Comment on "Statistical symmetries of the Lundgren-Monin-Novikov hierarchy"

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The article by M. Wacławczyk *et al.* [Phys. Rev. E **90**, 013022 (2014)] proposes two new statistical symmetries in the classical theory for turbulent hydrodynamic flows. In this Comment, however, we show that both symmetries are unphysical due to violating the principle of causality. In addition, they must get broken in order to be consistent with all physical constraints naturally arising in the statistical Lundgren-Monin-Novikov (LMN) description of turbulence. As a result, we state that besides the well-known classical symmetries of the LMN equations no new statistical symmetries exist. Finally, we criticize the relation between intermittency and global symmetries as it is presented throughout that study.

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Summary of key results in [1]. Based on the Lundgren-Monin-Novikov (LMN) hierarchy of the probability density functions (PDFs) f_n for unbounded turbulent flows

$$\left[\partial_t+\sum_{i=1}^{n}\boldsymbol{v}_{(i)}\cdot\nabla_{(i)}\right]f_n = -\sum_{j=1}^{n}\nabla_{\boldsymbol{v}_{(j)}}\cdot\left[\lim_{\boldsymbol{x}_{(n+1)}\to\boldsymbol{x}_{(j)}}\nu\Delta_{(n+1)}\int d\boldsymbol{v}_{(n+1)}\boldsymbol{v}_{(n+1)}f_{n+1} - \int d\boldsymbol{x}_{(n+1)}\left(\nabla_{(j)}\frac{1}{4\pi|\boldsymbol{x}_{(j)}-\boldsymbol{x}_{(n+1)}|}\right)\times\int d\boldsymbol{v}_{(n+1)}(\boldsymbol{v}_{(n+1)}\cdot\nabla_{(n+1)})^2 f_{n+1}\right], \tag{1}$$

along with several natural constraints to warrant a physical solution for the PDFs, two new statistical symmetries admitted by (1) were derived [Eqs. (42) and (63) in [1]]:

$$\bar{T}_2':\ t^*=t,\quad \boldsymbol{x}^*_{(l)}=\boldsymbol{x}_{(l)},\quad \boldsymbol{v}^*_{(l)}=\boldsymbol{v}_{(l)},\quad f_n^*=f_n+\psi(\boldsymbol{v}_{(1)})\delta(\boldsymbol{v}_{(1)}-\boldsymbol{v}_{(2)})\cdots\delta(\boldsymbol{v}_{(1)}-\boldsymbol{v}_{(n)}), \tag{2}$$

$$\bar{T}_s':\ t^*=t,\quad \boldsymbol{x}^*_{(l)}=\boldsymbol{x}_{(l)},\quad \boldsymbol{v}^*_{(l)}=\boldsymbol{v}_{(l)},\quad f_n^*=e^{a_s}f_n+(1-e^{a_s})\cdot\delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)}), \tag{3}$$

which were named as the "shape symmetry" and "intermittency symmetry," respectively. On the lower statistical level of the multipoint moments for $n\geqslant 0$

$$\mathbf{H}_{\{n+1\}}=\langle\mathbf{U}_{(1)}(\boldsymbol{x}_{(1)},t)\cdots\mathbf{U}_{(n+1)}(\boldsymbol{x}_{(n+1)},t)\rangle, \tag{4}$$

where $\mathbf{U}$ is the instantaneous (fluctuating) velocity field, the associated multipoint correlation (MPC) invariances

$$\bar{T}_2':\ t^*=t,\quad \boldsymbol{x}^*_{(l)}=\boldsymbol{x}_{(l)},\quad \mathbf{H}^*_{\{n\}}=\mathbf{H}_{\{n\}}+\mathbf{C}_{\{n\}}, \tag{5}$$

$$\bar{T}_s':\ t^*=t,\quad \boldsymbol{x}^*_{(l)}=\boldsymbol{x}_{(l)},\quad \mathbf{H}^*_{\{n\}}=e^{a_s}\mathbf{H}_{\{n\}}, \tag{6}$$

