Fractional quantum mechanics on networks: Long-range dynamics and quantum transport

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(Received 2 July 2015; revised manuscript received 17 September 2015; published 30 November 2015)

In this paper we study the quantum transport on networks with a temporal evolution governed by the fractional Schrödinger equation. We generalize the dynamics based on continuous-time quantum walks, with transitions to nearest neighbors on the network, to the fractional case that allows long-range displacements. By using the fractional Laplacian matrix of a network, we establish a formalism that combines a long-range dynamics with the quantum superposition of states; this general approach applies to any type of connected undirected networks, including regular, random, and complex networks, and can be implemented from the spectral properties of the Laplacian matrix. We study the fractional dynamics and its capacity to explore the network by means of the transition probability, the average probability of return, and global quantities that characterize the efficiency of this quantum process. As a particular case, we explore analytically these quantities for circulant networks such as rings, interacting cycles, and complete graphs.

DOI: 10.1103/PhysRevE.92.052814 PACS number(s): 89.75.Hc, 05.60.Gg, 05.40.Fb, 71.35.−y

I. INTRODUCTION

Dynamical processes that occur in networks have attracted the attention of researchers in diverse fields [1,2], particularly in the context of quantum transport [3,4]. In analogy with the classical random walk associated with diffusive processes on networks, continuous-time quantum walks (CTQW) are introduced in Ref. [5], a dynamical process that combines the transitions of the classical random walk with the quantum superposition of states [3,4,6–9]. CTQW have been applied to the study of transport processes on networks in diverse systems [4,10–14]. Commonly, the CTQW are defined as a local process with transitions from a node to one of its nearest neighbors in the network. CTQW and their efficiencies have been explored in diverse types of networks. In particular, the reviews [3,4,6] provide a general overview of CTQW on different structures.

On the other hand, fractional quantum mechanics has been studied in the continuum. This spatially nonlocal theory combines Lévy flights [15–18], a dynamics with random displacements of length l that asymptotically obey a power-law probability distribution of the form $P(l) \sim l^{-\beta}$, with quantum superposition of states [19–21]. Fractional quantum mechanics is a natural extension of the standard quantum mechanics when the Brownian trajectories in Feynman path integrals are replaced by Lévy flights. The theoretical consequences, possible implementations, and applications of this formalism developed using a fractional calculus approach [17,22–24] have been explored actively in past years [25–30].

In relation with the transport on networks, nonlocal processes of Lévy type are less common; in the context of diffusive transport, Lévy flights have been defined in general networks [31] and studied [32,33]; additionally, there is a connection between this dynamics and the fractional diffusion on networks [34,35]. In the quantum scenario, Lévy flights on a finite ring are analyzed in Ref. [36]; this type of long-range transport is motivated by recent findings in light-harvesting complexes and molecular aggregates [37–39], Rydberg atoms [40–43], and other applications for which the dynamics is nonlocal [44–47].

The aim of this paper is to introduce the fractional quantum mechanics on networks as a quantum counterpart of the fractional diffusion on networks recently introduced in Ref. [34]. We extend the existent formalism for CTQW, based on quantum mechanics, to a dynamical process that evolves with the fractional Schrödinger equation. In the first part we present a brief introduction to diverse quantities and definitions related with continuous-time quantum walks on networks and fractional quantum mechanics in continuous spaces; in particular, the fractional Schrödinger equation. Then, by using the equivalent on networks of the fractional Laplacian operator introduced within the context of fractional diffusion [34,35], we define the fractional quantum transport on networks that combines long-range displacements similar to Lévy flights with quantum superposition of states. We analyze the consequences of this dynamics and its capacity to explore networks; we calculate the transition probability, the average probability of return, and global quantities that describe the transport. The formalism introduced is general and applies to any type of connected undirected networks, including regular, random, and complex networks, and can be implemented from the spectral properties of the Laplacian matrix, providing an important tool to analyze the fractional quantum transport in these structures. In addition to the general theory, we study in detail the fractional quantum transport in some circulant networks, in particular, an infinite ring, where we found exact analytical results. Finally, from the analysis of the efficiency, we find that fractional quantum strategies define universal classes for the transport on this particular type of regular networks.

