Anomalous scaling in magnetohydrodynamic turbulence: Effects of anisotropy and compressibility in the kinematic approximation

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The field-theoretic renormalization group and the operator product expansion are applied to the model of passive vector (magnetic) field advected by a random turbulent velocity field. The latter is governed by the Navier-Stokes equation for compressible fluid, subject to external random force with the covariance $\propto \delta(t-t')k^{4-d-y}$, where d is the dimension of space and y is an arbitrary exponent. From physics viewpoints, the model describes magnetohydrodynamic turbulence in the so-called kinematic approximation, where the effects of the magnetic field on the dynamics of the fluid are neglected. The original stochastic problem is reformulated as a multiplicatively renormalizable field-theoretic model; the corresponding renormalization group equations possess an infrared attractive fixed point. It is shown that various correlation functions of the magnetic field and its powers demonstrate anomalous scaling behavior in the inertial-convective range already for small values of y. The corresponding anomalous exponents, identified with scaling (critical) dimensions of certain composite fields ("operators" in the quantum-field terminology), can be systematically calculated as series in y. The practical calculation is performed in the leading one-loop approximation, including exponents in anisotropic contributions. It should be emphasized that, in contrast to Gaussian ensembles with finite correlation time, the model and the perturbation theory presented here are manifestly Galilean covariant.

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I. INTRODUCTION

Much attention has been attracted to the problem of intermittency and anomalous scaling in developed magnetohydrodynamic (MHD) turbulence; see, e.g., Refs. [1–29] and references therein. It has long been known that in the so-called Alfvénic regime, the MHD turbulence demonstrates a behavior similar to that of the usual fully developed fluid turbulence: a cascade of energy from the infrared (IR) range towards smaller scales, where the dissipation effects dominate, and a self-similar (scaling) behavior of the energy spectra in the intermediate (inertial-convective) range. Moreover, a strongly non-Gaussian (intermittent) character of the fluctuations in the MHD turbulence is much more strongly pronounced than in ordinary turbulent fluids or in the passive scalar problem.

The solar wind, a conducting fluid expanding into the interplanetary space, covers a wide range of spatial and temporal scales and thus provides a kind of "laboratory" in which various models of the MHD turbulence can be tested [4–16]. In solar flares, highly energetic and anisotropic large-scale events (with the magnetic fields as intense as 500 Gauss) coexist with small-scale stochastic fluctuations and coherent structures, finally responsible for the dissipation. Thus modeling the way the energy is distributed and conveyed along the scales and eventually dissipated is a difficult task.

The intermittency strongly modifies the IR behavior of the higher-order correlation functions, leading to anomalous scaling with infinite sets of independent anomalous exponents [7].

A simplified description of the situation was proposed in Ref. [6]: the large-scale field $B^0_i = n_i B^0$ dominates the dynamics in the distinguished direction specified by a unit constant vector $\mathbf{n} = \{n_i\}$, while the fluctuations in the perpendicular plane are described as nearly two dimensional. This picture allows for precise numerical simulations, which show that turbulent fluctuations organize in rare coherent structures separated by narrow current sheets. On the other hand, the satellite observations [4] and numerical simulations [5,7] suggest that the scaling behavior in the solar wind is closer to the anomalous scaling in the three-dimensional fully developed hydrodynamic turbulence than to simple Iroshnikov-Kraichnan scaling [2,3] suggested by the two-dimensional picture with the inverse energy cascade.

Thus, further analysis of more realistic three-dimensional models is welcome.

In a number of papers, the problem was studied within the framework of the kinematic approximation, in which the magnetic field is passive in the sense that it does not affect the dynamics of the velocity field [17–29]. This approximation seems reasonable if the gradients of the magnetic fields are not too large. What is more, the renormalization group analysis of Ref. [30] suggests that such a "kinematic regime" can indeed describe the possible IR behavior of the full-scale problem. It is then possible to model the velocity field "by hands," that is, by simple statistical ensembles with prescribed properties. Most popular is the Kazantsev-Kraichnan ensemble [31,32]: The random velocity field is Gaussian and white in time and has a power-law spectrum.

Numerous analytical and numerical results were derived for the scalar and vector fields, advected by the Kazantsev-Kraichnan "flow," see Ref. [33] for the review and references. The main results concerning anomalous scaling for the magnetic case can be summarized as follows [17–20]:

(i) Anomalous scaling is present and appears already for the pair correlation function.

(ii) In the presence of large-scale anisotropy (brought about, e.g., by the constant background field B^0), the anomalous

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exponents for a given correlation function demonstrate a kind of hierarchy: in the expansion of correlation functions in the spherical harmonics Y_{lm}, the corresponding exponents increase with l, the degree of anisotropy. Thus, for the even-order functions, the leading terms of the inertial-range behavior are given by the isotropic contribution ($l = 0$). This gives quantitative support for Kolmogorov's hypothesis of the local isotropy restoration.

(iii) Nevertheless, the anisotropy survives at small scales and manifests itself in odd-order correlation functions or in dimensionless ratios involving such functions (like the skewness factor).

An important advantage of the Kazantsev-Kraichnan ensemble is the possibility to easily model anisotropy and compressibility. Importance of compressibility for the MHD turbulence was realized already in the classical study of Ref. [2]. Within the framework of the Kazantsev-Kraichnan ensemble, effects of compressibility were studied in Refs. [18,22,27]. It was shown that:

(iv) The anomalous exponents depend on the degree of compressibility. When it grows, the hierarchy of anisotropic contributions becomes less pronounced and the persistence of anisotropy in the depth of the inertial interval becomes more remarkable.

Of course, generalization of this analysis to more realistic velocity dynamics is necessary: Some of the aforementioned results can be artifacts of the oversimplified statistics.

It is possible to directly generalize the Kazantsev-Kraichnan ensemble to the case of finite correlation time; see, e.g., Refs. [34–36] for the passive scalar and [24] for the passive vector fields. However, such "synthetic" models with nonvanishing correlation time suffer from the lack of Galilean symmetry, which may lead to "interesting pathologies," quoting the authors of Ref. [37]. One such pathology manifests itself as ultraviolet (UV) divergence in the vertex [35], which in more realistic models is forbidden by Galilean invariance.

Thus, it is desirable to describe the advecting velocity field by the Navier-Stokes equations with a random stirring force and to work within Galilean covariant formalism. For the incompressible case, the analysis of the passive vector field was accomplished in Ref. [29].

In this paper, we study the anomalous scaling in the kinematic MHD problem and model the velocity dynamics by the non-Gaussian velocity field with finite correlation time, governed by the stochastic Navier-Stokes (NS) equation. We apply to the problem the approach based on the field-theoretic renormalization group (RG) and the operator product expansion (OPE), earlier applied to the passive scalar problem [38–40]. In that approach, the anomalous exponents are identified with the critical dimensions of certain Galilean-invariant composite fields ("operators"). It can be directly generalized to the cases of finite correlation time, the presence of anisotropy, non-Gaussianity, and so on. Passive advection of vector fields (and hence kinematic MHD problems) with various velocity ensembles were studied earlier within the RG+OPE approach in Refs. [21–29].

A general overview of the RG+OPE approach to the problem of anomalous scaling and more references can be found in Ref. [41]. Detailed exposition of earlier work on the field-theoretic RG approach to stochastic models of turbulence on the whole is presented in Ref. [42].

However, analysis of the compressible fluid on the base of the stochastic NS equation appears to be a difficult task in itself; see, e.g., Refs. [43–48]. In spite of some discrepancies, all of those studies support the existence of a "strongly compressible" scaling regime, different from the usual incompressible one.

In the present paper, we adopt the approach of Ref. [48], where, with the price of some natural approximations, the stochastic NS equation for a compressible fluid was reformulated as a multiplicatively renormalizable field-theoretic model. Then the standard field-theoretic RG was applied to the problem, and the resulting stationary scaling regime was associated with the IR attractive fixed point of the corresponding RG equations.

Recently, that ensemble was employed to study, within the RG+OPE framework, the problem of passive scalar advection in a turbulent compressible fluid [49]. In spite of close resemblance with the case of Kraichnan's model, some of the results appeared somewhat different. The present paper continues the study of Ref. [49] in connection with the MHD turbulence. For this reason, we will only briefly discuss the points, common to the scalar and vector problems, refer the reader to the papers [48,49] whenever possible, and focus on the points specific of the vector case.

