Synchronization of nearly identical dynamical systems: Size instability

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We study the generalized synchronization and its stability using the master stability function (MSF) in a network of coupled nearly identical dynamical systems. We extend the MSF approach for the case of degenerate eigenvalues of the coupling matrix. Using the MSF we study the size instability in star and ring networks for coupled nearly identical dynamical systems. In the star network of coupled Rössler systems we show that the critical size beyond which synchronization is unstable can be increased by having a larger frequency for the central node of the star. For the ring network we show that the critical size is not significantly affected by parameter variations. The results are verified by explicit numerical calculations.

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I. INTRODUCTION

In the physical world networks are ubiquities. Many practical complex systems in natural and social sciences and also humanities can be modeled as networks of interacting systems. Recently, the study of such complex networks has attracted much attention [1–4]. The complex networks can be grouped into different types depending on their structure into some universal categories, such as random networks [5], scale-free networks [6], small-world networks [7], etc.

When there are several interacting dynamical systems on networks, they can exhibit a rich variety of dynamical behavior such as synchronization [8–10], amplitude death [11], multistability [12], chimera states [13], phase flip [14], etc., which a single dynamical system is unable to show. Occurrence of synchronization between interacting dynamical units is an important and fundamental nonlinear phenomenon and the study of synchronization of coupled dynamical systems has attracted considerable attention in the past decades [8–10,15–23]. In particular, the nonlinear behavior of coupled chaotic systems tends to separate the nearby trajectories of the coupled systems, while a suitable coupling between them brings the trajectories back together. In this competition, when the latter wins the coupled systems undergo synchronization. Synchronization of coupled dynamical systems can be defined as a process where two or more coupled systems adjust their trajectories to a common behavior when they are coupled or driven by a common signal. In the context of coupled chaotic systems different types of synchronizations have been studied in the past years. These include complete or identical synchronization [15,16,18], phase synchronization [21,24], lag synchronization [25], anticipatory synchronization [26], imperfect phase synchronization [27], generalized synchronization [22,23], measure synchronization in Hamiltonian systems [28], etc. Among these the simplest and the most studied is the complete synchronization which occurs in two or more coupled identical dynamical systems and is characterized by the equality of state variables of the interacting systems. The stability of synchronization is normally determined by the negativity of the largest transverse Lyapunov exponent [19,29]. Pecora and Carroll have developed an elegant way, namely the master stability function (MSF), for analyzing the stability of complete synchronization for a network of identical dynamical systems [29]. The MSF allows one to study the stability of synchronization of different networks using a single function and has been used widely for a comparative study of synchronization of different networks of identical dynamical systems [30–36]. It is shown that the small-world network enhances synchronizability of a network of coupled identical systems [31]. In Ref. [32] it is shown that having smaller network distance is not sufficient for performing best synchronizability properties, it is also required to have homogeneous degree distribution among the coupled dynamical systems. Reference [37] introduces a new family of graphs, namely the entangled networks, which show better synchronizability. These entangled networks are interwoven and have extremely homogeneous structures, i.e., the degree distributions are very narrow.

The network of coupled identical dynamical systems is an ideal situation and in practical situations it is almost impossible to have a network of coupled identical dynamical systems. So it is important to study coupled nonidentical systems and see how their behavior compares with that of coupled identical systems. For the coupled nonidentical systems, one cannot get complete synchronization. Instead in this case the synchronization is of a generalized type where the state variables of the coupled dynamical systems are related with some functional relationship [22,23]. The nonidentical nature of the coupled systems can lead to desynchronization bursts which is known as the bubbling transition [38,39]. After the desynchronization burst the system returns to the synchronized state. Sun *et al.* [40] have extended the master stability approach for nearly identical systems to calculate the deviation from the average trajectory and the deviation is shown to be bounded.

In this paper, we extend the MSF formalism to a system of nearly identical systems. By coupled nearly identical systems we mean systems which have a node-dependent parameter (NDP). Preliminary results of this study were earlier reported in Ref. [41]. We then extend the MSF formalism to coupled nearly identical systems with degenerate eigenvalues of the coupling matrix. We also obtain MSF for systems with more than one NDP. We find that, in general, the stability

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of synchronization can be improved when we introduce a nonidentical nature into the coupled systems.

Next we use MSF to study size instability of synchronization in well-structured networks. By a well-structured network we refer to a network in which the number of nodes can be changed without changing the basic structure and symmetry of the network, e.g., star network, ring network, etc. The size instability is the phenomenon whereby there is a critical number of oscillators that can be coupled in a well-structured network to obtain synchronization and beyond this critical number no stable synchronization can be seen. The phenomena of size instability in identical oscillators is well known and has been studied widely [19,39,42–44]. Zamora-Munt *et al.* [45] have studied synchronization of delay-coupled lasers for a star network. For the star network of coupled nearly identical systems, we find that it is possible to increase the critical number of nodes beyond which synchronization is unstable by a judicious choice of NDP. In particular, for coupled Rössler systems the critical number can be increased by having a larger frequency for the central node. On the other hand, we find that for a ring network, the critical number of nodes is not significantly affected by an NDP. These results are verified by explicit numerical calculations.

II. STABILITY OF SYNCHRONIZATION OF COUPLED NEARLY IDENTICAL SYSTEMS

For networks of coupled identical systems the stability of complete synchronization has been well analyzed. As discussed in the Introduction, Pecora and Carroll (1998) [29] introduced a MSF which can be calculated from master stability equations. Using the master stability function one can calculate the largest transverse Lyapunov exponent for a network and study and compare the stability properties of synchronization of different networks. For coupled nearly identical systems, the synchronization is of a generalized type. We now extend the MSF approach to coupled nearly identical systems.

A. Master stability function for nearly identical systems

In Ref. [41], we have extended the formalism of MSF to coupled nearly identical systems and, in this subsection, we briefly review the same. This is done for the sake of completeness and also to establish the notation. We start by considering a network of N-coupled dynamical systems as

$$\dot{x}^i = f(x^i, r^i) + \varepsilon \sum_{j=1}^{N} g_{ij} h(x^j); \quad i = 1, \ldots, N, \tag{1}$$

where $x^i (\in R^m)$ is the m-dimensional state vector of system i; r^i is the node-dependent parameter (NDP) which makes the systems nonidentical; $f : R^m \to R^m$ and $h : R^m \to R^m$ give, respectively, the dynamical evolution of a single system and the coupling function; $G = [g_{ij}]$ is the coupling matrix; and ε is the coupling constant. The diagonal elements of the coupling matrix are $g_{ii} = -\sum_{j\neq i} g_{ij}$. Thus, the coupling matrix satisfies the condition $\sum_j g_{ij} = 0$, which fulfills the condition for invariance of the synchronization manifold [29]. Let the parameter $r^i = \tilde{r} + \delta r^i$, where $\tilde{r}$ is some typical value of the parameter and δr^i is a small mismatch.

When the coupled systems are identical, i.e., $r^i = r$; $\forall i$, then they can exhibit complete synchronization for a suitable coupling constant ε [18]. For complete synchronization all the state variables of the coupled systems become equal, i.e., $x^i = x$; $\forall i$ and the motion of the coupled systems are confined to the subspace defined by $x^i = x$ and this subspace is the synchronization manifold. The complementary space defines the transverse manifold. The synchronized state is stable when all the transverse Lyapunov exponents are negative. The Lyapunov exponents are calculated by expanding around the synchronous solution $x^i = x$.

