Stability of a flow down an incline with respect to two-dimensional and three-dimensional disturbances for Newtonian and non-Newtonian fluids

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Squire's theorem, which states that the two-dimensional instabilities are more dangerous than the three-dimensional instabilities, is revisited here for a flow down an incline, making use of numerical stability analysis and Squire relationships when available. For flows down inclined planes, one of these Squire relationships involves the slopes of the inclines. This means that the Reynolds number associated with a two-dimensional wave can be shown to be smaller than that for an oblique wave, but this oblique wave being obtained for a larger slope. Physically speaking, this prevents the possibility to directly compare the thresholds at a given slope. The goal of the paper is then to reach a conclusion about the predominance or not of two-dimensional instabilities at a given slope, which is of practical interest for industrial or environmental applications. For a Newtonian fluid, it is shown that, for a given slope, oblique wave instabilities are never the dominant instabilities. Both the Squire relationships and the particular variations of the two-dimensional wave critical curve with regard to the inclination angle are involved in the proof of this result. For a generalized Newtonian fluid, a similar result can only be obtained for a reduced stability problem where some term connected to the perturbation of viscosity is neglected. For the general stability problem, however, no Squire relationships can be derived and the numerical stability results show that the thresholds for oblique waves can be smaller than the thresholds for two-dimensional waves at a given slope, particularly for large obliquity angles and strong shear-thinning behaviors. The conclusion is then completely different in that case: the dominant instability for a generalized Newtonian fluid flowing down an inclined plane with a given slope can be three dimensional.

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I. INTRODUCTION

The hydrodynamics of film flows driven by gravity down an inclined plane have been studied for a long time. Such flows, which can be encountered in many industrial and geophysical situations as well as in everyday life, very often present singular wavy patterns which may become complex, depending on the slope, the flow speed, and the physical properties of the fluid. The waves that are triggered in such flows are initially quasi-plane waves with a large wavelength compared to the mean flow depth, and they are known as surface waves. Farther downstream, the waves grow in amplitude and quickly evolve towards a nonlinear regime. These waves appear for nonzero Reynolds number, and have then a convective characteristic. The onset of such waves in Newtonian fluids is well understood since the early linear stability studies by Benjamin [1] and Yih [2]. A long-wave approximation was adopted in these analytical approaches. They showed that the critical Reynolds number for the onset of the instabilities only depends on the inclination of the plate γ and is proportional to $\cot\gamma$. They also pointed out that inertia is required to trigger these free surface instabilities. The experimental works of Liu [3] have confirmed this dependence with the slope for the linear stability thresholds.

Most of the studies in the literature on this topic are based on a Newtonian fluid model. However, the liquids involved in many engineering applications such as coating processes, but also in some geophysical phenomena such as glaciers, mud, and debris flows, very often present complex rheological behaviors. Focusing more particularly on geophysical flows, the viscoplastic rheology was shown to describe quite well the behavior of the materials involved in debris flows [4,5]. Other experimental results show that the shear-thinning behavior is also suitable to describe this rheology [6]. Since most viscoplastic liquids are not ideal Bingham liquids, they often behave as shear-thinning liquids with a yield stress. For the sake of simplicity, we will not take this latter into account; thus we will focus, in the present paper, on non-Newtonian liquids for which the measured effective viscosity is always a scalar function of the shear rate: these fluids are called generalized Newtonian fluids. For this class of fluids, it is well understood that the interactions between particles create microscopic structures that may be deformed and gradually broken down (e.g., the shear-thinning behavior of clay suspensions or mud) or aggregated (e.g., the shear-thickening behavior of sand or cornstarch) when a shear stress is applied. The rheological behavior of many fluids cannot, then, be properly described by a Newtonian model and more sophisticated rheological models involving a nonlinear relationship between stress and strain would be more appropriate. Fewer studies have been carried out in the case of generalized Newtonian film flows. Ng and Mei [7] showed that a linear stability study with a power-law fluid is not sufficient to suggest a preferred wavelength for roll waves because the predicted growth rate of the unstable disturbances increases monotonically with the wave number. Then, using a long-wave approximation, they demonstrated a linear evolution of the critical Reynolds number as a function of both $\cot\gamma$ and the power-law exponent n. This study, however, is limited by the singularity introduced by the viscosity law in the model: a power law describes an infinite viscosity at the free surface characterized by a zero shear rate, which is not physically consistent. To remove this singularity, some authors considered a regularized power-law model: Ruyer-Quil *et al.* [8] by introducing a Newtonian plateau at small strain rate and Noble and Vila [9] by introducing a weaker formulation of the Cauchy momentum equations. They have shown the relevant influence of shear-thinning properties on the primary instability. Rousset *et al.* [10] studied the temporal stability of a Carreau fluid flow down an inclined plane. They performed an asymptotic analysis considering a weakly non-Newtonian behavior in the limit of very long waves and compared it with a more general numerical approach. It was found that the critical Reynolds number is lower for shear-thinning fluids than for Newtonian fluids, while the ratio between the critical wave celerity and the mean flow velocity at the free surface is more than twice larger [2]. Particular attention was paid to the situations with small angles of inclination. Indeed, in these cases, besides the long-wave free surface mode, another instability identified by Floryan *et al.* [11] as a shear mode can occur. It is characterized by a wavelength on the order of the layer thickness and a wave celerity lower than the free surface velocity. It was shown that taking into account the shear dependence of the viscosity can change the nature of the instability.

Superposed film layers flowing down an inclined plane can be subjected to interfacial instabilities even in the limit of zero Reynolds number according to the direction of viscosity stratification. This situation was first observed with Newtonian fluids by Kao [12]. Other studies led to the same observations for non-Newtonian viscosities, such as Balmforth *et al.* for power-law shear-thinning fluids [13] and Herschel-Bulkley viscoplastic fluids [14] and Millet *et al.* [15] for Carreau shear-thinning fluids.