The plan of the paper is the following. In Sec. II we give the detailed description of the model: the velocity ensemble, the stochastic MHD equation, and the field-theoretic formulation. In Sec. III we discuss canonical dimensions and renormalization of the field-theoretic model, demonstrate its multiplicative renormalizability, and calculate (in the leading one-loop approximation) the corresponding renormalization constant. In Sec. IV we derive the corresponding RG equations and show that they possess the only IR attractive fixed point in the physical region of parameters. This fact implies the scaling behavior in the IR range (long times, large distances); the corresponding critical dimensions of the basic fields and parameters are presented. In Sec. V we calculate the critical dimensions of the tensor composite fields (operators), constructed solely of the basic scalar fields; these will play the crucial role in the following. In Sec. VI we employ the OPE to derive the inertial-range asymptotic behavior of various correlation functions. Section VII is reserved for discussion and the conclusion.

II. THE MODEL

A. The velocity ensemble

Following Refs. [48,49], we describe the stochastic dynamics of a compressible fluid by a set of two equations:

$$\nabla_t v_i = \nu_0[\delta_{ik}\partial^2 - \partial_i\partial_k]v_k + \mu_0\partial_i\partial_k v_k - \partial_i\phi + f_i, \tag{2.1}$$

$$\nabla_t\phi = -c_0^2\partial_i v_i, \tag{2.2}$$

which are derived from the momentum balance equation and the continuity equation [50] with two assumptions: The kinematic viscosity coefficients ν_0 and μ_0 are assumed to be constant, that is, independent of $x = \{t,\mathbf{x}\}$, and the equation of state is taken in the simplest form of the linear relation $(p-\bar{p})=c_0^2(\rho-\bar{\rho})$ between the deviations of the pressure $p(x)$ and the density $\rho(x)$ from their mean values; then the constant c_0 has the meaning of the (adiabatic) speed of sound.

In Eqs. (2.1) and (2.2), $\boldsymbol{v}=\{v_i(x)\}$ is the velocity field and, instead of the density, we use the scalar field defined as $\phi(x)=c_0^2\ln[\rho(x)/\bar{\rho}]$. Furthermore,

$$\nabla_t=\partial_t+v_k\partial_k \tag{2.3}$$

is the Lagrangean (Galilean covariant) derivative, $\partial_t=\partial/\partial t$, $\partial_i=\partial/\partial x_i$, and $\partial^2=\partial_i\partial_i$ is the Laplace operator. The problem is studied in the d-dimensional (for generality) space $\mathbf{x}=\{x_i\}$, $i=1\ldots d$, and the summations over the repeated Latin indices are always implied.

In the Navier-Stokes equation (2.1), f_i is the density of the external force (per unit mass), which mimics the energy input into the system from the large-scale stirring. In order to apply the standard perturbative RG to the problem, it is taken to be Gaussian with zero mean, not correlated in time (this is dictated by the Galilean symmetry), with the given covariance

$$\langle f_i(x)f_j(x')\rangle=\delta(t-t')\int_{k>m}\frac{d\mathbf{k}}{(2\pi)^d}\,D_{ij}^f(\mathbf{k})\exp\{i\mathbf{k}\cdot\mathbf{x}\}, \tag{2.4}$$

with

$$D_{ij}^f(\mathbf{k})=D_0\,k^{4-d-y}\,\{P_{ij}^{\perp}(\mathbf{k})+\alpha P_{ij}^{\parallel}(\mathbf{k})\}. \tag{2.5}$$

Here $P_{ij}^{\perp}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2$ and $P_{ij}^{\parallel}(\mathbf{k})=k_ik_j/k^2$ are the transverse and the longitudinal projectors, respectively, $k=|\mathbf{k}|$ is the wave number (momentum), and D_0 and α are positive amplitudes. The parameter $g_0=D_0/\nu_0^3$ plays the part of the coupling constant (expansion parameter in the perturbation theory); the relation $g_0\sim\Lambda^y$ defines the typical UV momentum scale. The parameter $m\sim L^{-1}$, reciprocal of the integral turbulence scale, provides IR regularization; its precise form is unessential and the sharp cutoff is merely the simplest choice for calculational reasons. The exponent $0<y\leqslant 4$ plays is analogous to $\varepsilon=4-d$ in the RG theory of critical state [51,52]: It provides UV regularization (so the UV divergences have the form of the poles in y) and various scaling dimensions are calculated as series in y. The most realistic (physical) value is given by the limit $y\to 4$: Then the function (2.5) can be viewed as a powerlike representation of the function $\delta(\mathbf{k})$ and it corresponds to the idealized picture of the energy input from infinitely large scales.

As already mentioned, more detailed justification and discussion of the compressible model (2.1)–(2.5) is given in Refs. [48,49].

B. The MHD equation

In the presence of a constant background field $B_i^0=B^0n_i$ with a certain constant unit vector $\mathbf{n}=\{n_i\}$, the dynamic equation for the fluctuating part $\theta_i=\theta_i(t,\mathbf{x})$ of the full magnetic field $B_i=B^0(n_i+\theta_i)$ has the form

$$\partial_t\theta_i+\partial_k(v_k\theta_i-\theta_kv_i)=\kappa_0\partial^2\theta_i+n_k\partial_kv_i, \tag{2.6}$$

where $\kappa_0=c_l^2/4\pi\sigma$ is the magnetic diffusion coefficient. Equation (2.6) follows from the Maxwell equations neglecting the displacement current and the simplest form of Ohm's law for a moving medium $\mathbf{j}=\sigma(\mathbf{E}+c_l^{-1}[\boldsymbol{v},\mathbf{B}])$, where σ is the conductivity and c_l is the speed of light; see, e.g., Ref. [53].

The last term on the right-hand side of (2.6) maintains the steady state of the system and acts as the source of the anisotropy; in principle, it can be replaced with an artificial Gaussian noise with appropriate statistics. In the real problem, the field $\boldsymbol{v}$ satisfies the Navier-Stokes equation with the additional Lorentz force term $\sim(\mathbf{B}\times\operatorname{curl}\mathbf{B})$. In our kinematic approximation the back reaction of the magnetic field on the velocity dynamics is neglected and the latter is described by the stochastic problem (2.1)–(2.5) without the Lorentz term.

C. The field-theoretic formulation

It is well known that any stochastic problem of the type (2.1)–(2.5) can be reformulated, in a standard fashion, as a certain field-theoretic model; see, e.g., Refs. [51,52]. This means that various correlation and response functions of the original stochastic problem can be represented as functional integrals over the doubled set of fields Φ with the weight $\exp\mathcal{S}(\Phi)$, where $\mathcal{S}(\Phi)$ is the so-called De Dominicis-Janssen action functional. The action functional $\mathcal{S}_v(\Phi)$ for the problem (2.1)–(2.5) with $\Phi=\{\boldsymbol{v}',\phi',\boldsymbol{v},\phi\}$ looks too cumbersome, and we do not reproduce it here, as well as the elements of the corresponding Feynman diagrammatic techniques (bare propagators and vertices); they can be found in Refs. [48,49]. Below we only will need the velocity-velocity propagator at $c_0=0$; in the frequency-momentum (ω–$\mathbf{k}$) representation it has the form:

$$\langle v_iv_j\rangle_0=D_0\left\{\frac{P_{ij}^{\perp}(\mathbf{k})}{\omega^2+\nu_0^2k^4}+\frac{\alpha P_{ij}^{\parallel}(\mathbf{k})}{\omega^2+u_0^2\nu_0^2k^4}\right\}. \tag{2.7}$$

The full-scale stochastic problem (2.1)–(2.6) corresponds to the action functional

$$\mathcal{S}(\Phi)=\mathcal{S}_v(\boldsymbol{v}',\phi',\boldsymbol{v},\phi)+\mathcal{S}_\theta(\boldsymbol{\theta}',\boldsymbol{\theta},\boldsymbol{v}), \tag{2.8}$$

where

$$\mathcal{S}_\theta=\theta_i'\{-\partial_t\theta_i-\partial_k(v_k\theta_i-\theta_kv_i)+\kappa_0\partial^2\theta_i+n_k\partial_kv_i\} \tag{2.9}$$

is the De Dominicis-Janssen action for the problem (2.6) at fixed $\boldsymbol{v}$. It brings about the new propagator

$$\langle\theta_i'\theta_j\rangle_0=\frac{P_{ij}^{\perp}(\mathbf{k})}{-i\omega+w_0\nu_0k^2} \tag{2.10}$$

and the new vertex $V_{ijl}\theta_i'\theta_jv_l$ with the vertex factor

$$V_{ijl}(\mathbf{k})=i(\delta_{ij}k_l-\delta_{il}k_j). \tag{2.11}$$

A few remarks are in order here. First, the derivative at the vertex in (2.9) can be moved onto the auxiliary field $\boldsymbol{\theta}'$ using the integration by parts; thus $\mathbf{k}$ in (2.11) is the momentum argument of $\boldsymbol{\theta}'$. Second, the vertex factor satisfies the transversality condition

$$k_iV_{ijl}(\mathbf{k})=0 \tag{2.12}$$

that follows from its explicit form (2.11). We also note that another new mixed propagator $\langle\theta\boldsymbol{v}\rangle_0$ will not appear in relevant diagrams.

