Reply to "Comment on 'Statistical symmetries of the Lundgren-Monin-Novikov hierarchy'"

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In this Reply we respond to the criticism of Frewer *et al.* presented in the Comment on "Statistical symmetries of the Lundgren-Monin-Novikov hierarchy" by Wacławczyk *et al.* [Phys. Rev. E **90**, 013022 (2014)]. We discuss physical interpretation of the statistical symmetries, and respond to criticism on the violation of the causality principle. We derive the Lundgren-Monin-Novikov equations for a flow with boundaries. Last, we stress that our work addressed the phenomenon of "external intermittency" (separation between laminar and turbulent flow), and not "internal intermittency" (strong fluctuations at small scales).

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I. INTRODUCTION

We understand that besides less important points, the key criticism of Frewer *et al.* is that the statistical symmetries derived in Wacławczyk *et al.* [1] are unphysical, due to the fact that, although the probability density functions (PDFs) f_n transform, the sample space $\boldsymbol{v}_{(i)}$ remains invariant. According to Frewer *et al.* the new symmetries violate the principle of causality.

We agree with some of the minor points of the comment by Frewer *et al.*, and correct some of our statements. We also explain that the issue of boundary conditions raised in [2] is not a concern, by deriving the PDF equation in the presence of boundaries. Still, on the whole we do not at all agree with the conclusions of the comment. As we will show below, the proposed statistical symmetries are physically meaningful and the causality principle is not violated. In the following we would like to explain our view in more detail with a point-by-point reply to the criticism outlined in the Comment.

II. REPLY TO: VIOLATION OF THE CAUSALITY PRINCIPLE

In Ref. [1] we derived transformations of PDFs which follow from the scaling and translation transformations of multipoint correlation (MPC) equations from [3,4]. For the PDFs these transformations have the following form:

$$f_n^* = e^{a_s} f_n + (1 - e^{a_s})\delta(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(2)})\dots \cdots\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(n)}) \quad (1)$$

called the "intermittency symmetry" and

$$f_n^* = f_n + \psi(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(2)})\cdots\delta(\boldsymbol{v}_{(1)} - \boldsymbol{v}_{(n)}), \quad (2)$$

where $\int \psi \, d\boldsymbol{v}_{(1)} = 0$, called the "shape symmetry."

The authors of the *Comment on "Statistical symmetries of the Lundgren-Monin-Novikov hierarchy"* claim that the statistical transformations derived in Ref. [1] are unphysical as "deterministic equations due to their spatially nonlocal and temporally chaotic behavior induce the statistical equations, and not vice versa." The same criticism appears in another contribution of Frewer *et al.* [5].

Before we reply to this part of the Comment we would first like to explain the notion of "statistical symmetry." Such name has in fact been used in other areas of science, e.g., in the study of dynamical systems [6], statistical physics [7], or even sensory coding [8], to name only a few. The common observation of these works seems to be that even if one particular field (or image) does not verify the symmetry, it can be observed over a large ensemble of fields (or images). As an example, let us consider a steady laminar Poiseuille flow in a channel which is invariant under reflection about the center plane $y = 0$. This symmetry is broken when the flow becomes unsteady, i.e., $U(x,y,z) \neq U(x,-y,z)$; however, it is once again recovered for the ensemble averaged velocity, $\langle U\rangle(y) = \langle U\rangle(-y)$. The reason is that the instantaneous solutions $U(x,y,z)$ and $U(x,-y,z)$ appear in the ensemble with equal probabilities. In Ref. [1], by "statistical symmetry" we mean, generally, transformations of statistics (translations, reflections, scaling, etc.) which leave the considered hierarchy of equations invariant, which is the case for the transformations (1) and (2).

In Ref. [1] we claimed that constant fields (or laminar fields) can be solutions of the governing Navier-Stokes equations and that the transformations (1) and (2) are connected with the presence of such fields in the ensemble. Contrary to the criticism of Frewer *et al.*, we did not claim that the statistical equations induce anything on deterministic equations. We rather observed that the deterministic Navier-Stokes equations, due to the possibly different nature of their solutions (turbulent or nonturbulent), caused by small changes in the initial and boundary conditions, induced transformation of the PDF functions (1) and (2) and respective transformations of velocity moments.