For coupled nonidentical systems, the synchronization is of the generalized type, where the state variables of the coupled systems are related by a functional relationship [22]. Here we expand Eq. (1) around the solution $\tilde{x}$ of a system with some typical parameter $\tilde{r}$ [46]. In the expansion, we retain terms up to second order and we get [41]

$$\begin{aligned}\dot{z}^i &= D_x f(\tilde{x},\tilde{r}) z^i + D_r f(\tilde{x},\tilde{r}) \delta r^i + \frac{1}{2} D_r^2 f(\tilde{x},\tilde{r}) (\delta r^i)^2 \\ &\quad + D_r D_x f(\tilde{x},\tilde{r}) z^i \delta r^i + \varepsilon \sum_{j=1}^{N} g_{ij} D_x h(\tilde{x}) z^j,\end{aligned} \tag{2}$$

where $z^i = x^i - \tilde{x}$. In Eq. (2) we have dropped the term containing $(z^i)^2$ as we are interested in the solution $z^i \to 0$. Equation (2) contains both inhomogeneous and homogeneous terms. In Ref. [41], we had argued that the exponents in the expanding and contracting solutions are determined by the homogeneous terms and the inhomogeneous term does not contribute to these exponents. A similar observation was made in Ref. [47]. So to calculate Lyapunov exponents from Eq. (2) we drop the inhomogeneous terms to obtain

$$\dot{z}^i = D_x f(\tilde{x},\tilde{r}) z^i + D_r D_x f(\tilde{x},\tilde{r}) z^i \delta r^i + \varepsilon \sum_{j=1}^{N} g_{ij} D_x h(\tilde{x}) z^j. \tag{3}$$

Equation (3) can be put in the matrix form as

$$\dot{Z} = D_x f(\tilde{x},\tilde{r})\, Z + D_r D_x f(\tilde{x},\tilde{r})\, Z\, R + D_x h(\tilde{x})\, Z\, G^T, \tag{4}$$

where G^T is the transpose of the coupling matrix G and $Z = (z^1, \ldots, z^N)$ and $R = \mathrm{diag}(\delta r^1, \ldots, \delta r^N)$.

Let γ_j, $j = 1, \ldots, N$ be the eigenvalues of the coupling matrix G^T and the corresponding left and right eigenvectors be e_j^L and e_j^R, respectively. We multiply Eq. (4) by e_j^R from the right-hand side and use the m-dimensional vector $\phi_j = Z e_j^R$. Thus,

$$\dot{\phi}_j = [D_x f + \varepsilon \gamma_j D_x h] \phi_j + D_r D_x f\ Z\ R e_j^R. \tag{5}$$

Equation (5) is not in the diagonal form since, in general, e_j^R are not eigenvalues of R. To circumvent this problem, we use first-order perturbation theory and obtain the first-order correction due to the NDP as $\nu_j = e_j^L R e_j^R$. Thus, we can approximate Eq. (5) as

$$\dot{\phi}_j = [D_x f + \varepsilon \gamma_j D_x h] \phi_j + \nu_j D_r D_x \phi_j. \tag{6}$$

The above equation can be cast in the form of the master stability equation by introducing two complex parameters, namely the effective coupling parameter $\alpha = \varepsilon \gamma_j$ and the mismatch parameter $\nu_r = \nu_j$ [48] as

$$\dot{\phi} = [D_x f + \alpha D_x h + \nu_r D_r D_x f]\phi. \tag{7}$$

This equation reduces to the master stability equation of Pecora and Carroll [29] for identical systems when $\nu_r = 0$.

The MSF is defined as the largest Lyapunov exponent, $\lambda_{\max}$, of the above master stability equation, as a function of the parameters α and ν_r. The MSF for coupled nearly identical systems obtained using the above formalism is an approximation to the actual values. The accuracy of MSF and the Lyapunov exponents obtained using this formalism are discussed in Ref. [41] with numerical examples. It is found that the errors are small when the systems are synchronized.

B. Condition for stable synchronization

The MSF can be used to study the stability of synchronization of any network of N-coupled nearly identical systems. For a given network, one can determine the eigenvalues $\gamma_i, i = 1, \dots, N$ of G and the corresponding ν_i values. The eigenvalue $\gamma_1 = 0$ and it corresponds to the synchronization manifold. The remaining eigenvalues correspond to the transverse manifold. If the MSF for $\alpha = \gamma_i \varepsilon$ and $\nu_r = \nu_i$ is negative for all the transverse eigenvalues ($i = 2, \dots, N$), then the synchronization is stable.

As an example consider a network of coupled Rössler oscillators with the frequency ω as NDP and diffusive coupling in x variables. The dynamics of the network is

$$\begin{aligned} \dot{x}_i &= -\omega_i y_i - z_i + \varepsilon \sum_j G_{ij} x_j, \\ \dot{y}_i &= \omega_i x_i + a y_i, \\ \dot{z}_i &= b + z_i(x_i - c), \end{aligned} \tag{8}$$

The master stability equations for the above system of coupled Rössler oscillators are

$$\begin{aligned} \dot{\phi}_x &= -\omega\phi_y - \phi_z + \alpha\phi_x - \nu_\omega\phi_y, \\ \dot{\phi}_y &= \omega\phi_x + a\phi_y + \nu_\omega\phi_x, \\ \dot{\phi}_z &= z\phi_x + (x - c)\phi_z, \end{aligned} \tag{9}$$

where ω is a typical parameter. In Fig. 1 we plot the zero contours of MSF in the parameter plane $\alpha - \nu_\omega$ [49]. The region bounded by the zero contour curves corresponds to the region of negative values of MSF. If all the transverse Lyapunov exponents fall in this region, then the synchronization is stable.

Now let us consider a class of systems for which the master stability function is negative in a finite range of α values, say, (α_l^0, α_h^0) for identical systems and (α_l, α_h) for nearly identical systems. Rössler system considered above belongs to this class of systems (see Fig. 1). For these systems the condition for stable synchronization can be written as

$$\alpha_l < \varepsilon\gamma_2 \leqslant \cdots \leqslant \varepsilon\gamma_N < \alpha_h. \tag{10}$$

The above condition can be also be expressed as

$$\frac{\gamma_N}{\gamma_2} < \frac{\alpha_h}{\alpha_l}. \tag{11}$$

For small variations of NDP, we can treat the master stability function as linear near α_l^0 and α_h^0 and let $1/b_l$ and $1/b_h$ be the

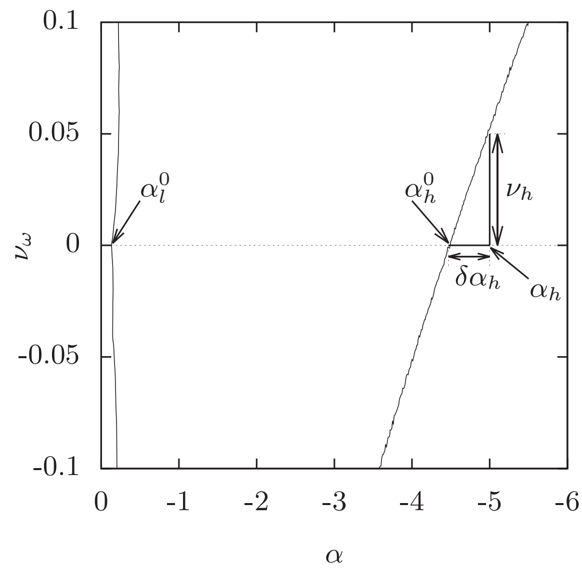

FIG. 1. The zero contour curves of the master stability function of Rössler oscillators with the frequency ω as NDP are plotted in the parameter plane $\alpha - \nu_\omega$. In the region bounded by the zero contour curves the MSF is negative, i.e., the region of stable synchronization. The range (α_l, α_h) for $\nu_\omega = 0$ corresponds to the range of stable synchronization for identical systems. The Rössler parameters are $a = b = 0.2$, $c = 7.0$, and $\alpha_l \sim -0.14$ and $\alpha_h \sim -4.48$. For nearly identical systems, a mismatch parameter ν_h corresponds to a change $\delta\alpha_h$ in the stability range as shown schematically in the figure.