III. UV DIVERGENCES AND THE RENORMALIZATION

A. Canonical dimensions, counterterms, and renormalizability

The analysis of UV divergences is based on the analysis of canonical dimensions; see, e.g., Refs. [51,52]. Dynamical models of the type (2.1)–(2.6) have two scales: the time scale T and the length scale L. The canonical dimension of any quantity F (a field or a parameter) is described by two numbers, the momentum dimension d_F^k and the frequency dimension d_F^ω, defined such that $[F] \sim [T]^{-d_F^\omega}[L]^{-d_F^k}$. In the renormalization theory, the central role is played by the total canonical dimension $d_F = d_F^k + 2d_F^\omega$, defined such that all the viscosity or diffusivity coefficients are dimensionless; see Refs. [42,52]. All the canonical dimensions in our model (2.1)–(2.6) are identical to their counterparts in the scalar case, and we refer the reader to Table 1 in Ref. [49].

The formal index of UV divergence of a certain 1-irreducible Green function Γ is given by its total canonical dimension:

$$\delta_\Gamma = d + 2 - \sum_\Phi N_\Phi d_\Phi, \tag{3.1}$$

where N_Φ are the numbers of the fields entering into the function Γ and d_Φ are their total canonical dimensions. Superficial UV divergences, whose removal requires counterterms, can be present only in the functions Γ with a non-negative integer δ_Γ. The counterterm is a polynomial in frequencies and momenta of degree δ_Γ, with the convention that $\omega \sim k^2$.

The dimensional analysis ("power counting") should be augmented by certain additional considerations:

(i) All the 1-irreducible Green functions without the auxiliary ("primed") fields vanish identically and thus require no counterterms.

(ii) If a number of external momenta occurs as an overall factor in all the diagrams of a certain Green function, then the real index of divergence δ'_Γ is smaller than δ_Γ by the corresponding number of unities. In the model $\mathcal{S}_v$ the field ϕ enters the corresponding vertex only in the form of a spatial derivative, which reduces the real index of divergence:

$$\delta'_\Gamma = \delta_\Gamma - N_\phi. \tag{3.2}$$

(iii) The Galilean invariance of the model requires that the counterterms be also invariant. In particular, this means that the covariant derivative (2.3) appears in the counterterms as a whole.

These considerations forbid superficial UV divergences in certain Green functions, allowed by dimensional analysis, and hence reduce the number of the counterterms needed for the renormalization of the model.

The analysis of the field-theoretic model with the action $\mathcal{S}_v$ in (2.8), performed in Ref. [48] (see also Ref. [49]), has shown that it is multiplicatively renormalizable (after a simple natural extension). This means that all the UV divergences can be removed from the Green functions by the renormalization of the fields $\phi \to Z_\phi \phi$, $\phi' \to Z_{\phi'}\phi'$ and of the parameters:

$$g_0 = g\mu^y Z_g, \quad \nu_0 Z_\nu, \quad c_0 = cZ_c, \tag{3.3}$$

and so on. Here the renormalization constants Z_i absorb all the UV divergences, so the Green functions are UV finite (that is, finite at $y = 0$) when expressed in terms of the renormalized parameters g, u, and so on; the reference scale (or the "renormalization mass") μ is an additional free parameter of the renormalized theory. No renormalization of the fields $\boldsymbol{v}', \boldsymbol{v}$ and of the parameters m, α is needed.

The inclusion of the new contribution $\mathcal{S}_\theta$ in the full model brings about the only new UV divergence in the 1-irreducible function $\langle\theta'\theta\rangle_{1-ir}$ with the counterterm $\theta'\partial^2\theta$. Two points are important here:

(iv) From the linerarity of the original stochatic model in the field θ it follows that $N_{\theta'} - N_\theta$ is a non-negative integer for any nontrivial 1-irreducible Green function: No other Feynman diagram can be drawn. This fact forbids the superficial divergences in all the 1-irreducible functions $\langle\theta'\theta\dots\theta\rangle_{1-ir}$, except for the first one and thus prevents our model from being nonrenormalizable, despite the fact that the magnetic field has a negative canonical dimension.

(v) For the full model (2.8), the items (ii) and (iii) require some additional discussion. The derivative at the vertex in $\mathcal{S}_\theta$ can be moved, using the integration by parts, onto the field θ'. Thus, the real index of divergence is reduced according to the item (ii) above, and θ' enters the countertems only as a spatial derivative. The expression (3.2) has to be replaced with

$$\delta'_\Gamma = \delta_\Gamma - N_\phi - N_{\theta'}. \tag{3.4}$$

Thus, the counterterm $\theta'\partial_t\theta$ is forbidden, and so is $\theta'(v_i\partial_i)\theta$ due to the Galilean symmetry (iii).

The only remaining counterterm $\theta'\partial^2\theta$ is naturally reproduced by the multiplicative renormalization of the magnetic diffusion coefficient: $\kappa_0 = \kappa Z_\kappa$. No renormalization of the fields θ', θ is needed.

The renormalized analog of the action functional (2.8) has the form

$$\mathcal{S}^R(\Phi) = \mathcal{S}_v^R(\Phi) + \mathcal{S}_\theta^R(\Phi), \tag{3.5}$$

where $\mathcal{S}^R(\Phi)$ is the renormalized analog of the action $\mathcal{S}(\Phi)$, given in Refs. [48,49], and

$$\mathcal{S}_\theta^R = \theta'_i\{-\partial_t\theta_i - \partial_k(v_k\theta_i - \theta_k v_i) + \kappa Z_\kappa \partial^2\theta_i + n_k\partial_k v_i\} \tag{3.6}$$

is the renormalized part of the full action that describes interaction with the magnetic field.

B. Leading-order calculation of the renormalization constant Z_κ

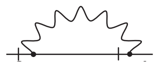
We performed the practical calculation of the renormalization constant Z_κ in the leading one-loop approximation, which is consistent with the accuracy of the calculation for the NS problem (2.1) and (2.2) made in Ref. [48]. Although this calculation is rather simple and similar to that for the Gaussian velocity ensemble [24], we will briefly discuss it for the sake of completeness and in order to stress some peculiarities.

The constant is found from the requirement that the 1-irreducible Green function $\langle\theta'\theta\rangle_{1-\mathrm{ir}}$ be UV finite (that is, finite at $y \to 0$) when expressed in renormalized parameters. In the frequency-momentum representation it has the form:

$$\langle\theta'_1\theta_2\rangle_{1-\mathrm{ir}}(\Omega,\mathbf{p}) = \{-\kappa_0 p^2 + i\Omega\}\, P_{12}^\perp(\mathbf{p}) + \Sigma_{12}(\Omega,\mathbf{p}), \tag{3.7}$$

where Σ_{12} is the "self-energy operator" given by infinite sum of 1-irreducible Feynman diagrams and $p = |\mathbf{p}|$. Because of the large number of tensor indices involved in our expressions, we use numbers (instead of Latin letters) to denote them, with the standard convention on the summation over repeated indices.

