Local-time representation of path integrals

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We derive a local-time path-integral representation for a generic one-dimensional time-independent system. In particular, we show how to rephrase the matrix elements of the Bloch density matrix as a path integral over x-dependent local-time profiles. The latter quantify the time that the sample paths $x(t)$ in the Feynman path integral spend in the vicinity of an arbitrary point x. Generalization of the local-time representation that includes arbitrary functionals of the local time is also provided. We argue that the results obtained represent a powerful alternative to the traditional Feynman-Kac formula, particularly in the high- and low-temperature regimes. To illustrate this point, we apply our local-time representation to analyze the asymptotic behavior of the Bloch density matrix at low temperatures. Further salient issues, such as connections with the Sturm-Liouville theory and the Rayleigh-Ritz variational principle, are also discussed.

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I. INTRODUCTION

The path integral (PI) has been used in quantum physics since the revolutionary work of Feynman [1], although the basic observation goes back to Dirac [2,3] who appreciated the role of the Lagrangian in short-time evolution of the wave function, and even suggested the time-slicing procedure for finite, i.e., noninfinitesimal, time lags. Since then, the PI approach has yielded invaluable insights into the structure of quantum theory [4] and has provided a viable alternative to the traditional operator-formalism-based canonical quantization. During the second half of the 20th century, the PI became a standard tool in quantum field theory [5] and statistical physics [6], often providing the easiest route to the derivation of perturbative expansions and serving as an excellent framework for (both numerical and analytical) nonperturbative analysis [7].

Feynman PI has its counterpart in pure mathematics, namely, in the theory of continuous-time stochastic processes [8]. There the concept of integration over a space of continuous functions (so-called fluctuating paths or sample paths) had already been introduced by Wiener [9] in the 1920s in order to represent and quantify the Brownian motion. Interestingly enough, this so-called Wiener integral (or integral with respect to Wiener measure) was formulated two years before the discovery of the Schrödinger equation and 25 years before Feynman's PI formulation.

The *local time* for a Brownian particle (in some literature also called *sojourn time*) has been of interest to physicists and mathematicians since the seminal work of Lévy in the 1930s [10]. In its essence, the local time characterizes the time that a sample trajectory $x(t)$ of a given stochastic process spends in the vicinity of an arbitrary point X. This in turn defines a sample trajectory L^X of a new stochastic process. A rich theory has been developed for local-time processes that stem from diffusion processes (see, e.g., Ref. [11] and citations therein). For later convenience, we should particularly highlight the Ray-Knight theorem, which states that the local time of the Wiener process can be expressed in terms of the squared Bessel process [12–14]. In contrast to mathematics, the concept of the local time is not uniquely settled in the physics literature. Various authors define essentially the same quantity under different names (local time, occupation time, traversal time, etc.), and with different applications in mind. For example, in Ref. [15], the *traversal time* is used to study quantum scattering and tunneling processes; in Ref. [16], the small-temperature behavior of the equilibrium density matrix is analyzed with a help of the *occupation time*; while in Ref. [17], the large-time behavior of path integrals that contain functionals of the *local time* is discussed.

The aim of this paper is to derive a local-time PI representation of the Bloch density matrix, i.e., the matrix elements $\langle x_b|e^{-\beta\hat{H}}|x_a\rangle$ of the Gibbs operator. This can serve not only as a viable alternative to the commonly used Feynman-Kac representation, but also as a powerful tool for extracting both large- and small-temperature behavior. Apart from the general theoretical outline, our primary focus here will be on the low-temperature behavior, which is technically more challenging than the large-temperature regime. In fact, the large-temperature expansion was already treated in some detail in our previous paper [18]. Last, but not least, we also wish to promote the concept of the local time, which is not yet sufficiently well known among path-integral practitioners.

The structure of the paper is as follows. To set the stage, we recall in the next section some fundamentals from the Feynman PI which will be needed in later sections. In Sec. III, we provide motivation for the introduction of a local time and construct a heuristic version of the local-time representation of PIs. The key technical part of the article is contained in Sec. IV, where we derive by means of the replica trick the local-time representation of the Bloch density matrix. Relation to the Sturm-Liouville theory is also highlighted in this context. A local-time analog of the Feynman-Matthews-Salam formula [19,20] is presented in Sec. V and its usage is illustrated with a computation of the one-point distribution of the local

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time. Since a natural arena for local-time PIs is in thermally extremal regimes, we confine our attention in Sec. VI to large- and small-β asymptotic behavior of the Bloch density matrix. There we also derive an explicit leading-order behavior in large-β (i.e., low-temperature) expansion. The analysis is substantially streamlined by using the Laplace asymptotic formula and the Rayleigh-Ritz variational principle. Finally, Sec. VII summarizes our results and discusses possible extensions, applications, and future developments of the present work. For the reader's convenience, the paper is supplemented with two appendices which clarify some finer technical details.

II. PATH-INTEGRAL REPRESENTATION OF THE BLOCH DENSITY MATRIX

Consider a nonrelativistic one-dimensional quantum-mechanical system described by a time-independent Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x})$, where $\hat{p}|x\rangle = -i\hbar\partial_x|x\rangle$. Throughout this paper, we will study the matrix elements

$$\rho(x_a,x_b,\beta) \equiv \langle x_b|e^{-\beta\hat{H}}|x_a\rangle \tag{1}$$

of the Gibbs operator $e^{-\beta\hat{H}}$, where $\beta = 1/(k_B T)$ is the inverse temperature and k_B is the Boltzmann constant. The matrix $\rho(x_a,x_b,\beta)$, known also as the Bloch density matrix, is a fundamental object in quantum statistical physics, as the expectation value of an operator $\hat{O}$ at the temperature T can be written in the form

$$\langle\hat{O}\rangle = \frac{1}{Z}\int_{\mathbb{R}}\int_{\mathbb{R}} dx_a dx_b \rho(x_a,x_b,\beta)\langle x_b|\hat{O}|x_a\rangle, \tag{2}$$

where $Z = \int_{\mathbb{R}} dx\ \rho(x,x,\beta)$ is the partition function of the system. If needed, ensuing quantum-mechanical transition amplitudes can be obtained from (1) via a Wick rotation which formally amounts to the substitution $\beta \to it/\hbar$, converting thus the Gibbs operator $e^{-\beta\hat{H}}$ to the quantum evolution operator $e^{-it\hat{H}/\hbar}$.

The matrix elements in Eq. (1) can be represented via the path integral as [4,7]

$$\rho(x_a,x_b,\beta) = \int_{x(0)=x_a}^{x(\beta\hbar)=x_b} \mathcal{D}x(\tau) \times \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau\left[\frac{M}{2}\dot{x}^2 + V(x)\right]\right\}. \tag{3}$$

This represents a "sum" over all continuous trajectories $x(\tau)$, $\tau \in [0,\beta\hbar]$, connecting the initial point $x(0) = x_a$ with the final point $x(\beta\hbar) = x_b$. It should be noted that the integral $\int_0^{\beta\hbar} d\tau[\frac{M}{2}\dot{x}^2 + V(x)]$ is the classical Euclidean action integral along the path $x(\tau)$ with $0 < \tau \leqslant \beta\hbar$. In the following, we will denote the Euclidean action as $\mathcal{A}$. The integrand in $\mathcal{A}$, i.e., $(M/2)\dot{x}^2(\tau) + V[x(\tau)]$, can be identified with the classical Hamiltonian function, in which the momentum p is substituted for $M\dot{x}$. One can also regard (3) as an expectation value of the functional $\exp\{-\int_0^{\beta\hbar} d\tau V[x(\tau)]/\hbar\}$ over the (driftless) Brownian motion with the diffusion coefficient $M/2\hbar$, and duration $\beta\hbar$, that starts at point x_a and terminates at x_b. The latter stochastic process is also known as the Brownian bridge.

III. LOCAL-TIME REPRESENTATION OF PATH INTEGRALS: HEURISTIC APPROACH

The purpose of this section is twofold. First, we would like to motivate a need for the reformulation of PIs in the language of the local-time stochastic process. In particular, we point out when such a reformulation can be more pertinent than the conventional "sum over histories" prescription. Second, we wish to outline a heuristic construction of the local-time representation of PIs. More rigorous and explicit (but less intuitive) formulation of PIs over an ensemble of local times will be presented in the subsequent section.

To provide a physically sound motivation for the local-time representation of PIs, we follow the exposition of Paulin *et al.* in Ref. [16]. To this end, we first consider the diagonal elements of the Bloch density matrix, i.e., $\rho(x_a,x_a,\beta)$ (often referred to as the Boltzmann density). Upon shifting $x \to x + x_a$, and setting $x = \lambda\xi$, $\tau = s\beta\hbar$ ($\lambda \equiv \sqrt{\beta\hbar^2/M}$ is the thermal de Broglie wavelength), the PI (3) can be reformulated in terms of dimensionless quantities s and $\xi(s)$ as

$$\rho(x_a,x_a,\beta) = \frac{1}{\lambda}\int_{x(0)=0}^{x(1)=0} \mathcal{D}\xi(s) \times \exp\left\{-\int_0^1 ds\left[\frac{1}{2}\dot{\xi}^2 + \beta V(x_a + \lambda\xi)\right]\right\}. \tag{4}$$

Note, in particular, that in contrast to $\mathcal{D}x(\tau)$, the measure $\mathcal{D}\xi(s)$ does not explicitly depend on β, and thus β-dependent parts in the PI are under better control. Such a rescaled representation is particularly useful when discussing large- and/or small-β behavior of the path integral in question. Path fluctuations in the potential are controlled by $\lambda \propto \sqrt{\beta}$, and the factor β quantifies the significance of the potential V with respect to the kinetic term.

