma}_1$.

Using the Jordan-Wigner transformations [20], it is possible to obtain the one- and two-site reduced density matrices for the disorder case. First, we map the Pauli spin operators to the spinless fermions via the Jordan-Wigner transformation, so that the Hamiltonian in Eq. (1) becomes

$$H = \kappa\left[\sum_{i,j=1}^{N} c_i^\dagger A_{ij} c_j + \frac{1}{2}\sum_{i,j=1}^{N}(c_i^\dagger B_{ij} c_{j+1}^\dagger + \text{H.c.})\right]. \tag{2}$$

Here A and B are symmetric and antisymmetric real $N \times N$ matrices, respectively, and are given by

$$A_{ij} = -h_i\delta_{ij} + \frac{J_i}{2}\delta_{i+1,j} + \frac{J_j}{2}\delta_{i,j+1},$$
$$B_{ij} = \frac{\gamma}{2}(J_i\delta_{i+1,j} - J_j\delta_{i,j+1}),$$

with $A_{1N} = A_{N1} = J_N$ and $B_{1N} = -B_{N1} = -\frac{\gamma}{2}J_N$ for the cyclic boundary condition. Here the $c_i^\dagger, c_i$ are spinless fermionic operators obtained via the Jordan-Wigner transformation. Note that the Hamiltonian in above Eq. (2) has a correction term proportional to $\exp(-i\pi\nu) + 1$, where ν is the sum of number operators over the sites. But for large systems, this correction term has a negligible contribution and we can then treat this as a "c-cyclic" problem. Defining Φ_k^T via the eigenequation

$$(A - B)(A + B)\Phi_k^T = \Lambda_k^2\Phi_k^T, \tag{3}$$

with eigenvalue Λ_k and obtaining the corresponding Ψ_k from the equation

$$\Psi_k^T = \Lambda_k^{-1}(A + B)\Phi_k^T, \tag{4}$$

we can calculate the correlation matrix G, defined as

$$G_{ij} = -\sum_k \psi_{ki}\phi_{kj} = -(\mathbf{\Psi}^T\mathbf{\Phi})_{ij}, \tag{5}$$

where $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are the matrices ϕ_{ki} and ψ_{ki}, with ϕ_{ki} (ψ_{ki}) being the ith element of Φ_k (Ψ_k). Finally, one can show that the magnetizations and two-point correlation functions of the zero-temperature state can be easily obtained from the correlation matrix G. We get $m_i^z = -G_{ii}$ and $m_i^x = m_i^y = 0$. The diagonal correlations are given by

$$T_{i,i+1}^{xx} = G_{i,i+1}, \tag{6}$$
$$T_{i,i+1}^{xx} = -G_{i+1,i}, \tag{7}$$
$$T_{i,i+1}^{zz} = G_{i,i}G_{i+1,i+1} - G_{i,i+1}G_{i+1,i}, \tag{8}$$

while all off-diagonal correlations vanish. The two-site reduced density matrix, ρ_{ij}, of the ground state can be easily constructed from the one- and two-point correlation functions as

$$\rho_{ij} = \frac{1}{4}\left[I\otimes I + m_i^z(\sigma^z\otimes I) + m_j^z(I\otimes\sigma^z) + \sum_{\alpha=x,y,z} T_{ij}^{\alpha\alpha}(\sigma^\alpha\otimes\sigma^\alpha)\right]. \tag{9}$$

C. Quantum XYZ spin glass

The general Hamiltonian for the quantum XYZ spin chain with nearest-neighbor interactions in an external magnetic field is given by

$$H = \kappa\left(\sum_i\left\{\frac{J_i}{4}\left[(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y\right] + \frac{\Delta}{4}\sigma_i^z\sigma_{i+1}^z\right\} - \frac{h}{2}\sum_i\sigma_i^z\right), \tag{10}$$

where κJ_i, κh, and $\kappa\Delta$ are, respectively, the coupling constants at the ith site, the magnetic field strength, and the nearest-neighbor coupling strength for the zz interaction. Here γ is the anisotropy constant. The constant κ has the units of energy, while J_i, h, Δ, and γ are dimensionless. Here our interest lies in the quenched averaged correlation lengths in the XYZ spin chain. The homogeneous quantum XYZ spin chain can be obtained from Eq. (10) by simply setting $J_i = J$ for $i = 1,\ldots,N$.