corresponding slopes. Thus,

$$\begin{aligned} \alpha_l &= \alpha_l^0 + \delta\alpha_l \approx \alpha_l^0 + b_l\nu_l \\ \alpha_h &= \alpha_h^0 + \delta\alpha_h \approx \alpha_h^0 + b_h\nu_h. \end{aligned} \tag{12}$$

Thus the condition for stable synchronization for nearly identical systems can be written as

$$\frac{\gamma_N}{\gamma_2} < \frac{\alpha_h^0 + b_h\nu_h}{\alpha_l^0 + b_l\nu_l}. \tag{13}$$

We can also use the above analysis to determine the ε range of synchronization, denoted by l_ε, by setting $\nu_l = \nu_2 = e_2^L R e_2^R$ and $\nu_h = \nu_N = e_N^L R e_N^R$, the mismatch parameters corresponding to the two extreme transverse eigenvalues γ_2 and γ_N. Thus, l_ε is given by

$$l_\varepsilon = \left| \frac{\alpha_h}{\gamma_N} - \frac{\alpha_l}{\gamma_2} \right| = l_\varepsilon^0 + \frac{\delta\alpha_h}{\gamma_N} - \frac{\delta\alpha_l}{\gamma_2} \approx l_\varepsilon^0 + \frac{b_h\nu_N}{\gamma_N} - \frac{b_l\nu_2}{\gamma_2}, \tag{14}$$

where $l_\varepsilon^0 = |\frac{\alpha_h^0}{\gamma_N} - \frac{\alpha_l^0}{\gamma_2}|$ is the ε range of synchronization for coupled identical systems.

When one studies synchronization of coupled nonidentical systems, in addition to the linear stability of synchronization, i.e., stability against small deviations, another important issue is the robustness of synchronization, i.e., how robust the synchronization is against the parameter mismatch. It is known that the parameter mismatch of the coupled dynamical systems can lead to bursting of the unstable periodic orbits, which is known as bubbling [39]. The bubbling is important when there are large deviations from the attractor. In this connection, Sun *et al.* [40] have derived a condition for the synchronization error (deviation) to be bounded and shown that for nearly identical systems, the error remains bounded to first order in parameter mismatch if the condition is satisfied. In Appendix A we derive similar conditions for nearly identical systems and show that the synchronization error remains bounded to second order in parameter mismatch if our conditions are satisfied.

C. Degenerate eigenvalues of coupling matrix *G*

In this section we will consider the case where the coupling matrix G has degenerate eigenvalues. Degenerate eigenvalues of the coupling matrix are observed in many networks with some symmetry property, such as a star network.

When the eigenvalues are degenerate, the first-order correction alone is not sufficient since the second-order correction diverges. In this case one needs to use the degenerate perturbation theory. Let the jth eigenvalue γ_j of G^T have p degeneracy and the left and right eigenvectors of G^T corresponding to eigenvalue γ_j be denoted by $e^L_{j1}, e^L_{j2}, \ldots, e^L_{jp}$ and $e^R_{j1}, e^R_{j2}, \ldots, e^R_{jp}$, respectively. For these p degenerate eigenvalues, we introduce the $p \times p$ matrix A_j as

$$A_j = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1p} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{p1} & \mu_{p2} & \cdots & \mu_{pp} \end{pmatrix},$$

where $\mu_{kl} = e^L_{jk} R e^R_{jl}$. Now we diagonalize matrix A_j to get the diagonal matrix $B_j = \mathrm{diag}[\nu_{j1}, \ldots, \nu_{jp}]$. Thus the linear stability equation (6) can be written as

$$\dot{\phi}_{jk} = [D_x f + \varepsilon\gamma_j D_x h]\phi_{jk} + \nu_{jk} D_r D_x \phi_{jk};\ k = 1, \ldots, p. \tag{15}$$

We note the above linear stability equation (15) has the same form as Eq. (6). Hence, the master stability equation for the degenerate case is the same as Eq. (7) and the master stability function for the degenerate case is the same as for the nondegenerate case.

As an example of the degenerate case we consider the star network of N nodes (see Fig. 2) [29]. The coupling matrix G is given by

$$\begin{aligned} G_{11} &= -(N-1) \\ G_{1i} &= G_{i1} = 1 \\ G_{ij} &= -\delta_{ij}, \end{aligned} \tag{16}$$

where $i,j = 2, \ldots, N$. The eigenvalues of G are $\gamma_1 = 0$, $\gamma_2 = \cdots = \gamma_{N-1} = -1$ and $\gamma_N = -N$. The eigenvalue -1 has $N-2$ degeneracy. The corresponding eigenvectors are given in Eq. (B3) from which the ν_{jk} can be obtained. Thus, we can evaluate Lyapunov exponents using Eq. (15).

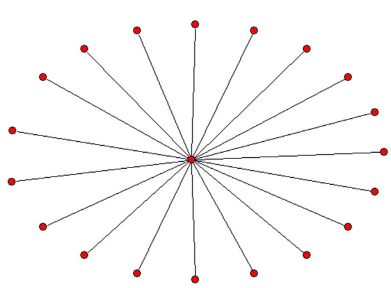

FIG. 2. (Color online) A simple star network with 20 nodes.

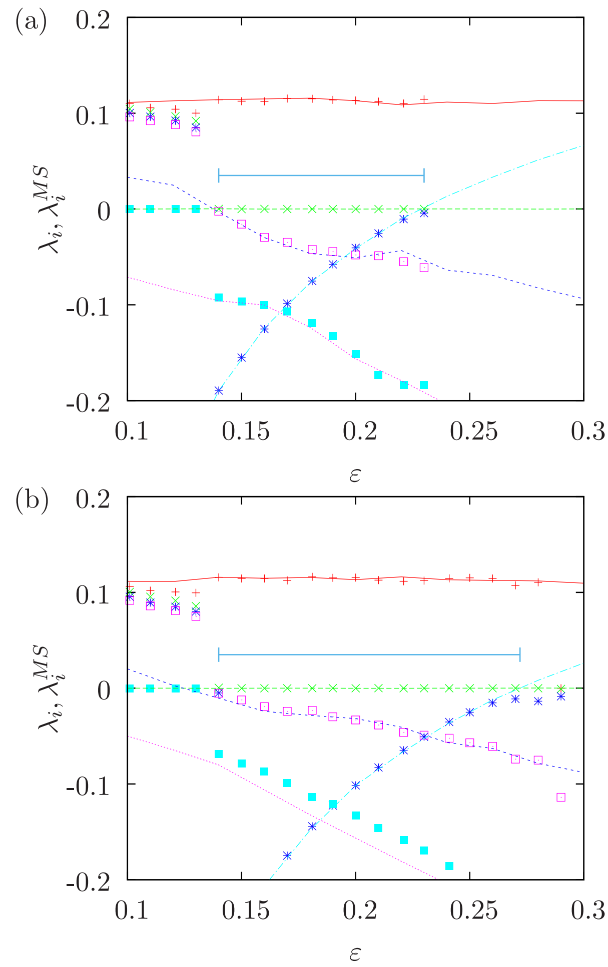

FIG. 3. (Color online) The five largest Lyapunov exponents (points with plus, cross, star, empty square, and filled square) and their estimated value (lines in red, green, blue, cyan, and magenta) using the master stability equation (7) are plotted as a function of the coupling constant ε for a 20-node star network of coupled Rössler systems with (a) 1% and (b) 10% mismatch in NDP ω. The epsilon range of synchronization is shown in both the figures by a horizontal line with vertical bars at the end. We note that the master stability function can be used to obtain five Lyapunov exponents which correspond to the largest Lyapunov exponents of the master stability equation. The other Lyapunov exponents are estimated using the other Lyapunov exponents of the master stability equation.

