Effective degrees of freedom of a random walk on a fractal

Alexander S. Balankin
Grupo "Mecánica Fractal," ESIME, Instituto Politécnico Nacional, México D.F., 07738, Mexico
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We argue that a non-Markovian random walk on a fractal can be treated as a Markovian process in a fractional dimensional space with a suitable metric. This allows us to define the fractional dimensional space allied to the fractal as the ν-dimensional space F^ν equipped with the metric induced by the fractal topology. The relation between the number of effective spatial degrees of freedom of walkers on the fractal (ν) and fractal dimensionalities is deduced. The intrinsic time of random walk in F^ν is inferred. The Laplacian operator in F^ν is constructed. This allows us to map physical problems on fractals into the corresponding problems in F^ν. In this way, essential features of physics on fractals are revealed. Particularly, subdiffusion on path-connected fractals is elucidated. The Coulomb potential of a point charge on a fractal embedded in the Euclidean space is derived. Intriguing attributes of some types of fractals are highlighted.

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I. INTRODUCTION

Scale-invariant fractal-like objects are ubiquitous in nature and technology [1]. Therefore, understanding the properties and dynamical behavior of fractals is of tremendous importance for science and engineering. Fractal geometry offers a helpful framework to deal with scale-invariant systems in which heterogeneities play an important role in almost all scales [1–3]. Within this framework, the fractal properties are characterized by a set of dimension numbers (see Table I). Specifically, the fractal mass distribution in the embedding Euclidean space E^n is governed by the mass dimension D [1–3]. The fractal mass is related to a box-counting (packing, capacity, Hausdorff, etc.) measure defined via the fractal covering by n-dimensional balls (cubes, tubes, etc.) of linear size ε, or at most ε [2]. In mathematics these covers are considered in the limit of $\varepsilon \to 0$ and not necessarily associated with the scale invariance of studied objects. In the case of scale-invariant patterns, the number of n-dimensional boxes needed to cover the pattern of linear size L scales as $N \propto (L/\varepsilon)^D$ [1–4]. This scaling relation allows to define the mass (box-counting) dimension D of prefractals and physical fractals possessing the scale invariance within a bounded range of length scale $\xi_0 \leqslant \varepsilon < L \leqslant \xi_C$ only, where ξ_0 and ξ_C are the lower and upper cutoffs of a physical nature [3,4].

The mass fractal dimension D is defined with respect to the Euclidean metric of the embedding space E^n (see Table I). On the other hand, there always exists the geodesic metric on the fractal that is independent of the embedding [5]. This metric is defined via the definition of the shortest (minimum) path between two randomly chosen points on a fractal $\ell_{\min}$ (see Table I). Accordingly, the fractal connectivity is defined as the number of fractal points $N \propto \ell_{\min}^{d_\ell}$ connected with an arbitrary point inside of the d_ℓ-dimensional ball of radius $\ell_{\min}$ around this point (see Table I), where d_ℓ is the connectivity dimension [6] also called as the intrinsic, chemical, or spreading dimension [7]. The scale invariance implies that the minimum path between two points on a fractal scales with the Euclidean distance between these points r is

$$\ell_{\min} \propto (r/\varepsilon)^{d_{\min}}, \tag{1}$$

at least in a statistical sense, where $0 < d_{\min} = D/d_\ell \leqslant D$ is the fractal dimension of the minimum path with respect to the Euclidean metric [8]. In this regard, it is pertinent to note that fractals which are totally disconnected in the embedding Euclidean space (e.g., Cantor dusts [2]) are path connected in their d_ℓ-dimensional chemical space Ω_ℓ [9]. This allows us to use Eq. (1) with $d_{\min} = D/d_\ell < 1$ to relate the minimum paths between points on a totally disconnected fractal to the Euclidean distances between these points in E^n [10].

The fractal topology also manages dynamical degrees of freedom. Specifically, the density of fractal vibrational modes scales with the vibration frequency ω as $\Lambda \propto \omega^{d_s-1}$, where d_s is the spectral (or fracton) dimension [11] equal to the number of effective dynamical degrees of freedom on the fractal [12]. Notice that for d-dimensional Euclidean objects the dimension numbers $d_s = d_\ell = D$ are equal to the topological dimension $d \leqslant n$ and $d_{\min} = 1$, whereas for path-connected fractals $d_{\min} \geqslant 1$ and $1 \leqslant d_s \leqslant d \leqslant d_\ell \leqslant D \leqslant n$ [13], while totally disconnected Cantor dusts in E^n are characterized by $0 = d < D \leqslant d_s = d_\ell = n$ and $d_{\min} = D/n < 1$ [10]. Fractal dimensionalities of some classic fractals are listed in Table II. Notice that the fractal dimension of the minimum path on a self-avoiding Koch curve is equal to D, while $d_\ell = d_s = 1$ [13], whereas Sierpinski gaskets and carpets in E^n have the connectivity dimension equal to the box-counting (mass) dimension, that is, $d_\ell = D$ and $d_{\min} = 1$ [13,14]. It is worth recalling that the connectivity dimension is invariant under invertible, continuous, but not necessarily differentiable, transformations (see Table I). Accordingly, the dimension of minimum path on fractals folded from two-dimensional self-avoiding sheets ($d_\ell = 2$) is equal to $d_{\min} = D/2$ [15].

Random walks on fractals have become a key paradigm for studies of complex dynamics of scale-invariant systems (see Refs. [10,16–18], and references therein). From the physical viewpoint, a normal diffusion in E^n is ruled by the Markovian motion of Brownian walkers. Accordingly, the central limit theorem states that the limiting distribution of random displacements of walkers in E^n is the Gaussian distribution with the variance proportional to the first power of time [19]. The scaling properties of fractals give rise to an anomalous diffusion, in the sense that the mean squared displacement of a random walker on a fractal obeys an asymptotic behavior,

$$\langle r^2 \rangle \propto t^{2/D_W}, \tag{2}$$

TABLE I. Topological properties, metrics, measures and scaling properties of fractals (topological properties are invariant under invertible, continuous (but not necessarily differentiable) transformations, whereas metrics and measures are invariant under quasi-isometries).