In Ref. [1] we first considered statistical turbulence motion in free space without boundaries. If f_n is a PDF of a turbulent flow [cf. Fig. 1(a)], both transformations, first (1) and next (2) with $\psi = \delta(\boldsymbol{v}_{(1)} - \boldsymbol{U}_0) - \delta(\boldsymbol{v}_{(1)})$ transform f_n into a PDF of an intermittent flow with a certain share of constant fields in the ensemble

$$f_n^* = e^{a_s} f_n + (1 - e^{a_s})\delta(\boldsymbol{v}_{(1)} - \boldsymbol{U}_0)\cdots\delta(\boldsymbol{v}_{(n)} - \boldsymbol{U}_0) \quad (3)$$

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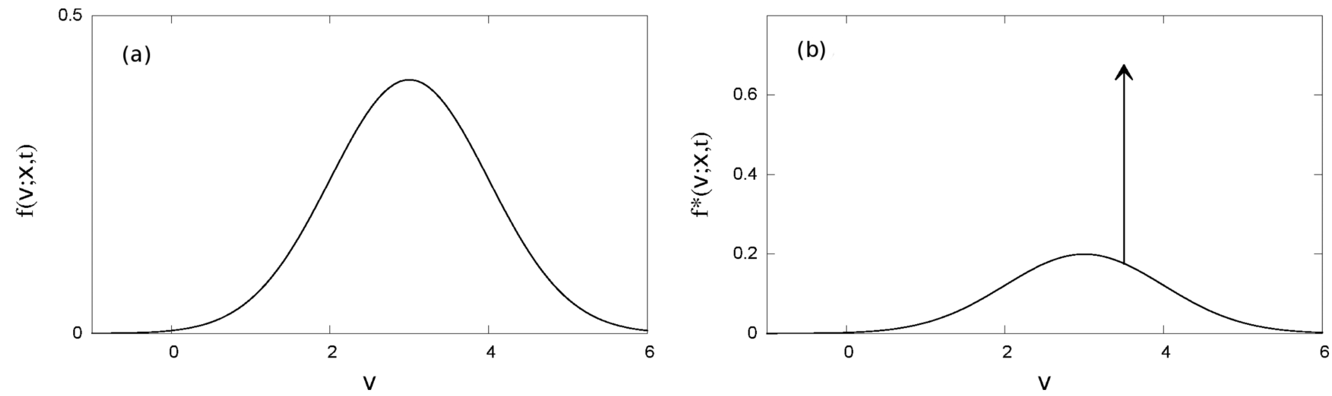

FIG. 1. (a) Turbulent PDF f_1. (b) Transformed PDF $f^*(\boldsymbol{v}) = e^{a_s} f_1 + (1 - e^{a_s})\delta(\boldsymbol{v} - \boldsymbol{U}_0)$.

[cf. Fig. 1(b)], where $\boldsymbol{U}_0$ is a laminar velocity. After such a transformation the moment statistics of velocities change to

$$\begin{aligned}\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots\ U_{i_{(n)}}(\boldsymbol{x}_{(n)},t)\rangle^* &\\ = e^{a_s}\langle U_{i_{(1)}}(\boldsymbol{x}_{(1)},t)\cdots U_{i_{(n)}}(\boldsymbol{x}_{(n)},t)\rangle &\\ +(1-e^{a_s})U_{0i_{(1)}}\cdots U_{0i_{(n)}}, &\end{aligned} \tag{4}$$

i.e., they are rescaled by the factor e^{a_s} (the same for all moments of any order) and translated.