The Hamiltonian for the quantum XY spin glass is obtained by setting $\Delta = 0$ in Eq. (10). The coupling strengths J_i are randomly chosen from independently and identically distributed (i.i.d.) Gaussian distributions with mean $\langle J\rangle$ and unit standard deviation.

Unlike the quantum XY disordered chain, one needs to resort to numerical techniques for both the homogeneous XYZ chain and the corresponding disordered system with random coupling strengths. In order to investigate the ground state for the system characterized by the Hamiltonian in Eq. (10), we employ the well-established numerical technique called the density-matrix renormalization group (DMRG) method [21]. After performing the standard infinite size DMRG, several finite size DMRG sweeps are also carried out in order to increase the accuracy of the calculations for the inhomogeneous chain.

III. QUANTUM CORRELATION MEASURES

The presence of quantum correlations between subsystems of a composite system helps in realizing many quantum information protocols. In order to explore these protocols, it is necessary to quantify the quantum correlations involved. In this work, we have mainly used two quantum correlations measures, namely, concurrence and QD. They belong to two different paradigms of quantum correlation; while the first corresponds to the entanglement-separability paradigm, the other is in the information-theoretic one. In the following sections, we briefly introduce both the quantum correlation measures considered here. We then provide a short introduction of the concept of quantum correlation length.

A. Concurrence

Concurrence [22] quantifies the amount of entanglement present in an arbitrary two-qubit state. Given a two-qubit density matrix, ρ_{AB}, the concurrence is defined as

$$C(\rho_{AB}) = \max\{0,\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{11}$$

where λ_i's are the eigenvalues of the Hermitian matrix $R = \sqrt{\sqrt{\rho}\widetilde{\rho}\sqrt{\rho}}$ and satisfy the order $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$. Here $\widetilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, with ρ^* being the complex conjugate of ρ in the computational basis.

B. Quantum discord

In classical information theory, the amount of ignorance about a probability distribution, p_i, is quantified by the Shannon entropy, defined as $H(\{p_i\}) = -\sum_i p_i \log_2 p_i$. The mutual information between two classical random variables i and j, having the marginal distributions $\{p_i\}$ and $\{p_j\}$, can be defined in two equivalent ways as

$$\begin{aligned}\mathcal{I}(\{p_{ij}\}) &= H(\{p_i\}) + H(\{p_j\}) - H(\{p_{ij}\}) \\ &= H(\{p_i\}) - H(\{p_{i|j}\}),\end{aligned} \tag{12}$$

where $\{p_{ij}\}$ and $\{p_{i|j}\}$ correspond to the joint probability distribution of the variables i and j, and the conditional probability distribution, respectively. In case of quantum systems, the quantity

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{13}$$

for a two-party quantum system ρ_{AB}, can be argued to quantify the total correlation present in the system where $S(\sigma) = -\text{tr}[\sigma \log_2 \sigma]$ and ρ_i, $i = A,B$ are the local density matrices of ρ_{AB} [23–25]. This quantity can be interpreted as the quantized version of the first expression of the classical mutual information in Eq. (12). To quantize the other expression, we consider that a measurement is performed on one party, say B, using a complete set of rank 1 projectors, $\{B_i\}$, satisfying the relations $B_iB_j = \delta_{ij}B_i$ and $\sum_i B_i = I_B$. The postmeasurement ensemble is given by $\{p_i,\rho^i_{AB}\}$, with $\rho^i_{AB} = (I_A \otimes B_i)\rho(I_A \otimes B_i)$ and $p_i = \text{tr}[(I_A \otimes B_i)\rho_{AB}(I_A \otimes B_i)]$, where I_A is the identity operator on the Hilbert space of the system with observer A. The corresponding quantum conditional entropy is given by

$$S(\rho_{A|B}) = \min_{\{B_i\}} \sum_i p_i S(\rho_{A|i}), \tag{14}$$

where $\rho_{A|i} = \text{tr}_B \rho^i_{AB}$. The quantum version of the second expression in Eq. (12) of classical mutual information then reads

$$\mathcal{J}(\rho_{AB}) = S(\rho_A) - S(\rho_{A|B}), \tag{15}$$

which turns out to be inequivalent to the expression in Eq. (13) and can be argued as a measure of classical correlation of the state ρ_{AB}. Quantum discord [9] is defined as the difference between these two inequivalent quantities and is given by

$$\mathcal{D}(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - \mathcal{J}(\rho_{AB}). \tag{16}$$