The only one-loop self-energy diagram looks as follows:

$$\Sigma_{12} =$$  . (3.8)

Here the wavy line denotes the bare propagator $\langle vv\rangle_0$, and the solid line with a slash denotes the bare propagator $\langle\theta\theta'\rangle_0$ from (2.10), with the slashed end corresponding to the field θ'. The dots with three attached fields θ', θ, v denote the vertex (2.11).

In this approximation, the renormalization constant in the bare term of (3.7) is taken to the first order in g, while in the diagram (3.8) all Z's are simply replaced with unities. Furthermore, we only need to know the divergent part of (3.8), which is quadratic in $\mathbf{p}$ (see the preceding subsection). Thus, we can put $\Omega = 0$ in (3.7) and retain only quadratic terms in the expansion of $\Sigma_{12}(\Omega = 0,\mathbf{p})$ in $\mathbf{p}$. Like for the original NS model, its divergent part is independent of $c_0 \sim c$ and can be calculated directly at $c = 0$; see the discussion in Ref. [49]. Thus, we can use the expression (2.7) for $\langle vv\rangle_0$.

Then the analytic expression for (3.8) takes on the form:

$$\begin{aligned}\Sigma_{12}(\Omega = 0,\mathbf{p}) = D_0 \int \frac{d\omega}{2\pi} \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} V_{143}(\mathbf{p}) V_{625}(\mathbf{p}+\mathbf{k}) \\ \times \left\{ \frac{P^{\perp}_{35}(\mathbf{k})}{\omega^2 + \nu^2 k^4} + \frac{\alpha P^{\parallel}_{35}(\mathbf{k})}{\omega^2 + u^2\nu^2 k^4} \right\} \\ \times \frac{P^{\perp}_{46}(\mathbf{k}+\mathbf{p})}{-i\omega + w\nu|\mathbf{k}+\mathbf{p}|^2},\end{aligned} \tag{3.9}$$

where $k = |\mathbf{k}|$. The simplifying replacement $P^{\perp}_{46} \to \delta_{46}$ can immediately be made due to the transversality condition (2.12) and the contraction with V_{625}.

Integrations over the frequency are easily performed, for example:

$$\begin{aligned}\int \frac{d\omega}{2\pi} \frac{1}{-i\omega + w\nu|\mathbf{p}+\mathbf{k}|^2} \frac{1}{\omega^2 + u^2\nu^2 k^4} \\ = \frac{1}{2u\nu^2 k^2 (uk^2 + w|\mathbf{p}+\mathbf{k}|^2)}.\end{aligned} \tag{3.10}$$

The numerators in the integrand of (3.9) contain the terms quadratic and linear in $\mathbf{p}$. For the first ones, one can immediately set $\mathbf{p} = 0$ in (3.10), while for the second ones, one should expand (3.10) up to the linear term in $\mathbf{p}$, for example,

$$\frac{1}{uk^2 + w|\mathbf{p}+\mathbf{k}|^2} = \frac{1}{(u+w)k^2} \left\{ 1 - \frac{2w}{(u+w)} \frac{(\mathbf{pk})}{k^2} \right\}.$$

With the aid of the formulas

$$\begin{aligned}\int d\mathbf{k} k_i f(k) = 0, \quad \int d\mathbf{k} \frac{k_i k_s}{k^2} f(k) = \frac{\delta_{is}}{d} \int d\mathbf{k}\, f(k), \\ \int d\mathbf{k} \frac{k_i k_s k_l k_p}{k^4} f(k) = \frac{\delta_{is}\delta_{lp} + \delta_{il}\delta_{sp} + \delta_{ip}\delta_{sl}}{d(d+2)} \int d\mathbf{k}\, f(k),\end{aligned} \tag{3.11}$$

where $f(k)$ is any function depending only on $k = |\mathbf{k}|$, all the resulting integrals are reduced to the scalar integral

$$J(m) = \int_{k>m} d\mathbf{k}\, \frac{1}{k^{d+y}} = S_d \frac{m^{-y}}{y}, \tag{3.12}$$

where

$$S_d = 2\pi^{d/2}/\Gamma(d/2) \tag{3.13}$$

is the surface area of the unit sphere in the d-dimensional space and $\Gamma(\cdots)$ is Euler's Gamma function.

The final result contains two types of terms, proportional to $P^{\perp}_{12}(\mathbf{p})$ and $P^{\parallel}_{12}(\mathbf{p})$, respectively. Due to the transversality of the fields θ, θ', the latter ones should be discarded. [This would happen automatically if we included the corresponding projector into the vertex (2.11), but we did not do that for brevity.] Practically, it is more convenient to collect only terms proportional to $\delta_{12}\, p^2$ and drop all the terms proportional to $p_1 p_2$ in the course of calculation. Then we express the amplitude D_0 in (3.9) in terms of renormalized variables: $D_0 = g\nu^3\mu^y$.

The final result reads:

$$\begin{aligned}\Sigma_{12}(\Omega = 0,\mathbf{p}) = -\,\nu p^2\, P^{\perp}_{12}(\mathbf{p})\, \frac{\hat{g}}{2dy} \left(\frac{\mu}{m}\right)^y \\ \times \left\{ \frac{(d-1)}{1+w} + \frac{\alpha(u-w)}{u(u+w)^2} \right\}.\end{aligned} \tag{3.14}$$

Here we passed to the new coupling constant

$$\hat{g} = g S_d/(2\pi)^d, \tag{3.15}$$

with S_d from (3.13).

Then in the MS scheme the renormalization constant Z_κ that cancels the pole of the expression (3.14) in the renormalized analog of the function (3.7) (that is, with the replacement $\kappa_0 \to \kappa Z_\kappa$ in the bare term) has the form:

$$Z_\kappa = 1 - \frac{\hat{g}}{2dwy} \left\{ \frac{(d-1)}{(1+w)} + \frac{\alpha(u-w)}{u(u+w)^2} \right\}, \tag{3.16}$$

while the corresponding anomalous dimension is

$$\gamma_\kappa = \frac{\hat{g}}{2dw} \left\{ \frac{(d-1)}{(1+w)} + \frac{\alpha(u-w)}{u(u+w)^2} \right\}, \tag{3.17}$$

with the corrections of the order $\hat{g}^2$ and higher.

It is interesting to note that the expression (3.17) literally coincides with its analog for the scalar fields advected by the same velocity ensemble; see Eq. (3.24) in Ref. [49]. Similar coincidence between the passive scalar and magnetic fields in the kinematic approximation was earlier observed for the incompressible case (see, e.g., Ref. [42]); sometimes it extends to the two-loop approximation [28].

IV. RG EQUATIONS, FIXED POINT, AND THE CRITICAL DIMENSIONS

Here we only briefly discuss the derivation of the IR scaling behavior from the RG equations in our model; it is nearly identical to the scalar case, discussed in Ref. [49] in great detail.

Multiplicative renormalizability of the field-theoretic model (2.8) allows one to derive, in a standard way, differential

RG equations for the renormalized Green functions

$$G(e,\mu,\dots) = \langle \Phi \dots \Phi \rangle_R.$$

Here $e = \{g,\nu,u,v,w,c,m,\alpha\}$ is the full set of renormalized parameters, μ is the reference momentum scale, and the ellipsis stands for the other arguments (times or frequencies and coordinates or momenta). For convenience, we introduced here three dimensionless ratios: $u_0 = \mu_0/\nu_0$ and $v_0 = \chi_0/\nu_0$ are related to the viscosity and diffusivity coefficients of the (properly extended) model (2.1), (2.2), while $w_0 = \kappa_0/\nu_0$ is related to the magnetic diffusivity coefficient; u, v, and w are their renormalized analogs.