are identified as the induced translation and scaling invariance of the LMN symmetries (2) and (3), respectively. Note that for reasons of space limitation we do not show here the transformation for the pressure related moments in (5) and (6) as it is done in the original article [1]. A further result obtained in [1] shows that in the intermittent flow regime of a fully developed plane Poiseuille channel flow the so-called shape symmetry (2) takes the form [Eq. (54) in [1]]

$$f_n^*=f_n+F(y_{(1)},\dots,y_{(n)})\psi(\boldsymbol{v}'_{(1)})\times\delta(\boldsymbol{v}'_{(1)}-\boldsymbol{v}'_{(2)})\cdots\delta(\boldsymbol{v}'_{(1)}-\boldsymbol{v}'_{(n)}), \tag{7}$$

where $v'_{(k)2}=v_{(k)2}$, $v'_{(k)3}=v_{(k)3}$, and

$$v'_{(k)1}=v_{(k)1}\left(1-\frac{y_{(k)}^2}{H^2}\right)^{-1},\quad \frac{1}{F}=\prod\nolimits_{i=1}^{n}\left(1-\frac{y_{(i)}^2}{H^2}\right), \tag{8}$$

with $y_{(k)}\equiv x_{(k)2}$ being the wall-normal coordinate and $2H$ the width of the channel. Symmetry (7) then gives rise to the following invariant translations of the moments:

$$\langle U_1(\boldsymbol{x}_{(1)},t)\cdots U_1(\boldsymbol{x}_{(n)},t)\rangle^* = \langle U_1(\boldsymbol{x}_{(1)},t)\cdots U_1(\boldsymbol{x}_{(n)},t)\rangle + C_{1\dots 1}\left(1-\frac{y_{(1)}^2}{H^2}\right)\cdots\left(1-\frac{y_{(n)}^2}{H^2}\right), \tag{9}$$

where, according to [1], the coefficients $C_{1\dots 1}$ satisfy the following restrictions:

$$-2U_{0\text{cr}}\leqslant C_1\leqslant 2U_{0\text{cr}},\quad -U_{0\text{cr}}^2\leqslant C_{11}\leqslant U_{0\text{cr}}^2,\ \cdots \tag{10}$$

with $U_{0\text{cr}}$ being the critical value up to which a laminar channel flow can be realized. A related invariant scaling law for the mean velocity profile is then finally derived:

$$\langle U\rangle=\frac{C_1}{k_2}\ln(y)+\frac{C_1}{2k_2}\left(1-\frac{y^2}{H^2}\right)+\mathcal{C}. \tag{11}$$

I. Violation of the causality principle. At first it should be noticed that the two new symmetries (2) and (3) (as well as the induced invariances (5) and (6)) only act in a pure statistical manner without having a transformational origin of any kind in the underlying deterministic set of equations, since they are "not reflected in the original [Euler and Navier-Stokes] equations" ([1], p. 2); neither as a distinctive symmetry, nor as any ordinary variable transformation. As also discussed further below from a different perspective, the reason is that in (3) *all* multipoint PDFs get scaled by the same constant factor e^{a_s}, while in (2) they *all* get translated by the same spatially

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constant shift ψ, which in both cases makes it mathematically impossible to establish a transformational link of *any* kind to the Euler and Navier-Stokes equations; inevitably, such an attempt even leads to an inherent contradiction. This problem is then transferred down to the corresponding invariances for the velocity moments (5) and (6) (see, e.g., the detailed study [2], particularly the proof in Appendix A).

It is obvious that such a property necessarily leads to unphysical behavior, because only the *deterministic* equations due to their spatially nonlocal and temporally chaotic behavior induce the *statistical* equations, and not vice versa. Consequently, *any* transformation which only acts on a purely statistical (averaged) level without having a deterministic (fluctuating) origin violates the classical principle of cause and effect, since the system would experience an effect (change in averaged dynamics) without a corresponding cause (change in fluctuating dynamics). Note that the cause itself need not be a symmetry in order to induce a symmetry as an effect; for example, as is the case for the well-known diffusion equation: Its underlying fine-grained discrete random walk does not admit the variable transformation $\mathsf{T}: t^* = e^{2\varepsilon}t,\ \boldsymbol{x}^* = e^{\varepsilon}\boldsymbol{x}$ as a symmetry transformation; only when coarse-graining this stochastic process to yield the continuous and diffusive Fokker-Planck equation will it turn into a scaling symmetry, resulting, however, from the cause of a (noninvariantly) transformed or changed random walk.