Contrary to the claim of Frewer *et al.*, the definition

$$f_n(\boldsymbol{v}_{(1)},\ldots,\boldsymbol{v}_{(n)}) = \left\langle \prod_{i=1}^{n} \delta[\boldsymbol{v}_{(i)} - \boldsymbol{U}(\boldsymbol{x}_{(i)},t)] \right\rangle \tag{5}$$

still applies to both f_n and f_n^* described by Eq. (3). Here, the single velocity fields $\boldsymbol{v}$ and $\boldsymbol{U}$ are not rescaled but the share of laminar solutions in the ensemble changes. This could be caused, e.g., by the change of initial conditions or external disturbances, hence the claim of Frewer *et al.* that "no cause at all exists from which (1) or (2) can emerge as a symmetry transformation" is not supported.

Lastly, Frewer *et al.* notice that "the cause itself need not to be a symmetry in order to induce a symmetry as an effect." This statement actually supports our result. Possibly different nature of solutions to the deterministic equations (turbulent or nonturbulent) induce an effect: scaling and translational symmetries of the velocity moments.

III. REPLY TO: NONCOMPATIBILITY WITH ALL LUNDGREN-MONIN-NOVIKO (LMN) CONSTRAINTS

Another central criticism of Frewer *et al.* is the incompatibility of the proposed statistical symmetries with the separation constraint $f_2(\boldsymbol{v}_{(1)},\boldsymbol{v}_{(2)};\boldsymbol{x}_{(1)},\boldsymbol{x}_{(2)},t) = f_1(\boldsymbol{v}_{(1)};\boldsymbol{x}_{(1)},t)f_1(\boldsymbol{v}_{(2)};\boldsymbol{x}_{(2)},t)$ for $|\boldsymbol{x}_{(1)} - \boldsymbol{x}_{(2)}| \to \infty$. We agree with this criticism. However, we have already explicitly noticed this inconsistency in our paper. In fact, we furthermore noticed in Ref. [1] that this property "is not satisfied by the corresponding symmetries of the MPC equations," which was supposed to be an observation, not an argument. We also discussed the case of a (purely) turbulent flow and end up with the conclusion that the separation constraint would reduce the "shape symmetry" (2) to the identity transformation $f_n^* = f_n$. Next, however, we presented the PDF as a sum of the laminar and turbulent parts $f = g_L + g_T$ and argued that the shape symmetry (2) can only modify the laminar part of the PDF. If we consider the PDF of the form (3), the two-point PDF in the limit $|\boldsymbol{x}_{(1)} - \boldsymbol{x}_{(2)}| \to \infty$ reads

$$\begin{aligned}\lim_{|\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(2)}|\to\infty} f_2^* &= e^{a_s} f_1(\boldsymbol{v}_{(1)};\boldsymbol{x}_{(1)},t) f_1(\boldsymbol{v}_{(2)};\boldsymbol{x}_{(2)},t)\\ &\quad + (1-e^{a_s})\delta(\boldsymbol{v}_{(1)} - \boldsymbol{U}_0)\delta(\boldsymbol{v}_{(2)} - \boldsymbol{U}_0).\end{aligned} \tag{6}$$

The moments of velocity calculated from (6) have the form

$$\begin{aligned}\langle U_i(\boldsymbol{x}_{(1)},t)\rangle^* &= e^{a_s}\langle U_i(\boldsymbol{x}_{(1)},t)\rangle + (1-e^{a_s})U_{0i},\\ \langle U_i(\boldsymbol{x}_{(1)},t)U_j(\boldsymbol{x}_{(2)},t)\rangle^* &= e^{a_s}\langle U_i(\boldsymbol{x}_{(1)},t)\rangle\langle U_j(\boldsymbol{x}_{(2)},t)\rangle\\ &\quad + (1-e^{a_s})U_{0i}U_{0j}\end{aligned}$$

and $\langle U_i(\boldsymbol{x}_{(1)},t)U_j(\boldsymbol{x}_{(2)},t)\rangle^* \neq \langle U_i(\boldsymbol{x}_{(1)},t)\rangle^*\langle U_j(\boldsymbol{x}_{(1)},t)\rangle^*$. Hence, although the intermittent PDFs are observed in nature, the separation constraint is not satisfied due to the presence of constant laminar fields in the ensemble which introduces correlation between different points in the flow even for $|\boldsymbol{x}_{(1)} - \boldsymbol{x}_{(2)}| \to \infty$. We also note here that when we consider the laminar and turbulent parts of the PDF separately, the separation constraint is satisfied for each of them.