The RG equation expresses the invariance of the renormalized Green function with respect to changing of the reference scale μ, when the bare parameters e_0 are kept fixed:

$$\left\{ \widetilde{\mathcal{D}}_\mu + \sum_\Phi N_\Phi \gamma_\Phi \right\} G(e,\mu,\dots) = 0. \tag{4.1}$$

Here and below we denote $\mathcal{D}_x \equiv x\partial_x$ for any variable x and $\widetilde{\mathcal{D}}_\mu$ is the operation $\mathcal{D}_\mu \equiv \mu\partial_\mu$ at fixed e_0. In terms of the renormalized variables, it takes the form

$$\widetilde{\mathcal{D}}_\mu = \mathcal{D}_\mu + \beta_g\partial_g + \beta_u\partial_u + \beta_v\partial_v + \beta_w\partial_w - \gamma_\nu\mathcal{D}_\nu - \gamma_c\mathcal{D}_c. \tag{4.2}$$

The anomalous dimension γ_F of a certain quantity F (a field or a parameter) is defined by the relation

$$\gamma_F = \widetilde{\mathcal{D}}_\mu \ln Z_F, \tag{4.3}$$

and the β functions for the dimensionless parameters ("coupling constants") are

$$\begin{aligned} \beta_g &= \widetilde{\mathcal{D}}_\mu g = g\,[-y - \gamma_g], \\ \beta_u &= \widetilde{\mathcal{D}}_\mu u = -u\gamma_u, \end{aligned} \tag{4.4}$$

and similarly for β_v, β_w. Here the second equalities result from the definitions and the relations of the type (3.3).

Note that from the definition of w_0 it follows that $Z_\kappa = Z_\nu Z_w$, so

$$\beta_w = w[\gamma_\nu - \gamma_w]. \tag{4.5}$$

The possible types of IR asymptotic behavior are associated with IR attractive fixed points of the RG equations. The coordinates $g_* = \{g_{i*}\}$ of the fixed points are found from the equations

$$\beta_i(g_*) = 0, \tag{4.6}$$

where $g = \{g_i\}$ is the full set of coupling constants and $\beta_i = \widetilde{\mathcal{D}}_\mu g_i$ are their β functions. The character of a fixed point is determined by the matrix

$$\Omega_{ij} = \partial\beta_i/\partial g_j|_{g=g_*}. \tag{4.7}$$

For the IR fixed points the matrix Ω is positive (that is, positive are the real parts of all its eigenvalues).

The analysis performed in Ref. [48] (see also Ref. [49]) on the base of the leading-order (one-loop) approximation has shown that the RG equations of the model $\mathcal{S}_v$, corresponding to the stochastic NS problem (2.1) and (2.2), possess the only IR attractive fixed point in the physical region of parameters ($g,u,v > 0$):

$$\hat{g}_* = \frac{4dy}{3(d-1)} + O(y^2), \quad u_* = 1 + O(y), \quad v_* = 1 + O(y). \tag{4.8}$$

From a certain exact relation between the renormalization constants [48], the exact result

$$\gamma_\nu^* = y/3 \tag{4.9}$$

follows (no corrections of the order y^2 and higher). Here and below, γ_i^* denotes the value of the anomalous dimension γ_i at the fixed point.

Now we substitute the one-loop expressions (3.17) and (4.8) and the exact result (4.9) into Eq. (4.5). Then the equation $\beta_w = 0$ yields, after simple algebra, the equation

$$(w-1)[(d-1)(w+1)(w+2) + 2\alpha] = 0, \tag{4.10}$$

which has the only positive solution $w_* = 1$, with possible corrections of order $O(y)$ and higher.

Since the functions $\beta_{g,u,v}$ do not depend on w, the new eigenvalue of the matrix (4.7) coincides with the diagonal element

$$\partial\beta_w/\partial w|_{g=g_*} = y\,\{3(d-1) + \alpha\}\,/6(d-1) > 0;$$

thus the fixed point (4.8) and $w_* = 1$ of the full model remains IR attractive.

Existence of an IR-attractive fixed point in the physical region of the parameters implies existence of scaling behavior in the IR range. The critical dimension of some quantity F (a field or a parameter) is given by the relation (see Refs. [42,52])

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \quad \Delta_\omega = 2 - \gamma_\nu^* = 2 - y/3. \tag{4.11}$$

Here d_F^k and d_F^ω are the canonical dimensions of F, γ_F^* is the value of the anomalous dimension γ_F at the fixed point, and Δ_ω is the critical dimension of the frequency.

The critical dimensions of the fields and parameters of the model described by the action $\mathcal{S}_v$ from Eq. (2.8) are presented in Ref. [48]; see also Ref. [49]. In addition, our full model involves two more critical dimensions:

$$\Delta_\theta = -1 + y/6, \quad \Delta_{\theta'} = d + 1 - y/6. \tag{4.12}$$

These expressions are exact because the fields θ and θ' are not renormalized.

V. COMPOSITE FIELDS AND THEIR DIMENSIONS

An important role in the following will be played by certain composite fields ("composite operators" in the quantum-field terminology). In general, a local composite operator is a monomial or polynomial built of the primary fields $\Phi(x)$ and their finite-order derivatives at a single space-time point x. In the Green functions with such objects, new UV divergences arise due to coincidence of the field arguments. They should be eliminated by additional renormalization procedure. As a rule, operators mix in renormalization: Renormalized operators (whose Green functions are UV finite) are given by finite sums of the original monomials. However, in the following only a simpler situation will be encountered, when the original operator $F(x)$ and its renormalized analog $F^R(x)$ are related by multiplicative renormalization $F(x) = Z_F F^R(x)$ with a single renormalization constant Z_F. Then the critical dimension Δ_F of the operator F is given by the same expression (4.11) and, in general, differs from the naive sum of the dimensions of the fields and derivatives that compose the operator.

We will focus on the irreducible tensor operators built solely of the fields θ. They have the forms

$$F_{nl} \equiv \theta_{i_1}(x)\cdots\theta_{i_l}(x)\,(\theta_i(x)\theta_i(x))^s + \cdots , \tag{5.1}$$

where $l \leqslant n$ is the number of free tensor indices and $n = l + 2s$ is the total number of the fields θ entering into the operator; the tensor indices and the argument x of the symbol F_{nl} are omitted. The ellipsis stands for the appropriate subtractions involving the Kronecker δ symbols, which ensure that the resulting expressions are traceless with respect to contraction of any given pair of indices, for example, $\theta_i\theta_j - \delta_{ij}(\theta_k\theta_k/d)$ and so on.

The total canonical dimension of any 1-irreducible Green function Γ with one operator $F(x)$ and arbitrary number of primary fields (the formal index of UV divergence) is given by

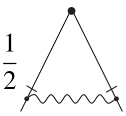
$$\delta_\Gamma = d_F - \sum_\Phi N_\Phi d_\Phi , \tag{5.2}$$

where N_Φ are the numbers of the fields entering into Γ, d_Φ are their total canonical dimensions, and d_F is the canonical dimension of the operator. Superficial UV divergences can be present only in the functions Γ with a non-negative integer δ_Γ. For the operators (5.1) from Table 1 in Ref. [49] we find $d_F = -n$. The linearity of Eq. (2.6) in the field θ imposes the restriction that N_θ in (5.2) cannot exceed the number of the fields θ in the operator F. The direct analysis shows that superficial UV divergences ($\delta_\Gamma \geqslant 0$) for F_{nl} can be present only in the 1-irreducible functions with $N_{\theta'} = N_v = 0$ and $N_\theta = n$; they are all logarithmic: $\delta_\Gamma = 0$. The simple inspection shows that the mixed propagator $\langle\theta v\rangle_0$ does not appear in the relevant Feynman diagrams; in other words, the last term in the right-hand side of Eq. (2.6) is unimportant here. Without that term, the model becomes $O(d)$ invariant. In turn, this means that irreducible operators with different values of l cannot mix with each other. We finally conclude that the operators (5.1) renormalize multiplicatively: $F_{nl} = Z_{nl} F^R_{nl}$ and turn to the one-loop calculation of the renormalization constant Z_{nl} and of the critical dimension of the operator (5.1), which will be denoted as Δ_{nl}.

Let $\Gamma(x;\theta)$ be the generating functional of the 1-irreducible Green functions with one composite operator $F(x) = F_{nl}$ and any number of fields θ. Here $x = \{t,\mathbf{x}\}$ is the argument of the operator and θ is the functional argument, the "classical analog" of the random field θ. We are interested in the n-th term of the expansion of $\Gamma(x;\theta)$ in θ, which we denote $\Gamma_n(x;\theta)$. It can be written as follows:

$$\Gamma_n(x;\theta) = \int dx_1 \cdots \int dx_n\, \theta(x_1)\cdots\theta(x_n) \times \langle F(x)\theta(x_1)\cdots\theta(x_n)\rangle_{1-\mathrm{ir}}. \tag{5.3}$$

In the one-loop approximation the function (5.3) is represented diagramatically as follows:

$$\Gamma_n(x;\theta) = F(x) + \frac{1}{2}$$  . (5.4)

The first term is the tree (loopless) approximation, and the thick dot with the two attached lines in the diagram denotes the operator vertex, to be specified later.

