Phenomenological Blasius-type friction equation for turbulent power-law fluid flows

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We propose a friction formula for turbulent power-law fluid flows, a class of purely viscous non-Newtonian fluids commonly found in applications. Our model is derived through an extension of the friction factor analysis based on Kolmogorov's phenomenology, recently proposed by Gioia and Chakraborty. Tests against classical empirical data show excellent agreement over a significant range of Reynolds number. Limits of the model are also discussed.

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I. INTRODUCTION

Since the pioneering works of Prandtl and Blasius in the early 1900's, the physics and the engineering communities have put a great effort into describing kinetic energy loss and pressure drop in the transportation of fluids along pipes and channels. This endeavor has been mainly driven by industrial applications, where the challenge is to maximize the fraction of the energy input into the system, usually through pumps, that is transferred to the mean axial momentum.

The main source of loss of the axial momentum is the viscous dissipation through frictional effects, mainly imposed by the no-slip boundary condition on rigid walls. In the vicinity of the wall, neighboring parallel layers with different characteristic velocities generate large velocity gradients, inducing energy dissipation through viscous (molecular) diffusion.

In turbulent flows, an important mechanism to describe the loss of the mean axial momentum is due to an interplay between diffusive and inertial effects. In the transitional region between the viscous and inertial sublayers, small instabilities between parallel layers, triggered by viscous stress, are amplified by inertial forces. As the Reynolds number, a nondimensional ratio of viscous to inertial forces, increases above a critical level, the flow develops an intermittent population of coherent vortical eddies transporting parcels of fluid from the high-momentum centerline region to the low-momentum near-wall region, and vice versa. This momentum transport, orthogonal to the main axial direction, retains part of the inputted energy. However, the main contribution of this turbulent-mixing mechanism is to enhance the viscous energy dissipation, by moving the high momentum parcels of the flow near the centerline to the highly dissipative viscous sublayer near the wall. This has led to several attempts to stabilize turbulent flows through the use of riblets on the rigid surfaces and through the addition of chemicals in the fluid.

Indeed, in Ref. [1], Toms showed that the addition of a minute amount of polymers in a turbulent Newtonian solvent, resulting in a non-Newtonian solution, can significantly reduce drag. The physical mechanism behind this remarkable experimental fact remains elusive, and most works concerning drag reduction mechanisms focus on the elastic nature of the solutions, see Refs. [2,3]. However, it has been shown that fluid flows described by purely viscous non-Newtonian models, such as the power-law model (Ostwald de Waele model), also displays drag reducing properties, see Ref. [4].

Power-law fluids are commonly used to model the transport of polymeric substances in industrial processes. They are characterized by a nonconstant and nonlinear relation between the applied stress, τ, a quantity expressing the internal forces that neighboring particles of a continuous media impart on each other, and the rate of strain, $\dot{\gamma} \equiv (\nabla \mathbf{u} + \nabla \mathbf{u}^t)$, a measure of the deformation of the fluid expressed in terms of gradients of velocity. The power-law model is built by replacing the Newtonian constant dynamic viscosity, μ, by a general viscosity, η, a function of the magnitude of the rate of strain. More precisely, for power-law fluids,

$$\tau = \eta(|\dot{\gamma}|)\,\dot{\gamma}, \tag{1}$$

where $\eta(|\dot{\gamma}|) = K|\dot{\gamma}|^{n-1}$, and $|\cdot|$ is a Euclidian tensor norm. The constant K is the proportionality consistency parameter, and n is the flow index, measuring the degree to which the fluid is shear thickening ($n > 1$), or shear thinning ($n < 1$). Although much of our subsequent analysis holds for general power-law fluids, emphasis will be placed on shear-thinning fluids for its drag-reducing properties, which results from the attenuation of both viscous and inertial effects. Indeed, experimental data shows that turbulent fluctuations are suppressed in directions orthogonal to the principal axial direction, see Refs. [4], [5], [6]. This happens because large velocity gradients lower local viscosity, so that neighboring parallel layers with different velocities transfer less momentum to each other. This reduces instability between the layers, and attenuates the flow's orthogonal motion. Figure 1 displays snapshots of power-law flows obtained from DNS data simulated by the authors. The ratio of inertial to viscous effects, as measured by Re_τ, are fixed around $\mathrm{Re}_\tau = 400$. One observes that the flow pattern is less heterogeneous, the lower the index n, indicating a reduced velocity fluctuation.