Most of the theoretical studies concerning flows down an incline make the assumption that the waves propagate in the same direction as the flow (waves denoted as two dimensional). Since the work of Squire [16], many people justify this simplification by saying that the two-dimensional waves are more dangerous than any oblique waves. This was in fact shown by Squire [16] for a Newtonian unidirectional forced flow between rigid boundaries. He showed that there was a relationship between the Reynolds numbers for an oblique wave and a two-dimensional wave, associated with a relationship between the wave numbers, so that the critical Reynolds number for a two-dimensional wave could be shown to be the smallest. Pearlstein [17] and Hesla *et al.* [18] reached the same conclusion for parallel flow of stratified Newtonian fluids. Yih [19] and, more clearly, Chang and Demekhin [20] extended these results to Newtonian flows with free surfaces, interfaces, or density stratification. For free-surface flows down an incline, Yih [19] showed that there were also relationships between the two-dimensional and oblique wave characteristics, allowing one to deduce the stability results for oblique waves from the results obtained for two-dimensional waves. These relationships, however, included a relationship between the slopes of the inclines. This means that the Reynolds number associated with a two-dimensional wave can be shown to be smaller than that for an oblique wave, but this oblique wave being obtained for a larger slope. In other words, using these Squire relationships, you can deduce any three-dimensional threshold for an oblique wave from the two-dimensional threshold for a longitudinal wave at another plane slope. This prevents the possibility to directly compare the thresholds at a given slope. Despite this, some studies as that of Benjamin [1] use the argument of Squire [16] to justify the focus on two-dimensional instabilities. Moreover, in the case of non-Newtonian liquid film flows down an incline, Gupta and Rai [21] for viscoelastic fluids and Sahu and Matar [22] for viscoplastic fluids found that, under certain circumstances, oblique wave instabilities may be the dominant instabilities, in opposition with Squire's theorem. In contrast, Nouar *et al.* [23] studied the three-dimensional temporal linear stability of shear-thinning fluid plane Poiseuille flows and showed that the two-dimensional instabilities seem dominant. They, however, remark that they cannot make use of Squire relationships, which only exist for a reduced problem neglecting some terms connected with the perturbation of the viscosity and not for the general eigenvalue problem.

In this study, we focus on non-Newtonian fluid film flows down an incline and want to compare the thresholds of oblique waves to those of two-dimensional waves for a given slope in order to reach the dominant instability in a given flow configuration. We will consider both Newtonian and generalized Newtonian fluids (in particular, Carreau fluids), with a particular distinction between the shear-thinning ($0 < n < 1$) and the shear-thickening cases ($n > 1$), n being the power-law index. We will also look for the existence or not of Squire relationships, and see whether they can be used in the comparison between the thresholds associated with the different waves.

II. MODEL AND EQUATIONS

We want to model a film flow developing down a plate inclined to the horizontal at an angle γ. We make use of a Cartesian coordinate system in which the origin is taken at the unperturbed free surface, the x axis—parallel to the plate—points down the slope, the z axis is horizontal, and the y axis is taken normal to the plate and oriented toward it. Let $y = \zeta(x,z,t)$ be the equation of the free surface at time t.

We will consider both Newtonian fluids with a constant viscosity η_0 and generalized Newtonian fluids with a viscosity $\bar{\eta}$ following the four-parameter Carreau inelastic model:

$$\frac{\bar{\eta} - \eta_\infty}{\eta_0 - \eta_\infty} = [1 + (\delta\dot{\gamma})^2]^{(n-1)/2}, \qquad (1)$$

with η_0 and η_∞ the limit Newtonian viscosities at low and high shear rate, respectively, δ a characteristic time, n a dimensionless parameter, and $\dot{\gamma}$ the local shear rate. With $0 < n < 1$ ($n > 1$) and $\eta_\infty < \eta_0$ ($\eta_\infty > \eta_0$), this law suitably describes the rheological behavior of shear-thinning (shear-thickening) fluids. Note that the Carreau model predicts a power-law behavior at moderate shear rate. However, unlike the power-law model, it predicts a viscosity that remains finite and tends to η_0 as the shear rate approaches zero. This feature makes the Carreau law particularly suitable for free-surface flow issues. For a given flow rate Q, the layer thickness d cannot be explicitly calculated and will depend on the different Carreau law parameters: therefore, it cannot be taken as a length scale as often done. Using the dimensionless variables proposed by Weinstein [24] [length scale $d_s = (\frac{\eta_0 Q}{\rho g \sin\gamma})^{1/3}$, velocity scale Q/d_s, time scale d_s^2/Q, viscosity scale η_0], the Carreau law for the dimensionless viscosity η becomes

$$\eta = I + (1-I)\left[1 + \left(L\frac{du_b}{dy}\right)^2\right]^{(n-1)/2}, \qquad (2)$$

where $I = \eta_0/\eta_\infty$, $L = \delta Q(\frac{\rho g \sin\gamma}{\eta_0 Q})^{2/3}$ and u_b is the dimensionless basic flow velocity.

The dimensionless equations governing the flow are

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,\\
&\mathrm{Re}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\\
&\quad = -\mathrm{Re}\frac{\partial p}{\partial x} + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right) + 1,\\
&\mathrm{Re}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\\
&\quad = -\mathrm{Re}\frac{\partial p}{\partial y} + \left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}\right) + \cot\gamma,\\
&\mathrm{Re}\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\\
&\quad = -\mathrm{Re}\frac{\partial p}{\partial z} + \left(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right),
\end{aligned} \qquad (3)$$

where u, v, and w are the dimensionless velocities along the x, y, and z directions, respectively; p is the dimensionless pressure (dimensionalized by $\rho Q^2/d_s^2$); σ_{ij} is the dimensionless viscous strain tensor; and Re is the Reynolds number defined by $\mathrm{Re} = \rho Q/\eta_0$. This system has to be solved with the associated boundary conditions at the bottom $y = d$ and at the free surface $y = 0$.

The system (3) has a steady solution u_b featuring a uniform longitudinal flow only depending on the normal direction y. For this basic solution, the equations are reduced to

$$\begin{aligned}
\frac{d\sigma_{xy}^b}{dy} &= \frac{d}{dy}\left(\eta\frac{du_b}{dy}\right) = -1,\\
\frac{dp_b}{dy} &= \frac{\cot\gamma}{\mathrm{Re}},
\end{aligned} \qquad (4)$$

and the boundary conditions correspond to no slip at the bottom and no friction stress at the flat free surface:

$$\begin{aligned}
u_b(y=d) &= 0,\\
\sigma_{xy}^b(y=0) &= 0,
\end{aligned} \qquad (5)$$

where $\sigma_{xy}^b = \eta\frac{du_b}{dy}$ is the only nonzero component of the viscous strain tensor for the basic flow. In the general case where η is given by the Carreau law (2), there is no analytical solution to these equations and u_b has to be obtained numerically.