The renormalization constant Z_{nl} for the operator (5.1) is found from the requirement that the renormalized analog $\Gamma^R_n = Z^{-1}_{nl}\Gamma_n$ of the function (5.3) be UV finite in terms of renormalized parameters.

For practical calculations, it is convenient to contract the tensors (5.1) with an arbitrary constant vector $\boldsymbol{\lambda} = \{\lambda_i\}$. The resulting scalar operator has the form

$$F^{(n,l)} = (\lambda_i\theta_i)^l(\theta_i\theta_i)^s + \cdots , \tag{5.5}$$

where the subtractions, denoted by the ellipsis, necessarily involve the factors of $\lambda^2 = \lambda_i\lambda_i$.

Within our accuracy, it is sufficient to replace all the renormalization constants in the diagram with unities, so $D_0 \to g\nu^3\mu^y$, $u_0 \to u$ and so on. Furthermore, we are eventually interested in the fixed-point value of the anomalous dimension, so we can set $u = w = 1$ in the following. Since the diagram is logarithmically divergent, we can set all the external frequencies and momenta equal to zero.

Like for the calculation of the self-energy diagram in Sec. III B, here and below we use numbers (instead of Latin letters) to denote the tensor indices. Then the diagram in (5.4) can be represented as follows:

$$V_{12}(\theta)\, C_{1278}\, \theta_7\theta_8, \tag{5.6}$$

where $V_{12}(\theta)$ is the operator vertex (denoted by the thick dot in the diagram and specified below), the fields $\theta_7\theta_8$ (denoted by wavy tails) are attached to the lower vertices (2.11) (small dots), and C_{1278} is the "core" of the diagram. It has the form

$$C_{1278} = \int \frac{d\omega}{2\pi} \int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{g\nu^3\mu^y R_{1278}}{(\omega^2 + \nu^2k^4)^2} \tag{5.7}$$

with the tensorial factor stemming from the vertices (2.11) and the projectors of the propagators:

$$R_{1278} = P^\perp_{13}(\mathbf{k})P^\perp_{24}(\mathbf{k})\big\{P^\perp_{56}(\mathbf{k}) + \alpha P^\parallel_{56}(\mathbf{k})\big\}V_{375}(\mathbf{k})V_{486}(\mathbf{k}). \tag{5.8}$$

Intergation over the frequency is easily performed:

$$\int \frac{d\omega}{2\pi} \frac{1}{(\omega^2 + \nu^2k^4)^2} = \frac{1}{4\nu^3k^6}. \tag{5.9}$$

Contraction of the vector indices in (5.8) leads to the following integrals over the momentum:

$$\int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+y}}\, P^\perp_{12}(\mathbf{k})\, P^\parallel_{78}(\mathbf{k}) \tag{5.10}$$

for the transverse contribution in (2.7) and

$$\int_{k>m} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+y}}\, P^\perp_{17}(\mathbf{k})\, P^\perp_{28}(\mathbf{k}) \tag{5.11}$$

for the longitudinal one. With the aid of the relations (3.11), all these integrals are reduced to the scalar integral (3.12).

Combining all these contributions and contracting the result with the fields $\theta_7\theta_8$ gives for (5.6) the following expression:

$$\frac{-D_0}{4d(d+2)\nu^3}\, V_{12}\,\{T_{12}+\alpha L_{12}\}, \tag{5.12}$$

where

$$T_{12}=(d+1)\delta_{12}\theta^2-2\theta_1\theta_2$$

and

$$L_{12}=\delta_{12}\theta^2+(d^2-2)\theta_1\theta_2.$$

Now let us turn to the vertex factor

$$V_{12}=\frac{\delta^2 F_{nl}(x)}{\delta\theta_1(x_1)\delta\theta_2(x_2)}. \tag{5.13}$$

Using the chain rule, it can be rewritten in the form

$$V_{12}=\frac{\partial^2 F_{nl}(w)}{\partial w_1\partial w_2}\,\delta(x-x_1)\delta(x-x_2), \tag{5.14}$$

where

$$F^{(n,l)}=(\lambda_i w_i)^l(w_iw_i)^s+\cdots, \tag{5.15}$$

with the subsequent substitution $w_i\to\theta_i(x)$.

The differentiation gives

$$\begin{aligned}\partial^2F^{(n,l)}/\partial w_1\partial w_2 &= 2s(w^2)^{s-2}(\lambda w)^l[\delta_{12}w^2+2(s-1)w_1w_2]\\ &\quad + l(l-1)(w^2)^s(\lambda w)^{l-2}\lambda_1\lambda_2\\ &\quad + 2ls(w^2)^{s-1}(\lambda w)^{l-1}(w_1\lambda_2+w_2\lambda_1),\end{aligned} \tag{5.16}$$

where $w^2=w_kw_k$ and $(\lambda w)=\lambda_kw_k$.

Now we have to contract the vertex factor (5.16) with the expression (5.12). In order to find the renormalization constant, it is sufficient to retain only the terms proportional to the principal monomial in (5.5) and discard all the terms containing the factors of $\lambda^2=\lambda_i\lambda_i$. Combining all the relevant factors finally gives

$$\Gamma_{nl}(x)=F_{nl}(x)\left\{1-\left(\frac{\mu}{m}\right)^y\frac{\hat g(Q_1+\alpha Q_2)}{8yd(d+2)}\right\}, \tag{5.17}$$

where

$$\begin{aligned}Q_1&=-n(n+d)(d-1)+(d+1)\,l(l+d+2),\\ Q_2&=-n(n+nd-d)(d-1)+l(l+d+2),\end{aligned} \tag{5.18}$$

and $\hat g$ is defined in (3.15).

The requirement that the renormalized analog of the function (5.3) be UV finite in the MS scheme gives:

$$Z_{nl}=\left\{1-\frac{\hat g}{8yd(d+2)}\,(Q_1+\alpha Q_2)\right\}, \tag{5.19}$$

and the anomalous dimension $\gamma_{nl}=\widetilde{\mathcal D}_\mu\ln Z_{nl}$ is

$$\gamma_{nl}=\frac{\hat g}{8d(d+2)}\,(Q_1+\alpha Q_2) \tag{5.20}$$

(we recall that we already set $u=w=1$).

Using the general expression (4.11), for the critical dimension Δ_{nl} at the fixed point (4.8) we obtain

$$\Delta_{nl}=n\Delta_\theta+\gamma^*_{nl}=\frac{ny}{6}+\frac{y(Q_1+\alpha Q_2)}{6(d-1)(d+2)} \tag{5.21}$$

with Δ_θ from (4.12). In particular, for the scalar operator one arrives at the expression

$$\Delta_{n0}=\frac{-yn}{6(d+2)}\left\{(n-2)+\alpha\frac{(3n+d-4)}{(d-1)}\right\}, \tag{5.22}$$

which is negative and decreases as α grows:

$$\partial\Delta_{n0}/\partial\alpha<0. \tag{5.23}$$

As we will see in the next section, this means that the anomalous scaling is indeed present in our model and becomes more strongly pronounced as the degree of compressibility increases.