II. FRICTION FACTOR FOR POWER-LAW FLOWS

The viscous effects of the wall vicinity on the mean axial momentum can be encapsulated in a single number, the friction factor, f, a dimensionless quantity defined as $f = 2\tau_w/\rho U^2$, where U is a characteristic velocity of the flow, usually the mean velocity, ρ is the fluid's density, and τ_w is the wall shear

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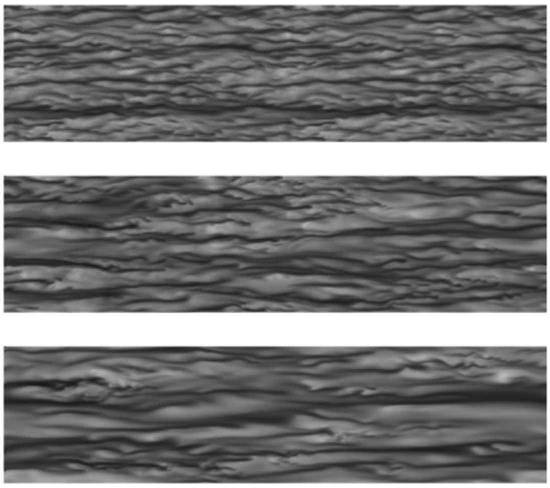

FIG. 1. Axial velocity for DNS data for power-law flows in a 3D channel with smooth walls at $y+ \approx 15$, and $\mathrm{Re}_\tau = 400$. From top to bottom, Newtonian ($n = 1.0$), PL ($n = 0.75$), PL ($n = 0.5$). The flow is from left to right. White represents high velocity and black low. Momentum imbalances decrease with n.

stress, a measure of force per unit area exerted by the fluid on the wall surface. In Ref. [7], Blasius established the empirical law $f \sim 1/\mathrm{Re}^{1/4}$, relating the friction factor and the Reynolds number, after the onset of turbulence, valid over the range Re ∼ 3000–100 000. In Ref. [8], Nikuradse extended Blasius' results, and included the dependence of the roughness of the wall, r, as a fraction of the radius of the pipe, R.

In Ref. [9], Metzner and Reed introduced a description of different power-law flows with a newly defined Reynolds number, based on a suitable definition of a global effective kinematic viscosity, ν, which takes into account the dependency of the local dynamic viscosity on the characteristics of the flow itself,

$$\mathrm{Re} = \mathrm{Re}_{MR} = \frac{UL}{\nu}, \quad \nu \equiv \frac{K((3n+1)/4n)^n 8^{n-1}}{\rho U^{1-n} L^{n-1}}. \tag{2}$$

As for the Newtonian case, this is a nondimensional ratio of viscous and inertial properties of the flow, where the numerical constants are chosen based on similarity arguments for friction factor of laminar flows. In Ref. [10], Dodge and Metzner published a semiempirical analysis of the friction factor of fully developed turbulent power-law fluid flows based on Re_{MR}. They have obtained the following relation

$$\frac{1}{\sqrt{f}} = \frac{4}{n^{0.75}} \log_{10}\left(\frac{\mathrm{Re}}{f^{\frac{n-2}{2}}}\right) - \frac{0.4}{n^{1/2}}. \tag{3}$$

For $n = 1$, (3) reduces to the celebrated Prandtl's equation for Newtonian flows. Indeed, the form of DM's equation can be obtained by assuming that dominant dissipation mechanisms stems from the logarithmic layer, see Ref. [11]. In practical applications, Eq. (3) is the most commonly used friction factor formula for power-law fluids among several available explicit formulas, see Ref. [12] for a statistical comparison of friction formulas. We remark that these formulas are mostly empirical models, lacking a solid physical basis.

III. KOLMOGOROV'S SCALING

Kolmogorov's theory of fully developed turbulent Newtonian fluid flows was originally formulated in purely dimensional grounds, and it is based on Richardson's energy cascade scenario. Fluctuating energy is injected at large scales, it cascades through intermediate scales, the so-called inertial range, and it is dissipated at small scales. The cascade hypothesis states that in the inertial range, the average energy flux is constant, independent of the kinematic viscosity, ν, and equals the mean energy dissipation rate, ϵ, see Refs. [11,13].

Denoting the energy flux at scale ℓ by Π_ℓ, it is assumed that in the inertial range, Π_ℓ is independent of ℓ, and it satisfies $\Pi_\ell \sim u_\ell^3/\ell \sim \epsilon$, which implies the Kolmogorov's spectral relation $u_\ell \sim \epsilon^{1/3}\ell^{1/3}$. This results in the expression $t_\ell^I \sim \epsilon^{-1/3}\ell^{2/3}$ for the inertial eddy turnover time.