For small β (i.e., high temperature), typical paths $x(s\beta\hbar) = x_a + \lambda\xi(s)$ stay in the vicinity of the point x_a, as depicted in Fig. 1, and therefore a systematic Wigner-Kirkwood expansion can be readily developed by Taylor expanding the potential part of the action [18].

When β is large (i.e., low temperature), the trajectories $x(s\beta\hbar)$ fluctuate heavily around the value x_a, and the potential term V dominates over the kinetic one. From the statistical physics point of view, the most important contribution to the low-temperature behavior of the path integral (4) should come from those paths that spend a sizable amount of time near the *global* minimum of the potential $V(x)$. For this reason, it is important to be able to keep track of the time which a given path spends in an infinitesimal neighborhood of an arbitrary point x.

Let us define, for each Wiener trajectory $x(\tau)$ present in the Feynman path integral (3), the ensuing *local time* as

$$L^X(\tau) = \int_0^\tau d\tau'\delta[X - x(\tau')]. \tag{5}$$

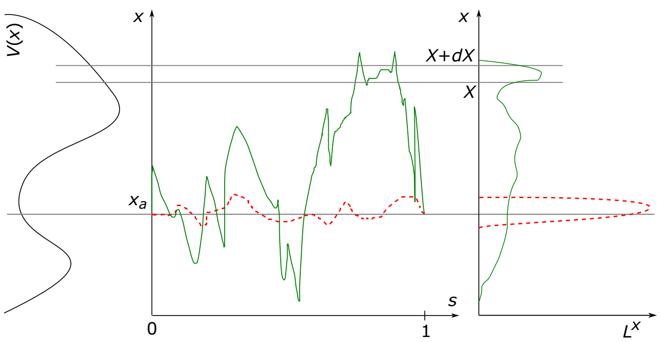

FIG. 1. (Color online) In the middle, two typical paths $x(s\beta\hbar) = x_a + \lambda\xi(s)$ are plotted as functions of the dimensionless time s. The solid green path, representing a typical trajectory with a high value of β, exhibits large fluctuations, whereas the dashed red path, corresponding to small β, stays in the vicinity of the initial and final point x_a. On the right, two local-time profiles $L(x)$ are shown. The broad one (solid green) arises from the violently fluctuating path $x(s\beta\hbar)$, whereas the narrow one (dashed red) corresponds to the path with small fluctuations. On the left, we depict a generic potential $V(x)$.

Since the local time $L^X(\tau)$ is a functional of the random trajectory $x(\tau')$ for $0 < \tau' < \tau$, it represents a random variable. From definition (5), we can immediately see that $L^X \geqslant 0$ for all $X \in \mathbb{R}$, $\int_{\mathbb{R}} dX L^X(\tau) = \tau$ and that L^X has a compact support. In addition, it can be proved [8,11] that local-time trajectories L^X are, with probability one, continuous curves which (similarly to trajectories in the underlying Wiener process) are nowhere differentiable. In Fig. 1, we depict two examples of representative local-time trajectories. An extensive mathematical discussion of properties of the local time can be found, e.g., in Refs. [11,12].

With definition (5), the potential part of the Euclidean action can be recast into the form $\int_{\mathbb{R}} dX L^X(\beta\hbar)V(X)$. A local-time representation of the Bloch density matrix $\rho(x_a,x_b,\beta)$ is then given by

$$\rho(x_a,x_b,\beta) = \int \mathcal{D}L^x \mathcal{W}[L;\beta,x_a,x_b]\delta\left(\int_{\mathbb{R}} dX L^X - \beta\hbar\right) \times \exp\left[-\frac{1}{\hbar}\int_{\mathbb{R}} dX L^X V(X)\right], \tag{6}$$

where the PI "sum" is taken over all local-time trajectories L^x, with x being the independent variable [not to be mistaken as the Wiener trajectory $x(\tau)$]. The δ function enforces the normalization constraint mentioned above. Basically, transition to the local-time description represents a change (or a functional substitution) of stochastic variables $x(\tau) \to L^x(\beta\hbar)$. The weight factor $\mathcal{W}$ appearing in Eq. (6) can be formally written in the form

$$\mathcal{W}[L;\beta,x_a,x_b] = \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \frac{M}{2}\dot{x}^2\right] \det\left[\frac{\delta L^x(\beta\hbar)}{\delta x(\tau)}\right]^{-1}. \tag{7}$$

It is a function of x_a, x_b [which are implicitly present in $L^x(\beta\hbar)$] and β, and a functional of the local time L^x. Of course, these cavalier manipulations do not have more than a heuristic nature, and it is, indeed, a nontrivial task to determine $\mathcal{W}$ directly from (7). For this reason, we will in the following section tackle this problem indirectly.

IV. LOCAL-TIME REPRESENTATION OF PATH INTEGRALS: DERIVATION

In this section, we present a derivation of the local-time representation of the Bloch density matrix (1). Initially, we limit ourselves to considering the case of the diagonal part, $x_b = x_a$, and arrive at the key result (16), which expresses the matrix elements in the ensuing Laplace picture with respect to β. This intermediate outcome is shown to agree with the Sturm-Liouville theory. In the next step, we generalize the latter result to the off-diagonal elements ($x_b \neq x_a$). The inverse Laplace transform will then yield the sought local-time representation of PI [cf. Eq. (28)].

A. Field-theoretic representation

It follows from definition (1) that the function $\rho(x_a,x_b,\beta)$ satisfies the heat equation

$$\left[\frac{\partial}{\partial\beta} - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial x_b^2} + V(x_b)\right]\rho(x_a,x_b,\beta) = 0, \tag{8}$$

with the initial condition $\rho(x_a,x_b,0_+) = \delta(x_a - x_b)$. This is merely a Wick-rotated ($t \to -i\hbar\beta$) analog of the Schrödinger equation. The Feynman-Kac formula [1,21,22] then ensures that the PI (3) can be calculated by solving the corresponding parabolic differential equation (8).

In the Laplace picture, Eq. (8) takes the form

$$\left[E - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial x_b^2} + V(x_b)\right]\widetilde{\rho}(x_a,x_b,E) = \delta(x_a - x_b), \tag{9}$$

with $\widetilde{\rho}(x_a,x_b,E) = \int_0^\infty d\beta e^{-\beta E}\rho(x_a,x_b,\beta)$. Equation (9) implies that $\widetilde{\rho}$ is nothing but the Green function of the operator $E + \hat{H}$. With the benefit of hindsight, we represent the Green function $\widetilde{\rho}$ as a path integral over fluctuating fields—the so-called functional integral [5]. This is rather standard strategy in quantum field theory [6,20], and in our case it yields

$$\widetilde{\rho}(x_a,x_b,E) = \frac{\int_{\psi(X_-)=0}^{\psi(X_+)=0} \mathcal{D}\psi(x)\psi(x_a)\psi(x_b)e^{-\frac{1}{2}\mathcal{A}^E[\psi]}}{\int_{\psi(X_-)=0}^{\psi(X_+)=0} \mathcal{D}\psi(x)e^{-\frac{1}{2}\mathcal{A}^E[\psi]}}, \tag{10}$$

where

$$\mathcal{A}^E[\psi] \equiv \int_{X_-}^{X_+} dx\,\psi(x)\left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2} + V(x) + E\right]\psi(x) = \int_{X_-}^{X_+} dx\left\{\frac{\hbar^2}{2M}\psi'(x)^2 + [V(x)+E]\psi(x)^2\right\} \tag{11}$$

is the Euclidean action functional of the field-theoretic path integral. The superindex E in $\mathcal{A}$ indicates the shift in the potential $V(x)$ by the amount E. Here, we have confined our quantum-mechanical system within a finite box $[X_-,X_+]$, with $X_- \ll \min\{x_a,x_b\}$ and $X_+ \gg \max\{x_a,x_b\}$. A real scalar field $\psi(x)$ satisfies Dirichlet boundary conditions

$\psi(X_-) = \psi(X_+) = 0$ so as to ensure the validity of the operations being performed.

B. Replica trick

Since we will ultimately want to invert the Laplace transform to regain from $\widetilde{\rho}(x_a,x_b,E)$ the original Bloch density matrix $\rho(x_a,x_b,\beta)$, we cannot treat the denominator in Eq. (10) as an irrelevant normalization constant (which is the usual practice in quantum field theory), but instead we have to take care of its E dependence. To this end, we take advantage of the formula

$$\frac{a}{b} = \lim_{D\to 0} ab^{D-1}, \tag{12}$$

which is a simple version of the *replica trick*. [The usual replica-trick formula (cf. e.g., Ref. [23]) can be obtained from (12) by integrating both sides with respect to b and subsequently dividing by a.] With the help of (12), we can rewrite (10) as a multidimensional functional integral,

$$\widetilde{\rho}(x_a,x_b,E) = \lim_{D\to 0}\frac{2}{D}\int_{\boldsymbol{\psi}(X_-)=\mathbf{0}}^{\boldsymbol{\psi}(X_+)=\mathbf{0}} \mathcal{D}\boldsymbol{\psi}(x)\boldsymbol{\psi}(x_a) \cdot \boldsymbol{\psi}(x_b) e^{-\sum_{\sigma=1}^{D}\mathcal{A}^E[\psi_\sigma]}, \tag{13}$$

where the multiplet $\boldsymbol{\psi} = (\psi_1,\ldots,\psi_D)$ is a D-component "replica" field in $1+0$ dimensions, $\boldsymbol{\psi}(x_a)\cdot\boldsymbol{\psi}(x_b)$ denotes the scalar product $\sum_{\sigma=1}^{D}\psi_\sigma(x_a)\psi_\sigma(x_b)$, and we have rescaled the fields by a factor of $\sqrt{2}$ in passing. The factor $1/D$ results from a PI generalization of the well-know mean-value identity $\langle x_i y_i\rangle = \langle \mathbf{x}\cdot\mathbf{y}\rangle/D$ valid for any two vectors in D-dimensional statistically isotropic environments.