In all the cases considered in this paper, the bipartite states are X states [26] with $|T^{xx}_{ij}| \geqslant |T^{yy}_{ij}|$, for which an analytical form of QD is available [27–30]. We have also checked the claim numerically. When the measurement is performed by the first party, i.e., on the ith spin of a two-party state ρ_{ij}, the classical correlation is given by $\mathcal{J}(\rho_{i,j}) = H_2(\frac{1+m^z_i}{2}) - H_2(\frac{1+p}{2})$, with $p = \sqrt{(m^z_i)^2 + (T^{xx}_{i,j})^2}$, and $H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$ being the binary entropy for $0 \leqslant x \leqslant 1$.

C. Quantum correlation length

As mentioned earlier, our primary aim is to investigate quantum correlation lengths in quenched disordered XY and XYZ spin models. Conventionally, the classical correlation length, ξ_C, associated with the classical correlation function, C, can be defined as $\xi_C^{-1} = \lim_{|i-j|\to\infty} \frac{(-\log_e\langle C_{i,j}\rangle)}{|i-j|}$, where $C_{i,j}$ is a classical correlation function between the sites i and j. However, as we discuss in subsequent sections, if we wish to characterize the many-body systems by using the corresponding quantum correlation lengths, it is useful to define the same as follow. For arbitrary homogeneous spin systems, let us first define the quantum correlation length. If the quantum correlation (QC), $Q_{i,j}$, between the ith and jth spins behaves as

$$Q_{i,j} = a + be^{-\frac{|i-j|}{\xi_Q}}, \tag{17}$$

then ξ_Q denotes the Q length, with a, b being arbitrary constants. It turns out that for concurrence, $a = 0$ since pairwise concurrence always vanishes at large distance. For quenched disordered systems, the QC length for the quenched averaged two-site QC measures, if the latter behaves as in Eq. (17), will be denoted by $\xi_{\langle Q\rangle}$. Note that this is different from the concept of *localizable entanglement*, the amount of entanglement that can be concentrated between two parties through measurements performed on the rest of the parties [31] (see also [32–34]).

IV. ENTANGLEMENT AND DISCORD LENGTHS IN ANISOTROPIC XY MODEL

In order to make a comparison between the disordered and homogeneous systems, we fix the means of the distributions of the disordered parameters in the disordered systems to be identical to the corresponding parameters of the homogeneous system. In particular, the quenched averaged physical quantity, $Q_{av}(\langle a\rangle,\langle b\rangle,\dots)$, corresponds to the disordered system with disordered parameters $a,b,\dots$ having means $\langle a\rangle,\langle b\rangle,\dots$, respectively. The corresponding physical quantity for the homogeneous system is then $Q(\langle a\rangle,\langle b\rangle,\dots)$, where the values of the system parameters $a,b,\dots$ are kept constant at $\langle a\rangle,\langle b\rangle,\dots$, respectively.

In the case of the homogeneous XY model, the ground state is multipartite entangled except at $J/h = 1/\sqrt{1-\gamma^2}$, which is known as the factorization point [35,36]. At this point, the ground state is doubly degenerate and both the degenerate states are factorized as a tensor product of quantum states corresponding to all the individual spins. Since we have taken the ground state to be a pure symmetric state by taking equal superposition of both the degenerate ground states, the two spin entanglement here vanishes at this point, while QD may have a nonzero value.