IV. REPLY TO: ON THE NEW "SHAPE SYMMETRY" IN CHANNEL FLOW

In the paper we argued that due to the interpretation of both symmetries, as connected with the laminar-turbulent flows in the case of channel flow, these symmetries would have a different form. The PDF of a laminar flow in the channel reads

$$\begin{aligned}f_L(\boldsymbol{v}_{(1)},\ldots,\boldsymbol{v}_{(n)};\boldsymbol{x}_{(1)},\ldots,\boldsymbol{x}_{(n)},t)&\\ = \left\langle \prod_{j=1}^{n} \delta\left[\boldsymbol{v}_{(j)} - \boldsymbol{U}_0^{\{\omega\}}\left(1 - \frac{y_{(j)}^2}{H^2}\right)\right]\right\rangle,&\end{aligned} \tag{7}$$

where $\boldsymbol{U}_0^{\{\omega\}} = [U_0^{\{\omega\}},0,0]$ is the streamwise velocity in the centerline which can, in general, be a random variable (for the case of a changing pressure gradient) and $y_{(j)} = x_{(j)2}$ is the wall-normal coordinate. This formula was further rewritten in Wacławczyk *et al.* [1] using properties of the delta function, and we presented the "shape" symmetry for the channel flow as

$$\begin{aligned}f_n^* &= f_n + F(y_{(1)},\ldots,y_{(n)})\psi(\boldsymbol{v}'_{(1)})\delta(\boldsymbol{v}'_{(1)} - \boldsymbol{v}'_{(2)})\\ &\quad \cdots\delta(\boldsymbol{v}'_{(1)} - \boldsymbol{v}'_{(n)}),\end{aligned} \tag{8}$$

where

$$F(y_{(1)},\ldots,y_{(n)}) = \prod_{j=1}^{n}\left(1-\frac{y_{(j)}^2}{H^2}\right)^{-1}, \tag{9}$$

and for each j

$$v'_{(j)1} = v_{(j)1}\left(1-\frac{y_{(j)}^2}{H^2}\right)^{-1}, \quad v'_{(j)2} = v_{(j)2}, \quad v'_{(j)3} = v_{(j)3}. \tag{10}$$

Moreover, as follows from Eq. (7) the flow occurs in the x direction only, hence the form of function ψ in Eq. (8) is

$$\psi(\boldsymbol{v}'_{(1)}) = \psi_1(v'_{(1)1})\delta(v_{(1)2})\delta(v_{(1)3}). \tag{11}$$

We agree that the LMN equations in the form presented in the paper are valid only in the infinite domain. We acknowledge this inconsistency and should have explicitly noted this problem in the paper. Below, we present the PDF equations for the bounded geometry of the channel flow and show its invariance under the transformation (8).

If we consider a domain Ω and its boundary Γ, then a solution of the Poisson equation for the pressure can be presented as

$$P(\boldsymbol{x}) = \int_\Omega G(\boldsymbol{x},\boldsymbol{x}')\nabla^2 P(\boldsymbol{x}')d\boldsymbol{x}' + \int_\Gamma P(\boldsymbol{x}')\frac{\partial G(\boldsymbol{x},\boldsymbol{x}')}{\partial \mathfrak{n}'}dS' - \int_\Gamma G(\boldsymbol{x},\boldsymbol{x}')\frac{\partial P(\boldsymbol{x}')}{\partial \mathfrak{n}'}dS', \tag{12}$$

where $G(\boldsymbol{x},\boldsymbol{x}')$ is a Green's function of a given geometry and $\partial/\partial\mathfrak{n}$ denotes a differentiation in the direction normal to the boundary Γ (the normal vector points outward). For the case of a laminar flow in the channel the first term in Eq. (12) vanishes as we have $\nabla^2 P = 0$. The remaining, in fact boundary terms, lead to the classical solution $P(x) = -2x\rho\nu U_0/H^2 + C$, where U_0 is the velocity at the channel centerline $y = 0$ and C is an arbitrary constant. The considerations presented by Frewer *et al.* in Ref. [5] would imply $P(x) = 0$, which of course, cannot be a solution for the pressure-driven channel flow.