II. CONTINUOUS-TIME QUANTUM WALKS

We consider undirected connected networks with N nodes $i = 1, \dots, N$ described by the adjacency matrix $\mathbf{A}$ with elements $A_{ij} = A_{ji} = 1$ if there is a link between the nodes i, j and $A_{ij} = A_{ji} = 0$ otherwise; also $A_{ii} = 0$ to avoid loops in the network. The degree of the node i is given by $k_i = \sum_{l=1}^{N} A_{il}$. From these concepts the Laplacian matrix $\mathbf{L}$ is defined with elements $L_{ij} = \delta_{ij} k_i - A_{ij}$, where δ_{ij} denotes the Kronecker delta; $\mathbf{L}$ is interpreted as a discrete version of the Laplacian operator $(-\nabla^2)$ [1]. Continuous-time quantum walks on networks are defined in terms of the Schrödinger equation (for convenience we use $\hbar = 1$) [6]:

$$\mathbf{H}|\Psi(t)\rangle = \mathrm{i}\frac{d}{dt}|\Psi(t)\rangle, \tag{1}$$

where $\mathrm{i} \equiv \sqrt{-1}$. The Hamiltonian is given by $\mathbf{H} = \mathbf{L}$ in order to establish the equivalent in networks of the Schrödinger equation $-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{x},t) = \mathrm{i}\hbar\frac{d}{dt}\Psi(\mathbf{x},t)$. The vector $|\Psi(t)\rangle$ represents the state of the system at time t and it is expressed in terms of the orthonormal canonical base $\{|j\rangle\}_{j=1}^N$ of $\mathbb{R}^N$ associated to each node of the network [6]. Therefore, the state of the system is $|\Psi(t)\rangle = \sum_{l=1}^N c_l(t)|l\rangle$, where the coefficients $c_l(t)$ are complex values determined by the initial state $|\Psi(0)\rangle$ and the temporal evolution governed by Eq. (1). The transition probability to pass from the state $|i\rangle$ at time $t = 0$ to the state $|j\rangle$ at time t is given by $\pi_{ij}(t) = |\langle j|e^{-\mathrm{i}\mathbf{H}t}|i\rangle|^2$ [6]. Also, the average probability of return to the initial node is defined as [6]

$$\bar{\pi}_0(t) \equiv \frac{1}{N}\sum_{l=1}^{N}\pi_{ll}(t)\,. \tag{2}$$

In order to compare the dynamics on different types of structures, the long-time average,

$$\chi_{ij} \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \pi_{ij}(t)dt, \tag{3}$$

is introduced and a similar definition in terms of the average probability of return is given by

$$\bar{\chi} \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \bar{\pi}_0(t)dt\,. \tag{4}$$

The quantity $\bar{\pi}_0(t)$ is the equivalent in quantum mechanics of the average probability of return $p_0(t)$ introduced in the context of diffusive transport on networks [6,34]. The quantum transport based on the local dynamics governed by Eq. (1) has been addressed by diverse authors for different types of structures [3,4,6,8]. In the following, we refer to the local quantum dynamics that evolves, with Eq. (1) as the normal quantum walk or normal quantum transport, in order to differentiate it from the fractional dynamics introduced in the following sections.