III. STABILITY ANALYSIS

We perform a temporal linear stability study on this problem. The basic flow is perturbed by fluctuations of the velocity, u', v', and w', and of the pressure p' and by the fluctuation of the free surface, ζ'. We obtain $u = u_b(y) + u'(x,y,z,t)$, $v = v'(x,y,z,t)$, $w = w'(x,y,z,t)$, $p = p_b(y) + p'(x,y,z,t)$, $\zeta = \zeta'(x,z,t)$, and $\sigma_{xy} = \sigma_{xy}^b(y) + \sigma'_{xy}(x,y,z,t)$. These expressions are substituted into the system (3) and after linearization, we get

$$\begin{aligned}
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0,\\
\mathrm{Re}\left(\frac{\partial u'}{\partial t} + u_b\frac{\partial u'}{\partial x} + v'\frac{du_b}{dy}\right) &= -\mathrm{Re}\frac{\partial p'}{\partial x} + \left(\frac{\partial \sigma'_{xx}}{\partial x} + \frac{\partial \sigma'_{xy}}{\partial y} + \frac{\partial \sigma'_{xz}}{\partial z}\right),\\
\mathrm{Re}\left(\frac{\partial v'}{\partial t} + u_b\frac{\partial v'}{\partial x}\right) &= -\mathrm{Re}\frac{\partial p'}{\partial y} + \left(\frac{\partial \sigma'_{yx}}{\partial x} + \frac{\partial \sigma'_{yy}}{\partial y} + \frac{\partial \sigma'_{yz}}{\partial z}\right),\\
\mathrm{Re}\left(\frac{\partial w'}{\partial t} + u_b\frac{\partial w'}{\partial x}\right) &= -\mathrm{Re}\frac{\partial p'}{\partial z} + \left(\frac{\partial \sigma'_{zx}}{\partial x} + \frac{\partial \sigma'_{zy}}{\partial y} + \frac{\partial \sigma'_{zz}}{\partial z}\right),
\end{aligned} \qquad (6)$$

where the strain perturbations are

$$\begin{aligned}
&\sigma'_{xx} = 2\eta\frac{\partial u'}{\partial x}; \quad \sigma'_{yy} = 2\eta\frac{\partial v'}{\partial y}; \quad \sigma'_{zz} = 2\eta\frac{\partial w'}{\partial z},\\
&\sigma'_{xz} = \eta\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right); \quad \sigma'_{yz} = \eta\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right); \quad \sigma'_{xy} = \theta\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right).
\end{aligned} \qquad (7)$$

Note that in the expression of the perturbed viscous strain in the (x,y) plane, a new viscosity θ appears [10,23]. This is due to the fact that, in this plane, the velocity perturbations induce viscosity perturbations. θ is given by

$$\theta = I + (1-I)\left[1 + n\left(L\frac{du_b}{dy}\right)^2\right]\left[1 + \left(L\frac{du_b}{dy}\right)^2\right]^{(n-3)/2}. \qquad (8)$$

The boundary conditions associated with this perturbation problem are the no-slip condition at the bottom:

$$u' = v' = w' = 0 \quad \text{at } y = d, \tag{9}$$

the kinematic condition at the free surface:

$$v' = \frac{\partial \zeta'}{\partial t} + U_b \frac{\partial \zeta'}{\partial x} = 0 \quad \text{at } y = 0, \tag{10}$$

the zero viscous strain at the perturbed free surface, which gives

$$\begin{cases} \sigma'_{xy} - \zeta' = 0 \\ \sigma'_{yz} = 0 \\ \sigma'_{xz} = 0 \end{cases} \quad \text{at } y = 0, \tag{11}$$

and the normal strain balance at the perturbed free surface, which gives

$$\zeta' \cot\gamma + \text{Re}\, p' - 2\eta \frac{\partial v'}{\partial y} + \frac{1}{\text{Ca}}\left(\frac{\partial^2 \zeta'}{\partial x^2} + \frac{\partial^2 \zeta'}{\partial z^2}\right) = 0 \quad \text{at } y = 0, \tag{12}$$

where $\text{Ca} = \frac{\eta_0 Q}{\sigma d_s}$ is the capillary number and σ is the surface tension.

The perturbations of velocity, u', v', and w', and of pressure p' and the fluctuation of the free surface, ζ', are expressed as three-dimensional normal modes:

$$[u',v',w',p'](x,y,z,t) = [\hat{u},\hat{v},\hat{w},\hat{p}](y)\, e^{i(\alpha x + \beta z - \alpha c t)},$$
$$\zeta'(x,z,t) = \hat{\zeta} e^{i(\alpha x + \beta z - \alpha c t)}, \tag{13}$$

where $(\hat{u},\hat{v},\hat{w},\hat{p},\hat{\zeta})$ are complex variables, $(\alpha,0,\beta)$ is the real wave vector (with $\alpha^2 + \beta^2 = k^2$), and c is the dimensionless complex celerity of the wave. Its real part c_r gives the dimensionless phase velocity and its imaginary part c_i gives the growth rate $\omega_i = \alpha c_i$. A stable (unstable) flow will correspond to negative (positive) values of c_i and the perturbations will be called three-dimensional if $\beta \neq 0$ and two-dimensional if $\beta = 0$. Introducing this formulation in the governing equations for the perturbations (6) and combining these equations appropriately, we obtain a system of two coupled equations for the unknowns $(\hat{v},\hat{w})$:

$$\begin{aligned} &i\alpha\, \text{Re}[(u_b - c)(D^2 - k^2) - D^2(u_b)]\hat{v} \\ &\quad = -4k^2 D(\eta D\hat{v}) + [D^2\theta + 2D\theta D + \theta(D^2 + k^2)] \\ &\qquad \times (D^2 + k^2)\hat{v} + \underline{i\beta(D^2+k^2)[(\theta-\eta)(D\hat{w} + i\beta\hat{v})]}, \end{aligned} \tag{14}$$

$$\begin{aligned} &i\alpha\, \text{Re}(u_b - c)(i\beta\hat{v} - D\hat{w}) - i\alpha\, \text{Re} D u_b \hat{w} \\ &\quad = \beta^2\theta D\hat{w} + k^2 D(\eta\hat{w}) - i\beta\theta(D^2 + \alpha^2)\hat{v} + 3i\beta D(\eta D\hat{v}) \\ &\qquad - i\beta(D^2 + \beta^2)(\eta\hat{v}) - (D^2 + \beta^2)(\eta D\hat{w}), \end{aligned} \tag{15}$$

where $D = \frac{d}{dy}$ is the derivative with respect to the normal direction y. The first equation is a generalized Orr-Sommerfeld equation for the normal velocity perturbation $\hat{v}$, which, however, cannot be solved alone in the general case due to the presence of the underlined term involving the perturbation $\hat{w}$. To satisfy the boundary conditions, we must note that, at the free surface, the basic flow shear is zero ($Du_b = 0$), so that $\eta = \theta = 1$ and $D\eta = D\theta = 0$. These boundary conditions are