A constant pressure gradient solution can be obtained from Eq. (12) if the Green's function satisfies a proper far-field condition. The Green's function for a channel was derived, e.g., in Ref. [9]. This function has the properties $\partial G/\partial y = 0$ at $y = \pm H$, $\partial G/\partial z = 0$ at $z = -L_z, L_z$ (in our case $L_z \gg H$), and has a far-field condition $G = |x|/(8L_z H)$ at $x = \pm L_x$ where $L_x \to \infty$. For the laminar flow in a channel, $\nabla^2 P(\boldsymbol{x}') = 0$ and we apply a constant pressure gradient at the inlet and outlet boundaries $\partial P/\partial x = \Delta P/(2L_x) = -2\rho\nu U_0/H^2$. From Eq. (12) for $L_z \gg H$ we obtain

$$\frac{\partial P(x)}{\partial x} = -\frac{1}{8L_z H}\frac{\partial}{\partial x}\int_{-H}^{H}\int_{-L_z}^{L_z}\left[\left(-\frac{\Delta P}{2L_x}|x-x'|\right)\Big|_{x'=-L_x} + \left(\frac{\Delta P}{2L_x}|x-x'|\right)\Big|_{x'=L_x}\right]dz'dy' = -2\rho\nu U_0/H^2. \tag{13}$$

If the Green's function in Eq. (12) is constructed such that it satisfies homogeneous Neumann boundary conditions $\partial G(\boldsymbol{x},\boldsymbol{x}')/\partial\mathfrak{n}' = 0$ at the wall boundaries, the nth equation of the LMN hierarchy for flow in a channel can be extended to

$$\left[\partial_t + \sum_{i=1}^{n}\boldsymbol{v}_{(i)}\cdot\nabla_{(i)}\right]f_n = \sum_{j=1}^{n}\nabla_{\boldsymbol{v}_{(j)}}\cdot\left\{\int_\Omega\left[\nabla_{(j)}G(\boldsymbol{x}_{(j)},\boldsymbol{x}_{(n+1)})\int(\boldsymbol{v}_{(n+1)}\cdot\nabla_{(n+1)})^2 f_{n+1}d\boldsymbol{v}_{(n+1)}\right]d\boldsymbol{x}_{(n+1)}\right\}$$
$$-\sum_{j=1}^{n}\nabla_{\boldsymbol{v}_{(j)}}\cdot\left\{\int_\Gamma\left[\nabla_{(j)}G(\boldsymbol{x}_{(j)},\boldsymbol{x}_{(n+1)})\right.\right.$$
$$\left.\left.\times\int\left\langle\frac{1}{\rho}\frac{\partial P(\boldsymbol{x}_{(n+1)})}{\partial\mathfrak{n}_{(n+1)}}\right|\boldsymbol{v}_{(1)} = \boldsymbol{U}(\boldsymbol{x}_{(1)},t),\ldots,\boldsymbol{v}_{(n+1)} = \boldsymbol{U}(\boldsymbol{x}_{(n+1)},t)\right\rangle f_{n+1}d\boldsymbol{v}_{(n+1)}\right]dS_{(n+1)}\right\}$$
$$-\sum_{j=1}^{n}\nabla_{\boldsymbol{v}_{(j)}}\cdot\left[\lim_{|\boldsymbol{x}_{(n+1)}-\boldsymbol{x}_{(j)}|\to 0}\nu\Delta_{n+1}\int\boldsymbol{v}_{(n+1)}f_{n+1}d\boldsymbol{v}_{(n+1)}\right]. \tag{14}$$