A. Anisotropic quantum XY spin glass: Entanglement length vs discord length

Let us first consider the quantum XY spin glass with N sites. All considerations are for the pure symmetric ground state. For site-independent coupling constants $J_i = J$ for $i = 1,\dots,N$, the system becomes the homogeneous quantum XY spin chain, which is exactly solvable via successive applications of the Jordan-Wigner, the Fourier, and the Bogoliubov transformations [20,37,38]. The corresponding disordered systems can also be handled up to relatively large system sizes by using the same transformations and one can obtain the one- and two-point correlation functions to generate the two-site reduced density matrix, using which one can eventually compute quantities like the concurrence and QD [39,40].

1. *Entanglement length*

We compare the two-site concurrence between the sites i and j of the homogeneous system, described by $H(J,h)$, with the quenched two-site concurrence between the same sites for the XY spin-glass system with $\langle J\rangle/h = J/h$. From Fig. 1, we notice that the concurrence for the homogeneous XY model vanishes for $|i-j| \gtrsim 3$, while in the disordered system it goes to zero for $|i-j| \gtrsim 4$, implying no significant enhancement or deterioration due to randomness. To compare entanglement lengths between systems with and without disorder, we find that, for example, with $\langle J\rangle/h = 0.5$, $\xi_{\langle C\rangle} = 0.69$ in the disordered case, whereas $\xi_C = 0.50$ for the homogeneous one. The numerical simulations seem to indicate the following: Away from criticality, the values of concurrence, for all pairs (i,j), are higher in the disordered system as compared to the corresponding homogeneous system, signaling order from disorder [17–19] as depicted in Figs. 1(a) and 1(d). The roles are reversed as we approach the quantum critical point [see Fig. 1(b)], except when the factorization point is also nearby [see Fig. 1(c)] (cf. [35,36]).

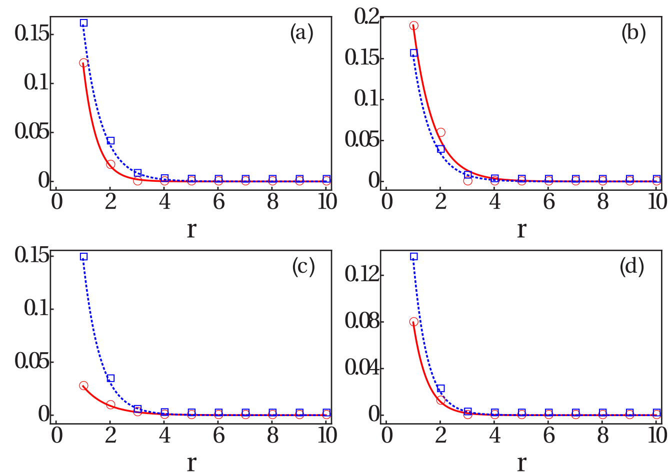

FIG. 1. Entanglement length in quantum XY spin glass vs the homogeneous XY model. In each panel, the blue squares denote the quenched averaged concurrence, $\langle C_{i,j}\rangle$, between sites i and j, plotted against the lattice distance $r = |i-j|$, for the quantum XY spin glass for $\langle J\rangle/h = 0.5$ (a), 0.9 (b), 1.1 (c), and 1.5 (d). The red circles denote corresponding homogeneous systems. The dotted and solid lines show the exponential fits for disordered and homogeneous systems, respectively. The vertical axis denotes the concurrences, while the horizontal ones denote the lattice distances. Here $N = 50$ and $\gamma = 0.5$. For the disordered case, the number of realizations of the random coupling is taken to be 10^4, for the quenched averaging. The lines are exponential fits and the obtained data at integer values of r. The vertical axis are measured in ebits, while the horizontal axis are in lattice length units.

Note that although the panels in Fig. 1 are plotted for system size $N = 50$, we have checked that increasing system size does not change the behavior of the entanglements and hence the entanglement lengths for both disordered and homogeneous systems.