$$\hat{v} = D\hat{v} = \hat{w} = 0 \quad \text{at } y = d, \tag{16}$$

$$[1 + (u_b - c)(D^2 + k^2)]\hat{v} = 0 \quad \text{at } y = 0, \tag{17}$$

$$i\beta\hat{v} + D\hat{w} = 0 \quad \text{at } y = 0, \tag{18}$$

$$-\beta D\hat{v} + i(\alpha^2 - \beta^2)\hat{w} = 0 \quad \text{at } y = 0, \tag{19}$$

$$\begin{aligned} &-ik^2(D^2+k^2)\left[\frac{1}{\alpha \tan\gamma} + \frac{k^2}{\alpha\, \text{Ca}}\right]\hat{v} + 4k^2 D\hat{v} + i\alpha\, \text{Re}(u_b - c) \\ &\quad \times D\hat{v} - D[(D^2 + k^2)\hat{v}] = 0 \quad \text{at } y = 0. \end{aligned} \tag{20}$$

In the case of two-dimensional perturbations ($\beta = 0$, $k = \alpha$), the system (14)-(15) for generalized Newtonian fluids is reduced to Eq. (14) without the underlined term, i.e., a generalized Orr-Sommerfeld equation given in [10]:

$$\begin{aligned} &i\alpha\, \text{Re}[(u_b - c)(D^2 - k^2) - D^2(u_b)]\hat{v} \\ &\quad = -4k^2 D(\eta D\hat{v}) + [D^2\theta + 2D\theta D + \theta(D^2+k^2)](D^2+k^2)\hat{v}, \end{aligned} \tag{21}$$

which, for two-dimensional perturbations, is solved with $\alpha = k$. Conversely, the three-dimensional perturbations for a Newtonian fluid ($\theta = \eta = 1$) satisfy the usual Orr-Sommerfeld equation deduced from (14):

$$i\alpha\, \text{Re}[(u_b - c)(D^2 - k^2) - D^2(u_b)]\hat{v} = (D^2 - k^2)^2\hat{v}. \tag{22}$$

IV. NUMERICAL PROCEDURE

A spectral Tau collocation method based on Chebyshev polynomials is used for the discretization of the generalized eigenvalue problem (14)–(20). The resulting system of algebraic equations, solved on the Gauss-Lobatto collocation points [$y_j = \cos(j\pi/N)$ for $j = 0,N$] in the layer, can be written in the abbreviated form

$$[A]X = \omega[B]X, \tag{23}$$

where X is the vector containing the algebraic values of $\hat{v}$ and $\hat{w}$ at each collocation point. The dimension of the square matrices $[A]$ and $[B]$ is twice the number of modes $N + 1$. The eigenvalues obtained when solving (23) are the complex angular frequencies $\omega = \alpha c$, and the imaginary part of ω is the growth rate ω_i.

From the spectra obtained by solving (23), we compute neutral curves (values of Re for which an eigenmode has a zero growth rate, whereas all the other eigenmodes have a negative growth rate) depending on the wave numbers α and β, from which critical Reynolds number Re_c can be obtained by minimization along α and β. This numerical procedure has been validated in former studies [10,15].

V. RESULTS FOR A NEWTONIAN FLUID

In the case of a Newtonian fluid, the three-dimensional stability problem is governed by the Orr-Sommerfeld equation (22) and the associated boundary conditions [those on $\hat{v}$ deduced from (16), (17), and (20)]. As already shown by Yih [19] and Chang and Demekhin [20], we have different relationships between the characteristics of the oblique waves and of the two-dimensional waves, which we will denote with the subscripts 3D and 2D, respectively. If the wave number for the two-dimensional waves, $\alpha_{2\mathrm{D}}$, is directly denoted as k ($\alpha_{2\mathrm{D}} = k$ in this case) and the wave numbers for the oblique waves simply as α and β, the different relationships are

$$\alpha^2 + \beta^2 = k^2, \tag{24}$$

$$\alpha\,\mathrm{Re}_{3\mathrm{D}} = k\,\mathrm{Re}_{2\mathrm{D}}, \tag{25}$$

$$\alpha\,\tan\gamma_{3\mathrm{D}} = k\,\tan\gamma_{2\mathrm{D}}, \tag{26}$$

$$\alpha\,\mathrm{Ca}_{3\mathrm{D}} = k\,\mathrm{Ca}_{2\mathrm{D}}. \tag{27}$$

Equations (24) and (25) come from the Orr-Sommerfeld equation (as for rigid boundaries), and (26) and (27) from the boundary conditions. These relationships indicate that the stability results for the oblique waves can be obtained from those for the two-dimensional waves, but for different involved plate inclinations. Then we cannot easily conclude for the comparison at a given plate inclination.

It is then interesting to compare numerically the three-dimensional and the two-dimensional instability thresholds. This comparison is given in Fig. 1 for a plate inclination $\gamma = 2^\circ$ and $1/\mathrm{Ca} = 0$. The neutral curves, Re versus α, have been first obtained for given values of the transverse wave number β [Fig. 1(a)] and then for given values of the wave obliquity angle i_{ob} defined as $\tan(i_{\mathrm{ob}}) = \beta/\alpha$ [Fig. 1(b)]. Remember that below (above) these neutral curves, the film flow is stable (unstable) with respect to the corresponding wave. The two-dimensional neutral curve decreases regularly when decreasing α and tends towards a minimum value when the longitudinal wave number α becomes small. In contrast, when decreasing α, the three-dimensional neutral curves obtained at given values of β decrease towards a minimum reached for a finite value of α and then strongly increase when α becomes small. A decrease of β induces a drift of the minimum towards lower values of both α and Re; the increase at small α is thus observed to be steeper. In any case, these neutral curves appear to be above the two-dimensional neutral curve. When obtained at given obliquity angle [Fig. 1(b)], the neutral curves look more similar to the two-dimensional curve, with a monotonous variation and a minimum reached asymptotically for small α. These curves, still above the two-dimensional curve, continuously tend to this curve when the obliquity angle i_{ob} is decreased to 0°. It is then interesting to see that the neutral curves obtained in Fig. 1 are physically equivalent, and that some properties of these three-dimensional neutral curves can be inferred from the Squire relationships (24)–(27).