So far, we did not specify the pressure gradient at the boundary, and hence, we wrote this term with the use of a conditional average. Alternatively, the pressure could be represented as a sum of three terms: the basic pressure $P_0 = x(d\mathcal{P}/dx)$, where $d\mathcal{P}/dx$ is a constant, complementary pressure $p_h(\boldsymbol{x},t)$ being a solution of the homogeneous Laplace equation satisfying the nonhomogeneous boundary conditions, and the Green's function pressure $p_G(\boldsymbol{x},t)$, which is a solution of the nonhomogeneous Poisson equation with homogeneous boundary conditions [10]. The boundary terms can be represented as functionals of the wall stress generated by the flow. Such representation leads to the Leray version of the Navier-Stokes equations where pressure is eliminated and $\nabla\cdot\boldsymbol{U} = 0$ enters the system as an initial condition [10].

We argued in Ref. [1] that the shape symmetries (2) and (8) could only modify the laminar part of the PDF, hence $f^* = g_T + g_L^*$. In the laminar case $\partial P^{\{\omega\}}/\partial x = -2\rho\nu U_0^{\{\omega\}}/H^2$ (where, similar as in Ref. [1], we assume that the pressure gradient may change and we treat the centerline velocity $U_0^{\{\omega\}}$ as a random variable). The PDF of laminar velocity is given by Eq. (7) and the boundary integral in (14) will read

$$\frac{2\nu}{H^2}\int F(y_{(1)},\dots,y_{(n)})\left\langle U_0^{\{\omega\}}\prod_{k=1}^{n+1}\delta\left(v'_{(k)}-U_0^{\{\omega\}}\right)\right\rangle dv'_{(n+1)} \tag{15}$$

$$=\frac{2\nu}{H^2}\frac{1}{1-y_{(j)}^2/H^2}v_{(j)}f_L(1,\dots,n), \tag{16}$$

where we used the sifting property of the delta function. Hence, after the transformation (8) we obtain an additional term in Eq. (14), namely,

$$-\frac{2\nu}{H^2}\frac{1}{1-y_{(j)}^2/H^2}\sum_{j=1}^{n}\nabla_{v_{(j)}}\cdot[v_{(j)}F(y_{(1)},\dots,y_{(n)})\psi(v'_{(1)})\times\delta(v'_{(1)}-v'_{(2)})\cdots\delta(v'_{(1)}-v'_{(n)})]. \tag{17}$$

On the other hand, the last, viscous term in Eq. (14), after the transformation (8) will give rise to the additional term

$$-\nu\sum_{j=1}^{n}\nabla_{v_{(j)}}\cdot\left[\lim_{|y_{(n+1)}-y_{(j)}|\to 0}\Delta_{n+1}\int v'_{(n+1)}\left(1-\frac{y_{(n+1)}^2}{H^2}\right)F(y_{(1)},\dots,y_{(n)})\psi(v'_{(1)})\delta(v'_{(1)}-v'_{(2)})\cdots\delta(v'_{(1)}-v'_{(n+1)})dv'_{(n+1)}\right]. \tag{18}$$

With the use of the sifting property of the delta function we change $v'_{(n+1)}$ to $v'_{(j)}$ and calculate the integral to obtain

$$-\nu\frac{1}{1-y_{(j)}^2/H^2}\sum_{j=1}^{n}\nabla_{v_{(j)}}\cdot\left[\lim_{|y_{(n+1)}-y_{(j)}|\to 0}\Delta_{n+1}\left(1-\frac{y_{(n+1)}^2}{H^2}\right)v_{(j)}F(y_{(1)},\dots,y_{(n)})\psi(v'_{(1)})\delta(v'_{(1)}-v'_{(2)})\cdots\delta(v'_{(1)}-v'_{(n)})\right]. \tag{19}$$

After calculating the Laplacian, (19) will cancel with (17). As other terms in (14) remain invariant under (8), this finally proves the invariance of (14) under the transformation (8).