As a side remark, note that we may now invert the Laplace transform, using the trivial identity

$$\int_0^\infty d\beta e^{-\beta E}\delta(\beta - c) = e^{-cE} \quad \text{for } c > 0, \tag{14}$$

to obtain the representation

$$\rho(x_a,x_b,\beta) = \lim_{D\to 0}\frac{2}{D}\int_{\boldsymbol{\psi}(X_-)=\mathbf{0}}^{\boldsymbol{\psi}(X_+)=\mathbf{0}} \mathcal{D}\boldsymbol{\psi}\,\boldsymbol{\psi}(x_a) \cdot \boldsymbol{\psi}(x_b)\delta\left[\int_{X_-}^{X_+}\boldsymbol{\psi}(x)^2 dx - \beta\right] e^{-\sum_{\sigma=1}^{D}\mathcal{A}^{E=0}[\psi_\sigma]}. \tag{15}$$

Upon scaling, $\boldsymbol{\psi} \to \sqrt{\beta}\boldsymbol{\psi}$, which agrees with the formula (2.9) in Ref. [17]. Though the result (15) generalizes to an arbitrary number of dimensions of the x space, i.e., $x\in\mathbb{R}^d$, our further development will be illustrated (for simplicity's sake) only on the one-dimensional case.

Let us remark that as an alternative to the replica trick, one can use the fermionic path-integral representation of the denominator in Eq. (10) to cast $\tilde{\rho}$ in the form of a supersymmetric path integral [24]. However, the ensuing expression containing Grassmann variables does not have a direct "visual" interpretation in terms of local times. On the other hand, as we shall see below, the replica approach allows one to relate local time directly to a radial part of the replica field ψ.

C. Connection with radial harmonic oscillator

In order to derive the weight factor $\mathcal{W}$ [cf. Eq. (7)], we have arrived at the representation (15) with D replica fields. This form is still not very transparent and a further simplification step is needed to get rid of an explicit dependence of the measure on D. To this end, we note that $\sum_{\sigma=1}^{D}\mathcal{A}^E[\psi]$ is, in fact, the action of a D-dimensional harmonic oscillator with the "time" variable x, "position" variable ψ, mass $\hbar^2/M$, and time-dependent "frequency" $V(x)+E$. Considering for a moment only diagonal matrix elements, $x_b = x_a$, spherical symmetry in the replica field space allows one to reduce the path integral (13) to its radial part. Due to the boundary conditions, $\boldsymbol{\psi}(X_-) = \boldsymbol{\psi}(X_+) = \mathbf{0}$, only the zero-angular-momentum (s-wave) contribution is nonvanishing (generally a weighted sum over radial PIs with different angular momenta would be required). This will be rigorously justified at the end of this section. The corresponding *radial* PI representation for (13) reads [6,7,25]

$$\widetilde{\rho}(x_a,x_a,E) = \lim_{D\to 0}\frac{2}{D\Omega(D)}\lim_{\eta_\pm\to 0}(\eta_-\eta_+)^{\frac{1-D}{2}} \times \int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+}\mathcal{D}\eta(x)\eta^2(x_a)e^{-\mathcal{A}_D^E[\eta]}. \tag{16}$$

Here, the radial part $\eta \equiv \sqrt{\boldsymbol{\psi}^2}$ of the D-dimensional replica field $\boldsymbol{\psi}$ is always non-negative, i.e., $\eta(x)\geqslant 0$; the area of a unit sphere in D dimensions, $\Omega(D) = 2\pi^{D/2}/\Gamma(D/2)$, may be replaced by its small-D asymptotic form $\Omega(D)\sim D$; and $\eta_\pm$ have been introduced to regularize the origin of the $\boldsymbol{\psi}$ space. The new action functional

$$\mathcal{A}_D^E[\eta] \equiv \mathcal{A}^E[\eta] + \int_{X_-}^{X_+} dx\,\frac{M}{\hbar^2}\frac{(D-1)(D-3)}{8\eta^2(x)} \tag{17}$$

is the Euclidean action functional of the *radial harmonic oscillator* [7,25,26]. It contains an additional centrifugal potential term (Edwards-Gulyaev or Langer term [7,27,28]), which emerges from Bessel function $I_{D/2-1}$ present in the finite sliced form of the radial PI (16). At this point, we should stress that in contrast to the quantum-mechanical radial PI, one can safely use the asymptotic expansion for the Bessel function $I_{D/2-1}$ (see, e.g. Ref. [29]):

$$I_\mu(y_j) \sim \frac{1}{\sqrt{2\pi y_j}}e^{y_j-(\mu^2-1/4)/2y_j} \quad (|y_j|\gg 1,\ \mathrm{Re}[y_j]>0),$$
$$y_j = (M/\varepsilon\hbar)r_j r_{j-1}, \tag{18}$$

in the Euclidean PI sliced form. Here the infinitesimal "time" slice ε is related to the number of slices N via the relation $\varepsilon = \hbar\beta/N$. In quantum mechanics, this is a problematic step because (18) requires $\mathrm{Re}[y_j]>0$, while there $\mathrm{Re}[y_j] = \mathrm{Re}[(M/i\varepsilon\hbar)r_j r_{j-1}] = 0$.

Fortunately, the PI for the radial harmonic oscillator is exactly solvable even in the case of x-dependent oscillator frequency. The solution reads [25]

$$(\eta_2 x_2|\eta_1 x_1)_D \equiv \int_{\eta(x_1)=\eta_1}^{\eta(x_2)=\eta_2} \mathcal{D}\eta(x) \exp\left(-\int_{x_1}^{x_2} dx\left\{\frac{\hbar^2}{2M}\eta'^2 + [V(x)+E]\eta^2 + \frac{M}{\hbar^2}\frac{(D-1)(D-3)}{8\eta^2}\right\}\right)$$
$$= \frac{\hbar^2}{M}\frac{\sqrt{\eta_1\eta_2}}{G(x_1)} I_{D/2-1}\left[\frac{\hbar^2}{M}\frac{\eta_1\eta_2}{G(x_1)}\right] \exp\left\{-\frac{\hbar^2}{2M}\left[\frac{F'(x_2)}{F(x_2)}\eta_2^2 - \frac{G'(x_1)}{G(x_1)}\eta_1^2\right]\right\}. \quad (19)$$

The functions $F(x)$ and $G(x)$ are two independent solutions of the differential equation

$$[\hat{H}+E]y(x) = \left[-\frac{\hbar^2}{2M}\frac{d^2}{dx^2} + V(x) + E\right]y(x) = 0, \quad (20)$$

with the initial conditions $F(x_1)=0$ and $F'(x_1)=1$, and $G(x_2)=0$ and $G'(x_2)=-1$. In addition, the Wronskian $W(F,G) \equiv F(x)G'(x) - F'(x)G(x)$ is independent of x, as can be proved by differentiation and by using the fact that F and G both satisfy Eq. (20). By equating the values of $W(F,G)$ at points x_1 and x_2, and taking into account the initial conditions for F and G, we find a useful identity $F(x_2) = G(x_1)$.

Now, the PI in Eq. (16) can be sliced at point x_a, and expressed as

$$\widetilde{\rho}(x_a,x_a,E) = \int_0^\infty d\eta_a (\eta_+ X_+|\eta_a x_a)_D \eta_a^2 (\eta_a x_a|\eta_- X_-)_D. \quad (21)$$

The limits in Eq. (16) are readily carried out with the help of the asymptotic formulas $I_{D/2-1}(z) \sim (z/2)^{D/2-1}/\Gamma(D/2)$ and $\Gamma(z) \sim 1/z$, valid for $z \to 0_+$. Subsequent integration over η_a brings (16) to the form

$$\widetilde{\rho}(x_a,x_a,E) = -\frac{2M}{\hbar^2}\frac{F_1(x_a)G_2(x_a)}{F_1(x_a)G_2'(x_a) - F_1'(x_a)G_2(x_a)}, \quad (22)$$

where $F_1(x)$ solves Eq. (20) with initial conditions $F_1(X_-)=0$ and $F_1'(X_-)=1$, and $G_2(x)$ solves the same equation with $G_2(X_+)=0$ and $G_2'(X_+)=-1$. The denominator in Eq. (22) is the Wronskian $W(F_1,G_2)$. The full derivation is given in Appendix A.

Although rather explicit, Eq. (22) is not well suited for the Laplace transform inversion, since functions $F(x)$ and $G(x)$ contain E in a nontrivial way, which, in addition, significantly hinges on the actual form of $V(x)$. For formal manipulations, it is still better the employ the PI representation (16). For instance, using Eq. (14), we can easily invert the Laplace transform to return from E back to the β variable, namely,

$$\rho(x_a,x_a,\beta) = \lim_{D\to 0}\frac{2}{D^2}\lim_{\eta_\pm\to 0}(\eta_-\eta_+)^{\frac{1-D}{2}}\int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+}\mathcal{D}\eta(x)$$
$$\times\, \eta^2(x_a)\delta\left(\int_{X_-}^{X_+}\eta^2 dx - \beta\right)e^{-A_D^{E=0}[\eta]}. \quad (23)$$

Note that we have utilized the asymptotic form $\Omega(D) \sim D$ which holds for $D \ll 1$. We shall see shortly that (23) can be straightforwardly related to the local-time PI representation of the Boltzmann density matrix.