Since surface tension only shifts the neutral curves without any influence on the associated thresholds for the long wavelength instabilities considered here [8], we will still assume for simplicity that $1/\mathrm{Ca} = 0$. If we first consider purely transverse instabilities, i.e., $\beta \neq 0$ and $\alpha = 0$ ($k = \beta$), the relation (26) gives $\gamma_{2\mathrm{D}} = 0$. As there is no instability in the horizontal situation, $\mathrm{Re}_{2\mathrm{D}} \to \infty$ and the film is linearly stable. Finally, the relation (25) gives $\mathrm{Re}_{3\mathrm{D}} \gg \mathrm{Re}_{2\mathrm{D}}$, so that $\mathrm{Re}_{3\mathrm{D}} \to \infty$. We then obtain that for any nonzero wave vector β, the neutral curve obtained for a given inclination $\gamma_{3\mathrm{D}}$ tends towards infinity when α decreases to zero, which is observed

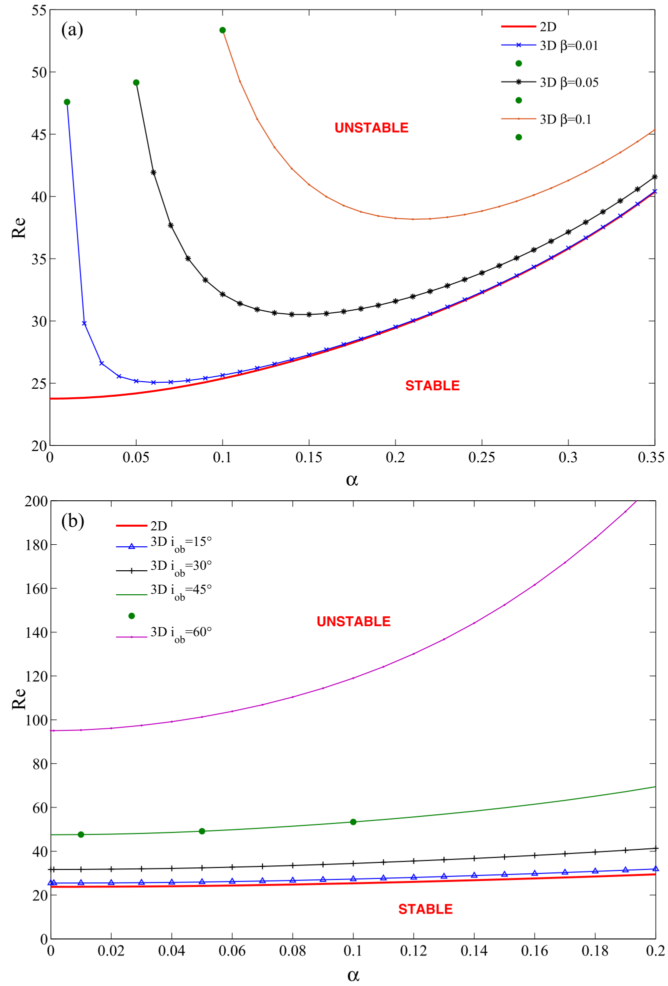

FIG. 1. (Color online) Three-dimensional wave stability results for the flow down an incline in the case of a Newtonian fluid and for a fixed inclination of the plate $\gamma = 2^\circ$ ($1/\mathrm{Ca} = 0$). The neutral curves, expressed as the Reynolds number Re versus the streamwise wave number α, are given for different values of the transverse wave number β (a), or for different obliquity angles of the waves i_{ob} (b). The film flow is stable (unstable) below (above) the curves, with respect to the different waves considered. In (a), for each value of β, the points corresponding to $\alpha = \beta$ are indicated. These points correspond to those shown in (b) for $i_{\mathrm{ob}} = 45^\circ$, showing the correspondence between the two plots. The two-dimensional wave neutral curve is given as a red heavy solid line for comparison.

in Fig. 1(a). Conversely, if we now consider that the waves have a given obliquity angle i_{ob} ($i_{\mathrm{ob}} \neq 90^\circ$), relation (26) gives $\tan(\gamma_{2\mathrm{D}}) = \tan(\gamma_{3\mathrm{D}})\cos(i_{\mathrm{ob}})$, so that $\gamma_{2\mathrm{D}}$ is nonzero and $\mathrm{Re}_{2\mathrm{D}}$ is finite. Relation (25) then gives $\mathrm{Re}_{3\mathrm{D}} = \mathrm{Re}_{2\mathrm{D}}/\cos(i_{\mathrm{ob}})$, indicating that $\mathrm{Re}_{3\mathrm{D}}$ also remains finite. We then obtain that for a given obliquity angle $i_{\mathrm{ob}} \neq 90^\circ$, the neutral curve obtained for a given inclination $\gamma_{3\mathrm{D}}$ remains finite when α decreases to zero, which is observed in Fig. 1(b).

The comparison shown above seems to indicate that the three-dimensional thresholds are larger than the two-dimensional thresholds. To justify this, it is interesting to consider the two-dimensional wave stability results which are shown in Fig. 2 as heavy dashed lines. When the inclination of the plate γ increases, the neutral curves [Fig. 2(a)] globally