In Figs. 3(a) and 3(b), six Lyapunov exponents (points) and their estimated values (lines) using the master stability equation (7) are plotted as a function of the coupling parameter ε for a star network of 20 coupled Rössler systems with 1% and 10% variation in NDP ω, respectively. We see a good agreement between the numerical and theoretical values of Lyapunov exponents in the synchronization region.

Figures 3 also show the stable range of synchronization l_ε. For the star network we can estimate l_ε using Eq. (14) as follows. The eigenvector e_N corresponding to the eigenvalue

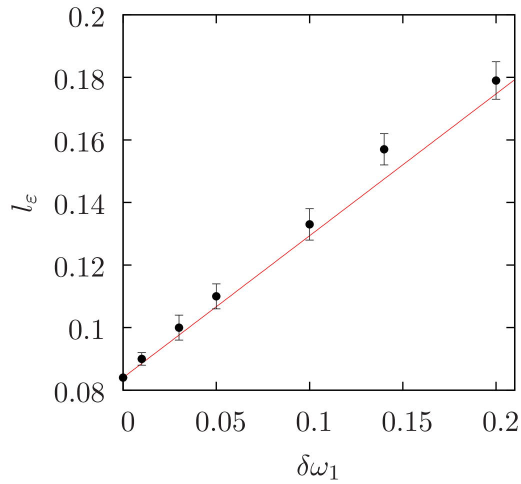

FIG. 4. (Color online) The interval of the stable synchronization l_ε is plotted as a function of $\delta\omega_1$ for the star network of 20 nodes of coupled nearly identical Rössler systems. The straight line is the theoretical result of Eq. (18).

$\gamma_N = -N$ is $e_N = \sqrt{\frac{1}{N(N-1)}}(N-1,-1,\ldots,-1)^T$. Thus,

$$\nu_N = e_N^T R_\omega e_N = \frac{N-2}{N-1}\delta\omega_1, \tag{17}$$

where $\delta\omega_1$ is the frequency mismatch of the central node. From Fig. 1, we can see that the location of α_h changes as a function of the mismatch parameter ν, while the location of α_l remains almost the same, i.e., $b_l \simeq 0$. Hence, using Eq. (14) we get

$$l_\varepsilon = l_\varepsilon^0 + \frac{b_h(N-2)}{N(N-1)}\delta\omega_1. \tag{18}$$

Figure 4 shows l_ε as a function of $\delta\omega_1$ for the star network of 20 nodes of coupled nearly identical Rössler systems. We see that the stability range of synchronization increases almost linearly with $\delta\omega_1$ for fixed N. The origin of the linear dependence is in the linear dependence of ν_N on $\delta\omega_1$ [Eq. (17)] and the linear dependence of $\delta\alpha_h$ on ν_N (see Fig. 1). The linear dependence of $\delta\alpha_h$ on ν_N is specific to the x-coupled Rossler systems with frequency as NDP and may not be true for other systems. The dependence of l_ε on N is more complicated since $l_\varepsilon^0 = |\frac{\alpha_h^0}{\gamma_N} - \frac{\alpha_l^0}{\gamma_2}| = \frac{\alpha_h^0}{N} - \alpha_l^0$. This is further discussed in Sec. III A.

D. MSF for coupled nearly identical oscillators with more than one NDP

In Sec. II A, we have derived the master stability equation for coupled nearly identical dynamical systems with one NDP. In this section we consider coupled nearly identical dynamical systems with more than one NDPs and derive the master stability equation for the same.

The dynamics of ith oscillator can be written as

$$\dot{x}^i = f(x^i, r_{i1}, \ldots, r_{iq}) + \varepsilon \sum_{j=1}^{N} g_{ij} h(x^j); \quad i = 1,\ldots,N, \tag{19}$$

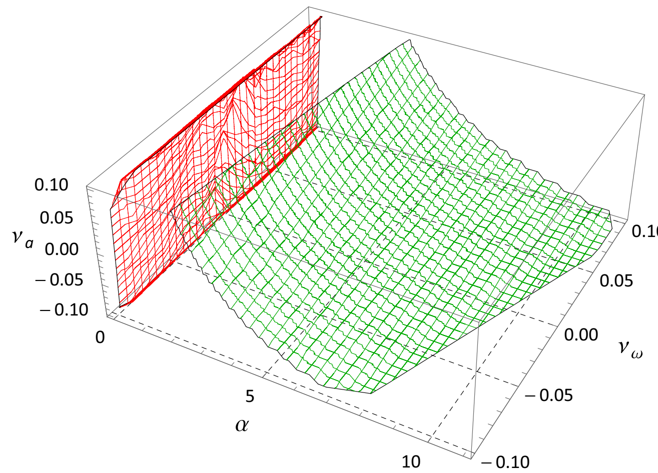

FIG. 5. (Color online) The zero contour surfaces [red (left) and green (right)] of the MSF are shown in the phase space $(\alpha,\nu_\omega,\nu_a)$ for Rössler system. The MSF is negative in the region covered by these surfaces and thus it gives the stable region.

where $r_{i1},\ldots,r_{iq}$ are the q independent NDPs of the ith node. Let the typical values of these NDP's be $\tilde{r} = (\tilde{r}_1,\ldots,\tilde{r}_q)$. The linearized equation is obtained following the procedure of Sec. II A and we get [see Eq. (3)]

$$\dot{z}^i = D_x f z^i + \varepsilon \sum_{j=1}^{N} g_{ij} D_x h z^j + \sum_{k=1}^{q} D_{r_k} D_x f z^i \delta r_{ik}, \tag{20}$$

where $z^i = x^i - \tilde{x}$ and $\delta r_{ik} = r_{ik} - \tilde{r}_k$.

Consider the jth eigenvalue, γ_j, of the coupling matrix G. For γ_j, the mismatch parameter for the NDP r_k using the first-order perturbation correction is $\nu_{jk} = e_j^L R_k e_j^R$ and hence Eq. (6) can be written as

$$\dot{\phi}_j = [D_x f + \varepsilon\gamma_j D_x h]\phi_j + \sum_{k=1}^{q} \nu_{jk} D_{r_k} D_x \phi_j. \tag{21}$$

We can write the master stability equation as before by introducing the effective coupling parameter $\alpha = \varepsilon\gamma_j$ and the mismatch parameters $\nu_k = \nu_{jk}$, $k = 0,\ldots,q$,

$$\dot{\phi} = \left[D_x f + \alpha D_x h + \sum_{k=1}^{q} \nu_k D_{r_k} D_x f \right]\phi. \tag{22}$$

As an example we consider coupled nearly identical Rössler oscillators with ω and a as NDPs. In Fig. 5 the zero contour surfaces of the MSF are plotted in the three-dimensional parameter space $(\alpha,\nu_\omega,\nu_a)$, where ν_ω and ν_a are the mismatch parameters corresponding to the NDPs ω and a, respectively. The MSF is negative in the region covered by these surfaces and thus it gives the stable region. From the figure we can see that the stable region for synchronization increases with mismatch parameter ν_ω while at the same time it decreases with mismatch parameter ν_a.

III. SIZE INSTABILITY

In this section we discuss the effect of NDP on the size instability of a network. As discussed in the Introduction, by size instability one refers to a critical number of oscillators that can be coupled in a well-structured network to obtain synchronization and beyond this critical number no stable synchronization can be seen.