III. FRACTIONAL SCHRÖDINGER EQUATION

In this section we present a brief review of the fractional quantum mechanics. In quantum mechanics, the dynamics of a particle is described by the Schrödinger equation that defines a local dynamics associated with the Laplacian operator ∇^2. Recently a generalization that incorporates the fractional Laplacian operator has been studied, combining a dynamics based on Lévy flights with the quantum superposition of states [19–21]. In fractional quantum mechanics, the Hamiltonian for one particle is [21]

$$H_\gamma = \mathcal{D}_\gamma|\mathbf{p}|^{2\gamma} + V(\mathbf{r},t)\,, \tag{5}$$

with $0 < \gamma \leqslant 1$, and $\mathcal{D}_\gamma$ is a constant. The value $\gamma = 1$ recovers the typical Hamiltonian that describes a single particle. Now, using the Fourier transform $\varphi(\mathbf{p},t)$ of the wave function $\psi(\mathbf{r},t)$ that satisfies

$$\varphi(\mathbf{p},t) = \int e^{-\mathrm{i}\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}}\psi(\mathbf{r},t)d\mathbf{r} \tag{6}$$

and its inverse

$$\psi(\mathbf{r},t) = \frac{1}{(2\pi\hbar)^3}\int e^{\mathrm{i}\frac{p\cdot r}{\hbar}}\varphi(\mathbf{p},t)d\mathbf{p}, \tag{7}$$

the fractional Schrödinger equation is obtained [21],

$$\mathrm{i}\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = D_\gamma(-\hbar^2\nabla^2)^\gamma\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t), \tag{8}$$

where the operator $(-\hbar^2\nabla^2)^\gamma$ satisfies

$$(-\hbar^2\nabla^2)^\gamma\psi(\mathbf{r},t) = \frac{1}{(2\pi\hbar)^3}\int e^{\mathrm{i}\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}}(|\mathbf{p}|^2)^\gamma\varphi(\mathbf{p},t)d\mathbf{p}\,.$$

Diverse studies have addressed the consequences and the mathematical structure behind the election of the Hamiltonian in Eq. (5) and the fractional Schrödinger equation (8). In the literature there are well-known solutions for the fractional case with different types of potentials; the path integral formalism of the theory based on Lévy flights has also been discussed, among other diverse studies [19–21,26–28].

IV. FRACTIONAL QUANTUM WALKS

Once we have introduced the concepts related with quantum dynamics on networks and the fractional quantum mechanics in continuous spaces, in this section we combine these two theories to define a dynamical process with a temporal evolution determined by the fractional Schrödinger equation in a network. We use the fractional Laplacian matrix $\mathbf{L}^\gamma$ with $0 < \gamma \leqslant 1$ as an equivalent of the operator $(-\nabla^2)^\gamma$ in networks. In our previous work [34,35], we explored the consequences of this election in the context of diffusive transport on different types of networks, finding a random walk with long-range displacements associated to the dynamics on the network.

Given that $\mathbf{L}$ is a symmetric matrix, by means of the Gram-Schmidt orthonormalization of the eigenvectors of $\mathbf{L}$, a set of eigenvectors $\{|\Psi_j\rangle\}_{j=1}^N$ is obtained that satisfies the eigenvalue equation,

$$\mathbf{L}|\Psi_j\rangle = \mu_j|\Psi_j\rangle, \tag{9}$$

for $j = 1,\ldots,N$ and $\langle\Psi_i|\Psi_j\rangle = \delta_{ij}$. Also, for connected networks, the Laplacian spectrum is real and satisfies $\mu_1 = 0$ and $0 < \mu_m$ for $m = 2,\ldots,N$ [48]. We define the orthogonal matrix $\mathbf{Q}$ with elements $Q_{ij} = \langle i|\Psi_j\rangle$ and the diagonal matrix $\Lambda = \mathrm{diag}(0,\mu_2,\ldots,\mu_N)$. These matrices satisfy $\mathbf{L}\,\mathbf{Q} = \mathbf{Q}\,\Lambda$; therefore $\mathbf{L} = \mathbf{Q}\Lambda\mathbf{Q}^\dagger$, where $\mathbf{Q}^\dagger$ denotes the Hermitian transpose of $\mathbf{Q}$. Now, in terms of these matrices, we have [34]