Let us finally comment on the higher-angular-momentum terms which, as claimed, should not contribute to expression (16). For arbitrary angular momentum $\ell \geqslant 0$, we employ formula (19) with a slight modification, $D \to D + 2\ell$. Now, for example, in the limit $\eta_- \to 0$, this becomes $(\eta_a x_a|\eta_- X_-)_{D+2\ell} \propto \eta_-^{\ell+D/2-1/2}$, which, multiplied by the prefactor $\eta_-^{1/2-D/2}$, implies the behavior $\sim \eta_-^\ell$. That is, only the ($\ell = 0$) term can give a nonvanishing contribution.

D. Connection with the Sturm-Liouville problem

Consider again Eq. (9) and a finite interval $x \in [X_-,X_+]$. The corresponding Green function of the operator $\hat{H}+E$ can be easily constructed (at least formally) with the help of the Sturm-Liouville theory [30,31]. An immediate consequence of the latter is that for $x_a < x_b$, the Green function has the form

$$\widetilde{\rho}(x_a,x_b,E) = -\frac{2M}{\hbar^2}\frac{F(x_a)G(x_b)}{W(F,G)}, \quad (24)$$

where the functions $F(x)$ and $G(x)$ satisfy Eq. (20) with the initial conditions $F(X_-)=0$ and $F'(X_-)=1$, and $G(X_+)=0$ and $G'(X_+)=-1$, respectively. In addition, the above Green function should be symmetric due to the Hermitian nature of $\hat{H}$.

The Sturm-Liouville theory ensures that the solution to the second-order differential equation (20) is unique, when specifying the values of $y(x_0)$ and $y'(x_0)$ at some point x_0. Therefore, the functions F and G must coincide with F_1 and G_2 of Eq. (22), and the diagonal part of (24), i.e., $\widetilde{\rho}(x_a,x_a,E)$, reduces to expression (22). This is an important consistency check of our representation (16).

E. Extension to off-diagonal matrix elements

Let us now generalize the PI representation (16) to the full Bloch density matrix, i.e., we wish to also include the off-diagonal matrix elements, $x_b \neq x_a$. If we go back to the replica representation (13), we realize that the requirement $x_b \neq x_a$ spoils rotational symmetry in the replica field space, and thus precludes straightforward reduction to a radial path integral. Instead of refining the reduction procedure, we simply make a guess, which, as we prove in Appendix A, coincides with the well-established Sturm-Liouville formula (24). Our guess is based on mathematical results presented in Eq. [12]. In particular, we claim that the extension of the representation (16) to off-diagonal matrix elements should read

$$\widetilde{\rho}(x_a,x_b,E) = \lim_{D\to 0}\frac{2}{D^2}\lim_{\eta_\pm\to 0}(\eta_-\eta_+)^{\frac{1-D}{2}}$$
$$\times \int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+}\mathcal{D}\eta(x)\eta(x_a)\eta(x_b)e^{-\mathcal{A}_\Delta^E[\eta]}, \quad (25)$$

with the action functional

$$\mathcal{A}^E_\Delta[\eta] \equiv \mathcal{A}^E[\eta] + \int_{X_-}^{X_+} dx \frac{M}{\hbar^2}\frac{\Delta(x)}{8\eta^2(x)}, \qquad (26)$$

where $\mathcal{A}^E[\eta]$ is defined in Eq. (11), and $\Delta(x)$ is a piecewise constant function,

$$\Delta(x) = \begin{cases} -1 & \text{for } x \in [x_a,x_b] \\ (D-1)(D-3) & \text{otherwise.} \end{cases} \qquad (27)$$

At this point, we can invert the Laplace transform with the help of Eq. (14). As a result, we obtain the sought local-time PI representation of the Bloch density matrix (1), namely,

$$\rho(x_a,x_b,\beta) = \lim_{D\to 0}\frac{2}{D^2}\lim_{\eta_\pm \to 0}(\eta_-\eta_+)^{\frac{1-D}{2}} \times \int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+} \mathcal{D}\eta\, \eta(x_a)\eta(x_b) \times \delta\left(\int_{X_-}^{X_+}\eta^2 dx - \beta\right) e^{-\mathcal{A}^{E=0}_\Delta[\eta]}. \qquad (28)$$

Here, integrations over $\eta(x)$ run from 0 to $+\infty$, i.e., the paths $\eta(x)$ are non-negative. Comparing this result with the anticipated heuristic form (6), we can identify $\eta^2(x) = L^X(\beta\hbar)/\hbar$. Representation (28) allows us to identify the weight factor (7) with

$$\mathcal{W}_D[\eta;\beta,x_a,x_b] = \frac{2}{D^2}(\eta_-\eta_+)^{\frac{1-D}{2}}\eta(x_a)\eta(x_b) \times \exp\left\{-\int_{X_-}^{X_+} dx\left[\frac{\hbar^2}{2M}\eta'^2 + \frac{M}{\hbar^2}\frac{\Delta(x)}{8\eta^2}\right]\right\}. \qquad (29)$$

Contrary to expectation, the right-hand side of this expression does not depend on β. Subindex D in $\mathcal{W}_D$ indicates that the weight factor must be regularized when we pull it out of the PI (28). By analogy with quantum mechanics, one can represent (28) in the discretized time-sliced form. In such a case, the weight $\mathcal{W}_D$ would be a product of terms involving the Bessel functions $I_{D/2-1}$, if $\Delta(x) = (D-1)(D-3)$, or I_0, if $\Delta(x) = -1$ (see Ref. [25]).

Last, but not least, expressions (28) and (29) indicate that the square root of L^X is (at least from a physicist's point of view) a more convenient variable to describe local-time trajectories than L^X alone. From a mathematical standpoint, the local-time representation of the density matrix (28) can be regarded as a PI variant of the Ray-Knight theorem [12–14], which plays a prominent role in the theory of stochastic processes.

V. FUNCTIONALS OF THE LOCAL TIME

Formula (28) provides a way of rewriting the PI (3) in terms of the local time. In this section, we consider a more general scenario in which the initial path integral is of the form

$$\bar{F}(x_a,x_b,\beta) \equiv \int_{x(0)=x_a}^{x(\beta\hbar)=x_b} \mathcal{D}x(\tau) F[L] \times \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left[\frac{M}{2}\dot{x}^2 + V(x)\right]\right\}, \qquad (30)$$

where F is an arbitrary functional of the local time $L^X(\beta\hbar)$, which itself is (as seen in Sec. III) a functional of the paths $x(\tau)$. Relation (30) represents a local-time analog of the Feynman-Matthews-Salam formula [19,20].

To bring it into more manageable form, we may observe that for any X, the action of L^X in the PI (30) can be taken over by the functional derivative $-\hbar\delta/\delta V(X)$, acting on the exponential. This becomes transparent after rewriting the potential part as $\int_0^{\beta\hbar} d\tau\, V[x(\tau)] = \int_{\mathbb{R}} dX\, V(X)L(X)$. The entire functional $F[L]$ can therefore be pulled out of the path integral, which then allows one to write

$$\bar{F}(x_a,x_b,\beta) = F\left[-\hbar\frac{\delta}{\delta V}\right]\rho(x_a,x_b,\beta). \qquad (31)$$

When we employ the local-time representation of PI for $\rho(x_a,x_b,\beta)$ [cf. Eq. (28)], each functional derivative $-\hbar\delta/\delta V(x)$ will produce the term $\hbar\eta^2(x)$. In such a way, $\bar{F}$ can be written as

$$\bar{F}(x_a,x_b,\beta) = \lim_{D\to 0}\frac{2}{D^2}\lim_{\eta_\pm\to 0}(\eta_-\eta_+)^{\frac{1-D}{2}}\int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+}\mathcal{D}\eta\,\eta(x_a)\eta(x_b) \times \delta\left(\int_{X_-}^{X_+}\eta^2 dx - \beta\right) F[\hbar\eta^2] e^{-\mathcal{A}^{E=0}_\Delta[\eta]}, \qquad (32)$$

where, strictly speaking, the functional $F[\cdots]$ is regularized in such a way that it depends on L^X only for $X \in [X_-,X_+]$, and $X_\pm$ are sent to $\pm\infty$ only at the end of the calculation.

First, let us make the simple observation that formula (32) reduces to (28) for the choice $F[L] = 1$. One of the most important mean values of a local-time functional, as evaluated with Eq. (32), is the mean of $\exp[-\int_{\mathbb{R}} dX L^X j(X)]$, which gives the moment-generating functional. The local-time moment structure is particularly pertinent in various perturbative expansions, including low- and high-temperature expansions (see Sec. VI). Another important example, namely the case of a one-point distribution function, will be discussed in the following section.

In passing, we should note that should we have started from (15) and repeated the above procedure, an analog of Eq. (32) for higher-dimensional spaces, $x \in \mathbb{R}^d$, could be easily obtained. This would include the D-dimensional replica field $\boldsymbol{\psi}$ in the d-dimensional Euclidean configuration space.

Example: One-point distribution function at the origin

A simple, though quite important, consequence of Eq. (32) is that it readily provides the N-point distribution functions of the local time. This is achieved when we set $F[L] = \prod_{n=1}^N \delta(L^{X_n} - L_n)$. In order to see what is involved, let us now illustrate the calculation for $N = 1$ (with $L_1 \equiv L$). Our discussion will be greatly simplified by considering only a free particle [i.e., $V(x) = 0$] that starts and ends at the origin, i.e., $x_a = x_b = 0$. This corresponds to a stochastic process known as Brownian bridge. Our goal is to derive the one-point distribution function, denoted $p(L;\beta)$, of the local time at $X = 0$. We define $p(L;\beta)$ by Eq. (30) with $F[L] = \delta(L^0 - L)$, and calculate it from the representation (32) as follows.