2. *Discord length*

We now show that QD length behaves in a qualitatively different way than the entanglement length both in homogeneous and disordered systems. To investigate it, we divide the entire range of $\langle J\rangle/h$ (for the spin-glass system, which is the same as J/h for the homogeneous system) into three portions, namely $0 < \langle J\rangle/h < \lambda_c$, $\langle J\rangle/h > \lambda_c$, and the neighborhood of λ_f, with λ_c and λ_f being the quantum critical and the factorization points of the homogeneous system, respectively. Since we investigate the system by varying $\langle J\rangle/h$, the factorization point lies always in the second region and hence the behavior of discord length in the second region discussed below is excluding the neighborhood of factorization point. The whole discussion will be carried out for $\gamma = 0.5$ and hence the factorization point is $\lambda_f = 1.1547$. The quantitative feature remain unchanged for other values of γ.

Case when $\langle J\rangle/h < \lambda_c = 1$. In the homogeneous system, QD $D_{i,j}$, decays exponentially as in Eq. (17) [11,12]. In the XY spin-glass model, QD also decays exponentially, but with a different decay rate. As an exemplary case, let us

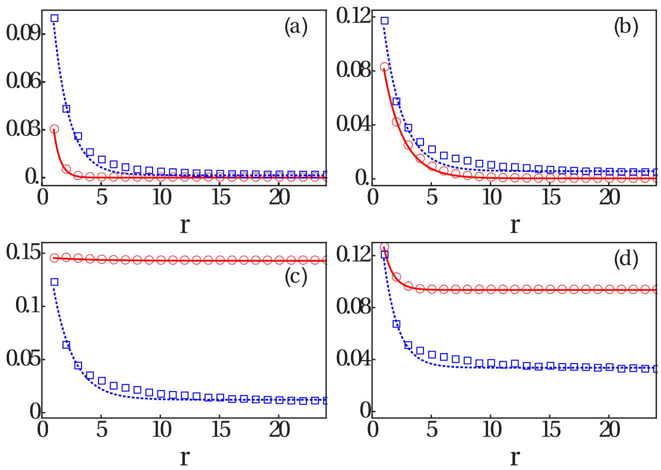

FIG. 2. All considerations here are the same as in Fig. 1, except that QD is considered instead of concurrence and that the former is measured in ebits.

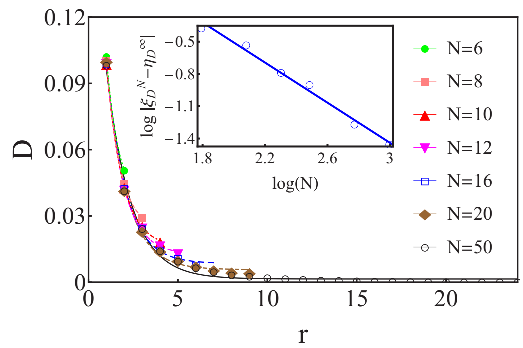

FIG. 3. Scalings of QD length for the 1D quantum XY spin glass. We plot the quenched averaged QD on the vertical axis against lattice distance on the horizonal one for different values of N. In the main figure, the vertical axis is in bits, while the horizontal one is in lattice length units. In the inset, the unit of the horizontal and vertical axes are, respectively, logarithms of the total numbers of lattice sites and of lattice length. The logarithms are with base e.

consider $\langle J\rangle/h = 0.5$ for which the quenched averaged QD of the XY spin-glass model behaves as $\langle D_{i,j}\rangle = a + be^{-\frac{r}{\xi_{\langle D\rangle}}}$, with $a = 1.4\times10^{-3}$, $b = 0.20$, $\xi_{\langle D\rangle} = 1.36$, while $a = 4.1\times10^{-3}$, $b = 0.18$, $\xi_D = 0.56$, for the homogeneous XY model, implying $\xi_{\langle D\rangle} = 2.4\xi_D$. Therefore, unlike entanglement, we observe significant enhancement of discord length in the disordered system as also depicted in Fig. 2. The increment of length by introducing randomness in the system can therefore only be viewed for QC measures which are different from entanglement. Moreover, we find that in this region, $\langle D_{i,j}\rangle > D_{i,j}$, exhibiting thereby an order-from-disorder phenomenon.