Kolmogorov's phenomenology implies the existence of a transitional scale, ℓ_d, called the Kolmogorov's dissipative scale, between the inertial and dissipative ranges, where both viscous and inertial effects are important, meaning that the eddy turnover time t_ℓ^I equals $t_\ell^\nu \sim \ell^2/\nu$, the characteristic time that diffusion takes to dissipate energy at scale ℓ. Equating both times, one obtains $\ell_d \sim (\nu^3/\epsilon)^{1/4}$. At the top of the inertial range, the global energy flux relation $\epsilon \sim U^3/L$ holds. Inserting this relation into the relation for ℓ_d results in $\ell_d/L \sim 1/\mathrm{Re}^{3/4}$. Because at this scale, the inertial and the dissipative time scales are equal, we obtain $u_d/U \sim (\ell_d/L)^{1/3} \sim 1/\mathrm{Re}^{1/4}$.

Despite of the complex nature of non-Newtonian rheology, Kolmogorov's phenomenology can be recovered for turbulent power-law fluid flows, including the inertial range spectral relation $u_\ell \sim \epsilon^{1/3}\ell^{1/3}$, see Refs. [4,6,14]. In order to obtain Kolmogorov's dissipative scales for power-law fluid flows, one has only to modify the dimensional analysis for the viscous time scale, replacing the constant kinematic viscosity by the effective kinematic viscosity, ν, defined in (2). The transitional Kolmogorov's dissipative length and time scales become

$$\ell_d^{(n)} \sim K^{\frac{3}{2(n+1)}} \epsilon^{\frac{n-2}{2(n+1)}}, \quad t_d^{(n)} \sim K^{\frac{1}{(n+1)}} \epsilon^{\frac{-1}{(n+1)}}. \tag{4}$$

As for Newtonian flows, the hypothesis of constant energy flux on the top of the inertial range implies $\epsilon \sim U^3/L$. Inserting this relation into (4), we obtain

$$\frac{u_d^{(n)}}{U} \sim \left(\frac{\ell_d^{(n)}}{L}\right)^{1/3} \sim \frac{1}{\mathrm{Re}^{\frac{1}{2(n+1)}}}. \tag{5}$$

The mean-field arguments above assume that for turbulent flows, the only relevant global physical quantities at the transitional scale, ℓ_d, are the effective kinematic viscosity, ν, and the global energy dissipation rate ϵ. This is the reason why classical Kolmogorov's framework is frequently applied only under the assumptions of homogeneity and isotropy of the flow, where no other physical scales are available. However, it has been recently shown that much of the theory can be extended to inhomogeneous and anisotropic flows in wall-bounded flows, even in the presence of roughness elements, see Refs. [15,16].

IV. PHENOMENOLOGICAL FRICTION FACTOR

In Ref. [16], Gioia and Chakaborty introduced a phenomenological framework to model Nikuradse's experiments on turbulent friction in rough pipes. We follow their approach to model the Reynolds stress tensor of power-law flows bounded by smooth walls.

Let us decompose the velocity field, $\mathbf{u}(\mathbf{x},t)$, in mean and fluctuating components, $\mathbf{u}(\mathbf{x},t) = \langle \mathbf{U} \rangle(\mathbf{x}) + \mathbf{u}'(\mathbf{x},t)$, where the symbol $\langle \cdot \rangle$ represents a suitable average. We denote by U the norm of the mean flow $\|\langle \mathbf{U} \rangle\|$. The Reynolds equation describing the mean momentum of incompressible power-law fluids, see, e.g., Ref. [17], is

$$\rho\left(\frac{\partial\langle U_i\rangle}{\partial t} + \langle U_j\rangle\frac{\partial\langle U_i\rangle}{\partial x_j}\right) = \frac{\partial}{\partial x_j}[\langle(\eta(|\dot{\gamma}|)|\dot{\gamma}|)_{ij}\rangle - \langle p\rangle\delta_{ij} - \rho\langle u_i' u_j'\rangle]. \quad (6)$$

This is a momentum conservation equation, and the terms in brackets represent three distinct stress tensors. The first term represents the power-law viscous stress, the second one is an isotropic stress arising from the mean pressure field, and the third term, $\tau_R = \rho\langle u_i' u_j'\rangle$, is the Reynolds stress tensor, which arises from the fluctuating velocity components communicating excitation with the mean flow.