In the Laplace picture, $\widetilde{p}(L;E) = \int_0^\infty d\beta e^{-\beta E} p(L;\beta)$, the path integral (32) can be sliced at $x_a = x_b = 0$ so that

$$\widetilde{p}(L;E) = \lim_{D\to 0}\frac{2}{D^2}\lim_{\eta_\pm\to 0}(\eta_-\eta_+)^{\frac{1-D}{2}}\int_0^\infty d\eta_0\eta_0^2 \times\delta\big(\hbar\eta_0^2 - L\big)(\eta_+X_+|\eta_0 0)_D(\eta_0 0|\eta_-X_-)_D. \tag{33}$$

The η_0 integration can be done easily by realizing that $\delta(\hbar\eta_0^2 - L) = \delta(\eta_0 - \sqrt{L/\hbar})/2\sqrt{\hbar L}$. Furthermore, the limits in $\eta_\pm$ and D can be carried out with the help of formulas (A2) and (A3) from Appendix A. Consequently, we obtain

$$\widetilde{p}(L;E) = \exp\left\{-\frac{L\hbar^2}{2M}\left[\frac{F_1'(0)}{F_1(0)} - \frac{G_3'(0)}{G_3(0)}\right]\right\}, \tag{34}$$

where, for the free-particle case, $F_1(x) = \sinh[\sqrt{2ME/\hbar^2}(x - X_-)]/\sqrt{2ME/\hbar^2}$, and $G_3(x) = \sinh[\sqrt{2ME/\hbar^2}(X_+ - x)]/\sqrt{2ME/\hbar^2}$, as one can straightforwardly verify. In the limit $X_\pm \to \pm\infty$, Eq. (34) reduces to

$$\widetilde{p}(L;E) = e^{-\sqrt{2\hbar^2 E/M}L}, \tag{35}$$

and its inverse-Laplace transform yields

$$p(L;\beta) = \frac{L\exp\left(-\frac{L^2\hbar^2}{2\beta M}\right)}{\sqrt{2\pi M\beta^3/\hbar^2}}. \tag{36}$$

We stress that $p(L;\beta)$ thus obtained is, in fact, the (unnormalized) *joint* probability density for stochastic events $x(0) = 0 \rightsquigarrow x(\beta\hbar) = 0$ and $L^0 = L$. By Bayes' theorem of the probability calculus, the desired conditional probability density $p[L^0 = L|x(0) = 0 \rightsquigarrow x(\beta\hbar) = 0]$ is obtained from (36) by dividing $p(L;\beta)$ by the Brownian-bridge probability density $p[x(0) = 0 \rightsquigarrow x(\beta\hbar) = 0]$, which is (omitting again normalization) $(2\pi\beta\hbar^2/M)^{-1/2}$ (see, e.g., Ref. [4]). Normalization factors mutually cancel in the fraction and we arrive at

$$p[L^0 = L|x(0) = 0 \rightsquigarrow x(\beta\hbar) = 0] = \frac{\hbar^2 L\exp\left(-\frac{L^2\hbar^2}{2\beta M}\right)}{\beta M}, \tag{37}$$

which is clearly normalized to 1. One can proceed along the same lines also in more complicated higher-dimensional ($N > 1$) cases. Our result agrees with the one found through other means in Ref. [12].

VI. ASYMPTOTIC BEHAVIOR OF THE BLOCH DENSITY MATRIX

A compelling feature of the local-time representation (28) is that it naturally captures both small- and large-β asymptotic regimes. This should be compared with the Feynman-Kac PI representation (3), which is typically suitable only for the small-β (i.e., large-temperature) analysis. The latter is epitomized either by the WKB approximation [6,7] or Wigner-Kirkwood expansion [18]. In the large-β (small-temperature) limit, the spectral representation of the Gibbs operator, $e^{-\beta\hat{H}} = \sum_n e^{-\beta E_n}|\phi_n\rangle\langle\phi_n|$, reduces the Bloch density matrix to the ground-state contribution,

$$\rho(x_a,x_b,\beta) \overset{\beta\to\infty}{\sim} e^{-\beta E_0}\psi_{gs}^*(x_a)\psi_{gs}(x_b), \tag{38}$$

which is not evident from the Feynman-Kac PI representation [32]. In connection with Eq. (38), it is useful to remember that in $d = 1$, the discrete bound states can all be chosen to be real [33], so that the Bloch density matrix is real and symmetric and can be written in the form

$$\rho(x_a,x_b,\beta) = \sum_{n=0} e^{-\beta E_n}\psi_n(x_a)\psi_n(x_b) \overset{\beta\to\infty}{\sim} e^{-\beta E_0}\psi_{gs}(x_a)\psi_{gs}(x_b). \tag{39}$$

Let us first comment on the small-β regime of the local-time representation, assuming $x_b = x_a$ for simplicity. This case was discussed in detail in our previous article [18]. There, one should first Taylor expand the potential $V(x)$ around the point x_a, and then expand the exponential part containing the structure $\int \eta^2(x)O(x - x_a)dx$, where

$$O(x - x_a) = \beta\sum_{m\neq 0}\frac{V^{(m)}(x_a)}{m!}[\lambda(x - x_a)]^m. \tag{40}$$

After the term $e^{-\beta V(x_a)}/\lambda$ is factored out of the integral, the individual summands of the ensuing series are of the form (32) with the potential $V(x) = 0$, and functional $F[L] \propto \prod_n L^{x_n}/\hbar$. The latter can be related to the power expansion in β presented in Ref. [18] through the equality of representations (30) and (32). The whole Bloch density matrix (containing also off-diagonal elements) can be treated similarly in a full analogy with Ref. [18].

Let us now turn to the second and more interesting situation, namely, the large-β regime. In doing so, we will also highlight some pertinent technical issues related to the radial PI involved. The large-β expansion of Eq. (28) can be conveniently studied after rescaling $\eta \to \sqrt{\beta}\eta$, in which case we can write

$$\begin{aligned}\rho(x_a,x_b,\beta) &= \lim_{D\to 0}\frac{2}{D^2}\lim_{\eta_\pm\to 0}(\eta_-\eta_+)^{\frac{1-D}{2}}\int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+}\mathcal{D}\eta(x)\eta(x_a)\eta(x_b)\\ &\quad\times\delta\left(\int_{X_-}^{X_+}\eta^2 dx - 1\right)\exp\left\{-\int_{X_-}^{X_+}dx\left[\frac{\beta\hbar^2}{2M}\eta'^2 + \beta V(x)\eta^2 + \frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2}\right]\right\}\\ &= \lim_{D\to 0}\frac{2}{D^2}\lim_{\eta_\pm\to 0}(\eta_-\eta_+)^{\frac{1-D}{2}}\frac{\beta\delta^2}{\delta J(x_a)\delta J(x_b)}\int_{c-i\infty}^{c+i\infty}\frac{d\kappa}{2\pi i}\int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+}\mathcal{D}\eta(x)\\ &\quad\times\exp\left(-\beta\left\{\int_{X_-}^{X_+}dx\left[\frac{\hbar^2}{2M}\eta'^2 + V(x)\eta^2 - \kappa\eta^2\right] + \kappa\right\}\right)\exp\left\{-\int_{X_-}^{X_+}dx\left[\frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2} + J\eta\right]\right\}\Bigg|_{J=0},\end{aligned} \tag{41}$$

where c is an arbitrary real number. With the *method of images* [7,25,34], we can rewrite the radial PI involved as a superposition of two genuine one-dimensional PIs [26],

$$\int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+} \mathcal{D}^R\eta(x) \exp\{-\beta[\langle\eta|\hat{H}|\eta\rangle - \kappa(\langle\eta|\eta\rangle - 1] - \langle J|\eta\rangle\} \exp\left[-\int_{X_-}^{X_+} dx \frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2}\right]_R$$
$$= \int_{\eta(X_-)=\eta_-}^{\eta(X_+)=\eta_+} \mathcal{D}\eta(x) \exp\{-\beta[\langle\eta|\hat{H}|\eta\rangle - \kappa(\langle\eta|\eta\rangle - 1)] - \langle J||\eta|\rangle\} \exp\left[-\int_{X_-}^{X_+} dx \frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2}\right]$$
$$+ \sin(\pi D/2)\int_{\eta(X_-)=-\eta_-}^{\eta(X_+)=\eta_+} \mathcal{D}\eta(x) \exp\{-\beta[\langle\eta|\hat{H}|\eta\rangle - \kappa(\langle\eta|\eta\rangle - 1)] - \langle J||\eta|\rangle\} \exp\left[-\int_{X_-}^{X_+} dx \frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2}\right], \quad (42)$$

where Dirac's notation was employed [35]. A few comments are in order about the right-hand side of the above relation. First, the presence of the parity-even terms $\langle J||\eta|\rangle$ in PIs should be noticed. Second, PIs differ by their respective Dirichlet boundary conditions. Third, the superscript R was used to stress the restricted nature of the fluctuations in the radial PI measure, while the measure without R represents a usual one-dimensional PI measure, i.e.,

$$\mathcal{D}^R\eta(x) \doteq \lim_{N\to\infty}\left(\frac{\beta\hbar^2}{2\pi\varepsilon M}\right)^{N/2}\prod_{k=1}^{N-1}\int_0^\infty d\eta_k, \quad \mathcal{D}\eta(x) \doteq \lim_{N\to\infty}\left(\frac{\beta\hbar^2}{2\pi\varepsilon M}\right)^{N/2}\prod_{k=1}^{N-1}\int_{-\infty}^\infty d\eta_k\,. \quad (43)$$