But for (2) and (3) one observes that in both cases only the coarse-grained PDF f_n gets transformed, while the sampled values $\boldsymbol{v}_{(n)}$ for the fine-grained instantaneous (fluctuating) velocity field $\mathbf{U}$ at point $\boldsymbol{x}_{(n)}$ and time t stay invariant. Hence, although the dynamical system $(t,\boldsymbol{x},\mathbf{U})$ stays unchanged under both symmetry transformations (2) and (3) on its fine-grained level, it nevertheless undergoes a *global* change $f_n \to f_n^*$ on its induced coarse-grained level, which is unphysical and not realized in nature, simply as no cause *at all* exists from which (2) or (3) can emerge as a symmetry transformation. Keep in mind that here the coarse-grained multipoint PDF f_n is defined as an ensemble average over all possible fine-grained realizations of the flow (see, e.g., [3])

$$f_n(\boldsymbol{x}_{(1)},\boldsymbol{v}_{(1)};\dots;\boldsymbol{x}_{(n)},\boldsymbol{v}_{(n)};t) = \left\langle \prod_{i=1}^{n} \delta(\boldsymbol{v}_{(i)} - \mathbf{U}(\boldsymbol{x}_{(i)},t)) \right\rangle, \tag{12}$$

where $\langle\cdot\rangle$ denotes the averaging or coarse-graining process, and all $\boldsymbol{v}_{(i)}$ the sampled values of the fluctuating or fine-grained instantaneous velocity field $\mathbf{U}$ at every point $\boldsymbol{x}_{(i)}$ and time t. Hence, when additionally also regarding the *invariance* of equality (12), both transformations (2) and (3) are even incompatible to this defining relation which is symmetry breaking, because the left-hand side of (12) changes while the right-hand side stays unchanged (for the obvious fact that the coarse-graining process $\langle\cdot\rangle$ has no transformational effect on a kernel which itself stays invariant). Note that the opposite conclusion is not the rule, i.e., a change on the fluctuating level can occur without inducing an effect on the averaged level. A macroscopic or coarse-grained (averaged) observation might be insensitive to many microscopic or fine-grained (fluctuating) details in that they get averaged out to zero, a property of nature widely known as universality (see, e.g., [4]).

II. Noncompatibility with all LMN constraints. It is obvious that the symmetry transformations (2) and (3) must be compatible or consistent to *all* constraints of the LMN equations (1), in particular also to the separation constraint (shown here only for the two-point PDF)

$$\lim_{|\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(2)}|\to\infty} f_2(1,2) = f_1(1)f_1(2), \tag{13}$$

otherwise physical solutions get mapped to unphysical ones. In [1] (see Secs. III D 1 and III D 2) it is correctly shown that both symmetries (2) and (3) are compatible with the so-called normalization constraint "(19)" [up to a natural condition "(46)" for the function ψ in (2)] and the coincidence constraint "(21)". At the same time, however, a misleading impression is given that, in particular, for symmetry transformation (2) the separation constraint (13) can be ignored without any important consequences, because, apparently, this property "is never used in the derivation of the [LMN] equations," and that it is "not satisfied by the corresponding symmetries of the MPC equations, either" ([1], p. 6). In this opinion we cannot agree: (i) Although the property (13) is not directly used to construct the LMN equations (1), it nevertheless is an inherent part of these equations to warrant the outcome of physical solutions, and (ii) it is not expedient to relate the MPC equations to the LMN equations, especially since they exhibit a fundamental disadvantage over the LMN equations when performing an invariance analysis upon them, in particular as the MPC system is *not* equivalent to the LMN system (for more details see [2]).