As for Newtonian flows in smooth pipes, for low Reynolds number, velocity fluctuations are relatively small and, thus, the viscous stress tensor is dominant. We only remark that for power-law fluids, because of the nonlinear nature of its viscosity term, the mean viscous stress possesses correlations of fluctuating components as well. This has been associated to a delay in transition to turbulence observed in numerical simulations, see, e.g., Refs. [5,6].

Our phenomenology assumes that for moderately large Reynolds number, there exists a viscous wet surface $\mathcal{W}$ of constant thickness in the flow, parallel to the wall, such that above it the velocity of the flow is $\sim U$. In this upper region, the fluid flow carries a high horizontal momentum per unit volume (ρU). Below $\mathcal{W}$, the velocity of the flow is small, and the fluid has a negligible horizontal momentum per unit volume. We also assume that over the wet surface $\mathcal{W}$, stress is mainly induced by vertical fluctuations of horizontal momentum, so that the Reynolds stress is the dominant stress term over $\mathcal{W}$. Below $\mathcal{W}$, the Reynolds stress contribution decays fast, so that in the immediate vicinity of the wall, stress is manly induced by viscous forces. This description suggests a momentum balance so that the wall shear stress, τ_w, is of the order of the momentum transport in the vertical direction across $\mathcal{W}$, implying τ_w scales as $\tau_R\restriction_{\mathcal{W}}$. This hypothesis, also assumed by Gioia and Chakraborty in Ref. [16], is key to our subsequent analysis, and it is supported by moderately large Re numerical simulations, see, e.g., Ref. [6].

Now, the analysis proceeds with the pictorial description of eddies that straddle the wet surface $\mathcal{W}$. The eddies transport fluid of high horizontal momentum across $\mathcal{W}$ in the wall direction, and fluid of negligible horizontal momentum across $\mathcal{W}$ in the centerline direction. The net rate of momentum transport across $\mathcal{W}$ is set by v_N, the eddy's velocity normal to $\mathcal{W}$. Our key hypothesis is that $v_N \sim u_d^{(n)}$.

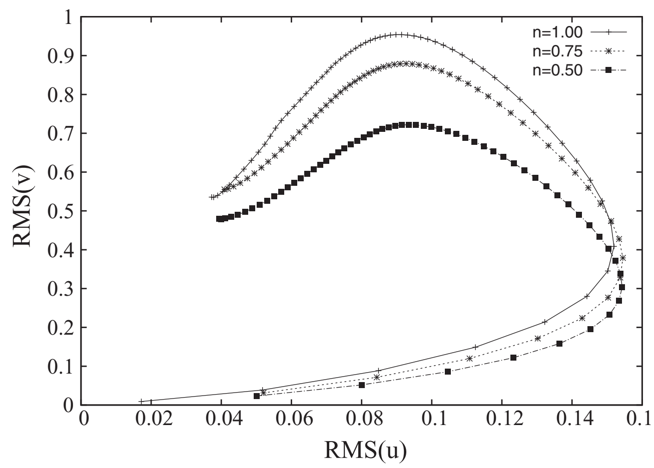

FIG. 2. Axial and vertical velocity fluctuations (renormalized by the shear velocity u_τ) for the same DNS data shown in Fig. 1, from channel's bottom wall to its center in the counterclockwise direction.

In Fig. 2, we exhibit the fluctuating velocities in the axial direction, u', and in the vertical direction, v', obtained from direct numerical simulations of 3D channel power-law flows for the moderate frictional Reynolds number $\mathrm{Re}_\tau \sim 400$. One can notice that the r.m.s. of the axial velocity component u has the same scale in the center of the channel, and in a region very close to the wall, whereas v_N decreases much faster.

Since we have assumed that the wall shear stress scales as the momentum transport in the vertical direction across $\mathcal{W}$, we obtain $\tau_w \sim \rho\, U\, v_n$, which is reminiscent from the analysis presented in Ref. [16]. Inserting this relation into the definition of friction coefficient, and using (5), we propose the following friction formula:

$$f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = g(n)\frac{u_d^{(n)}}{U} = g(n)\frac{1}{\mathrm{Re}^{\frac{1}{2(n+1)}}}, \quad (7)$$

where the nondimensional parameter, $g(n)$, is dependent only on the flow index n. Equation (7) is our main result. Notice that for $n = 1$, one recovers Blasius' scaling for Newtonian flows. It