A simple explanation of size instability can be obtained from Eq. (14) which says that for a given network we can find the stable interval of coupling parameter as $l_\varepsilon(N) = |\alpha_h/\gamma_N - \alpha_l/\gamma_2|$. The synchronization is stable when the eigenvalues satisfies the condition, $\alpha_h/\alpha_l > \gamma_N/\gamma_2$, i.e., $l_\varepsilon(N) > 0$. For some networks, the stable interval decreases as we keep on adding nodes and it becomes zero when the network reaches the critical size N_c. Thus, the critical number of oscillators up to which synchronization is possible is given by

$$l_\varepsilon(N_c) = 0. \tag{23}$$

Hence, the critical size N_c can be determined by solving the equation

$$\frac{\gamma_{N_c}}{\gamma_2} = \frac{\alpha_h}{\alpha_l}. \tag{24}$$

From the plot of MSF in Fig. 1, we see that the NDP can play a crucial role in the size instability since the stability range $l_\varepsilon(N)$ changes with the mismatch parameter ν. We have obtained the mismatch parameter using the first-order perturbation correction which requires the eigenvectors of G. Thus, the nature of the eigenvectors of G plays a crucial role in determining the effect of NDP on size instability. We will see that this leads to different effects of NDP on size instability in star and ring networks.

A. Star network

For a star network of N nodes where all the nodes on the peripherals are only connected with a central nodes, the coupling matrix $G_s^{(0)}$ is given by

$$G_s^{(0)} = \begin{pmatrix} 1-N & 1 & 1 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 1 & 0 & 0 & & -1 \end{pmatrix}, \tag{25}$$

and the eigenvalues of the coupling matrix are $\gamma_1 = 0, \gamma_2 = \cdots = \gamma_{N-1} = -1, \gamma_N = -N$. Thus, the critical number of identical oscillators that can be coupled in a star network to achieve synchronization is [Eq. (24)]

$$N_c^0 = \frac{\alpha_h^0}{\alpha_l^0}. \tag{26}$$

From Fig. 1, we get $N_c^0 \sim 32$ for coupled identical Rössler systems [50].

Now we consider the coupled Rössler oscillators with NDP ω. Using Eqs. (26), (13), and (17), and noting that $b_l \simeq 0$ from Fig. 1, we obtain the maximum number of oscillators that can be coupled in a star network for stable synchronization for nearly identical systems as

$$N_c = \frac{\alpha_h}{\alpha_l} = \frac{\alpha_h^0 + b_h \nu_N}{\alpha_l^0} = N_c^0\left[1 + \frac{b_h(N_c-2)}{\alpha_h^0(N_c-1)}\delta\omega_1\right]. \tag{27}$$

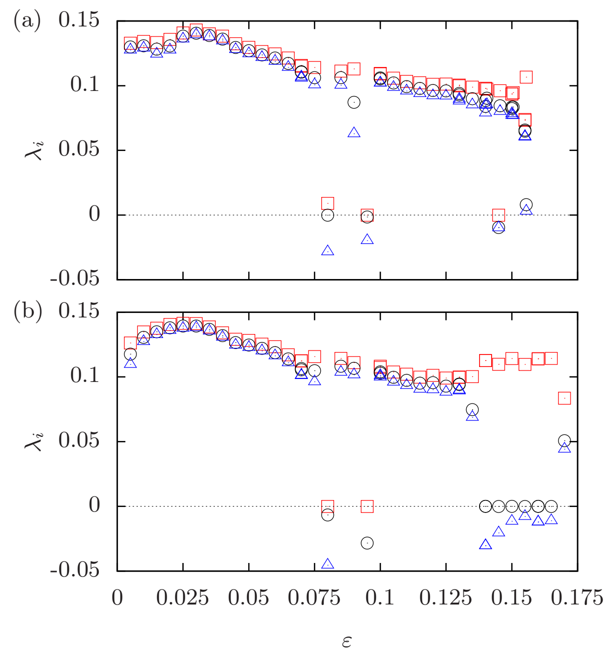

FIG. 6. (Color online) (a) Three largest Lyapunov exponents (red square, black circle, and blue triangle) are shown as a function of ε for 32 coupled identical Rössler oscillators on a star network. The frequency $\omega = 1.0$ for all the oscillators. There are some windows of periodic motion between $\varepsilon = 0.075$ to 0.1. No range of ε showing stable synchronization could be detected. (b) Three largest Lyapunov exponents are shown as a function of ε for 32 coupled nearly identical Rössler oscillators on a star network. The central node has the largest frequency $\omega = 1.05$, and the other nodes have smaller frequencies such that the average frequency of the network is $\bar{\omega} = 1.0$. A definite range of stable synchronization $\varepsilon \in (0.139, 0.169)$ where the third-largest Lyapunov exponent becomes negative can be seen.

Approximating N_c as N_c^0 in the right-hand side, and using $\delta\omega_1 = 0.05$, we get $N_c \sim 35.4$. Thus, the maximum number of oscillators that can synchronize for coupled nearly identical Rössler systems with $\delta\omega_1 = 0.05$ is 35.

We have verified the above result by explicit numerical calculations. Figures 6(a) and 6(b) show the three largest Lyapunov exponents as a function of ε for 32 coupled identical Rössler oscillators and 32 coupled nearly identical Rössler oscillators, respectively, for a star network. For identical systems we do not see any finite range of coupling constant ε showing stable synchronization while for nearly identical systems there is a finite range of ε showing stable synchronization where the third largest Lyapunov exponent becomes negative. This stable range of ε decreases as we increase N. Figures 7(a), 7(b), 7(c), and 7(d) show the three largest Lyapunov exponents of coupled nearly identical Rššler oscillators for $N = 33$, 34, 35, and 36, respectively. Thus, we see that synchronization is possible up to $N = 35$ oscillators and $N = 36$ does not show any stable synchronization.

From Eq. (27) we see that if we choose the frequency of the central node smaller than the average, i.e., $\delta\omega_1 < 0$, then the critical number of oscillators for stable synchronization will decrease. For example, if $\delta\omega_1 = -0.05$, we get $N_c = 28.6$. We have verified this result numerically.

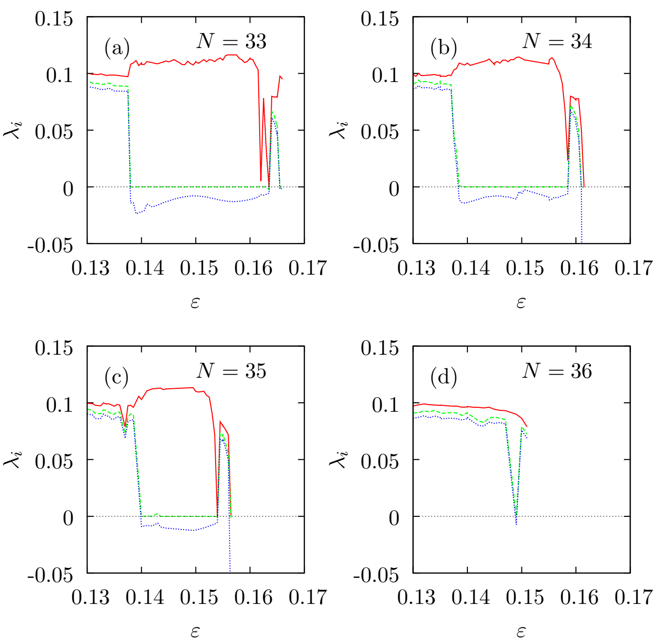

FIG. 7. (Color online) (a) The three largest Lyapunov exponents (red line, green dashed line, and blue dotted line) are shown as a function of ε for 33 coupled nearly identical Rössler oscillators on a star network. A finite range of ε of stable synchronization is seen where the third largest Lyapunov exponent becomes negative. Panels (b), (c), and (d) are similar figures for $N = 34$, 35, and 36. We can see a finite range of ε of stable synchronization for $N = 34$ and 35 but not for $N = 36$. In all the figures the central node has $\omega = 1.05$ and the other frequencies are chosen such that the average $\bar{\omega} = 1.0$.