On the other side, for symmetry transformation (3) an example is given [see Eq. (65) in [1]], which should show that for a *certain specification* of the PDF this symmetry is compatible with the separation constraint (13). But, this proof by example is a *tautology*: By specifying the PDF f_n as the spatially independent and zero-valued δ function $f_n = \delta(\boldsymbol{v}_{(1)})\cdots\delta(\boldsymbol{v}_{(n)})$, the symmetry transformation (3) leads to the expression $f_n^* = f_n$, i.e., this particular specification turns the symmetry (3) into a trivial identity transformation, which, of course in a trivial manner, is always compatible to any thinkable constraint, also to the separation constraint (13). Moreover, as can be readily recognized, this tautological specification of a trivial zero-valued δ function is also the only possible PDF specification which allows for such a compatibility.

Hence, the interpretation in [1] on the noncompatibility of symmetry (2) and the proof by example on the compatibility of symmetry (3) regarding the separation constraint (13) is misleading. It is obvious that both symmetry transformations (2) and (3) are, without exceptions, *incompatible* with the separation constraint (13), and that they thus violate one of the most intuitive physical constraints of the LMN equations: For *all times* every PDF solution should show the spatial property of statistical independence when any two points are infinitely far apart, that is, any two infinitely distant points should not influence each other. But exactly this property cannot be constantly maintained when transforming the system's variables according to (2) and (3). The reason is that since both transformations are true symmetry transformations which map solutions to new solutions, and since both at the same time are not compatible with the physical constraint (13) of the LMN equations, we thus obtain the unwanted effect that an initially physical solution can get mapped to an unphysical solution in which a nonzero correlation between infinitely distant points will be induced, and this for all times since the transformations (2) and (3) are time independent. This is definitely not physical and even is strictly avoided in every PDF-modeling technique (see, e.g., [5,6]).

Note that both symmetry transformations (2) and (3) can also not be interpreted as a valid approximation for moderate separations when the constraint (13) is violated, as was done by the authors in their preceding work [7], in particular for symmetry (2). This is due to the fact that every joint-PDF (f_n for $n \geqslant 2$) is a spatially connected quantity, meaning that if it does not show the correct behavior for large separations, there is no guarantee that the same PDF will then show a realistic or physical behavior for moderate separations. This situation can be formally compared to a boundary value problem for a differential equation with at least one infinite extension, in that, if the boundary condition at infinity is not satisfied, it will effect the solution in the whole domain. In this sense both symmetries (2) and (3) are unphysical and only turn into physical transformations if they also satisfy the separation constraint, but which, however, can only be achieved if $\psi = 0$ and $a_s = 0$. Hence, the separation constraint (13) breaks the LMN symmetries (2) and (3) in a nonapproximative manner.

III. On the new "shape symmetry" in channel flow. In the course of adapting the original "shape symmetry" (2) into form (7), in order to access the intermittent flow regime of a fully developed Poiseuille channel flow, the authors in [1] seem to have overlooked that this process is symmetry breaking, that is, transformation (7) in its general form does not leave the LMN equations (1) invariant anymore. For the proof of this statement, and to see how this adapted shape symmetry (7) induces an inconsistent transformation in the statistical transport equations, in particular in the one-point momentum equations, we again refer to the detailed study [2].

The explanation for this transformational failure already lies in the equational structure of the LMN hierarchy itself. While system (1) by construction only applies for spatially *unbounded* flows, the authors in [1], however, consider PDFs for *bounded* flows by using the profile of a laminar channel flow (including its boundary region at $y = \pm H$; see Eq. (49) in [1]) in order to convert their shape symmetry (2) into the form (7). A loss of symmetry is thus the consequence. Because, when formally removing the boundary condition again, in letting $H \to \infty$, will turn the *non*invariant transformation (7) back into the symmetry (2) again (since for any finite values of the wall-normal coordinate the symmetry-breaking factors in (7) will neutralize to $F = 1$ and $\boldsymbol{v}' = \boldsymbol{v}$).

Hence, in order to properly address PDFs for *bounded* flows, the LMN hierarchy must be rederived such as to incorporate the considered boundary conditions into the integrodifferential equations. However, this will result into fundamentally different equations than those given by the hierarchy (1), which only applies for unbounded flows, and to the best of our knowledge such PDF evolution equations for bounded flows have not been derived yet. But if, then it is also straightforward to show that the adapted shape symmetry (7) is *not* admitted as a symmetry transformation by the bounded LMN equations for plane channel flow either (see [2]).