Case when $\langle J\rangle/h > \lambda_c = 1$. In this antiferromagnetic phase, QD saturates to a constant value and hence indicates long-range order in the system even in the thermodynamic limit as also predicted in Refs. [11,12]. For example, if one fixes J/h as 1.5, QD of the XY model without disorder behaves as in Eq. (17) with $a = 0.093$, $b = 0.115$, and $\xi_D = 0.80$. Note that we chose $J/h = 1.5$ since we are interested in the behavior of discord length which is far from λ_f, which is 1.1594 in this case. In the XY spin-glass model, quenched averaged QD again shows long-range order. Specifically, after an initial decay, it saturates to a constant value. We find $\xi_{\langle D\rangle} = 1.21 > \xi_D = 0.80$ for $\langle J\rangle/h = 1.5$, although the order-from-disorder phenomenon is absent [see Fig. 2(d)].

Neighborhood of $\lambda_f = 1/\sqrt{1-\gamma^2}$. At the factorization point, QD remains constant for all pairs of (i,j) for the homogeneous system and hence ξ_D goes to ∞ for all nontrivial b. In contrast, discord length is finite in the disordered case. Therefore, at the factorization point, $\xi_D > \xi_{\langle D\rangle}$. The decrement of discord length due to disorder can also be observed in the vicinity of the factorization point. Hence, enhancement of discord length is seen in the entire region of $\langle J\rangle/h$ except at the neighborhood of the factorization point.

As a by-product, in both the homogeneous and the disordered systems, we can prove that QD does not follow any monogamy relation [41] for arbitrary N, in any of the three regions [42]. Quantitatively, for a given N-party state $\rho_{1,\ldots,N}$, a bipartite QC measure $\mathcal{Q}$ is said to be monogamous, with the party 1 as the "nodal observer," if $\mathcal{Q}(\rho_{1:2,\ldots,N}) \geqslant \sum_{i=2}^{N}\mathcal{Q}(\rho_{1i})$, where $\mathcal{Q}(\rho_{1:2,\ldots,N})$, and $\mathcal{Q}(\rho_{1i})$ are, respectively, the QC in the 1 : rest and the 1 : i bipartition. Suppose, if possible, that the QD of the ground state satisfies monogamy. That would imply $1 \geqslant \mathcal{D}(\rho_{1,\ldots,N}) \geqslant \sum\mathcal{D}(\rho_{1i}) \geqslant \cdots \geqslant N\mathcal{D}(\rho_{1N})$. The first and the last inequalities are due to the fact that each local system is a qubit and $\mathcal{D}(\rho_{12}) \geqslant \mathcal{D}(\rho_{13}) \geqslant \mathcal{D}(\rho_{1N})$, respectively. As argued, $\mathcal{D}(\rho_{1N})$ can tend to a nonzero constant as $N\to\infty$, in both the homogeneous and the disordered systems. Therefore, in the thermodynamic limit, $N\mathcal{D}(\rho_{1N})\to\infty$, giving us a contradiction. Since $D(\rho_{1N})$ can have a nonzero value for large N, it is easy to see that any monogamy-type relations would be violated for QD for those states for sufficiently large N [43,44]. In particular, a similar argument will imply that also the squares of QD also cannot be monogamous for these states with sufficiently large N [45].

It is also interesting to check the behavior of discord length with different N. We observe that the behavior of QD freezes for $N \geqslant 50$ and hence we can safely assume that the results obtained for $N = 50$ will mimic those of an infinite spin chain (see Fig. 3). Therefore, we take $\xi_{\langle D\rangle}^{N=50} = \xi_{\langle D\rangle}^{\infty}$ and we find that $\xi_{\langle D\rangle}$ scales as $N^{-0.932}$.

B. Random field quantum XY spin chain

Here we consider an N-site quantum XY spin chain with uniform nearest-neighbor exchange interactions, $J_i = J$, but with field strengths, h_i, randomly chosen from i.i.d., the Gaussian probability distributions with mean $\langle h\rangle$, and unit standard deviation. In the corresponding homogeneous system, h_i assumes a constant value, $\langle h\rangle$, at each site. Again we consider the pure symmetric ground state.