Attributes	Property	Definition and scaling relation	Dimension number
Topological	Connectivity [6]	The number of fractal points (N) connected with an arbitrary point inside of the d_ℓ-dimensional ball of radius $\ell_{\min}$ around this point ($N \propto \ell_{\min}^{d_\ell}$).	d_ℓ
	Loop topology	$\langle \ell_{\min}^2 \rangle \propto s^{2/d_W}$, where s is the number of steps in d_ℓ--dimensional (chemical) space with the geodesic metric.	d_W
Metrics	Geodesic metric [5,6]	$\ell_{\min} := \inf\{\ell(\gamma) : \gamma$ is a path joining x to $y\}$, where $\ell(\gamma)$ is the length of γ.	–
	Metric induced by the fractal topology	Eqs. (8b), (8c), and (9)	d_ℓ/ν
	Intrinsic time metric	Eqs. (10)–(12)	β
Measures	Hausdorff measure with respect to the Euclidean metric [2]	$H_\varepsilon^s(F) = \inf\{\sum_i^\infty \lvert U_i \rvert^s : \{U_i\}$ is an ε cover of F$\}$; $\lim_{\varepsilon\to 0} H_\varepsilon^s(F) = 0$, if $s > D$, or ∞, if $s < D$. Number of n-dimensional covers, $N \propto \varepsilon^{-D}$.	D
	Box-counting (Hausdorff-like) measure with respect to the topology induced metric	$H_{\ell_\nu}^s(F) = \inf\{\sum_i^\infty \lvert U_i \rvert^s : \{U_i\}$ is a ℓ_ν cover of F$\}$; $\lim_{\ell_\nu\to 0} H_{\ell_\nu}^s(F) = 0$, if $s > \nu$, or ∞, if $s < \nu$. Number of ν -dimensional ℓ_ν covers, $N \propto \ell_\nu^{-\nu}$	ν
Degrees of freedom	Number of effective spatial degrees of freedom	The number of independent directions in which a walker can move without violating any constraint imposed on it.	ν
	Number of effective dynamical degrees of freedom[11,12]	The probability of a random walker to return to its starting point scales as $P(s \to \infty,0) \propto s^{-d_s/2}$, where s is the number of steps in d_ℓ-dimensional (chemical) space with the geodesic metric.	d_s
Scaling behavior	Shortest distance [8]	Eq. (1)	$d_{\min}$
	The squared displacement of random walker	Eq. (2)	$D_W = d_{\min} d_W$
	Electrical resistance [20]	$\rho \propto r^\zeta$, if $d_s < 2$ ($\zeta > 0$), or $\rho \propto \text{const} - r^\zeta$, if $d_s > 2$ ($\zeta < 0$); $2\zeta = D_W(2 - d_s)$	ζ

where the scaling exponent D_W is commonly called the random walk dimension [16–18]. The subdiffusion on path-connected fractals is characterized by $D_W > 2$ [16–18], whereas the superdiffusion on totally disconnected Cantor dusts is characterized by $D_W < 2$ [10]. Using the Einstein relation between diffusion and conductivity, it was shown that, generally, $D_W(2 - d_s) = 2\zeta$ [20], where ζ is the resistance exponent (see Table I). Accordingly, for fractals obeying the Alexander-Orbach relationship,

$$d_s = \frac{2D}{D_W} = \frac{2d_\ell}{d_W}, \tag{3}$$

the resistance exponent is equal to $\zeta = D_W - D = d_{\min}(d_W - d_\ell)$, where $d_W = D_W/d_{\min}$ is the random walk dimension (exponent) with respect to the geodesic metric [16,21] which depends on the loop topology of the fractal (see Table I). A loop is defined as a sequence of links starting and terminating in the same node and passing only once through each node [22]. Accordingly, the presence of loops increases the number of paths available for spreading processes on the fractal. Consequently, the resistance exponents of fractals with loops (e.g., Sierpinski gaskets and carpets) are less than the fractal dimensions of minimum paths, and so $d_W = d_\ell + \zeta/d_{\min} < d_\ell + 1$ [16]. On the other hand, for fractals without loops, such as Koch curves, branched trees, and fractals folded from self-avoiding sheets, $\zeta = d_{\min}$, and so the spectral dimension of the loopless fractal is related to its connectivity dimension as

$$d_s = \frac{2d_\ell}{d_\ell + 1}, \tag{4}$$

while $d_W = d_\ell + 1$ [16,23]. Therefore, the spectral dimension of fractal self-avoiding curves (e.g., Koch curves) is $d_s = d_\ell = 1$, whereas fractals folded from self-avoiding two-dimensional sheets ($d_\ell = 2$) have the universal spectral dimension $d_s = 4/3$.

The problem of Brownian motion on fractals has stimulated the development of approaches to analysis and probabilistic techniques on fractals [24,25]. In this way, it has been rigorously proved that the conditional probability density to find the walker at distance $r \in E^n$ from the origin $P_E(r,t)$ exists [25], but it is discontinuous at all scales, because the

TABLE II. Fractal dimensionalities (from references) and numbers of effective spatial degrees of freedom (ν) for some types of fractals and fractional dimensional spaces.

		D	d_ℓ	d_s	ν, Eq. (20)
Euclidean lattices		n	n	n	n
Self-avoiding curves, e.g., Koch-like curves[1]		$1 < D \leqslant n$	1	1	1
Sierpinski gasket [11,13,14a]		ln3/ln2	D	2ln3/ln5	1.805
Sierpinski carpet [14b,66]		ln8/ln3	D	1.805	≈2
Menger sponge [13,66]		ln20/ln3	D	2.5 ± 0.2	≈3
Cantor dusts in E^n [10]		$0 < D < n$	n	n	n
Diamond fractals [66]		D	D	D	D
Fractional space F^γ [34,41]		$D = \gamma$	D	D	D
Fractional space F^ν with the topology induced metric(9)		$D = d_\ell d_{\min}$	d_ℓ	d_s	$2d_\ell - d_s$
Percolation	$n = 2$	91/48	≈1.67	1.30 ± 0.05	≈2
clusters [16]	$n = 3$	2.53	1.841	≈1.33	2.35
	$n = 4$	3.05	1.90	≈1.33	2.46
	$n = 5$	3.54	1.95	≈1.33	2.57
	$n \geqslant 6$	4	2	4/3	8/3
	Cayley tree	4	2	4/3	8/3
Two-dimensional self-avoiding sheets folded in fractal balls in E^n[Eq. (25)]		$2 < D < n$	2	4/3	8/3