Zamora-Munt *et al.* [45] have reported a similar observation for a system of delay-coupled lasers on a star network where they find that the critical size N_c increases with the width of the frequency distribution.

B. Ring network

For a ring network, with nearest-neighbor coupling, the coupling matrix is

$$G_r^{(1)} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & & 0 \\ \vdots & \vdots & & \ddots & \\ 1 & 0 & 0 & & -2 \end{pmatrix}. \quad (28)$$

The eigenvalues of $G_r^{(1)}$ are $\gamma_k = -4\sin^2\theta_k$, $\theta_k = \pi k/N$, $k = 1,\ldots,N$. The corresponding eigenvectors are $e_k = \sqrt{\frac{1}{N}}\{1, \exp(i2\theta_k), \ldots, \exp[i2\theta_k(N-1)]\}^T$.

For an eigenvalue γ_k, the mismatch term due to NDP is $\nu_k = e_k^\dagger R e_k = \sum_j \delta r_j = 0$, where we choose the average parameter as the typical value. Thus, the mismatch term is zero for all the eigenvalues γ_k. Hence, for the ring network, the NDP does not have any significant effect on the size instability. We have verified this result numerically.

As noted earlier, the different effect of NDP on the star and ring networks is because of the nature of eigenvectors of G. In the star network this can lead to either increase or decrease

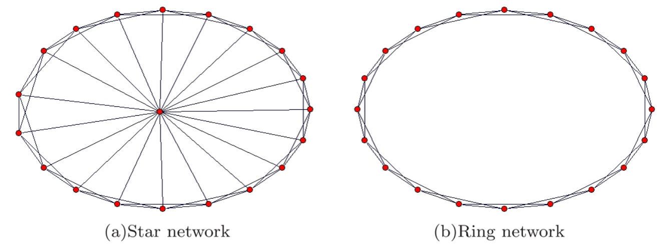

FIG. 8. (Color online) (a) The figure shows a star network with the nodes on the periphery connected to four ($k = 2$) of its nearest neighbors. (b) The figure shows a ring network with the nodes connected to four ($k = 2$) of its nearest neighbors.

of the critical number of nodes for synchronization; while in the ring network it annuls the effect of NDP at least to first order and hence does not have significant effect on the critical number of nodes.

C. Star and ring networks with additional links

We now consider star and ring networks with additional links to $2k$ nearest neighbors. For the star network, the additional links are for the peripheral nodes (see Fig. 8). We find that the critical number of nodes increases considerably with k. The analytical results for the critical number for both identical and nearly identical systems are given in Appendix B. Here we give the numerical results for coupled Rössler oscillators.

We have already seen that in a simple star network of identical chaotic Rössler oscillators, a maximum of $N_c^0 = 32$ oscillators can be coupled to obtain synchronization. Beyond this number the star network loses its synchronizability. This critical number increases with k. In Fig. 9, we show the critical size N_c^0 of the star network (solid red line) for identical chaotic Rössler oscillators as a function of k. We see an almost linear increase with k [see Eq. (B5)].

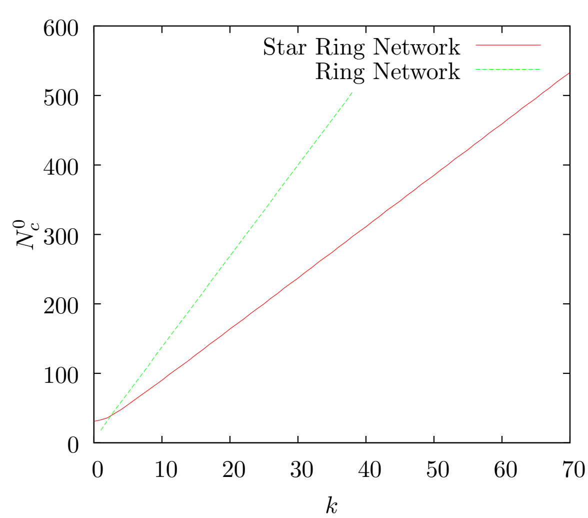

FIG. 9. (Color online) The critical size N_c^0 of the star network (red line) and ring network (green dashed line) with peripheral nodes which are connected to their $2k$ nearest neighbors are shown as a function of k for identical Rössler oscillators.

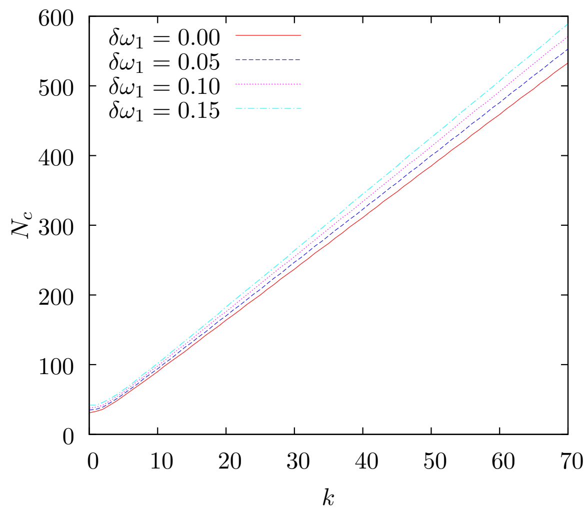

FIG. 10. (Color online) The critical size N_c for star network with peripheral nodes connected to their $2k$ nearest neighbors are shown as a function of k. The solid red line is for identical Rössler oscillators, and the dashed blue, dotted magenta, and dashed-dotted cyan lines are for Rössler oscillators with, respectively, 5%, 10%, and 15% mismatch in frequency parameter.

For nearly identical systems, Eq. (27) for N_c is valid for all k (see Appendix B). In Fig. 10, we show the critical size N_c as a function of k for the star network with peripheral nodes connected to $2k$ nearest neighbors for different ω_1 values for the central node. The solid, dashed, dotted, and dashed dotted lines are for, respectively, $\delta\omega_1 = 0\%$, 5%, 10%, and 15%. From the figure we can observe that the critical size of the star network increases with increasing $\delta\omega_1$ for fixed k and is consistent with Eq. (27).

In a simple ring network with nearest-neighbor couplings of identical chaotic Rössler oscillators, a maximum of $N_c^0 = 17$ oscillators can be coupled to obtain synchronization. Beyond this number the ring network loses its synchronizability. As in the star network the critical number increases with k. Figure 9 plots N_c^0 as a function of k for a ring network of identical oscillators (dashed green line). Linear dependence of N_c^0 for large k can be seen as given by Eq. (B10).

For the ring network with links to $2k$ nearest neighbors of nearly identical oscillators, the NDP does not have any significant effect on the critical size N_c and this result is independent of $2k$, the number of nearest neighbors (see Appendix B). The result is verified numerically.

IV. CONCLUSION

In this paper we have studied the stability of synchronization of coupled nearly identical systems on a network using MSF. We extend the study to the case of degenerate eigenvalues of the coupling matrix G and to more than one NDP. The main result of the paper is about the effect of NDP on size instability. The nature of the eigenfunctions of the coupling matrix G play a crucial role in deciding the effect of NDP on size instability. For coupled nearly identical Rössler systems on a star network, we show that the critical number of nodes beyond which synchronization is not possible can be increased by having a larger frequency for the central node. This result can be of importance for any real system which has a leader-follower structure. For a ring network, the NDP does not have any significant effect on the critical number of nodes.