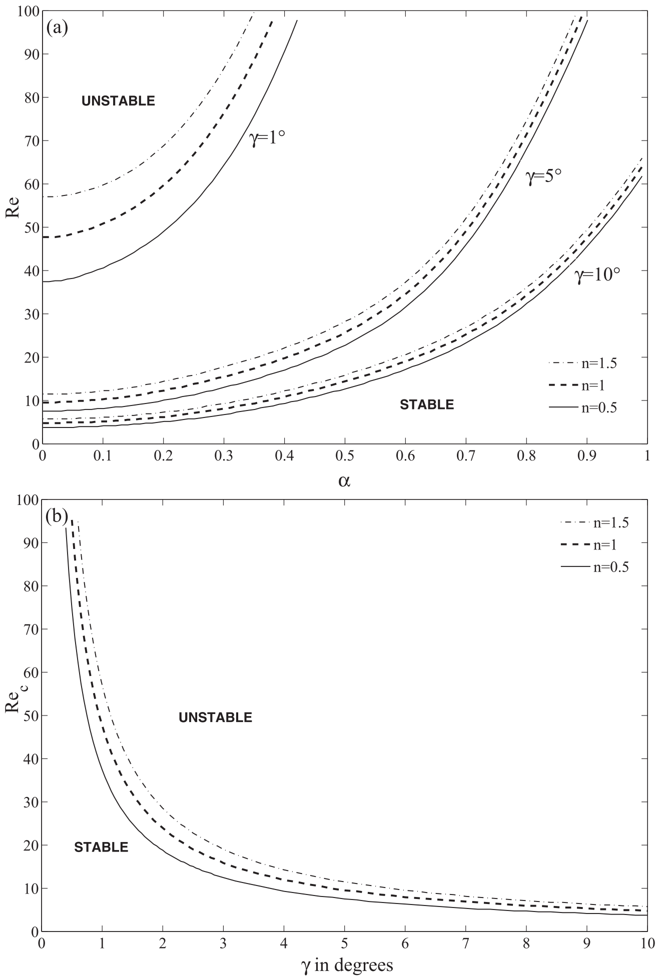

FIG. 2. Two-dimensional wave stability results for the flow down an incline in the case of generalized Newtonian fluids ($1/\mathrm{Ca} = 0$): (a) neutral curves expressed as the Reynolds number Re versus the streamwise wave number α for different inclinations γ of the plate; (b) stability curve showing the critical Reynolds number Re_c as a function of the inclination γ. The film flow is stable (unstable) below (above) the curves, with respect to the different waves considered. In (a) and (b), the cases corresponding to a Newtonian fluid (heavy dashed lines, $n = 1$) are compared to those corresponding to a shear-thinning fluid ($L = 0.5$, $n = 0.5$, $I = 0$) (solid lines) and those corresponding to a shear-thickening fluid ($L = 0.5$, $n = 1.5$, $I = 0$) (dashed-dotted lines).

decrease, as well as the minimum value Re_c reached for small α. The stability curve showing Re_c versus γ [Fig. 2(b), heavy dashed line] is then a continuously decreasing curve, below (above) which the film flow is stable (unstable) for these two-dimensional instabilities. Returning to the Squire relationships used for $1/\mathrm{Ca} = 0$, we can write

$$\mathrm{Re}_{3\mathrm{D}}(\gamma_{3\mathrm{D}}) > \mathrm{Re}_{2\mathrm{D}}(\gamma_{2\mathrm{D}}), \tag{28}$$

from (25) and $\gamma_{3\mathrm{D}} > \gamma_{2\mathrm{D}}$ from (26). The decrease of the curves in Fig. 2 then indicates that $\mathrm{Re}_{2\mathrm{D}}(\gamma_{2\mathrm{D}}) > \mathrm{Re}_{2\mathrm{D}}(\gamma_{3\mathrm{D}})$, and the combination with (28) finally gives that

$$\mathrm{Re}_{3\mathrm{D}}(\gamma_{3\mathrm{D}}) > \mathrm{Re}_{2\mathrm{D}}(\gamma_{3\mathrm{D}}). \tag{29}$$

Using the Squire relationships together with the results obtained for two-dimensional waves, it is thus possible to show that, for a Newtonian fluid flowing down an incline at a given slope, the two-dimensional waves are always the more dangerous. Note that this proof uses a property of the two-dimensional stability study result, namely, that $\mathrm{Re}_{2\mathrm{D}}(\gamma_{2\mathrm{D}})$ monotonically decreases. In this case, Squire's theorem is thus extended to the dominance of two-dimensional instabilities at a given slope. However, this property can only be deduced *a posteriori* rather than stated prior to the two-dimensional study.

VI. RESULTS FOR A GENERALIZED NEWTONIAN FLUID

The general system we have to solve for a generalized Newtonian fluid flowing down an incline is given by Eqs. (14) and (15) with associated boundary conditions. As already shown, for two-dimensional waves, the system is reduced to Eq. (21) with $\alpha = k$. The two-dimensional wave results obtained in the shear-thinning case for $L = 0.5$, $n = 0.5$, and $I = 0$ and in the shear-thickening case for $L = 0.5$, $n = 1.5$, and $I = 0$ are also given in Fig. 2 with solid lines and dashed-dotted lines, respectively. These stability results obtained for generalized Newtonian fluids look similar to those obtained for Newtonian fluids, with similar neutral curves decreasing as a whole when γ is increased [Fig. 2(a)] and a stability curve decreasing as well [Fig. 2(b)]. All these values, however, are smaller (larger) for the shear-thinning fluid (shear-thickening fluid) than for the Newtonian fluid. The influence of L on the critical Reynolds number for these generalized Newtonian fluids is shown in Fig. 3 for a fixed inclination angle $\gamma = 2^\circ$. As expected, we observe a decrease of the critical Reynolds number with L for the shear-thinning fluids ($0 < n < 1$) and an increase for the shear-thickening

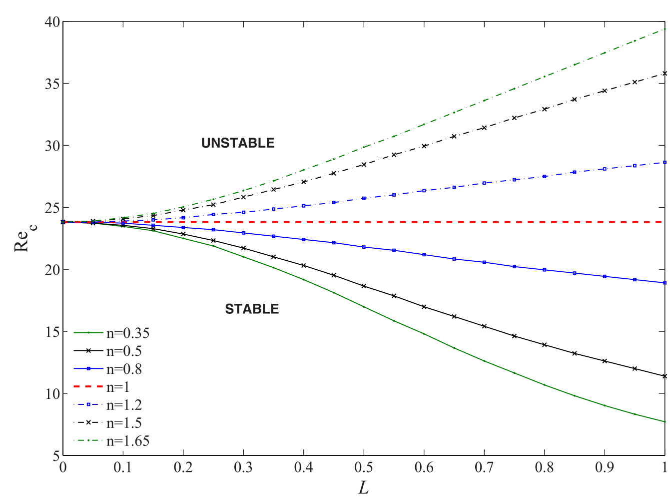

FIG. 3. (Color online) Two-dimensional wave stability results for the flow down an incline for a fixed inclination of the plate, $\gamma = 2^\circ$ ($I = 0$, $1/\mathrm{Ca} = 0$): stability curves showing the critical Reynolds number Re_c as a function of L. The results are obtained for different values of the power-law index for shear-thinning (solid lines) and shear-thickening (dashed-dotted lines) fluids. The constant value obtained in the Newtonian case is given as a heavy (red) dashed line for comparison.