fractal density is almost everywhere discontinuous in E^n [26]. To deal with nonanalytic functions $P_E(r,t)$, O'Shaughnessy and Procaccia [26] have introduced the concept of analytic envelope defined as a smoothed approximation $P(r,t)$ of nonanalytic $P_E(r,t)$. In this context, the scaling relation (2) gives rise to formally define the time and/or space dependent coefficient of diffusion $\Xi_n(r,t) \propto \langle r^2\rangle/t$ on the fractal embedded in E^n [26,27].

Furthermore, it was proved that in the d_ℓ-dimensional chemical space Ω_ℓ with the metric defined by the minimum paths (see Table I), the conditional probability density function $P(\ell_{\min},t)$ is continuous and obeys the two-sided sub-Gaussian estimate

$$P_1(\ell_{\min},t,c_1,c_2) \leqslant P(\ell_{\min},t) \leqslant P_2(\ell_{\min},t,c_3,c_4), \quad (5)$$

where $c_i > 0$ are some positive constants which are independent of $\ell_{\min}$ and t [25]. Besides, for some types of fractals and metric spaces it was established [25] that

$$P_j(\ell_{\min},t,c_i,c_{i+1}) = c_i t^{-d_s/2} \exp\left[-c_{i+1}\left(\frac{\ell_{\min}}{t^{1/d_W}}\right)^u\right], \quad (6a)$$

where

$$u = \frac{d_W}{d_W - 1}, \quad (6b)$$

but very little is known about how $P(\ell_{\min},t)$ behaves between bounds defined by Eqs. (5) and (6), although it was recognized that $P(\ell_{\min},t)$ is a log-periodic function of y^u, where $y = \ell_{\min}/t^{1/d_W}$ [28]. On the other hand, it has been argued that the probability density in the chemical space can be approximated by a stretched Gaussian function of the form

$$P(\ell_{\min},t) \propto t^{-d_s/2} \exp\left[-c\left(\frac{\ell_{\min}}{t^{1/d_W}}\right)^u\right], \quad (6c)$$

where c is the constant, while the scaling exponent u is defined by Eq. (6b) [12,29]. Consequently, it was suggested that the analytic envelope of $P_E(r,t)$ can be obtained from $P(\ell_{\min},t)$ using the scaling relation (1), such that the propagator on the fractal $F^D \subset E^n$ acquires the form $P(r \in F^D,t) \propto t^{-d_s/2} \exp[-c(r/t^{1/D_W})^{d_{\min}u}]$ [12]. Accordingly, the analytic envelope of the conditional probability density function $P_E(r,t)$ is expected to have the following form:

$$P(r \in E^n,t) \propto r^{D-n} t^{-d_s/2} \exp\left[-c\left(\frac{r}{t^{1/D_W}}\right)^{\upsilon}\right], \quad (7a)$$

with

$$\upsilon = d_{\min} u, \quad (7b)$$

where the term r^{D-n} accounts for the spatial distribution of fractal density in E^n (see Ref. [30]). Besides, it was also suggested that the data of numerical simulations in the short time regime ($Y = r/t^{1/D_W} \gg 1$) can be more precisely fitted by adding a potential term Y^η to the right side of Eq. (7a) [31]. However, even the sign of the scaling exponent η is still a matter of discussion (see Refs. [30–32]). Furthermore, in spite of enormous efforts to understand the anomalous diffusion on fractals, the diffusion equations leading to $P(r,t)$ defined by Eq. (7), or to $P(\ell_{\min},t)$ of the form of Eq. (6c), were not established yet. So, the subdiffusion on path-connected fractals remains a challenging problem [33].

In this background, it is a straightforward matter to understand that in order to model physical phenomena on fractals, it is crucial to know the effective degrees of freedom governed by the fractal geometry. Accordingly, the main goal of this paper is to identify the spatial and temporal metrics induced by the fractal topology and define the effective degrees of freedom of walkers on a fractal. In this way, the diffusion equation on fractals is derived and the relation between the number of effective spatial degrees of freedom ν and fractal dimensionalities is established. Furthermore, we recall the axiomatic definition of fractional dimensional spaces suggested by Stillinger [34] and define the fractional dimensional space F^{ν} allied to the fractal as the ν-dimensional space endowed with the metric induced by the fractal topology. This allows us to map the problems on fractals, which are essentially nondifferentiable in the conventional sense, into the problems in F^{ν}. The developed framework is employed to elucidate some physical phenomena on fractals.

The paper is organized as follows. In Sec. II we define the metric induced by the fractal topology and the intrinsic time of random walk on fractals. Section III is devoted to diffusion on fractals. The diffusion equation is derived and its solution is compared with the available data of numerical simulations on deterministic and random (pre)fractals. The Laplacian operator in the fractional dimensional space F^{ν} allied to the fractal is constructed in Sec. IV. In Sec. V we derive the Coulomb potential on fractals embedded in the Euclidean space. Intriguing features of some types of fractals are revealed. Two relatively simple experiments are suggested to verify theoretical predictions. A brief summary and some relevant conclusions are outlined in Sec. VI.