$$\mathbf{L}^\gamma = \mathbf{Q}\Lambda^\gamma\mathbf{Q}^\dagger = \sum_{m=2}^{N}\mu_m^\gamma|\Psi_m\rangle\langle\Psi_m|, \tag{10}$$

where $\Lambda^\gamma = \mathrm{diag}(0,\mu_2^\gamma,\ldots,\mu_N^\gamma)$. In this way, Eq. (10) gives the spectral form of the fractional Laplacian matrix. In Refs. [34,35] it is shown that the fractional Laplacian matrix $\mathbf{L}^\gamma$ can be used to define a long-range dynamics on networks of any kind, not only involving the shortest paths, but also trajectories of different lengths in the network. Also, in some cases a connection between this matrix and Lévy flights has been probed in the context of diffusive transport [31,34,35].

Having defined the fractional Laplacian matrix, in analogy with Eq. (8), we introduce the fractional Schrödinger equation for a network,

$$\mathbf{H}_\gamma|\Psi(t)\rangle = \mathrm{i}\frac{d}{dt}|\Psi(t)\rangle\,, \tag{11}$$

where the Hamiltonian is given by $\mathbf{H}_\gamma = \mathbf{L}^\gamma$ with $0 < \gamma \leqslant 1$. For convenience we take $D_\gamma = 1$, $\hbar = 1$; this choice results in a redefinition of the temporal scales of the problem without loss of generality in the mathematical treatment.

In the following part we generalize all the treatment introduced for the normal transport on networks to the fractional case in order to investigate the consequences of the dynamics determined by Eq. (11). By using the eigenvectors and eigenvalues of the Laplacian matrix $\mathbf{L}$, we obtain the probability amplitude $\alpha_{ij}^{(\gamma)}(t)$ to start in the state $|i\rangle$ at time $t = 0$ and measure the state $|j\rangle$ at time t:

$$\alpha_{ij}^{(\gamma)}(t) = \langle j|e^{-\mathrm{i}\mathbf{H}_\gamma t}|i\rangle = \sum_{l=1}^{N} e^{-\mathrm{i}\mu_l^\gamma t}\langle j|\Psi_l\rangle\langle\Psi_l|i\rangle. \tag{12}$$

The corresponding transition probability is given by the square of the norm of the amplitude; in this way $\pi_{ij}^{(\gamma)}(t) = |\alpha_{ij}^{(\gamma)}(t)|^2$. Therefore,

$$\pi_{ij}^{(\gamma)}(t) = \left|\sum_{l=1}^{N} e^{-\mathrm{i}\mu_l^\gamma t}\langle j|\Psi_l\rangle\langle\Psi_l|i\rangle\right|^2. \tag{13}$$

Also, from Eq. (13) we deduce the average probability of return $\bar{\pi}_0^{(\gamma)}(t) \equiv \frac{1}{N}\sum_j \pi_{jj}^{(\gamma)}(t)$ that takes the form

$$\bar{\pi}_0^{(\gamma)}(t) = \frac{1}{N}\sum_{j,l,m=1}^{N} e^{-\mathrm{i}(\mu_l^\gamma - \mu_m^\gamma)t}|\langle j|\Psi_l\rangle|^2|\langle j|\Psi_m\rangle|^2. \tag{14}$$

The fractional dynamics introduced in Eq. (11) and the results in Eqs. (12)–(14) are general and can be applied to any connected undirected network, including random and complex networks [49,50]. In the next section we analyze in detail a particular type of regular networks.

V. FRACTIONAL DYNAMICS ON CIRCULANT NETWORKS

In this section we study in detail some characteristics of the fractional quantum dynamics on a particular type of regular networks (networks with a constant degree k, i.e., $k_i = k$ for $i = 1,2,\dots N$). We are interested in this kind of networks, because for regular networks we can obtain exact analytic results using our general fractional formalism, as shown in detail in Ref. [35]. These exact results allow us to interpret and understand more clearly the dynamics and the different measures associated to the fractional quantum dynamics, as we will show in the following part of this paper.