ACKNOWLEDGMENTS

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APPENDIX A: ROBUSTNESS OF SYNCHRONIZATION

Here we derive the conditions for the synchronization error to remain bounded to second order in parameter mismatch. We follow a procedure similar to that of Sun *et al.* [40] who derive a condition for the synchronization error to remain bounded to first order in parameter mismatch.

In matrix form, Eq. (2) can be written as

$$\dot{Z} = D_x f Z + \tfrac{1}{2} Z^T D_x^2 f Z + D_r f R + \tfrac{1}{2} D_r^2 f R^2 + D_x D_r f Z R + \varepsilon D_x h Z G^T + \varepsilon Z^T D_x^2 h Z G^T, \quad \text{(A1)}$$

where the matrices Z, G, and R are defined in Sec. II A and Z^T is the transpose of Z. Following the method of Sec. II A, we can define the dynamics of an m-dimensional vector $\phi_j = Z e_j^R$ which are transverse to the synchronization manifold as

$$\dot{\phi}_j = [D_x f + \varepsilon\gamma_j D_x h]\phi_j + \tfrac{1}{2}\big[Z^T D_x^2 f + \varepsilon\gamma_j Z^T D_x h\big]\phi_j + D_r f R e_j^R + \tfrac{1}{2} D_r^2 R^2 e_j^R + D_x D_r f Z R e_j^R, \quad \text{(A2)}$$

where $j = 2,\dots,N$.

Let us introduce three parameters as $\zeta_j = R e_j^R$, $2\eta_j = R^2 e_j^R$, and $\theta_j = Z R e_j^R$. Equation (A2) can be written in the generic form

$$\dot{\phi} = [D_x f + \alpha D_x h]\phi + \tfrac{1}{2}\big[Z^T D_x^2 f + \alpha Z^T D_x h\big]\phi + D_r f \zeta + D_r^2 f \eta + D_x D_r f \theta. \quad \text{(A3)}$$

We note that the inhomogeneity in Eq. (A3) is due the parameter mismatch of the coupled systems. Let us consider that without any parameter mismatch the coupled systems undergo stable synchronization as a function of coupling parameter ε, i.e., in this case the solution of the homogeneous part of Eq. (A3) can be written as

$$\phi^{h*}(t) = \mathcal{M}(t,0)\phi^{h*}(0); \quad j = 2,\dots,N, \quad \text{(A4)}$$

where $\mathcal{M}(t,\tau)$ is the transition matrix [51] and

$$\|\mathcal{M}(t,\tau)\| \leqslant c e^{-\kappa(t-\tau)}, \quad \text{(A5)}$$

where c and κ are finite positive constants and $t \geqslant \tau$.

The full solution of Eq. (A3) can then be written as [52]

$$\phi^{f*}(t) = \phi^{h*}(t) + \int_0^t \mathcal{M}(t,\tau) b_1(\tau) d\tau + \int_0^t \mathcal{M}(t,\tau) b_2(\tau) d\tau + \int_0^t \mathcal{M}(t,\tau) b_3(\tau) d\tau, \quad \text{(A6)}$$

where $b_1(\tau) = D_r f[\tilde{x}(\tau),\tilde{r}]\zeta$, $b_2(\tau) = D_r^2 f[\tilde{x}(\tau),\tilde{r}]\eta$, and $b_3(\tau) = D_x D_r f[\tilde{x}(\tau),\tilde{r}]\theta$.

Under the condition of Eq. (A5), we can show that the solution $\phi^{f*}(t)$ of Eq. (A6) is bounded by the following inequality:

$$\begin{aligned}\|\phi^{f*}(t)\| &\leqslant \|\phi^{h*}(t)\| + \int_0^t \|\mathcal{M}(t,\tau)\|d\tau.\sup_t |b_1(t)| \\ &\quad + \int_0^t \|\mathcal{M}(t,\tau)\|d\tau \sup_t |b_2(t)| \\ &\quad + \int_0^t \|\mathcal{M}(t,\tau)\|d\tau \sup_t |b_3(t)|, \qquad \text{(A7)}\\ &\leqslant ce^{-\kappa t}\|\phi^{h*}(0)\| + \frac{c}{\kappa}(1-e^{-\kappa t})\sup_t \|b_1(t)\| \\ &\quad + \frac{c}{\kappa}(1-e^{-\kappa t})\sup_t \|b_2(t)\| \\ &\quad + \frac{c}{\kappa}(1-e^{-\kappa t})\sup_t \|b_3(t)\|, \qquad \text{(A8)}\\ &\xrightarrow{t\to\infty} \frac{c}{\kappa}\sup_t \|b_1(t)\| + \frac{c}{\kappa}\sup_t \|b_2(t)\| \\ &\quad + \frac{c}{\kappa}\sup_t \|b_3(t)\|. \qquad \text{(A9)}\end{aligned}$$

Thus, the full solution of the Eq. (A3) is bounded if (i) the homogeneous part is exponentially stable and (ii) the inhomogeneous parts $b_1(\tau) = D_r f[\tilde{x}(\tau),\tilde{r}]\zeta, b_2(\tau) = D_r^2 f[\tilde{x}(\tau),\tilde{r}]\eta$, and $b_3(\tau) = D_x D_r f[\tilde{x}(\tau),\tilde{r}]\theta$ are bounded.

APPENDIX B: SYNCHRONIZATION OF LARGE NUMBER OF OSCILLATORS IN STAR AND RING NETWORKS

In this Appendix, we consider star and ring networks with additional links to $2k$ nearest neighbors and obtain analytical results for the critical size of the network.

1. Star network

For the star network with links to $2k$ nearest neighbors of the peripheral nodes, the coupling matrix $G_s^{(k)}$ has the form

$$\begin{aligned}&[G_s^{(k)}]_{11} = -(N-1) \quad [G_s^{(k)}]_{1i} = (G_s^{(k)})_{i1} = 1\\ &[G_s^{(k)}]_{ii} = -2k-1\\ &[G_s^{(k)}]_{ij} = 1, \quad j = i\pm 1,\ldots,i\pm k\\ &[G_s^{(k)}]_{ij} = 0, \text{ otherwise,}\end{aligned} \qquad \text{(B1)}$$

where $i = 2,\ldots,N$ and j is periodic with $2 \leqslant j \leqslant N$.

The eigenvalues of the coupling matrix of Eq. (B1) are

$$\begin{aligned}&\gamma_1 = 0 \quad \gamma_N = -N\\ &\gamma_j = -2k-1+2\sum_{l=1}^{k}\cos\left[\frac{2\pi(j-1)l}{N-1}\right];\ j = 2,\ldots,N-1\end{aligned} \qquad \text{(B2)}$$

and the corresponding eigenvectors are

$$\begin{aligned}&e_1 = (1,\ldots,1)^T\\ &e_N = \sqrt{\frac{1}{N(N-1)}}(N-1,-1,\ldots,-1)^T \qquad \text{(B3)}\\ &e_j = [0,1,\omega_j,\omega_j^2,\ldots,\omega_j^{(N-2)}]^T,\end{aligned}$$

where $j = 2,\ldots,N-1$ and $\omega_j = \exp(i\frac{2\pi j}{N-1})$ are the $N-1$th roots of unity. For large k,N and $k/N \ll 1$, by series expansion, we can get

$$\gamma_2 \approx -1 - \frac{2\pi^2 k(k+1)(2k+1)}{3(N-1)^2}. \qquad \text{(B4)}$$

From Eqs. (26), (B4), and (B2), the critical size of the star network with its peripheral nodes connected to $2k$ nearest neighbors, for large k, N, and $k/N \ll 1$, can be written as

$$N_c^0 \approx k\sqrt[3]{\frac{4\pi^2}{3}\frac{\alpha_h^0}{\alpha_l^0}}. \qquad \text{(B5)}$$

Let us now consider nearly identical systems on a star network with peripheral nodes connected to $2k$ nearest neighbors. As argued in Sec. III A, the eigenvalue γ_2 does not lead to any change in N_c since $b_l = 0$. The eigenvector corresponding to γ_N [Eq. (B3)] is the same as in Sec. III A. Hence, Eq. (27) for N_c is also valid for nearly identical systems.