IV. The nonconnectedness to random Galilean invariance. As an aside, we want to remark that the statistical translation group (5), when specified to $\mathbf{C}_{\{1\}} \neq 0$ and $\mathbf{C}_{\{k\}} = 0$, $k \geqslant 2$ [Eq. (15) in [1]], was erroneously identified in [1] as the random Galilean invariance group, first introduced by Kraichnan 1965 [8]. This new translation group "(15)" is not in conformance with the random Galilean group as defined by Kraichnan (for details see [2,9]). The reason is that the random Galilean group acts on the fine-grained (fluctuating) level, while the statistical translation group "(15)" only acts on the coarse-grained (averaged) level without having a cause of any kind on the lower fluctuating level; a property which even violates the classical principle of causality (see Sec. I). Moreover, (i) the spatial coordinates of the random Galilean group get transformed in contrast to "(15)", and (ii) the transformation rule for all n-point velocity correlations beyond the mean velocity ($n \geqslant 2$) do not transform invariantly under the random Galilean group as given in "(15)".

V. On capturing intermittency with global symmetries. In our opinion, the method of Lie groups, when used as a constructive method to generate scaling laws in particular from *global* symmetry transformations, is not the appropriate method to capture the complex spatiotemporal phenomenon of intermittency in dynamical systems, irrespective of whether internal (small scale) or external (large scale) intermittency is considered. In general, intermittency is a property which rather breaks than restores symmetries, not only on the fine-grained (fluctuating) but also on the coarse-grained (averaged) level [10–13]. A prominent historical example is the failure of Kolmogorov's K41 theory [14–16], which has been found to be increasingly inaccurate for higher order statistics (see, e.g., [17,18]). Instead of statistically restoring the deterministic scaling symmetry of the Navier-Stokes system [17,19,20], an induced turbulent flow will always statistically evolve such by showing the property of anomalous scaling and the breaking of global self-similarity, which both interdependently can be attributed to the complex property of intermittency [13,17,18]. Although this example only addresses the effect of global symmetry breaking in the process of internal intermittency, it nevertheless serves as a representative example for any type of intermittency.

The point is, that even if we would only consider an isotropic turbulent flow (which is a highly idealized flow), the statistical solutions, in particular the higher order correlations, due to their increasingly pronounced intermittent behavior, are by far more complicated than we currently can imagine and that it is actually unrealistic to believe that this complicated behavior can be captured by some *global* scaling symmetries. The complexity even increases when considering wall-bounded flows showing external intermittency [10], which, in contrast to internal intermittency, is a strongly nonuniversal process where the mechanism of symmetry breaking will be even more pronounced. Hence, proposing (11) as an "invariant solution" to the intermittent scaling behavior of the wall-bounded plane channel flow as done in [1], it is highly questionable if this is really the case when (11) is matched to numerical or physical experiments, and furthermore, whether (11) is really consistent also to all higher order moments when trying to involve them accordingly.

Despite the fact that the two symmetries (2) and (3), along with their induced invariant transformations (5) and (6) which then lead to (11), are unphysical in that they not only violate the fundamental principle of causality of classical mechanics (as shown in Sec. I) but also must get broken in order to be compatible with all physical constraints of the underlying dynamical equations (as shown in Sec. II), the further and more general problem is that the deterministic Navier-Stokes theory itself, unfortunately, only allows for spatially *global* and not for spatially *local symmetries* [21]: All physical symmetries of the deterministic Euler and Navier-Stokes equations listed in [1] [Sec. II A, Eqs. (8)–(13)] are only *spatially global* symmetries. No other, more general symmetries for this theory exist or are known yet. And it is exactly due to this fact, that a Lie-group based symmetry analysis for the unclosed statistical Navier-Stokes theory did not achieve the same great breakthrough as it did, for example, for the theory of quantum fields (see, e.g., [22,23]), which is based on a spatially local symmetry, the local gauge symmetry which successfully predicts the unknown functional structure of the interacting fields between the various elementary particles.

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