II. SPATIAL METRIC INDUCED BY FRACTAL TOPOLOGY AND INTRINSIC TIME OF RANDOM WALK ON FRACTALS

The motion of a random walker is governed by its effective degrees of freedom. A random motion in the Euclidean space E^n is characterized by equal numbers of walkers' spatial and dynamical degrees of freedom. The scale invariance imposes a constraint which reduces the numbers of admissible degrees of freedom of the walker on a path-connected fractal [35]. Specifically, the number of dynamical degrees of freedom becomes equal to the spectral dimension of the fractal $d_s \leqslant D \leqslant n$ [11,12]. The number of effective spatial degrees of freedom is the number of independent directions in which the walker can move without violating any constraint imposed on it. In other words, the number of effective spatial degrees of freedom ν can be viewed as the number of directions that someone who lived on the fractal would experience. So, the number of effective spatial degrees of freedom can be defined via the fractal covering by the ν-dimensional covers of size ℓ_ν (see Table I). It is pertinent to note that the definition of ν given in Table I concurs with the definition of the intrinsic Hausdorff dimension suggested in [36] up to the dimension and metric of covering balls. Specifically, to define the number of effective spatial degrees of freedom we use the ν-dimensional covers with metric induced by the fractal topology (ℓ_ν) instead of d_ℓ-dimensional covers with the geodesic metric ($\ell_{\min}$) used in Ref. [36] to define the intrinsic Hausdorff dimension. In this regard, it is worth recalling that the intrinsic Hausdorff dimension defined in [36] is, in fact, the connectivity dimension d_ℓ with respect to the geodesic metric [37] (see Ref. [38]). So, it is a straightforward matter to understand that, generally, the number of effective spatial degrees of freedom (ν) can be larger or equal to the connectivity dimension (d_ℓ).

The fractal density scales in the embedding space E^n as $\rho \propto (r/\xi_c)^{D-n}$ [1]. Accordingly, the scaling of fractal mass can be defined as

$$M \propto \int_{E^n} r^{D-n} d^n r \propto \int_{E^n} r^{D-1} dr \propto \int_{F\subset E^n} d^D r \propto (r/\xi_c)^D, \tag{8a}$$

where the third integral is a fractional (e.g., Riemann-Liouville or Riesz) integral of order D [39]. On the other hand, from the definitions of dimension numbers d_ℓ and $d_{\min}$ (given in Table I) it follows that the fractal mass scales in the d_ℓ-dimensional chemical space Ω_ℓ as

$$M \propto \int_{\Omega_{\text{chem}}} d^{d_\ell} \ell_{\min} \propto \ell_{\min}^{d_\ell}, \tag{8b}$$

and so the scaling relation (1) holds. Taking into account that $\nu \geqslant d_\ell$, Eq. (8b) can be mapped into the ν-dimensional space F^ν with metric ℓ_ν, such that

$$M \propto \int_{F^\nu} d^\nu \ell_\nu \propto \ell_\nu^\nu. \tag{8c}$$

From Eqs. (8b) and (8c) it immediately follows that the metric induced by the fractal topology can be related to the geodesic metric via the scaling relation

$$\ell_\nu = a\ell_{\min}^{d_\ell/\nu}, \tag{9}$$

where a is the geometric constant and $d_\ell/\nu \leqslant 1$. In this regard, it should be emphasized that the metric induced by the fractal topology and the geodesic metric are both independent of the fractal embedding in E^n (see Table I). This allows us to define the fractional dimensional space F^ν allied to the fractal as the ν-dimensional space (as it was axiomatically defined by Stillinger [34]) equipped with the topology induced metric (9). In this context, it is pertinent to recall that Stillinger [34] has defined the fractional dimensional space F^γ by five structural axioms: (1) F^γ is a metric space; (2) F^γ is dense in itself; (3) F^γ is metrically unbounded; (4) any two points in F^γ are connected by a continuous line embedded in that space; (5) the result produced by integrating any rooted Gaussian depends only on distances between root points (which can be anywhere in F^γ), and not in any way on absolute position in F^γ. Axiom (5) confers the overall uniformity on F^γ and defines the space dimension γ via the postulated form of Gaussian integral. So, it is a straightforward matter to understand that γ is equal to the number of effective spatial degrees of freedom in F^γ. Although Stillinger [33] has stated that the metric in F^γ can be a non-Euclidean [40], only the Euclidean metric was used in works [33,41–43]. Further, in Ref. [44] it was shown that the dimension of the fractional dimensional space defined in [34] is equal to the box-counting dimension with respect to the Euclidean metric, and so $\gamma = \nu = D = d_\ell$ (see Table II). In contrast to this, the fractional dimensional space F^ν allied to the fractal has the metric defined by Eq. (9) with $\nu \geqslant d_\ell$, whereas $D \geqslant d_\ell$ if the fractal is path connected, or $D < d_\ell$ for totally disconnected connected Cantor dusts [45].

Now, let us consider a random walk on a fractal with ν spatial and d_s dynamical degrees of freedom. A random walk is a sequence of random steps with independent and identically distributed increments. However, if $\nu < n$ and/or

$d_s < n$, the random walk on the fractal embedded in E^n becomes non-Markovian due to the geometric restrictions. At the same time, in the fractional ν-dimensional space F^ν allied to the fractal random walkers can move in any of ν admissible directions with equal probability in each step. Therefore, the non-Markovian random walk on the fractal can be treated as a Markovian process in the ν-dimensional space F^ν with the metric induced by the fractal topology.

The Markovian nature of random walking in F^ν implies that the limiting distribution of random walker displacements with respect to the topology induced metric (9) is the Gaussian distribution with the variance proportional to the first power of the intrinsic time of the walk τ. That is, the mean squared displacement from the origin in F^ν scales with τ as

$$\langle \ell_\nu^2 \rangle \propto \tau, \tag{10}$$

whereas the mean squared minimum path between the walker position and the origin scales with the natural time in the embedding space E^n as $\langle l_{\min}^2 \rangle \propto t^{2/d_W}$ [12]. From Eqs. (9) and (10) together with Eqs. (1) and (2) it follows that the intrinsic time of the random walking on fractal scales with the natural time in embedding space E^n as

$$\tau = b t^\beta, \tag{11}$$

where b is constant in the statistical sense and

$$\beta = \frac{2d_\ell}{\nu d_W} = \frac{D(2-d_s)}{\nu\zeta}, \tag{12a}$$

while $D_W = 2\zeta/(2-d_s)$. For fractals obeying the Alexander-Orbach relation (3) the intrinsic time scaling exponent becomes equal to

$$\beta = d_s/\nu \leqslant 1. \tag{12b}$$

Hence, the intrinsic time of random walk on a fractal obeying relation (3) coincides with the natural time in the embedding space, if and only if the numbers of effective spatial and dynamical degrees of freedom are equal.