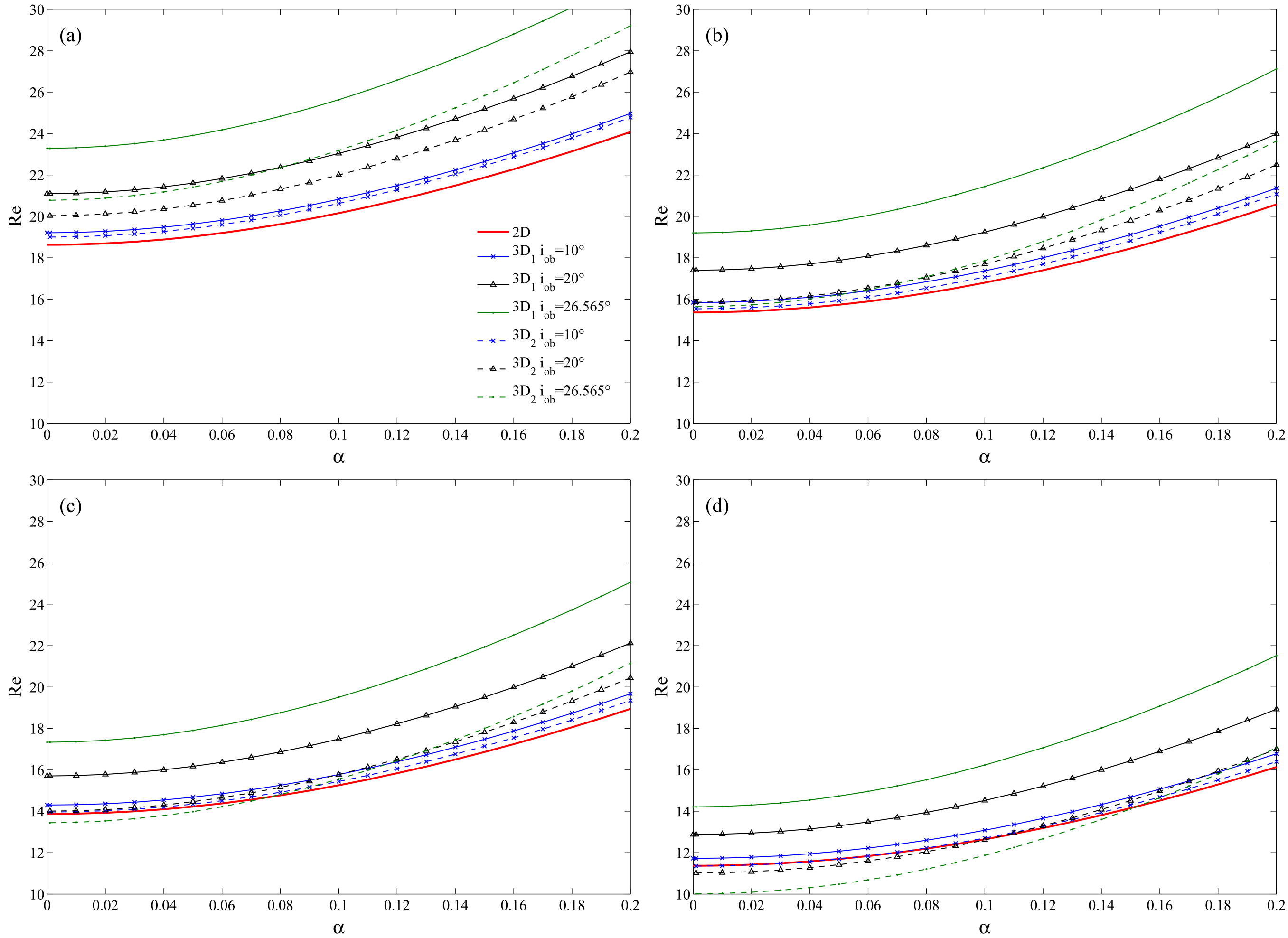

FIG. 4. (Color online) Three-dimensional wave stability results for the flow down an incline in the case of a shear-thinning fluid and for a fixed inclination of the plate, $\gamma = 2^\circ$ ($n = 0.5$, $I = 0$, $1/\mathrm{Ca} = 0$). The neutral curves, expressed as the Reynolds number Re versus the streamwise wave number α, are given for different obliquity angles of the wave, i_{ob}, and for different values of L, $L = 0.5$ (a), $L = 0.7$ (b), $L = 0.8$ (c), and $L = 1$ (d). The results obtained in the general case ($3D_2$, dashed lines) are compared with those obtained with the reduced model ($3D_1$, solid lines). The curve for $i_{\mathrm{ob}} = 26.565^\circ$ corresponds to $\beta = \alpha/2$. For each value of L, the two-dimensional wave neutral curve is given as a heavy (red) solid line for comparison.

fluids ($n > 1$). Note also the symmetry around the Newtonian case ($n = 1$) in the neutral curves between the shear-thickening ($n = 1.2$, 1.5, and 1.65) and the shear-thinning ($n = 0.8$, 0.5, and 0.35) cases.

Following the same assumptions as Nouar *et al.* [23], a mathematical simplification of the three-dimensional wave problem can be obtained if we consider that $\theta = \eta$ in the underlined term in Eq. (14). This assumes that the perturbations of viscosity are neglected in this term, which could be thought as a weak assumption as the instability is known to be driven by the shear at the free surface, where $\theta = \eta = 1$. With this assumption, the system to solve is reduced to a single Orr-Sommerfeld equation given by (21). For this equation and the associated boundary conditions, the same Squire relationships as in (24)–(27) can be derived, indicating that the same type of neutral curves as in Fig. 1 can be obtained for this three-dimensional wave reduced problem. Moreover, the Squire relationships and the decrease of the two-dimensional wave stability curves [Fig. 2(b)] also indicate that, for shear-thinning fluids as well as for shear-thickening fluids, the two-dimensional wave thresholds are smaller than the three-dimensional wave thresholds obtained for this reduced problem.

Before claiming that Squire's theorem is extended to generalized Newtonian fluid film flows, it is now useful to solve the general system (14)–(15) and check if the indications obtained with the reduced problem can be confirmed or not. The neutral curves obtained for an inclination $\gamma = 2^\circ$, $n = 0.5$ (shear-thinning case), $I = 0$, and different obliquity angles of the three-dimensional waves are shown in Fig. 4, both for the general problem (denoted as $3D_2$) and the reduced problem (denoted as $3D_1$). The two-dimensional results ($i_{\mathrm{ob}} = 0^\circ$) are also given for comparison. Each graph corresponds to a different value of the parameter L, $L = 0.5$, 0.7, 0.8, and 1. As expected, the neutral Reynolds number values obtained with the reduced problem are above those obtained in the two-dimensional case. The values for the general problem, however, are below those for the reduced problem, sometimes much below, indicating that the assumption used to reduce the three-dimensional problem is not so weak in the shear-thinning