In regular networks with constant degree k, the series expansion of $\mathbf{L}^\gamma$ is given by [35]

$$\mathbf{L}^\gamma = (k\mathbb{I} - \mathbf{A})^\gamma = \sum_{m=0}^{\infty}\binom{\gamma}{m}(-1)^m k^{\gamma-m}\mathbf{A}^m. \tag{15}$$

In this equation, $\mathbb{I}$ denotes the $N \times N$ identity matrix, $\binom{x}{y} \equiv \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$, where $\Gamma(x)$ denotes the Γ function [51]. Therefore, the elements of $\mathbf{L}^\gamma$ have the form

$$(\mathbf{L}^\gamma)_{ij} = \sum_{m=0}^{\infty}\binom{\gamma}{m}(-1)^m k^{\gamma-m}(\mathbf{A}^m)_{ij}. \tag{16}$$

The expression (16) relates the fractional Laplacian matrix with the integer powers of the adjacency matrix $\mathbf{A}^m$ ($m = 1,2,\dots$), for which the element $(\mathbf{A}^m)_{ij}$ is the number of all the possible paths connecting the nodes i, j with m links [1,52]. Therefore, in Eq. (16) it is shown explicitly how in regular networks the fractional Laplacian matrix $\mathbf{L}^\gamma$, that defines the Hamiltonian in Eq. (11), contains information about all the possible paths connecting all the nodes of the network. In this ways a long-range dynamics emerge, revealing what lies inside the fractional formalism.

In what follows, we analyze circulant networks that are a special type of regular networks for which the corresponding adjacency matrix, and thus the Laplacian matrix $\mathbf{L}$, are circulant matrices [48,53]. The eigenvalues and eigenvectors of circulant matrices are well known [48,53] and, in this case, the Laplacian matrix $\mathbf{L}$ has eigenvectors $\{|\Psi_m\rangle\}_{m=1}^N$ with components given by $\langle j|\Psi_l\rangle = \frac{1}{\sqrt{N}}\xi^{(j-1)(l-1)}$ with $\xi = e^{-\frac{2\pi}{N}\mathrm{i}}$ [48]. Therefore, in circulant networks, we can rewrite Eq. (13) as

$$\begin{aligned}\pi_{ij}^{(\gamma)}(t) &= \frac{1}{N^2}\sum_{l,m=1}^{N} e^{-\mathrm{i}(\mu_l^\gamma - \mu_m^\gamma)t}\xi^{(l-m)(j-i)}\\ &= \frac{1}{N^2}\sum_{l,m=1}^{N}\cos\left[(\mu_l^\gamma - \mu_m^\gamma)t + \frac{2\pi(j-i)(l-m)}{N}\right].\end{aligned} \tag{17}$$

In a similar way, the average probability of return in Eq. (14) for circulant networks takes the form

$$\bar{\pi}_0^{(\gamma)}(t) = \frac{1}{N^2}\sum_{l,m=1}^{N}\cos\left[(\mu_l^\gamma - \mu_m^\gamma)t\right]. \tag{18}$$

In Eqs. (17) and (18), the values $\{\mu_m\}_{m=1}^N$ are the eigenvalues of the Laplacian matrix $\mathbf{L}$ of the corresponding circulant graph. A common set of circulant graphs are the interacting cycles, for which initially N nodes form a ring (one-dimensional lattice with periodic boundary conditions), then each node is connected to its left J and right J nearest neighbors, and $2J$ is the degree of the resulting structure [35]. The value J is the interaction parameter, and in this structure any two nodes whose distance in the initial ring is smaller than or equal to J are connected by additional bonds (see Fig. 1). The Laplacian matrix of interacting cycles is circulant, its unordered Laplacian spectrum is $\mu_1 = 0$, and [48]

$$\mu_m = 2J + 1 - \frac{\sin\left[\frac{\pi}{N}(m-1)(2J+1)\right]}{\sin\left[\frac{\pi}{N}(m-1)\right]} \tag{19}$$

for $m = 2,\dots,N$. From the particular value $J = 1$, we obtain the Laplacian spectra of a ring with N nodes.