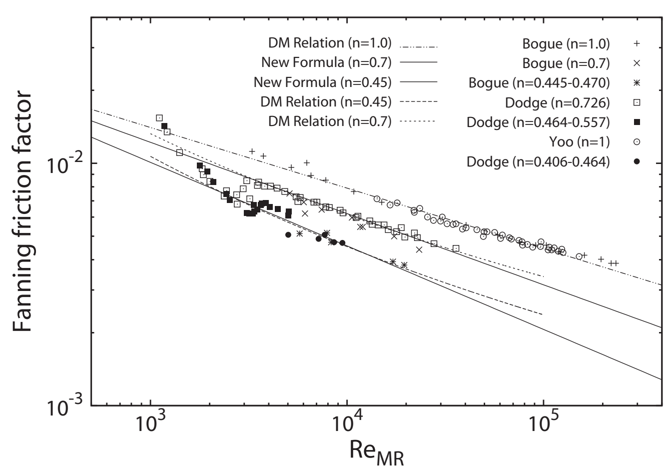

FIG. 3. Comparison of generalized Blasius formula (7), Dodge and Metzner's formula, and empirical data from Bogue [18] and Dodge [19]

is possible to derive different expressions for $g(n)$, and fit them with experimental data. Assuming a rational expression for $g(n)$, we use an extensive set of data available in the literature to fit $g(n)$, resulting in

$$f = \left(0.102 - 0.033n + \frac{0.01}{n}\right)\frac{1}{\mathrm{Re}_{MR}^{\frac{1}{2(n+1)}}}. \tag{8}$$

In Fig. 3, we compare (8) with 80 points drawn from experimental data not used in the fitting procedure. We also display the Dodge and Metzner's empirical law (3) for comparison. The accuracy is very good, with a relative error less than 1.5% for all points in the fully developed region.

V. DISCUSSION

We have derived (7), a Blasius-type formula for the friction factor of power-law fluids, based on Kolmogorov's phenomenology adapted to power-law fluid scaling, and inspired by the stress tensor analysis proposed by Gioia and Chakraborty in Ref. [16]. In contrast to Dodge and Metzner's equation, the equation is explicit and easy to use. As for the Newtonian Blasius' relation, its domain of validity extends over a limited range of Reynolds number, before wall effects have to be taken into account.

For low Reynolds number, velocity fluctuations are relatively small, the inertial range is immature, and the viscous stress tensor, defined in (6), is dominant in the entire flow. Therefore, τ_w scales as $\rho\nu U/L$. As for the Newtonian case, it implies that the laminar friction coefficient satisfies $f \sim 1/\mathrm{Re}$, as observed in several experiments and numerical simulations, see, e.g., Refs. [4,5,6]. We remark, however, that due to suppression of flow instabilities, it is well known that shear-thinning flows present delayed transition to turbulence, see Ref. [5]. This suggests that Blasius-type range starts only for higher Re. Transition to turbulence in non-Newtonian flows is an interesting subject in itself, and further investigations are necessary to characterize the beginning of the Blasius-type range.

For sufficiently high Re, after transition to turbulence, the effective viscosity, ν, is still an important parameter in near wall regions, alongside the wall shear stress, τ_w. As for Newtonian flows, we can define the friction velocity $u_\tau \equiv (\tau_w/\rho)^{1/2}$, and the wall viscous length scale, $\delta_\nu^{(n)} \equiv \nu/u_\tau$.

For the analysis in Ref. [16], the roughness scale r was crucial. Here we modify their analysis in terms of Kolmogorov's length scale $\ell_d^{(n)}$ and the wall viscous length scale, δ_ν. Let s denote the characteristic size of the eddies transporting momentum over $\mathcal{W}$, and let $a_n\,\delta_\nu^{(n)}$ denote the characteristic thickness of the viscous surface, where a_n is a $O(1)$ constant that needs to be estimated from experiments (in turbulent Newtonian flows, a_1 is typically 5, see Ref. [11]).