III. NUMBER OF EFFECTIVE SPATIAL DEGREES OF FREEDOM AND DIFFUSION EQUATION ON FRACTALS

The conservation of probability implies that the conditional probability to find the walker at distance ℓ_ν from the origin after the elapsed time τ should satisfy the continuity equation

$$\frac{\partial}{\partial\tau}P(\ell_\nu,\tau) = -\ell_\nu^{1-\nu}\frac{\partial}{\partial\ell_\nu}J(\ell_\nu,\tau), \tag{13}$$

where

$$J(\ell_\nu,\tau) = -\Xi_\nu \ell_\nu^{\nu-1}\frac{\partial}{\partial\ell_\nu}P(\ell_\nu,\tau) \tag{14}$$

is the net current through the shell between ℓ_ν and $\ell_\nu + d\ell_\nu$ on the fractal [46], and Ξ_ν is the coefficient of diffusion, while $\partial/\partial\tau$ and $\partial/\partial\ell_\nu$ denote the metric derivatives (see Refs. [47,48]) with respect to the intrinsic time (11) and topology induced metric (9), respectively. Furthermore, the scaling relation (10) implies that the diffusion coefficient $\Xi_\nu \propto \langle \ell_\nu^2\rangle/\tau$ is independent of ℓ_ν and τ, at least in the statistical sense. Accordingly, the diffusion equation on the fractal can be presented in the form

$$\frac{\partial}{\partial\tau}P = \Xi_\nu\left(\frac{\partial^2}{\partial\ell_\nu^2}P + \frac{\nu-1}{\ell_\nu}\frac{\partial}{\partial\ell_\nu}P\right), \tag{15}$$

where the term in parentheses represents the radial part of the Laplacian in the ν-dimensional space F^ν with the metric defined by Eq. (9). The point source solution of Eq. (15) reads as

$$P(\ell_\nu,t) = C\tau^{-\nu/2}\exp\left[-\left(\ell_\nu^2/4\Xi_\nu\tau\right)\right], \tag{16}$$

while the constant C is defined by the normalization condition

$$\int P(\ell_\nu,t)d^\nu\ell_\nu = \pi^{\nu/2}\Gamma^{-1}(\nu/2)\int P(\ell_\nu,t)\ell_\nu^{\nu-1}d\ell_\nu = 1, \tag{17}$$

where $\Gamma(\cdots)$ denotes the gamma function. Accordingly, in the d_ℓ-dimensional chemical space Ω_ℓ the conditional probability density function (16) gets the following form:

$$P(\ell_{\min},t) = \frac{2(4\Xi_\nu b)^{-\nu/2}}{\nu\Gamma(\nu/2)}t^{-d_s/2}\exp\left[-\frac{a^2}{4b\Xi_\nu}\left(\frac{\ell_{\min}}{t^{d_s/2d_\ell}}\right)^{2d_\ell/\nu}\right], \tag{18a}$$

independently of whether or not the Alexander-Orbach relation (3) holds. For fractals obeying relation (3), Eq. (18a) gets the form

$$P(\ell_{\min},t) = \frac{2(4\Xi_\nu b)^{-\nu/2}}{\nu\Gamma(\nu/2)}t^{-d_\ell/d_W}\exp\left[-\frac{a^2}{4b\Xi_\nu}\left(\frac{\ell_{\min}}{t^{1/d_W}}\right)^{2d_\ell/\nu}\right], \tag{18b}$$

while the coefficient of diffusion in the chemical space, defined as $\Xi_\ell = \langle \ell_{\min}^2\rangle/t$, scales with the natural time as

$$\Xi_\ell = (2\nu b a^{-1}\Xi_\nu)^{\nu/d_\ell}t^{-\alpha}, \tag{19a}$$

where $\alpha = 1 - 2/d_W$, whence the apparent coefficient of diffusion on the fractal embedded in E^n ($\Xi_n \propto \langle r^2\rangle/t$) also scales with r as

$$\Xi_n \propto r^{2-D_W}, \tag{19b}$$

as it was suggested in [26]. However, the diffusion equation (15) coincides with the diffusion equation suggested in Ref. [26] only in the limiting case of $\nu = D = d_s$, when $\beta = 1$, $D_W = 2$, and so $\Xi_n = \text{const}$, even if $D = \nu = d_s < n$, as this is in the case of diamond fractals (see Table II). Notice that in this case the diffusion is not anomalous. More generally, if $\nu = D \geqslant d_s$, Eq. (15) gets the form $\frac{\partial}{\partial t}P = \Xi_n(r)(\frac{\partial^2}{\partial r}P + \frac{D-1}{r}\frac{\partial}{\partial r}P)$, while the scaling behavior of $\Xi_n(r)$ is given by Eq. (19b).

From the comparison of Eqs. (18) and (6), it immediately follows that the number of effective spatial degrees of freedom on the fractal obeying relation (3) is equal to

$$\nu = 2d_\ell - d_s = d_s(d_W - 1), \tag{20}$$

whereas, more generally, one can expect that $\nu = 2d_\ell(d_W - 1)/d_W = 2d_\ell - D(2-d_s)/\zeta$, while $D_W(2-d_s) = 2\zeta$.

From Eq. (20) it follows that the numbers of effective spatial and dynamical degrees of freedom are equal only for fractals with $d_s = d_\ell$. The diffusion on such fractals is characterized by the universal dimension of random walk with respect to the geodesic metric $d_W = 2$, while $D_W = 2d_{\min}$, as this is, for example, for self-avoiding Koch curves ($d_s = d_\ell = 1$) and Cantor dusts in E^n ($d_s = d_\ell = n$). Numbers of effective spatial and dynamic degrees of freedom for some classic types of fractals are summarized in Table II along with other parameters characterizing the fractal topology, metrics, and measures. Notice that relation (20) implies that the spectral dimension of the fractional dimensional space F^γ defined in [34] is $d_s = \nu = \gamma = D$, as it is stated in Table II.