Here, $\doteq$ denotes De Witt's "equivalence" symbol [36]. Finally, the correct time-sliced form of the exponential with the centrifugal potential is (cf. e.g., Refs. [7,26])

$$\exp\left[-\int_{X_-}^{X_+} dx \frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2}\right]_R \doteq \lim_{N\to\infty}\prod_{k=1}^{N}\sqrt{2\pi\frac{\beta\hbar^2}{M}\frac{\eta_k\eta_{k-1}}{\varepsilon}\tilde{\Delta}_k}\exp\left(-\frac{\beta\hbar^2}{M}\frac{\eta_k\eta_{k-1}}{\varepsilon}\tilde{\Delta}_k\right)I_{\frac{D-2}{2}}\left(\frac{\beta\hbar^2}{M}\frac{\eta_k\eta_{k-1}}{\varepsilon}\tilde{\Delta}_k\right),$$
$$\exp\left[-\int_{X_-}^{X_+} dx \frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2}\right] \doteq \lim_{N\to\infty}\prod_{k=1}^{N}\psi_{\frac{D-2}{2}}\left(-\frac{\beta\hbar^2}{M}\frac{\eta_k\eta_{k-1}}{\varepsilon}\tilde{\Delta}_k\right), \quad (44)$$

with

$$\tilde{\Delta}_k \equiv \tilde{\Delta}(x_k) = \begin{cases}-(D-1)(D-3) & \text{for } x_k \in [x_a,x_b]\\ 1 & \text{otherwise,}\end{cases} \quad (45)$$

and (see, e.g., Refs. [26,37])

$$\psi_p(-x) = e^{-x}\sqrt{\frac{\pi x}{2}}[I_{-p}(x) + I_p(x)],$$
$$\psi_p(x) = \frac{e^x}{\sin(\pi p)}\sqrt{\frac{\pi x}{2}}[I_{-p}(x) - I_p(x)] = e^x\sqrt{\frac{2x}{\pi}}K_p(x). \quad (46)$$

(I_p and K_p are the modified Bessel functions of the first and the second kind, respectively.) In cases when $x \gg 1$, meaning that $|\eta_k\eta_{k-1}| \gg \varepsilon$ (e.g., "typical situation" for very fine time slicings), the asymptotic form of $\psi_p(\pm x) \sim 1 \mp (1-4p^2)/8x + O(1/x^2)$ holds. With the help of the preceding asymptotic behavior, one obtains

$$\prod_{k=1}^{N}\psi_{\frac{D-2}{2}}\left(-\frac{\beta\hbar^2}{M}\frac{\eta_k\eta_{k-1}}{\varepsilon}\tilde{\Delta}_k\right) \sim \prod_{k=1}^{N}\left[1 - \frac{M}{\beta\hbar^2}\frac{(D-3)(D-1)}{8\eta_k\eta_{k-1}\tilde{\Delta}_k}\varepsilon + O(\varepsilon^2)\right] \sim \prod_{k=1}^{N}\exp\left[-\frac{M}{\beta\hbar^2}\frac{\Delta_k}{8\eta_k\eta_{k-1}}\varepsilon + O(\varepsilon^2)\right]$$
$$\sim \exp\left[-\int_{X_-}^{X_+} dx \frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2}\right]. \quad (47)$$

Potential singularities of the integral at $\eta = 0$ can be regularized, e.g., by a principal value prescription. Unfortunately, the formula (47) cannot be directly used in our case because the boundary values η_- and η_+ are arbitrarily close to zero, and hence the assumed asymptotic behavior for ψ_p is not fulfilled. This situation can be rectified by factorizing out the problematic boundary points as

$$\prod_{k=1}^{N}\psi_{\frac{D-2}{2}}\left(-\frac{\beta\hbar^2}{M}\frac{\eta_k\eta_{k-1}}{\varepsilon}\tilde{\Delta}_k\right) \sim \frac{\sin^2(\pi D/2)}{\pi}\left(\frac{\beta\hbar^2}{2M\varepsilon}\right)^{D-1}[(\eta_-\eta_1)(\eta_+\eta_{N-1})]^{(D-1)/2}\prod_{k=2}^{N-1}\exp\left[-\frac{M}{\beta\hbar^2}\frac{\Delta_k}{8\eta_k\eta_{k-1}}\varepsilon + O(\varepsilon^2)\right]. \quad (48)$$

Here we have utilized the asymptotic form $\psi_p(-x) \sim -(x/2)^{p+1/2}\sin(\pi p)/[\sqrt{\pi}\,\Gamma(-p)] + O(x^{p+3/2})$ valid for $0 < x \ll 1$ and $p < 0$ ($p \notin \mathbb{Z}^-$). For the second PI in Eq. (42), which has negative lower Dirichlet boundary condition (namely, $\eta_- \mapsto -\eta_-$), we need to employ the asymptotic expansion $\psi_p(x) \sim (x/2)^{p+1/2}\,\Gamma(-p)/\sqrt{\pi} + O(x^{p+3/2})$ (again, $0 < x \ll 1$) instead. This implies that in the second PI, we get in contrast to (48) only $\sin(\pi D/2)$ rather than $\sin^2(\pi D/2)$.

The passage from the radial PI (41) to the ordinary (one-dimensional) PI brings about an important advantage, namely, one can perform the WKB approximation. In particular, one can use Laplace's formula of the asymptotic calculus [38,39],

$$\int_{-\infty}^{\infty} dt f(t,\beta)\exp[-\beta g(t)] = \sqrt{\frac{2\pi}{\beta g''(t_0)}} f(t_0,\beta)\exp[-\beta g(t_0)] + O\left\{\frac{\exp[-\beta g(t_0)]}{\beta^{3/2}}\right\}, \tag{49}$$

with t_0 being a solution of $g'(t) = 0$ [provided $g(t)$ has a smooth absolute minimum at the interior point $t = t_0 (\neq \pm\infty)$]. The function $f(t,\beta)$ is assumed to be bounded as $\beta \to \infty$. If needed, the full asymptotic expansion can be systematically generated via conventional Laplace's method; see, e.g., Ref. [38]. In Appendix B, we show that

$$\sqrt{\frac{2\pi}{\beta g''(t_0)}} \mapsto \left(\frac{\hbar^2}{2\pi M}\right)\left(\det{}'\left\{-\frac{d^2}{dx^2} + \frac{2M}{\hbar^2}[V(x) - E_0]\right\}\right)^{-1/2}. \tag{50}$$

Here, E_0 is the ground-state energy and the prime in $\det'\{\cdots\}$ indicates that the zero mode is factored out from the determinant. In fact, there is a quick way to compute $\det'\{\cdots\}$, by using either the Wronski construction [40,41] or the contour integration method [42]. In both of these approaches, one arrives at the result

$$\det{}'\left\{-\frac{d^2}{dx^2} + \frac{2M}{\hbar^2}[V(x) - E_0]\right\} = -\frac{1}{\dot\eta_0(X_+)\dot\eta_0(X_-)} = -\frac{\varepsilon^2}{[\eta_+ - (\eta_0)_{N-1}][(\eta_0)_1 - \eta_-]} \sim \frac{\varepsilon^2}{(\eta_0)_{N-1}(\eta_0)_1}. \tag{51}$$

Here, η_0 is the (normalized) ground-state wave function of the Hamiltonian $\hat{H}$ [or, equivalently, the zero-mode eigenvector of $(\hat{H} - E_0)$] with the Dirichlet conditions $\eta_0(X_-) = \eta_- \sim 0$ and $\eta_0(X_+) = \eta_+ \sim 0$. A similar result would hold also in the second PI in Eq. (42) where $\eta_- \mapsto -\eta_-$. The only difference in this case would be the presence of a minus sign in front of the last three expressions in Eq. (51).

To complete the WKB approximation, we substitute for $f(t_0,\beta)$ in Eq. (49) the functional expression

$$\beta \exp\left[-\langle J||\eta_0|\rangle - \int_{X_-}^{X_+} dx \frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta_0^2}\right], \tag{52}$$

in its explicit time-sliced form (48). Note that such $f(t_0,\beta)$ is bounded for $\beta \to \infty$ as required by Laplace's formula. In Eq. (52), we have denoted the WKB solution that minimizes the functional $\langle\eta|\hat{H}|\eta\rangle - \kappa(\langle\eta|\eta\rangle - 1)$ as η_0, i.e., with the same symbol as in Eq. (51). This is because according to the Rayleigh-Ritz variation principle (see, e.g., Refs. [33,43]), such a WKB function $\eta(x)$ is the ground-state wave function of the Hamiltonian $\hat{H}$, i.e., $\eta_0(x) = \psi_{gs}(x)$ with $\kappa_0 = E_0$. Notice also that the stationary point in κ is real, but the integration contour in κ is parallel to the imaginary axis. Both the reality and positivity of $\eta(x)$ pose no restriction in the Rayleigh-Ritz principle because the ground state can always be chosen real and positive [44]. By substituting for $g(t_0)$ the expression $\langle\eta_0|\hat{H}|\eta_0\rangle - \kappa_0(\langle\eta_0|\eta_0\rangle - 1) = E_0$ and using the Laplace asymptotic formula (49), it is easy to see that for the first PI in Eq. (42), we get (cf. also Appendix B)

$$\frac{\sin^2(\pi D/2)}{\pi^2}(\eta_-\eta_+)^{\frac{D-1}{2}} \exp\left(-\langle J|\eta_0\rangle - \beta E_0\right), \tag{53}$$

while for the second PI, we have

$$\frac{\sin(\pi D/2)}{\pi^2}(\eta_-\eta_+)^{\frac{D-1}{2}} \exp(-\langle J|\eta_0\rangle - \beta E_0). \tag{54}$$

By plugging this into (41) and performing the $\eta_\pm \to 0$ and $D \to 0$ limits, respectively, we get the leading large-β behavior of the Bloch density matrix in the form

$$\rho(x_a,x_b,\beta) = e^{-\beta E_0}\psi_{gs}(x_a)\psi_{gs}(x_b), \tag{55}$$

as expected from the spectral expansion; cf. also Eq. (39).