For a fixed n, the dimensions (5.21) exhibit a kind of hierarchy with respect to the rank l (which measures the "degree of anisotropy"):

$$\partial\Delta_{nl}/\partial l>0. \tag{5.24}$$

In contrast to the Gaussian model (see, e.g., Refs. [22,35]), this hierarchy becomes more strongly pronounced as α increases:

$$\partial^2\Delta_{nl}/\partial l\partial\alpha>0. \tag{5.25}$$

VI. OPERATOR PRODUCT EXPANSION AND THE ANOMALOUS SCALING

A. General discussion and isotropic case

The quantities of interest are, in particular, the pair correlation functions of the (renormalized analogs of the) operators (5.1). In the following, we restrict ourselves with the equal-time correlations because they are Galilean invariant and do not bear strong dependence on the IR scale $m=L^{-1}$ caused by the sweeping of small-scale vortices by the large-scale ones. Then one can write

$$\begin{aligned}\langle F_{nl}(t,\mathbf x)F_{qj}(t,\mathbf x')\rangle &= \mu^{d_F}\nu^{d_F^\omega}\eta_{nl,qj}(\mu r,mr,c/\mu\nu)\\ &\simeq \mu^{d_F}\nu^{d_F^\omega}(\mu r)^{-\Delta_{nl}-\Delta_{qj}}\zeta_{nl,qj}(mr,c(r)),\end{aligned} \tag{6.1}$$

where $r=|\mathbf r|=|\mathbf x'-\mathbf x|$; it is also assumed that $n,q\geqslant 1$.

The first equality follows from simple dimensionality considerations; then d_F^ω and d_F are the canonical dimensions of the correlation function, given by simple sums of the corresponding dimensions of the operators, and $\eta(\dots)$ is some function of completely dimensionless parameters. We have expressed the right-hand side in renormalized variables, when the reference mass μ is the substitute of the UV momentum scale Λ. The second (approximate) equality is valid in the IR asymptotic range $\mu r\gg 1$ and follows from solving the RG equation in the presence of the IR attractive fixed point; Δ_{nl} and Δ_{qj} being the critical dimensions of the operators from the left-hand side, given by Eq. (5.21). In the following, we omit the RG invariant variable $c(r)=c(\mu r)^{\Delta_c}/(\mu\nu)$, which is restricted in the IR range; for more explanations, see Ref. [49]. We will also omit the indices of the scaling functions η and ζ and do not display the dependence on the parameters ν, μ that are fixed for a given physical setup.

The inertial range corresponds to the additional inequality $mr\ll 1$. The form of the functions ζ in (6.1) is not determined by the RG equations alone; they should be augmented by the OPE. In the case at hand, the OPE states that the equal-time product $F_{nl}(x)F_{qj}(x')$ at $\mathbf{x} = (\mathbf{x} + \mathbf{x}')/2 = \text{const}$ and $\mathbf{r} = \mathbf{x}' - \mathbf{x} \to 0$ can be represented in the form

$$F_{nl}(x)F_{qj}(x') \simeq \sum_F C_F(\mathbf{r})F(t,\mathbf{x}). \tag{6.2}$$

Here the functions C_F are the Wilson coefficients, regular in m^2, and F are, in general, all possible renormalized local composite operators allowed by the symmetry of the model and of the left-hand side. In the case at hand this implies that only Galilean invariant operators contribute. If these operators have additional vector indices, they are contracted with the corresponding (additional) indices of the coefficient functions C_F.

Without loss of generality, it can always be assumed that the expansion (6.2) is made in the irreducible operators with definite critical dimensions Δ_F. The correlation functions (6.1) are obtained by averaging the expression (6.2) with the weight $\exp \mathcal{S}(\Phi)$, where $\mathcal{S}(\Phi)$ is the (renormalized) action functional (3.5). Then the quantities $\langle F\rangle$ appear on the right-hand sides. Consider first the isotropic case, then only the contributions from scalar operators survive. Their asymptotic behavior for $m \to 0$ is found from the RG equations for the operators F and has the form $\langle F\rangle \propto m^{\Delta_F}$ (we recall that $\Delta_m = 1$).

Thus, combining the expressions (6.1) and (6.2) gives the following inertial-range asymptotic representation for the scaling functions ζ:

$$\zeta(mr) \simeq \sum_F A_F\,(mr)^{\Delta_F}, \tag{6.3}$$

where all the coefficients $A_F = A_F(mr)$ are regular in $(mr)^2$.

Singularities for $mr \to 0$ (and thus the anomalous scaling) result from the contributions in (6.3) of the operators with *negative* critical dimensions, termed "dangerous" in Ref. [42]. Clearly, if the number of such operators were finite, the leading contribution would be determined by the operator with the lowest dimension. However, one can argue that if at least one dangerous operator exists in a model, the latter necessarily involves an infinite set of dangerous operators, and the spectrum of their dimensions is not bounded from below; see Appendix for discussion. In our case, from the expression (5.22) we can see that all the scalar operators F_{n0} are dangerous, and their dimensions Δ_{n0} increase without bound as n grows.

Fortunately, the linearity of the original equation (2.6) in the field θ imposes the restriction that the number of the fields θ in all the composite operators in the expansion (6.2) cannot exceed their number in the left-hand side; cf. the remark below Eq. (5.2) in Sec. V. In turn, this means that only a finite number of the operators of the type F_{k0} can contribute to any given OPE. For the product (6.2), these are the operators with $k \leqslant n + q$. Thus,

$$\zeta(mr) \simeq \sum_{k=0}^{n+q} A_k(mr)\,(mr)^{\Delta_{k0}} + \cdots \tag{6.4}$$

with Δ_{k0} from (5.22); the ellipsis stands for the "more distant" corrections to the small-mr behavior, given by the operators with derivatives and other types of fields. The leading term in (6.4) is determined by the operator with the maximum possible $k = n + q$, so the final leading-order asymptotic expression for the correlation function (6.1) in the inertial range $\mu r \gg 1$, $mr \ll 1$ has the form

$$\langle F_{nl}F_{qj}\rangle \simeq (\mu r)^{-\Delta_{nl}-\Delta_{qj}}(mr)^{\Delta_{n+q,0}}. \tag{6.5}$$

As already mentioned in the preceding section, the inequality (5.23) means that the anomalous scaling becomes more strongly pronounced as the degree of compressibility grows. We also note that the inequality

$$\Delta_{n+q,0} < \Delta_{nl} + \Delta_{qj}, \tag{6.6}$$

which follows from the explicit expressions (5.21) and, in fact, is required by the probabilistic theory, shows that the expression (6.5) diverges for $r \to 0$.

B. Effects of large-scale anisotropy

Consider effects of the anisotropy, introduced into the system at large scales $\sim L$ through, say, the large-scale field $B_i^0 = n_i B^0$ or through the correlation function of the artificial force. Then the irreducible tensor composite operators acquire nonzero mean values, built of the vector $\mathbf{n}$: for example, the mean value of the second-rank operator is proportional to the irreducible tensor $n_i n_j - \delta_{ij}/d$. In general, the mean value of any l-th rank irreducible operator is proportional to the tensor $n_{i_1} \ldots n_{i_l} + \cdots$, where the ellipsis stands for the appropriate subtractions with the Kronecker δ symbols that make it irreducible. Upon substitution into the OPE for the product of two scalar operators, their tensor indices are contracted with the corresponding indices of the coefficient functions $C_F(\mathbf{r})$. This gives rise to the Gegenbauer polynomials, the d-dimensional analogs of the Legendre polynomials $P_l(\cos\vartheta)$, where ϑ is the angle between the vectors $\mathbf{r}$ and $\mathbf{n}$. For general anisotropy, the d-dimensional spherical harmonics appear on the right-hand side, while for products of tensor operators, their tensor analogs arise.

Consider, as the simplest example, the pair correlation function (6.1) of two scalar operators in the inertial range:

$$\langle F_{n0}(t,\mathbf{x})F_{q0}(t,\mathbf{x}')\rangle \simeq r^{-\Delta_{n0}-\Delta_{q0}} \sum_{l=0}^{N} c_l\, P_l(\cos\vartheta)(mr)^{\Delta_{Nl}} + \cdots, \tag{6.7}$$

with $N = n + q$ and Δ_{Nl} from (5.21); c_l are numerical coefficients and the ellipsis stands for the "distant" contributions with $l > N$. The inequality (5.24) means that the anisotropic contributions in (6.7) exhibit a kind of hierarchy related to the degree of anisotropy l: the leading contribution is given by the isotropic "shell" ($l = 0$), while the contributions with $l > 1$ give only corrections which become relatively weaker as $mr \to 0$, the faster the higher the degree of anisotropy l is. A similar hierarchy, observed earlier in numerous models, e.g., Refs. [19–24,34–36,49], gives quantitative support for Kolmogorov's hypothesis of the local isotropy restoration.