The panels of Figs. 4 and 5 illustrate the features of concurrence and QD, respectively, for different choices of $\langle h\rangle$, against r. We find that the physics remain qualitatively unchanged in the random field XY chain if one compares with the XY spin glass model. We observe that disorder, in general, does not help in establishing long-range entanglement, while it can significantly increase the other QC length, specifically discord

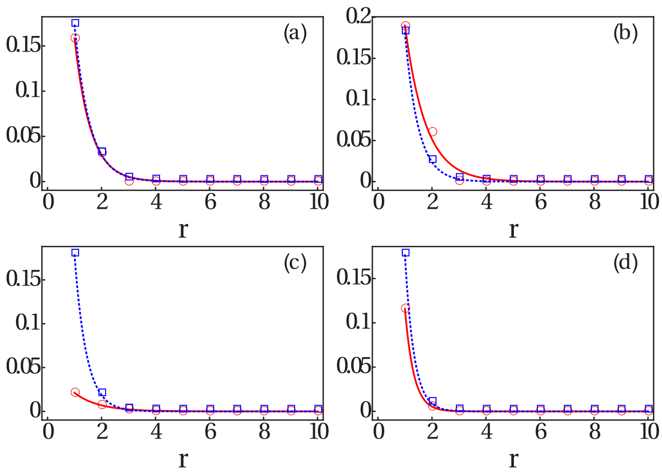

FIG. 4. Entanglement length in random field quantum XY spin chain vs homogeneous XY chain for $J/\langle h\rangle =$ (a) 0.67, (b) 0.91, (c) 1.1, and (d) 2.0. All considerations, except for the model considered and for the fact that the different panels are now for different values of $J/\langle h\rangle$, remain the same as in Fig. 1.

length, almost everywhere in the parameter space except near the factorization point. As observed for the XY spin–glass model, although discord length gets enhanced due to disorder, the value of quenched averaged QD decreases in the region $J/\langle h\rangle > 1$. On the other hand, the value of entanglement increases in this region, showing a complementarity between the two types of QC measures.

V. ENTANGLEMENT LENGTH AND DISCORD LENGTH IN XYZ SPIN GLASS

Our results showed that even though the disorder-driven systems are only minimally benefited in terms of the enhancement in concurrence length, a noteworthy endowment occurs in discord length. It is natural to inquire whether the findings are generic in one dimensional systems. Specifically, one can extend the analysis to the disordered quantum XYZ spin glass [see Eq. (10)].

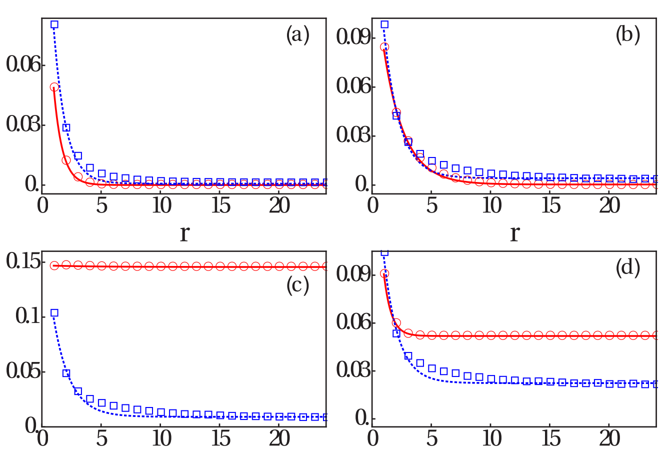

FIG. 5. All considerations here are the same as in Fig. 4, except that QD is considered instead of concurrence and that the former is measured in ebits.

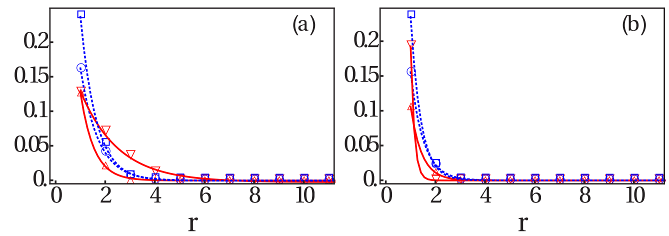

FIG. 6. The homogeneous and disordered XYZ models. Scaling of entanglement as a function of distance for the homogeneous and disordered spin chains with $N = 24$ and $\gamma = 0.5$ for $\langle J\rangle/h =$ (a) 0.5 and (b) 1.5. The vertical axes denote the concurrences, while the horizontal ones denote the lattice distances. The up and down triangles correspond to the homogeneous system with $\Delta = 0.1$ and $\Delta = 0.5$, respectively. The circles and the squares correspond to the disordered system for $\Delta = 0.1$ and $\Delta = 0.5$, respectively. The solid and dotted lines show exponential fits for different cases. For the disordered case, the number of random realizations taken is 8000.