We conclude the discussion of the low-temperature expansion by noting that the Rayleigh-Ritz variation principle states that *all* eigenvalues and (normalized) eigenvectors of $\hat{H}$ come from stationary solutions of $\langle\eta|\hat{H}|\eta\rangle - \kappa(\langle\eta|\eta\rangle - 1)$, and conversely [33]. In the spirit of the WKB approximation, one should sum over all path integrals evaluated about *all* stationary solutions. It is, however, only the ground-state configuration $\{\psi_{gs}(x), E_0\}$ that acquires the global minimum and which gives the largest contribution to the WKB approximation. This fact was implicitly used in our preceding reasonings. Should we have also included other stationary solutions, we would recover higher-order terms in the spectral expansion of the Bloch density matrix (39).

So what we have just demonstrated is that the WKB expansion of the local-time PI (in contrast to the Feynman-Kac PI) representation picks up the correct asymptotic behavior known from spectral theory. As mentioned in Sec. III, this should be expected because the most important contribution to the low-temperature behavior of $\rho(x_a,x_a,\beta)$ stems from those paths that spend a sizable amount of time near the global minimum, and the WKB expansion of local-time PIs is organized precisely in terms of the local time of a stationary configuration and ensuing fluctuations. Of course, the usefulness of the local-time PI approach lies in the situations where neither energy spectrum nor associated eigenvalues are explicitly known and various direct PI techniques can be conveniently employed to probe the low-temperature regime.

VII. CONCLUSION AND OUTLOOK

In this paper, we have derived the local-time PI representation of the Bloch density matrix. We have shown that the result obtained, apart from being of interest in pure mathematics (stochastic theory, Sturm-Liouville theory, etc.), can serve as a useful alternative to the traditional

Feynman-Kac PI representation of Green functions of Fokker-Planck equations. Furthermore, by analytically continuing the result back to the real time via the inverse Wick rotation, $\beta \to it/\hbar$, one obtains the local-time PI representation of quantum-mechanical transition amplitudes, i.e., matrix elements of the evolution operator $e^{-it\hat{H}/\hbar}$. From a physics point of view, perhaps the most important application of local-time PIs lies in statistical physics, and, namely, in the low- and high-temperature treatments of the Bloch density matrix. This is because in conventional PIs only a very tiny subset of paths gives a relevant contribution in these asymptotic regimes. In particular, the high-temperature regime of the Boltzmann density function $\rho(x,x,\beta)$ is dominated by paths that spend a sizable amount of time in the vicinity of the point x. Similarly, the low-temperature regime is controlled by paths with a large local time near the global minimum of the potential. Here we have exemplified the conceptual convenience of the local-time formulation by providing a generic analysis of the low-temperature behavior of the Bloch density matrix. Our formulation proved to be particularly instrumental in obtaining the correct asymptotic behavior (known from the spectral theory), which is otherwise notoriously difficult to obtain within the Feynman-Kac PI framework [4,16]. As a byproduct, we have uncovered an interesting connection between a low-temperature PI expansion, the Laplace asymptotic formula, and the Rayleigh-Ritz variational principle.

In order to further reinforce our analysis, we formulated a local-time analog of the Feynman-Matthews-Salam formula which is [similarly to its quantum field theory counterpart] expedient in a number of statistical-physics contexts. The prescription obtained was substantiated by an explicit calculation of a one-point distribution function of the local time. In addition, the obtained relationship between the local-time representation of PI and the radial PI provides a practical illustration of the Ray-Knight theorem of the stochastic calculus.

It appears worthwhile to stress that our local-time representation (with its built-in replica field trick) is in its present form applicable only to one-dimensional quantum-mechanical systems. With hindsight, we reflected this fact already in our choice of the incipient PI (3) where we assumed $\tau \in \mathbb{R}$ and $x \in \mathbb{R}$. Though one may easily proceed up to Eq. (15) without any restriction on the value of d [in fact, Eq. (15) is valid for any $x \in \mathbb{R}^d$ with $d \geqslant 1$], further progress in this direction is hindered by the fact that the replica fields depend on a d-dimensional argument x, and thus the PI in Eq. (15) can no longer be regarded as a quantum-mechanical PI (i.e., PI over fluctuating paths). In effect, we cannot use existing mathematical techniques of the PI calculus (e.g., transformation of PIs to polar coordinates) that we have employed to get the radial PI (16). The issue of the extension of our local-time PI representation to higher-dimensional configuration space is currently under active investigation.

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APPENDIX A: OFF-DIAGONAL MATRIX ELEMENTS

In this Appendix, we show that the representation (25) reduces to the well-established result (24) of the Sturm-Liouville theory. Since $\widetilde{\rho}(x_a,x_b,E)$ is symmetric in x_a and x_b, we will assume, without loss of generality, that $x_a < x_b$.

The path integral in Eq. (25) can be expressed via Eq. (19) as

$$\int_0^\infty d\eta_a d\eta_b (\eta_+ X_+|\eta_b x_b)_D \eta_b (\eta_b x_b|\eta_a x_a)_2 \eta_a (\eta_a x_a|\eta_- X_-)_D. \tag{A1}$$

The limits in Eq. (25) can be carried with the help of the asymptotic formulas $I_{D/2-1}(z) \approx (z/2)^{D/2-1}/\,\Gamma(D/2)$, and $\Gamma(z) \approx 1/z$, valid for $z \to 0$. We obtain

$$\frac{1}{D}\eta_-^{\frac{1-D}{2}}(\eta_a x_a|\eta_- X_-)_D \xrightarrow{\eta_-\to 0} \left[\frac{\hbar^2\eta_a}{2MF_1(x_a)}\right]^{D/2}\frac{\exp\left[-\frac{\hbar^2}{2M}\frac{F_1'(x_a)}{F_1(x_a)}\eta_a^2\right]}{\frac{D}{2}\Gamma\left(\frac{D}{2}\right)\sqrt{\eta_a}} \xrightarrow{D\to 0} \frac{1}{\sqrt{\eta_a}}\exp\left[-\frac{\hbar^2}{2M}\frac{F_1'(x_a)}{F_1(x_a)}\eta_a^2\right], \tag{A2}$$

where $F_1(x)$ satisfies Eq. (20) with initial conditions $F_1(X_-) = 0$ and $F_1'(X_-) = 1$, and similarly, we find

$$\frac{1}{D}\eta_+^{\frac{1-D}{2}}(\eta_+ X_+|\eta_b x_b)_D \xrightarrow{\eta_+\to 0, D\to 0} \frac{1}{\sqrt{\eta_b}}\exp\left[\frac{\hbar^2}{2M}\frac{G_3'(x_b)}{G_3(x_b)}\eta_b^2\right], \tag{A3}$$

where $G_3(x)$ satisfies Eq. (20) with initial conditions $G_3(X_+) = 0$ and $G_3'(X_+) = -1$. Formula (25) then reduces to

$$\widetilde{\rho}(x_a,x_b,E) = \frac{2\hbar^2}{M}\int_0^\infty d\eta_a d\eta_b \frac{\eta_a\eta_b}{G_2(x_a)} I_0\left[\frac{\hbar^2}{M}\frac{\eta_a\eta_b}{G_2(x_a)}\right]\exp\left\{-\frac{\hbar^2}{2M}\left[\frac{W(G_2,F_1)\eta_a^2}{F_1(x_a)G_2(x_a)} + \frac{W(G_3,F_2)\eta_b^2}{F_2(x_b)G_3(x_b)}\right]\right\}, \tag{A4}$$

where $F_2(x)$ and $G_2(x)$ satisfy Eq. (20) with initial conditions $F_2(x_a) = 0$ and $F_2'(x_a) = 1$, and $G_2(x_b) = 0$ and $G_2'(x_b) = -1$, respectively. The Wronskian $W(F,G) \equiv F(x)G'(x) - F'(x)G(x)$ is independent of x, as discussed in Sec. IV C, and antisymmetric, i.e., $W(F,G) = -W(G,F)$.