Furthermore, it is worth recalling that the spectral dimension of loopless fractals is related to their connectivity dimension by Eq. (4), and so $\nu = d_\ell d_s$. Specifically, the number of effective spatial degrees of freedom on self-avoiding one-dimensional strings folded in E^n up to the fractal dimension $1 < D = d_{\min} < n$ is $\nu = 1$ for any $n \geqslant 2$. Consequently, the mean squared displacement of a random walker on a self-avoiding fractal curve is expected to scale as $\langle r^2 \rangle \propto t^{1/D}$, while the probability density function (18b) gets the form

$$P \propto t^{-1/2} r^{D-n} \exp(-cr^{2D}/t), \tag{21}$$

which coincides with the probability density distribution on Koch curves derived by other means and supported by numerical simulations in Ref. [49]. In the case of totally disconnected Cantor dust in E^n (see Table II) Eq. (18a) together with Eq. (20) recover the probability density function derived in [10] in another fashion.

Equation (18a) with ν defined by Eq. (20) is also consistent with numerical simulations on the Sierpinski gasket [50] for which $2d_\ell/\nu = 1.76$, while $\ell_{\min} \propto r$ (see Table II). For a random walker on a percolation cluster generated on a Cayley tree at criticality it was found that the probability density function has the form of Eq. (6c) with $u = 1.5$ [16b], that is consistent with $2d_\ell/\nu = 3/2$ (see Table II). Furthermore, although for diffusion in the chemical space of critical percolation clusters in $n = 2$ the scaling exponent $2d_\ell/\nu = 1.67$ (see Table II) is quite smaller than the value of $u = 1.9 \pm 0.1$ found in numerical simulations [51]; it is consistent with the finding of $u = 1.71 \pm 0.1$ in more recent numerical simulations [32].

However, the mapping of Eq. (18) into the embedding Euclidean space is not straightforward, because the scaling relation (1) generally holds only in the statistical sense. Therefore, in order to map the diffusion in the chemical space into diffusion on the fractal embedded in E^n, one should account for the statistical distribution of the minimum paths on the fractal (see Refs. [52]). This can be made using the integral relation

$$P(r,t) = \int_0^\infty p(r|\ell_{\min}) P(\ell_{\min},t) d\ell_{\min}, \tag{22}$$

where $p(r|\ell_{\min})$ is the conditional density for the distribution of r given the value of $\ell_{\min}$ [51], while $P(\ell_{\min},t)$ is defined by Eq. (18a).

In the case of percolation clusters, it was suggested that

$$p(r|\ell_{\min}) = A\left(\frac{r}{l_{\min}^{1/d_{\min}}}\right)^g \exp\left[-\phi\left(\frac{r}{l_{\min}^{1/d_{\min}}}\right)^\delta\right], \tag{23}$$

where A and ϕ are geometric constants, and the scaling exponent g is the constant ($g = 2.5 \pm 0.3$ for $n = 2$), while the scaling exponent δ is governed by the geodesic metric as $\delta = d_{\min}/(d_{\min} - 1)$ [52]. Consequently, the analytic envelope of the conditional probability density function on percolation clusters gets the form of Eq. (7a), but with the scaling exponent defined as

$$\upsilon = \frac{d_{\min} u}{(d_{\min} - 1)u + 1}, \tag{24a}$$

instead of υ given by Eq. (7b) [51]. From Eq. (24a), together with Eqs. (18b) and (20), it follows that for diffusion on percolation clusters the analytical envelope of the conditional probability density function in the embedding Euclidean space has the form of Eq. (7a) with the scaling exponent

$$\upsilon = \frac{2D}{\nu + 2d_\ell(d_{\min} - 1)} = \frac{2D}{2D - d_s} = \frac{D_W}{D_W - 1}. \tag{24b}$$

Notice that the value $\upsilon = 1.54$ calculated with Eq. (24b) is consistent with scaling exponents found by numerical simulations: $\upsilon = 1.65 \pm 0.1$ in Ref. [51] and $\upsilon = 1.56 \pm 0.1$ in more recent work [32]. In this regard, it is pertinent to point out that Eq. (24b) is not general, but it is valid only for the sub-Gaussian form (23) of the minimum path distribution.

As was noted in the Introduction, the connectivity and spectral dimensions of fractals folded from self-avoiding two-dimensional sheets are universal (see Table II). Therefore, the number of effective spatial degrees of freedom on these fractals is also universal. Namely

$$\nu = 2d_s = 8/3 \tag{25}$$

for any $n \geqslant 3$, while $2 < D = 2d_{\min} < n$, $D/\nu = 3D/8 < d_{\min} = D/2$, and $\beta = d_s/\nu = 1/2$ [53]. In this background, it is interesting to note that experimentally it was found that equilibrium configurations of balls folded from randomly patterned papers have the local mass fractal dimension equal to $D = 8/3$ within the measurement uncertainty [54]. Accordingly, the subdiffusion on randomly folded paper balls is expected to obey the universal scaling behavior,

$$\langle r^2 \rangle \propto t^{1/2}, \tag{26a}$$

while

$$P(r,t) \propto r^{-1/3} t^{-2/3} \exp\left[-c\left(\frac{r}{t^{1/4}}\right)^\upsilon\right], \tag{26b}$$

where $\upsilon = 2$, if Eq. (1) holds strictly, whereas $\upsilon = 1$, if the distribution of minimum path on the folded sheet is a Gaussian distribution ($\delta = 2$), or $\upsilon = 4/3$, if the distribution of minimum path on the folded sheet has the form of Eq. (23) with $\delta = d_{\min}/(d_{\min} - 1) = D/(D - 2) = 4$. We expect that the theoretical prediction (26) can be experimentally verified in studies of ink diffusion from a point source on a randomly crumpled paper ball.