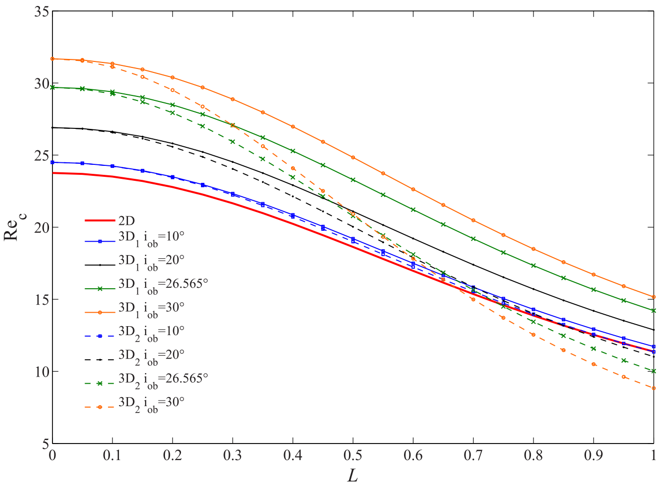

FIG. 5. (Color online) Three-dimensional wave stability results for the flow down an incline in the case of a shear-thinning fluid and for a fixed inclination of the plate, $\gamma = 2^\circ$ ($n = 0.5$, $I = 0$, $1/\mathrm{Ca} = 0$): stability curves showing the critical Reynolds number Re_c as a function of L. The results are obtained for different obliquity angles of the waves (from $i_{\mathrm{ob}} = 0$ to 30°), for the general problem ($3\mathrm{D}_2$, dashed lines) and for the reduced model ($3\mathrm{D}_1$, solid lines). The two-dimensional wave stability results are given as a heavy (red) solid line for comparison.

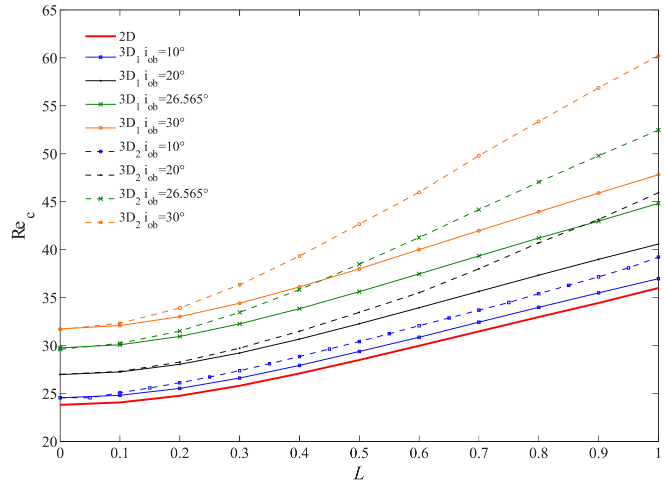

FIG. 6. (Color online) Three-dimensional wave stability results for the flow down an incline in the case of a shear-thickening fluid and for a fixed inclination of the plate, $\gamma = 2^\circ$ ($n = 1.5$, $I = 0$, $1/\mathrm{Ca} = 0$): stability curves showing the critical Reynolds number Re_c as a function of L. The results are obtained for different obliquity angles of the waves (from $i_{\mathrm{ob}} = 0$ to 30°), for the general problem ($3\mathrm{D}_2$, dashed lines) and for the reduced model ($3\mathrm{D}_1$, solid lines). The two-dimensional wave stability results are given as a heavy (red) solid line for comparison.

case. For small values of L as $L \leqslant 0.7$, the two-dimensional neutral values seem to remain the smallest. In contrast, for larger values of L, the neutral curves of the general problem can be below the two-dimensional neutral curves, particularly for large values of the obliquity angle i_{ob}. The critical curves, Re_c versus L, obtained in the different approximations of the problem are also shown in Fig. 5 for different obliquity angles. In any case, the curves decrease when L is increased. The critical curves for the general problem, however, decrease more rapidly than those obtained for the reduced problem. This effect is particularly important for the large obliquity angles, where the departure between the two curves corresponding to the different approximations strongly increases with L. As a result, the critical curves for the general problem can decrease below the two-dimensional wave curves when L is increased, and this effect seems to occur at smaller L values when the obliquity angle i_{ob} is increased. These results thus indicate that three-dimensional wave instabilities can be the more dangerous in shear-thinning fluids, particularly those with a large obliquity angle, and for sufficient values of L. Another conclusion is that the perturbations of viscosity cannot be neglected in this three-dimensional wave instability problem for shear-thinning fluids.

The three-dimensional wave instability curves obtained by solving either the general problem or the reduced problem in the shear-thickening case are also given in Fig. 6. For the shear-thickening case, both types of critical curves increase when L is increased, but the curves obtained for the general problem increase more rapidly than those obtained for the reduced problem. As was previously shown that the instability thresholds for the reduced problem are higher than the two-dimensional thresholds, the critical curves for the general problem in the shear-thickening case will remain above the corresponding two-dimensional critical curve for all values of the obliquity angle i_{ob}. These results thus indicate that, in the shear-thickening case, the oblique wave instabilities are never the dominant instabilities for any value of L, even if the Squire relationships cannot be derived for the general problem in this case.

VII. CONCLUSION

This study has been focused on the possible occurrence of three-dimensional wave instabilities as the dominant instability in flows down an incline. The two cases of Newtonian fluids and generalized Newtonian fluids have been considered. For Newtonian fluids, it was possible to extend Squire's theorem and show that the three-dimensional wave instabilities are never the dominant instabilities at a given inclined plane slope, so that the well known long wavelength free-surface two-dimensional waves remain the more dangerous. This result cannot be obtained from the Squire relationships alone, but needs to make use of the particular variation of the two-dimensional critical curve with regard to the slope. The result was further confirmed by some three-dimensional wave stability results. In contrast, for generalized Newtonian fluids, Squire relationships only exist for a reduced problem neglecting some terms connected with the perturbation of the viscosity and not for the general problem. For this nonphysical reduced problem, we can still conclude that the long wavelength free-surface two-dimensional waves are the more dangerous. For the general problem, however, no conclusion can be obtained from Squire considerations as Squire relationships cannot be derived for these generalized Newtonian fluids. Nevertheless, some numerical stability calculations have shown that the thresholds for oblique waves can be smaller than the thresholds for two-dimensional waves obtained at the same inclined plane slope, particularly for large obliquity angles and strong shear-thinning properties, whereas the thresholds for oblique waves are always higher than the two-dimensional thresholds for shear-thickening fluids.

In conclusion, Squire's theorem, which says that the two-dimensional instabilities are the more dangerous and is known to be valid in a channel flow of Newtonian fluid, has been shown to remain valid for a flow down an incline for a Newtonian fluid, considering longitudinal and oblique waves at the same inclination angle. It seems to be also valid for shear-thickening fluids. In contrast, for shear-thinning fluids, this theorem is no more valid, as cases have been found in which the oblique waves are the more dangerous instabilities at a given inclination angle.

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