Wronskians $W(G_2,F_1)$ and $W(G_3,F_2)$ assume a particularly simple form when evaluated at points x_b and x_a, respectively, due to the initial conditions satisfied by G_2 and F_2. We find $W(G_2,F_1) = F_1(x_b)$ and $W(G_3,F_2) = G_3(x_a)$. Rescaling

$\eta_a \to \sqrt{G_2(x_a)M/\hbar^2}\eta_a$, $\eta_b \to \sqrt{F_2(x_b)M/\hbar^2}\eta_b$, and using the relation $F_2(x_b) = G_2(x_a)$, we obtain

$$\widetilde{\rho}(x_a,x_b,E) = \frac{2M}{\hbar^2}G_2(x_a)\int_0^\infty d\eta_a d\eta_b \eta_a\eta_b I_0(\eta_a\eta_b)\exp\left[-\frac{F_1(x_b)}{2F_1(x_a)}\eta_a^2 - \frac{G_3(x_a)}{2G_3(x_b)}\eta_b^2\right]. \tag{A5}$$

The integrations are readily performed using the formula [29]

$$\int_0^\infty dz I_0(bz)\exp\left(-\frac{a}{2}z^2\right) = \frac{1}{a}\exp\left(\frac{b^2}{2a}\right), \tag{A6}$$

yielding

$$\widetilde{\rho}(x_a,x_b,E) = \frac{2M}{\hbar^2}\frac{F_1(x_a)G_3(x_b)G_2(x_a)}{F_1(x_b)G_3(x_a) - F_1(x_a)G_3(x_b)}. \tag{A7}$$

To prove equality with (24), we only have to show that

$$F_1(x_a)G_3(x_b) - F_1(x_b)G_3(x_a) = G_2(x_a)W(F_1,G_3). \tag{A8}$$

This is done by realizing that $G_2(x)$, being a solution of the second-order linear differential equation (20), can be uniquely composed as a linear combination of two other solutions $F_1(x)$ and $G_3(x)$,

$$G_2(x) = \frac{F_1(x)G_3(x_b) - F_1(x_b)G_3(x)}{W(F_1,G_3)}. \tag{A9}$$

Indeed, thus defined, G_2 satisfies the initial conditions $G_2(x_b) = 0$ and $G_2'(x_b) = -1$.

We conclude that

$$\widetilde{\rho}(x_a,x_b,E) = -\frac{2M}{\hbar^2}\frac{F_1(x_a)G_3(x_b)}{W(F_1,G_3)}, \tag{A10}$$

which coincides with the Sturm-Liouville result (24).

APPENDIX B: PROOF OF IDENTITY (50)

In this Appendix, we derive the identity (B6). According to Laplace's formula (49), we may assume that the dominant contribution to the PI (42) comes from the extremization of

$$g(t) \mapsto \langle\eta|\hat{H}|\eta\rangle - \kappa(\langle\eta|\eta\rangle - 1) \equiv s[\eta,\kappa], \tag{B1}$$

while the role of $f(t,\beta)$ is played by the functional expression

$$\exp\left[-\langle J||\eta|\rangle - \int_{X_-}^{X_+} dx\, \frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2}\right]. \tag{B2}$$

Let η_0 and k_0 be corresponding extremizers of $s[\eta,\kappa]$ and let $\delta\eta$ and $\delta\kappa$ describe fluctuations around η_0 and k_0. Then the expansion of $s[\eta,\kappa]$ reads

$$s[\eta,\kappa] = s[\eta_0,\kappa_0] + \langle\delta\eta|\hat{H} - \kappa_0|\delta\eta\rangle - \delta\kappa(\langle\delta\eta|\eta_0\rangle + \langle\eta_0|\delta\eta\rangle) + \cdots, \tag{B3}$$

(recall that the linear terms are absent due to the extremalization condition $\delta s = 0$). Notice that according to the Rayleigh-Ritz variation principle (see, e.g., Refs. [33,43]), η_0 and κ_0 must correspond to the (normalized) ground-state wave function of the Hamiltonian $\hat{H}$ and to the ground-state energy E_0, respectively. For the case at hand, the leading WKB approximation [i.e., if we include only quadratic terms in the expansion (B3)] becomes [cf. Laplace's formula (49)]

$$\begin{aligned}
&\int_{\eta(X_-)}^{\eta(X_+)} \mathcal{D}\eta\, \exp\{-\beta s[\eta,\kappa]\}\exp\left[-\langle J||\eta|\rangle - \int_{X_-}^{X_+} dx\,\frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta^2}\right]\\
&= \exp[-\beta E_0]\exp\left[-\langle J||\eta_0|\rangle - \int_{X_-}^{X_+} dx\,\frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta_0^2}\right]\\
&\quad\times \frac{1}{2\pi}\int_{-\infty}^{\infty} d\delta\kappa \int_{\delta\eta(X_-)=0}^{\delta\eta(X_+)=0}\mathcal{D}\delta\eta\,\exp\left[-\beta(\langle\delta\eta|,\delta\kappa)\begin{Vmatrix}\hat{H}-E_0, & i|\eta_0\rangle\\ i\langle\eta_0|, & 0\end{Vmatrix}\begin{pmatrix}|\delta\eta\rangle\\ \delta\kappa\end{pmatrix}\right]\\
&= \exp[-\beta E_0]\exp\left[-\langle J||\eta_0|\rangle - \int_{X_-}^{X_+} dx\,\frac{M}{\beta\hbar^2}\frac{\Delta(x)}{8\eta_0^2}\right]\left\{\det\hat{H}_0^{-1}\det\begin{Vmatrix}\hat{H}-E_0, & i\eta_0\\ i\eta_0^{\intercal}, & 0\end{Vmatrix}\right\}^{-1/2}.
\end{aligned} \tag{B4}$$

Here we have used that $s[\eta_0,\kappa_0] = E_0$. We have also set $c = E_0$ in the κ integration and subsequently rotated the integration contour so that the $\delta\kappa$ integration would run along the real axis. The (formal) expression inside of curly parentheses is a customary shorthand notation for the correct time-sliced form

$$\{\cdots\} = (2\pi)^2 \det\left\| \begin{matrix} -\varepsilon^2\nabla\overline{\nabla} + \varepsilon^2 2M[V(x)-E_0]/\hbar^2, & i2M\eta_0\varepsilon^2/\hbar^2 \\ i2M\eta_0^{\intercal}\varepsilon^2/\hbar^2, & 0 \end{matrix}\right\|, \tag{B5}$$

where the difference operators (lattice derivatives) ∇ and $\overline{\nabla}$ are defined as [7]

$$\nabla\eta(x) = \frac{1}{\varepsilon}[\eta(x+\varepsilon)-\eta(x)], \quad \overline{\nabla}\eta(x) = \frac{1}{\varepsilon}[\eta(x)-\eta(x-\varepsilon)], \tag{B6}$$

with $\nabla\overline{\nabla} = \overline{\nabla}\nabla$ being the Hermitian operator on the space of "time-sliced" functions with vanishing end points, i.e., $\eta(x_N) = \eta(x_0) = 0$. Equation (B5) can be further simplified by using the *Schur complement* technique for the calculation of determinants of partitioned matrices [45]. In particular, we have

$$\{\cdots\} = \left(\frac{2\pi M\varepsilon^2}{\hbar^2}\right)^2 \det\left\{-\varepsilon^2\nabla\overline{\nabla} + \varepsilon^2\frac{2M}{\hbar^2}[V(x)-E_0]\right\}\eta_0^{\intercal}\left\|-\varepsilon^2\nabla\overline{\nabla} + \varepsilon^2\frac{2M}{\hbar^2}[V(x)-E_0]\right\|^{-1}\eta_0. \tag{B7}$$

In order to find a finite expression for the indeterminate form 0/0 (caused by the presence of the *zero mode*), we must properly regularize the numerator and denominator in Eq. (B7). This can be done by introducing a small parameter k^2 which moves the zero mode away from zero. In this way, we can write

$$\begin{aligned}\{\cdots\} &= \lim_{k\to 0}\varepsilon\left(\frac{2\pi M}{\hbar^2}\right)^2 \frac{\det\left\{-\varepsilon^2\nabla\overline{\nabla} + \varepsilon^2\frac{2M}{\hbar^2}[V(x)-E_0] + \varepsilon^2k^2\right\}}{k^2} \\ &= \lim_{k\to 0}\varepsilon\left(\frac{2\pi M}{\hbar^2}\right)^2 \frac{\det\left\{-\varepsilon^2\nabla\overline{\nabla} + \varepsilon^2\frac{2M}{\hbar^2}[V(x)-E_0] + \varepsilon^2k^2\right\}}{k^2\det\left[-\varepsilon^2\nabla\overline{\nabla}\right]}\det[-\varepsilon^2\nabla\overline{\nabla}] \\ &= \varepsilon\left(\frac{2\pi M}{\hbar^2}\right)^2 \frac{\det'\left\{-d^2/dx^2 + \frac{2M}{\hbar^2}[V(x)-E_0]\right\}}{\det\left[-d^2/dx^2\right]}\det[-\varepsilon^2\nabla\overline{\nabla}] \\ &= \left(\frac{2\pi M}{\hbar^2}\right)^2 \det'\left\{-\frac{d^2}{dx^2} + \frac{2M}{\hbar^2}[V(x)-E_0]\right\}. \end{aligned} \tag{B8}$$

The prime in $\det'\{\cdots\}$ indicates that the zero mode is divided out from the determinant. On the first line of (B8), we have used the fact that in the continuum limit,

$$\eta_0^{\intercal}\left\|-\varepsilon^2\nabla\overline{\nabla} + \varepsilon^2\frac{2M}{\hbar^2}[V(x)-E_0] + \varepsilon^2k^2\right\|^{-1}\eta_0 \sim \frac{1}{\varepsilon^3}\int_{\mathbb{R}^2} dxdy\,\eta_0(x)G^k(x,y)\eta_0(y) = \frac{1}{\varepsilon^3k^2}, \tag{B9}$$

where $G^k(x,y)$ is the Green's function satisfying the equation

$$\left\{-\frac{d^2}{dx^2} + \frac{2M}{\hbar^2}[V(x)-E_0] + k^2\right\}G^k(x,y) = \delta(x-y). \tag{B10}$$

On the last line of (B8), we have employed the well-known formulas [7,41]

$$\det[-\varepsilon^2\nabla\overline{\nabla}] = N = (X_+ - X_-)/\varepsilon, \quad \det[-d^2/dx^2] = X_+ - X_-. \tag{B11}$$

Comparison of (B4) and (B8) with (49) yields the result (50).

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