Finally, it is worth recognizing that percolation clusters in $n \geqslant 6$ (see Table II) belong to the universality class (25) of the folded self-avoiding sheets. Therefore the critical percolation cluster in $n \geqslant 6$ can be viewed as a two-dimensional sheet ($d_\ell = 2$) folded in such a way that its box-counting dimension with respect to the Euclidean metric in E^n is $D = 4$. This is consistent with topological properties of percolation clusters in $n \geqslant 6$ reported in Ref. [55]. Furthermore, it has been proved that the distribution of shortest paths on percolation clusters in $n \geqslant 6$ is a Gaussian distribution (see Refs. [52,56]). Therefore, the subdiffusion on critical percolation clusters in $n \geqslant 6$ is expected to obey the scaling relation (26a) and the probability density function is expected to behave as

$$P(r,t) \propto r^{4-n} t^{-2/3} \exp\left[-c\left(\frac{r}{t^{1/4}}\right)\right], \tag{27}$$

while $d_W = 3$, $d_{\min} = 2$, and $D_W = 6 \leqslant n$ (see Table II). It will be interesting to verify Eq. (27) in numerical simulations.

IV. LAPLACIAN IN THE FRACTIONAL DIMENSIONAL SPACE ALLIED TO THE FRACTAL

The knowledge of effective spatial degrees of freedom allows us to map the problems on fractals into the corresponding problems in the fractional dimensional space F^ν endowed with the metric (9). In this regard, it is worth noting that in the case of $\nu = D$ the term in parentheses in Eq. (16) is similar to the radial component of the fractional Laplacian introduced axiomatically by Stillinger [34] to deal with problems in the fractional dimensional space F^γ with the Euclidean metric ($\gamma = \nu = D < n$). It is also worth mentioning that the Stillinger's Laplacian was widely used to study diverse phenomena in low-dimensional systems (see, for example, Refs. [42] and references therein) and fractional dimensional spaces [43].

In order to deal with physical problems on anisotropic fractals, we need to define the angular components of the Laplacian operator in F^ν. Following to Ref. [41], it is a straightforward matter to deduce that in the case of $\nu > 2$ the Laplacian of a scalar field f in F^ν can be defined as follows:

$$\Delta_\ell^\nu = \frac{1}{\ell_\nu^{\nu-1}}\frac{\partial}{\partial \ell_\nu}\left(\ell_\nu^{\nu-1}\frac{\partial}{\partial \ell_\nu}f\right) + \frac{1}{\ell_\nu^2 \sin^{\nu-2}\theta}\frac{\partial}{\partial\theta}\left(\sin^{\nu-2}\theta\frac{\partial}{\partial\theta}f\right) + \frac{1}{\ell_\nu^2\sin^2\theta}\nabla^2_{S^{\nu-2}}f, \tag{28}$$

where $0 \leqslant \ell_\nu \leqslant \infty$ is the radial distance measured with the metric defined by Eq. (9), $0 \leqslant \theta \leqslant \pi$ is the zenith angle measured relative to an axis passing through the origin in F^ν, and $\nabla^2_{S^{\nu-2}}$ is the Laplace-Beltrami operator on the unit fractional $(\nu-2)$ sphere in F^ν (see [57]). In the case of $2 < \nu \leqslant 3$ the Laplace-Beltrami operator gets the form

$$\nabla^2_{S^{\nu-2}}f = \frac{\partial^2 f}{\partial\phi^2} + \frac{(\nu-3)}{\tan\phi}\frac{\partial f}{\partial\phi}, \tag{29}$$

where $0 \leqslant \varphi \leqslant 2\pi$ is the azimuthal angle measured from a fixed reference direction in F^ν (see Ref. [41]).

It is imperative to recall that physical problems on fractals can be mapped into the corresponding problems in the allied fractional dimensional space F^ν without any reference to the embedding Euclidean space. On the other hand, the scaling relations (1) and (9) imply that

$$\ell_\nu = a\ell_{nun}^{d_\ell/\nu} = \widehat{a}\left(\sum_i^n x_i^2\right)^{d_{\min}d_\ell/2\nu}, \tag{30}$$

where $\widehat{a}$ is constant in the statistical sense, while $x_i \in E^n$ are the Cartesian coordinates in the embedding space. Equation (30) allows us to define the fractional coordinates $\ell_i \in F^\nu$ which are related to the Cartesian coordinates in E^n as

$$\ell_i = a_i x_i^{D/\nu}, \tag{31a}$$

where a_i are constants accounting for the fractal anisotropy, and so

$$\ell_\nu = \widehat{a}\left[\sum_i^n (\ell_i/a_i)^{2\nu/D}\right]^{D/2\nu}. \tag{31b}$$

It is worth noting that the norm in $F^\nu \subset E^n$ defined by Eq. (31b) has the structure of the conventional p norm in E^n with $p = 2\nu/D$ (see Ref. [58]). Therefore, it satisfies all conventional criteria required of a norm if $2\nu \geqslant D$. Notice that the last condition holds for all fractals with $d_{\min} \leqslant 2$, because $\nu \geqslant d_\ell$. In this regard, it is pertinent to point out that, in contrast to the metric induced by the fractal topology (9) which is independent of embedding, the norm (31b) in $F^\nu \subset E^n$ is associated with the embedding Euclidean space. Accordingly, Eq. (31) allows us to map the Laplacian operator (28) onto the Cartesian coordinates $x_i \in E^n$. Specifically, following Ref. [41], we find that the Laplacian of a scalar field f gets the following form:

$$\Delta^\nu f = \sum_i^n a_i^{-2} x_i^{2(1-D/\nu)}\left(\frac{\partial^2}{\partial x_i^2}f + \frac{\nu/n - D/\nu}{x_i}\frac{\partial}{\partial x_i}f\right), \tag{32}$$

where the coefficient $\nu/n - D/\nu \geqslant 0$, if $\nu > \sqrt{nD}$'; otherwise it is negative. The Laplacian of a vector field in $F^\nu \subset E^n$ can be derived in a straightforward manner (see, for example, Ref. [59]).

Notice that in the special case of $\nu = D$ the norm defined by Eq. (31) is essentially the Euclidean norm, even if $\nu = D < n$. Accordingly, in this case, the Laplacian operator (32) converts into the Stillinger's Laplacian in the fractional dimensional space F^γ, as it was defined in Ref. [41]. On the other hand, in the case of $\nu = n > D$ the Laplacian operator defined by Eq. (32) becomes similar to the Hausdorff Laplacian defined in Ref. [60], as well as to the metric Laplacian defined in [48]. Furthermore, in the case of $\nu = d_\ell \leqslant D$ Laplacian (32) concurs with the Laplacian operator used in Ref. [61] up to the definition of the fractal metric. Notice that both operators coincide if $D - D_S = D/\nu = d_{\min}$, where D_S is the fractal dimension of intersection between the fractal and two-dimensional Cartesian plane in E^n.