At the end we may consider a case with a constant, fixed pressure gradient $\partial P/\partial x=\Delta P/(2L_x)$ in the channel. By analogy to Eq. (3), both, the intermittency and the shape symmetry can transform f_n of a turbulent flow into

$$f_n^*=e^a f_n+(1-e^a)F(y_{(1)},\dots,y_{(n)})\delta(v'_{(1)}-U_0)\times\delta(v'_{(1)}-v'_{(2)})\cdots\ \delta(v'_{(1)}-v'_{(n)}). \tag{20}$$

Such a transformation can be caused by a change of a level of external disturbances. As a result, the character of the flow can change from turbulent to laminar or intermittent (laminar or turbulent) [11]. Hence, we want to reject once again the criticism of Frewer *et al.* of the violation of the causality principle. It should be noted that in Wacławczyk *et al.* [1] the effect of boundaries was ignored indeed. Still, the analysis given above rectifies this problem and, most important, is fully consistent with the argument of statistical symmetries.

V. REPLY TO: THE NONCONNECTEDNESS TO RANDOM GALILEAN INVARIANCE

In Ref. [12], Kraichnan considered a case when "Each flow in the ensemble is subject to translation velocity, constant in space and time, but which has Gaussian ensemble distribution," $U^*=U+a\xi$, $x^*=x+at\xi$ where ξ is a Gaussian random number. He argued that for such a case "the internal dynamics of the turbulence is unaffected." Although it first seemed to us that some analogy may exist between the statistical symmetries and Kraichnan's "random Galilean invariance," we admit that such statement was not justified. Still, we think this is only a matter of wording and has nothing to do with the mathematical or physical content of our paper. In this context, we should mention that the far-field condition for $|x-x'|\to\infty$ is not satisfied for the random Galilean invariance, either, $\langle U_i(x,t)U_j(x',t)\rangle^*=\langle U_i(x,t)\rangle\langle U_j(x',t)\rangle+a_ia_j\neq\langle U_i^*(x,t)\rangle\langle U_j^*(x,t)\rangle$. This is, similar as in our case, due to the presence of the constant fields.

VI. REPLY TO: ON CAPTURING INTERMITTENCY WITH GLOBAL SYMMETRIES

Frewer *et al.* discuss our results in the context of the anomalous scaling and breaking of global self-similarity. These phenomena are connected with the phenomenon of "internal intermittency" of small turbulence scales. We neither mentioned this phenomenon nor aimed to capture it in Wacławczyk *et al.* [1]. In our paper we wrote explicitly that "by *intermittency* we understand a flow with subsequently changing turbulent and nonturbulent regimes" (i.e., "external intermittency").

VII. SUMMARY

We admit that certain inexactnesses can be found in our paper, i.e., there is no clear analogy between the statistical symmetries and Kraichnan's random Galilean invariance. Moreover, we should have discussed the fact that the considered LMN system is valid for boundary-free flows. In the present "Reply to the Comment…" we derived the LMN equation in a bounded domain.

We do not agree, however, with the main objection of Frewer *et al.* that the transformation of a PDF can only follow from a transformation of sample space variables v.

We stated in Ref. [1] that statistical symmetries first found in the MPC equations ([3,4]) are connected with turbulent or non turbulent flows. The intermittency symmetry changes the intermittency factor while the shape symmetry modifies the laminar part of the PDF. After such transformations the statistics change, although the instantaneous velocities U in separate realizations of the flow are not transformed (i.e., they are not rescaled or translated).

The statistical symmetries of MPC equations [Eqs. (6) and (7) in the "Comment on . . . ." by Frewer *et al.*) have explicitly been used for the construction of group invariant solutions (turbulent scaling laws) in Refs. [3,13,14] and are clearly validated therein. Hence, there is clear physical evidence for the existence and importance of the statistical symmetries. We note in passing that the present interpretation involving turbulent or nonturbulent flows has only become clear after considering PDF equations in Ref. [1].

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