In our moderate Reynolds number scenario, as Re increases, eddies in the transitional region become both slower and smaller, as well as the thickness of the viscous layer. Indeed, it is easy to derive from (5) and (7) that

$$\frac{\ell_d^{(n)}}{L} \sim \frac{1}{\mathrm{Re}^{\frac{3}{2n+2}}}, \qquad \frac{\delta_\nu^{(n)}}{L} \sim \frac{1}{\mathrm{Re}^{\frac{4n+3}{4n+4}}}. \tag{9}$$

The ratio, H, between the transitional Kolmogorov scale, $\ell_d^{(n)}$, and the thickness of the viscous layer, $a_n\delta_\nu^{(n)}$, therefore, satisfies

$$H(\mathrm{Re},n) \equiv \frac{a_n\delta_\nu^{(n)}}{\ell_d^{(n)}} = \frac{2^{3-n}\sqrt{2}}{(3n+1)^n g^{1/2}(n)} \times \frac{a_n}{\mathrm{Re}^{\frac{4n-3}{4n+4}}}. \tag{10}$$

For a range of Re so that $s \sim \ell_d^{(n)} < a_n\delta_\nu^{(n)}$, i.e., $H > 1$, the flow near the wall is well described by the phenomenology of last section, and (7) is a good approximation. However, if $\ell_d^{(n)} \geqslant a_n\delta_\nu^{(n)}$, i.e., $H < 1$, then transitional eddies become too large to get absorbed in the viscous layer, so that our phenomenology breaks down, and wall properties have to be considered.

Notice that for $n > 0.75$, the ratio H decays with increasing Re. This implies that for this class of flows, our phenomenology is no longer valid for sufficiently large Reynolds number. For Newtonian flows, $n = 1$, the valid range extends from around 3000 to around 100 000. For shear-thickening flows, let us take $n = 1.2$, for example, the ratio H is already $O(1)$ for Re $\sim$ 6000, so that our phenomenology seems to be valid only on a very restricted range of Re. However, for shear-thinning flows, let us take $n = 0.9$, for example, H approaches 1 only for Re above 10^9, indicating that validity of Blasius-type formula is substantially extended for higher Re for this class of flows.

For $n < 0.75$, an interesting phenomenon occurs. For this class of shear-thinning flows, the ratio H increases with increasing Re and is always above $O(1)$, suggesting that our phenomenology never breaks down due to wall effects, although in real fluid flows, degradation of polymers and roughness effects may eventually become relevant. Figure 4 displays the behavior of H against Re for some values of n. An extensive set of experiments or simulations of power-law fluid flows with very high Re is not available in the literature, and further investigations are needed to confirm the predictions for very high Reynolds number.

In the literature, some empirical equations have similar functional form, such as $f = 0.0682n^{-1/2}\mathrm{Re}^{-1/(1.87+2.39n)}$,

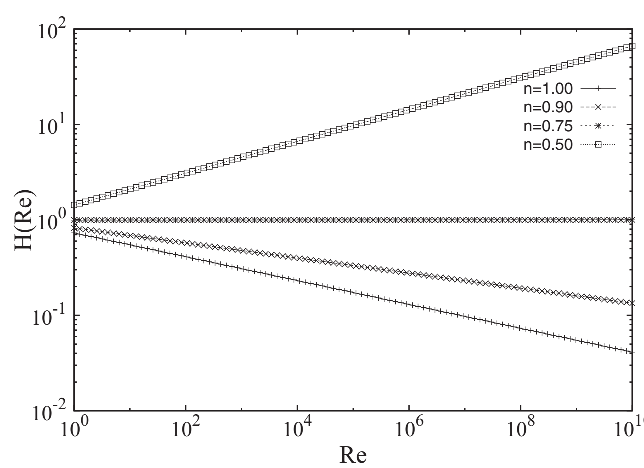

FIG. 4. Plot of $H \equiv \delta_\nu^{(n)}/\ell_d^{(n)}$ versus Re. For $n > 0.75$, $\delta_\nu^{(n)}$ becomes eventually of the order of $\ell_d^{(n)}$, when our phenomenological scaling breaks down. For $n = 1$ (Newtonian), the ratio is of order $O(1)$ in the range of Re $\sim 10^5$, which is a well-known upper limit for Blasius' law. For $n = 0.9$, the domain of validity is pushed above Re $\sim 10^9$. We set $a_n = 1$.

proposed by Hanks and Ricks in Ref. [20]. The reason behind possible deviations may come from one of our central hypothesis, $\tau_w \sim \tau_R \restriction_{\mathcal{W}}$, which neglects the effects of the viscous stress tensor on the wet surface $\mathcal{W}$. Other reasons may be the absence of wall roughness effects, or even because of intermittency not included in our phenomenology, in the spirit of Landau's criticism of Kolmogorov's theory. A thorough empirical study is therefore necessary in order to compare Eq. (7) with other proposed friction equations.

ACKNOWLEDGMENTS

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