V. COULOMB POTENTIAL ON FRACTALS

The Poisson's equation for a point charge q at the origin in the fractional dimensional space allied to the fractal reads as

$$\Delta_\ell^\nu \phi_C = -4\pi q\delta(\ell_\nu), \tag{33}$$

where $\delta(\cdots)$ denotes the delta function in F^ν. The solution of Eq. (33) is $\varphi_C(\ell_\nu,\theta) = \lambda q \ell_\nu^{2-\nu}$ (see [62]), while the constant λ is defined by the requirement that the total electric flux on the ν-dimensional sphere must satisfy Gauss's law,

$$\oint_\nu \vec{E}\cdot d\vec{S} = \oint_\nu E_{\rm radial}(\ell_\nu,\theta)dS = -\frac{2\pi^{(\nu-1)/2}}{\Gamma(\nu-1/2)}\int_0^\pi\left[\frac{\partial\phi}{\partial\ell_\nu}\ell_\nu^{\nu-1}\right]\sin^{\nu-2}\theta d\theta = \frac{q}{\varepsilon_f\varepsilon_0}, \tag{34}$$

where ε_0 is the vacuum permittivity and ε_f is the dielectric constant of the fractal. Accordingly, the Coulomb potential of a point charge on a fractal embedded in E^n gets the form

$$\varphi_C(r) = \frac{q\Gamma(\nu/2)}{\pi^{\nu/2}(\nu-2)\varepsilon_0}a^{2-\nu}r^{-\zeta}, \tag{35a}$$

where the scaling exponent

$$\zeta = D\left(1-\frac{2}{\nu}\right) > 0 \tag{35b}$$

depends on the box-counting dimension with respect to the Euclidean metric, as well as on the number of effective spatial degrees of freedom on the fractal. Notice that, if

$$2 < \nu < \nu_C = 2D/(D-1) \leqslant 3, \tag{36}$$

the Coulomb potential (35) decays more slowly than the Coulomb potential of point charge in E^3, whereas it falls off more quickly than r^{-1}, if $\nu_C < \nu \leqslant 3$. In both cases the dynamical symmetry is broken (see [63]).

For example, the Coulomb potential of a point charge on a folded self-avoiding sheet is expected to decay as $r^{-D/4}$. In E^3 this can be experimentally verified for balls folded from thin dielectric sheets, e.g., a paper. Taking into account that randomly folded paper balls have the local mass fractal dimension $D \approx 8/3$ [54], one may expect that the Coulomb potential of a point charge placed in the center of a ball will decay as $r^{-2/3}$, that can be detected in experiments with balls of different radius.

However, if the number of effective spatial degrees of freedom is equal to ν_C, as this is in the case of percolation clusters in $n \geqslant 6$ and four-dimensional balls folded in $n \geqslant 6$ from two-dimensional sheets (see Table II), the Coulomb potential (35) decays as r^{-1} and so the dynamical symmetry $SO(n+1)$ is not violated, despite the fact that $D = 4 > 3$ [64].

The scalar gravitational potential of a point mass m in F^ν obeys the Poisson's equation $\Delta_\ell^\nu\phi_G = -4\pi Gm\delta(\ell_\nu)$, where G is the usual gravitational universal constant. The solution of this equation obeying Gauss's law reads as $\varphi_G = \lambda r^{-\zeta}$, where $\lambda = 2\Gamma(\nu/2)Gma^{\nu-2}/\pi^{\nu/2-1}(\nu-2)$, while the scaling ζ is defined by Eq. (35b). Therefore, on two-dimensional sheets folded up to $D = 4$ and percolation clusters in $n \geqslant 6$ the scalar gravitational potential also decays as r^{-1}. We believe that this attribute has a deep physical origin. In this regard, it is worth noting that Riemann has used a crumpled ball of paper with bookworms to explain the hidden dimensions in non-Euclidean geometry [65].

VI. CONCLUSIONS

Summarizing, the effective degrees of freedom of a random walker on a fractal are defined and the relation between the number of effective spatial degrees of freedom and the fractal dimensionalities is established. The metric and norm in the fractional dimensional space F^ν allied to the fractal are defined. The intrinsic time of random walking on the fractal and the Laplacian operator in F^ν are deduced. This allows us to map problems on fractals into the problems in the ν-dimensional space F^ν with the spatial and temporal metrics governed by the fractal topology. This framework is used to reveal some relevant features of physics on fractals.

Particularly, the diffusion on fractals is highlighted. The diffusion equation in F^ν is derived. The nature of anomalous subdiffusion on path-connected fractals is elucidated. It is pointed out that the subdiffusion in the chemical space is caused by the difference between the numbers of effective spatial and dynamical degrees of freedom of random walkers, whereas the subdiffusion in the embedding Euclidean space also reflects the nonlinear relation between the geodesic and Euclidean metrics. The mapping from the chemical space to the embedding Euclidean space is discussed. It is demonstrated that solutions of diffusion equation in F^ν are consistent with the available data of numerical simulations on deterministic and random fractals.

The Coulomb and gravitational potentials of point sources on fractals with $2 < \nu \leqslant 3$ are derived. A possible experimental validation of theoretical predictions is briefly discussed. Intriguing features of some types of fractals are revealed. Specifically, we found that fractals folded from self-avoiding two-dimensional sheets and percolation clusters in $n \geqslant 6$ have the same universal numbers of effective spatial and dynamical degrees of freedom (25), such that the scalar gravitational potential on these fractals decays as r^{-1}. This means that the scale invariance of percolation clusters in $n \geqslant 6$ does not cause a violation of the dynamical symmetry in the embedding space.

Our findings provide deeper insight into the physics on fractals and fractional dimension spaces, so we expect that this paper will stimulate experimental studies, numerical simulations, and theoretical research in these areas.

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