In order to obtain the ground state of the XYZ spin chain, we employ the DMRG technique [21], which is best suited for studying the spin chains with open boundary conditions in order to achieve high accuracy[1] [46]. However, with the open boundary condition, the drawback is that the measurement of the observables on the fringes would experience boundary effects. In order to investigate the correlations encapsulated between two sites, we consider the central spin on the $(N/2)$th site and another site which is positioned at a distance r from the $(N/2)$th site, but is still far from the boundary.

We notice that in the homogeneous system, increasing Δ raises the concurrence length for $\langle J\rangle/h < 1$, while there is no notable change in the length, when $\langle J\rangle/h > 1$ (see Fig. 6). The situation is true also when disorder in introduced in the system. However, the disordered system fares better than the homogeneous system in the region $\langle J\rangle/h > 1$ compared to that of the $\langle J\rangle/h < 1$ for higher values of Δ. Similar to the XY model, QD behaves quite differently than entanglement in the XYZ model with small values of Δ, essentially mimicking the results obtained for the XY model. Enhancements of discord length are observed both in $\langle J\rangle/h < 1$ and $\langle J\rangle/h > 1$ regions in the presence of disorder.

However, in the region $\langle J\rangle/h < 1$, the advantage of discord length obtained in the XY spin glass over the corresponding homogeneous system is faded out with increase of the zz interaction. For fixed $\langle J\rangle/h$, with the increase of Δ, the discord length decreases in the disordered system. This is illustrated in Table I, where we consider a chain of 24 spins. See also Fig. 7.

[1]This work has been developed by using the DMRG code released within the "Powder with Power" project.

TABLE I. Comparison of discord length for both the homogeneous and the disordered systems in quantum XYZ model for different values of $\langle J\rangle/h$ and Δ/h, with $N = 24$.

	$\Delta/h = 0.1$	$\Delta/h = 0.5$
$\langle J\rangle/h = 0.5$	$\xi_D = 0.64$	$\xi_D = 4.05$
	$\xi_{\langle D\rangle} = 1.26$	$\xi_{\langle D\rangle} = 0.86$
$\langle J\rangle/h = 1.5$	$\xi_D = 1.04$	$\xi_D = 0.68$
	$\xi_{\langle D\rangle} = 1.26$	$\xi_{\langle D\rangle} = 0.73$

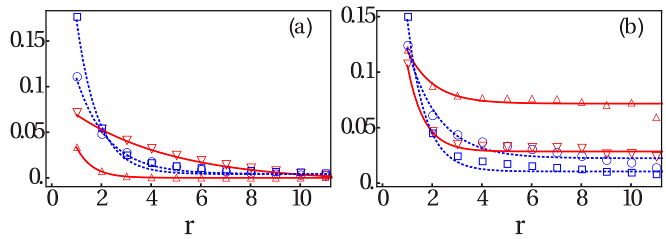

FIG. 7. All considerations here are the same as in Fig. 6, except that QD is considered instead of concurrence and that the vertical axes are measured in bits.

VI. CONCLUSION

In summary, we have studied the effects of disorder on QC lengths in spin chains. The paradigmatic models considered are XY spin glass in which coupling strengths are chosen randomly, the XY spin chain with random field, and the XYZ model with random xx and yy couplings. We find that entanglement length shows neither significant reduction nor enhancement with the introduction of disorder in the system. In sharp contrast, the discord length is significantly higher in the XY disordered models in comparison to the corresponding homogeneous ones. The results remain unchanged in the XYZ model for low values of the zz interaction.

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