2. Ring network

For the ring network with links to $2k$ nearest neighbors, the coupling matrix $G_r^{(k)}$ has the form

$$\begin{aligned}&[G_r^{(k)}]_{ii} = -2k\\ &[G_r^{(k)}]_{ij} = 1, \quad j = i\pm 1,\ldots,i\pm k, \bmod(N)\\ &[G_r^{(k)}]_{ij} = 0, \text{ otherwise,}\end{aligned} \qquad \text{(B6)}$$

where $i = 1,\ldots,N$ and j is periodic in N. The eigenvalues of this coupling matrix $G_s^{(k)}$ are given by

$$\gamma_j = -2k + 2\sum_{l=1}^{k}\cos\left[\frac{2\pi(j-1)l}{N}\right], \qquad \text{(B7)}$$

where $j = 1,\ldots,N$. The corresponding eigenvectors are

$$e_j = [1,\omega_j,\omega_j^2,\ldots,\omega_j^{(N-1)}]^T \qquad \text{(B8)}$$

where $\omega_j = \exp(i\frac{2\pi j}{N})$ is the Nth root of unity.

For large k, N, and $k/N \ll 1$, by series expansion, we get

$$\begin{aligned}\gamma_2 &\approx -\frac{2\pi^2 k(k+1)(2k+1)}{3N^2}\\ \gamma_N &\approx -(2k+1)(1+2/3\pi).\end{aligned} \qquad \text{(B9)}$$

From Eqs. (26) and (B9), the critical size of the ring network with $2k$ nearest neighbors and identical oscillators for large k, N, and $k/N \ll 1$ is given by

$$N_c^0 \approx k\sqrt{\frac{2\pi^2}{3(1+2/3\pi)}\frac{\alpha_h^0}{\alpha_l^0}}. \qquad \text{(B10)}$$

We now consider the ring network with links to $2k$ nearest neighbors of nearly identical oscillators. We note that the eigenvectors (B8) are the same as for $k = 1$ considered in Sec. III B. Hence, $\nu_j = e_j^\dagger R e_j = \sum_l \delta r_l = 0$. Hence, for the ring network, the NDP does not have any significant effect on the critical size N_c and this result is independent of $2k$, the number of nearest neighbors.

[1] R. Albert and A. L. Barabäsi, Rev. Mod. Phys. **74**, 47 (2002).
[2] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, Oxford, 2003).
[3] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[4] S. Boccalettia, V. Latorab, Y. Morenod, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[5] P. Erdös and A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
[6] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[7] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[8] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001).
[9] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, Phys. Rep. **366**, 1 (2002).
[10] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep. **469**, 93 (2008).
[11] D. V. Ramana Reddy, A. Sen, and G. L. Johnston, Phys. Rev. Lett. **80**, 5109 (1998).
[12] S. Kim, S. H. Park, and C. S. Ryu, Phys. Rev. Lett. **79**, 2911 (1997).
[13] D. M. Abrams and S. H. Strogatz, Phys. Rev. Lett. **93**, 174102 (2004).
[14] A. Prasad, J. Kurths, S. K. Dana, and R. Ramaswamy, Phys. Rev. E **74**, 035204(R) (2006).
[15] H. Fujisaka and T. Yamada, Progr. Theor. Phys. **69**, 32 (1983).
[16] T. Yamada and H. Fujisaka, Progr. Theor. Phys. **70**, 1240 (1983).
[17] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, Radiophys. Quant. Electron. **29**, 795 (1986).
[18] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[19] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994).
[20] L. M. Pecora, T. L. Carroll, G. A. Johnson, D. J. Mar, and J. F. Heagy, Chaos **7**, 520 (1997).
[21] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
[22] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).
[23] H. D. I. Abarbanel, N. F. Rulkov, and M. M. Sushchik, Phys. Rev. E **53**, 4528 (1996).
[24] E. Rosa, E. Ott, and M. H. Hess, Phys. Rev. Lett. **80**, 1642 (1998).
[25] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).
[26] G. Ambika and R. E. Amritkar, Phys. Rev. E **79**, 056206 (2009).
[27] M. A. Zaks, E.-H. Park, M. G. Rosenblum, and J. Kurths, Phys. Rev. Lett. **82**, 4228 (1999).
[28] A. Hampton and D. H. Zanette, Phys. Rev. Lett. **83**, 2179 (1999).
[29] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[30] K. S. Fink, G. Johnson, T. Carroll, D. Mar, and L. Pecora, Phys. Rev. E **61**, 5080 (2000).
[31] M. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[32] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).
[33] M. Zhao, T. Zhou, B.-H. Wang, and W.-X. Wang, Phys. Rev. E **72**, 057102 (2005).
[34] C. Zhou and J. Kurths, Phys. Rev. Lett. **96**, 164102 (2006).
[35] L. Huang, Q. Chen, Y.-C. Lai, and L. M. Pecora, Phys. Rev. E **80**, 036204 (2009).
[36] J.-F. Zhu, M. Zhao, W. Yu, C. Zhou, and B.-H. Wang, Phys. Rev. E **81**, 026201 (2010).
[37] L. Donetti, P. I. Hurtado, and M. A. Muñoz, Phys. Rev. Lett. **95**, 188701 (2005).
[38] P. Ashwin, J. Buescu, and I. Stewart, Phys. Lett. A **193**, 126 (1994).
[39] J. G. Restrepo, E. Ott, and B. R. Hunt, Phys. Rev. Lett. **93**, 114101 (2004).
[40] J. Sun, E. M. Bollt, and T. Nishikawa, Europhys. Lett. **85**, 60011 (2009).
[41] S. Acharyya and R. E. Amritkar, Europhys. Lett. **99**, 40005 (2012).
[42] T. Bohr, Phys. Rev. Lett. **63**, 2161 (1989).
[43] M. A. Matías and J. Güémez, Phys. Rev. Lett. **81**, 4124 (1998).
[44] D. V. Senthilkumar, P. Muruganandam, M. Lakshmanan, and J. Kurths, Phys. Rev. E **81**, 066219 (2010).
[45] J. Zamora-Munt, C. Masoller, J. Garcia-Ojalvo, and R. Roy, Phys. Rev. Lett. **105**, 264101 (2010).
[46] The choice of typical parameter value as the average value, i.e., $\tilde{r} = \bar{r} = 1/N \sum_i r^i$, gives a good approximation to the Lyapunov exponents (see Fig. 1 of Ref. [41]) though some other choices of the typical value also give reasonable results.
[47] F. Sorrentino and M. Porfiri, Europhys. Lett. **93**, 50002 (2011).
[48] Note that in Ref. [41] the parameter ν_r was called Δ.
[49] The contours for other values of MSF are shown in Fig. 4 of Ref. [41].
[50] The size instability for star network of coupled identical systems was noted by Pecora and Carroll [29].
[51] W. J. Rugh, *Linear Systems Theory* (Prentice Hall, Upper Saddle River, NJ, 1992).
[52] L. Perko, *Differential Equations and Dynamical Systems* (Springer-Verlag, New York, 1996).