The inequality (5.25) means that the hierarchy (5.24) becomes stronger as the degree of compressibility α grows: The anisotropic corrections are getting further from one another and from the isotropic term, contrary to the situation observed earlier for the passive vector field, advected by Kraichnan's ensemble [22]. A similar discrepancy for the scalar field was encountered recently in Ref. [49]. This means that the results obtained on the basis of simplified "synthetic" ensembles should be taken with some precaution.

C. Structure functions

The phenomenon of anomalous scaling is often discussed in terms of structure functions; see, e.g., Ref. [1] and references therein. For the vector case they can be defined as

$$S_p(\mathbf{r}) = \langle [\theta_r(t,\mathbf{x}) - \theta_r(t,\mathbf{x}')]^{2p} \rangle, \tag{6.8}$$

where θ_r is the component of the field θ parallel to the vector $\mathbf{r} = \mathbf{x}' - \mathbf{x}$. For simplicity, in this subsection we restrict ourselves with the isotropic model and therefore consider only even-order functions.

After the binomial expansion and the decomposition in irreducible tensors, the function (6.8) is represented as a linear combination of two types of terms. The first type involves the pair correlation functions (6.1) of the operators (5.1) with $n + q = p$ and all possible values of l, j. Their inertial-range behavior is described by the expression (6.5).

The second type involves the mean values $\langle F_{pk} \rangle$ of the operators from the same family; these are independent of $\mathbf{r}$. The RG analysis shows that the analog of (6.5) for them has the form

$$(m/\mu)^{\Delta_{pk}}; \tag{6.9}$$

for the isotropic case, only the term with $k = 0$ survives. It can be rewritten in a somewhat artificial form:

$$(m/\mu)^{\Delta_{p0}} = (mr)^{\Delta_{p0}} (\mu r)^{-\Delta_{p0}}. \tag{6.10}$$

Now comparison with (6.5) along with the inequality (6.6) shows that the contribution of the mean value is the leading term of the inertial-range asymptotic behavior (we recall than $n + q = p$).

The structure functions in the kinematic MHD model are superpositions of a constant term and a number of power-law corrections should be taken into account in the interpretation of experimental data, while they are usually fitted by single power-law terms. The structure functions, at least from a theoretical point of view, are more convenient objects for the models with the symmetry with respect to the shift $\theta \to \theta$, like the scalar tracer model; cf. the discussion, e.g., in Refs. [35,49]. It should be stressed, however, that the RG+SDE scenario of anomalous scaling proved to be successful for linear models (kinematic dynamo model in the present case), while its relevance for the full-scale nonlinear MHD problem is far from obvious.

VII. CONCLUSION

We have studied the model of the passive vector field, advected by a turbulent flow. The latter is described by the Navier-Stokes equations for a strongly compressible fluid (2.1), (2.2) with an external stirring force with the correlation function $\propto k^{4-d-y}$; (2.4) and (2.5). From physics viewpoints, the model describes magnetohydrodynamic turbulence in the so-called kinematic approximation, where the effects of the magnetic field on the dynamics of the fluid are neglected.

The full stochastic problem can be cast as a field-theoretic model with the action functional specified in (2.8) and (2.9). That model appears multiplicatively renormalizable, so the corresponding RG equations can be derived in a usual way. They have the only IR attractive fixed point in the physical range of parameters, so various correlation functions reveal scaling behavior in the IR region.

Their inertial-range behavior was studied by means of the OPE; existence of anomalous scaling (singular power-like dependence on the integral scale L) was established. The corresponding anomalous exponents were identified with the scaling (critical) dimensions of certain composite fields (composite operators), namely powers of the magnetic field. They can be systematically calculated as series in the exponent y. The practical calculation was accomplished in the leading order; the results are presented in (5.21).

The results obtained are quite similar to those derived earlier for the vector fields advected by synthetic velocity ensembles [22,24]: The anomalous scaling becomes more remarkable as the degree of compressibility α increases; the anisotropic contributions form a hierarchy related to the degree of anisotropy l, so the leading inertial-range contribution is the same as for the isotropic case. However, that hierarchy becomes stronger as the degree of compressibility grows, in contrast to what was observed in Ref. [22] for the Kraichnan's rapid-change ensemble. In this respect, our results are close to what was recently observed for the scalar field, advected by the same velocity ensemble [49].

From a more theoretical point of view, it is important that in our case, the anomalous exponents are associated with the critical dimensions of certain individual composite operators, exactly as in the RG+OPE treatment of the rapid-change models; see, e.g., Refs. [21,22,38,40,41]. In the zero-mode approach to the latter, the anomalous exponents are related to the so-called zero modes (unforced solutions) of the exact differential equations satisfied by the equal-time correlation functions; see, e.g., Refs. [17–19,33]. In a more general sense, zero modes can be interpreted as certain statistical conservation laws in the dynamics of particle clusters [33]. The close resemblance in the RG+OPE pictures of the origin of anomalous scaling for the present model and its rapid-change predecessors suggests that the concept of zero modes (and thus that of statistical conservation laws) is also applicable in much more realistic models.

Admittedly, our results are derived only in the leading order of the expansion in a parameter, which is not very small. So it is hard to expect a good quantitative agreement between the theory and experiment. On the other hand, to the best of our knowledge, the dependence of the anomalous exponents on the degree of compressibility has not been studied experimentally.

Further theoretical investigation should include, in particular, an account of the reaction of the magnetic field on the fluid dynamics. Existing works of this problem, based on the RG techniques, were concerned with the incompressible fluid and did not discuss the anomalous multiscaling. Thus, much work remains for the future.

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APPENDIX: INFINITE NUMBER OF NEGATIVE DIMENSIONS

Let $F(\mathbf{x})$ be a certain renormalized composite operator in a certain multiplicatively renormalizable field-theoretic model with an IR attractive fixed point of the RG equations. Assume that F has a definite negative critical dimension, $\Delta_F < 0$, and assume that it is the lowest dimension in the model (that is, F is the "most dangerous" operator).

Consider its pair correlation function:

$$\langle F(\mathbf{x})F(\mathbf{x}')\rangle = \int \frac{d\mathbf{k}}{(2\pi)^d} D_F(k) \exp\{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})\}, \qquad (A1)$$

where $k = |\mathbf{k}|$. Our assumption implies that in the IR range the function $D_F(k)$ has the asymptotic form

$$D_F(k) \simeq k^{-d+2+\Delta_F} f(m/k), \qquad (A2)$$

where m is some typical IR momentum scale, $f(m/k)$ is some dimensionless scaling function, and we assumed that $\Delta_m = 1$ (like in our model). Now consider the mean value

$$\langle F^2(\mathbf{x})\rangle = \int \frac{d\mathbf{k}}{(2\pi)^d} D_F(k). \qquad (A3)$$

The function $f(m/k)$ provides IR regularization of the integral (A3). The question is whether this integral remains convergent for large k if the exact (unknown) function $D_F(k)$ is replaced with its asymptotic form (A2). According to the OPE, the asymptotic behavior of the function $f(m/k)$ for large k or, equivalently, for small m is determined by the contribution of the most dangerous operator, which by assumption is F itself:

$$f(m/k) \simeq (m/k)^{\Delta_F}. \qquad (A4)$$

Thus, for large k we have

$$D_F(k) \simeq k^{-d+\Delta_F} \qquad (A5)$$

and the integral (A3) remains convergent upon the substitution of (A2). In turn, this means that it is expressed only in terms of the IR parameter m (UV regularization Λ can be removed). Then it is easily found from the dimension:

$$\langle F^2(\mathbf{x})\rangle \simeq m^{2\Delta_F}. \qquad (A6)$$

Expression (A6) means, however, that the operator F^2 has the negative dimension $2\Delta_F < \Delta_F < 0$ which is smaller than that of F. We arrive at the contradiction with our initial assumption about F.

To avoid possible misunderstanding we stress that our consideration does not mean that F^2 is necessarily dangerous and its dimension is exactly $2\Delta_F$ (although this indeed happens, e.g., for the powers of the velocity field in the stochastic NS problem; see Ref. [42]). It means that operators with negative dimensions, if any, always appear in a model as infinite families, with the spectrum of dimensions not bounded from below. This fact should be taken into account in axiomatic or phenomenological implementations of the OPE to models of turbulence [54].

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