In Fig. 2 we depict the results obtained from Eq. (17) for interacting cycles with $N = 21$ nodes. We plot the probability to start in $i = 11$ and reach node j at time t. The results show the oscillatory behavior described by Eq. (17) related to the

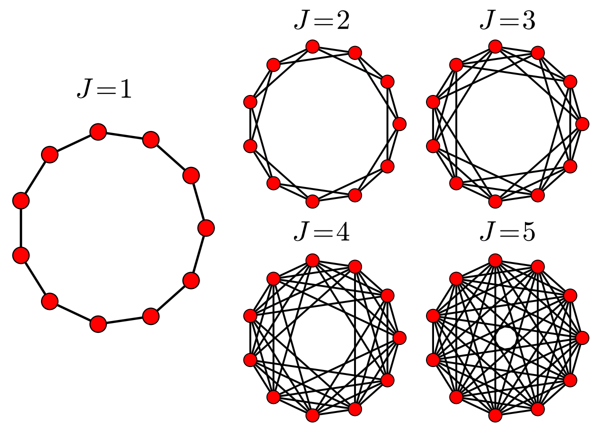

FIG. 1. (Color online) Interacting cycles with $N = 11$ nodes. For the value $J = 1$ we obtain the initial ring; in this network we add links in order to connect each node to its J left and J right nearest nodes. In this case $J = 5$ defines a complete graph.

temporal evolution and the cyclic structure of the network. It is observed how, after some time, the oscillations interfere constructively and it is highly probable to measure the initial state. Also, contrary to intuition and the result observed for the fractional diffusion on networks [34,35], the introduction of the fractional Laplacian in the quantum dynamics slows the exploration of the network. In order to study this result, in Fig. 3 we depict the average probability of return $\bar{\pi}_0^{(\gamma)}(t)$ given by Eq. (18) for different circulant networks. We analyze a ring (one-dimensional lattice with periodic boundary conditions), interacting cycles, and a two-dimensional (2D) square lattice. In Fig. 3(a), the results reveal how the different types of fractional quantum transport decay as a power law. A similar behavior for a ring is reported for the long-range quantum walk studied in Ref. [36].

VI. QUANTUM TRANSPORT ON AN INFINITE RING

In this section we study the fractional quantum mechanics for the transport on a ring in the limit $N \to \infty$. For a ring with N nodes, the spectrum of the Laplacian matrix is [48]

$$\mu_l = 2 - 2\cos\left[\frac{2\pi}{N}(l-1)\right], \qquad (20)$$

and for this circulant matrix $\langle m|\Psi_l\rangle = \xi^{(l-1)(m-1)}/\sqrt{N}$ with $\xi = \exp[-\mathrm{i}2\pi/N]$. Therefore the probability amplitude in Eq. (12) takes the form

$$\alpha_{ij}^{(\gamma)}(t) = \sum_{l=1}^{N} e^{-\mathrm{i}(2-2\cos[\frac{2\pi}{N}(l-1)])^{\gamma} t} e^{-\mathrm{i}\frac{2\pi}{N}(l-1)(j-i)}. \qquad (21)$$

In the limit $N \to \infty$, the sum in Eq. (21) can be approximated by an integral; by using the variable $\theta = \frac{2\pi}{N}(l-1)$ and $d\theta = \frac{2\pi}{N}$, we have

$$\alpha_n^{(\gamma)}(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}t\,(2-2\cos\theta)^{\gamma}} e^{\mathrm{i}n\theta} d\theta, \qquad (22)$$

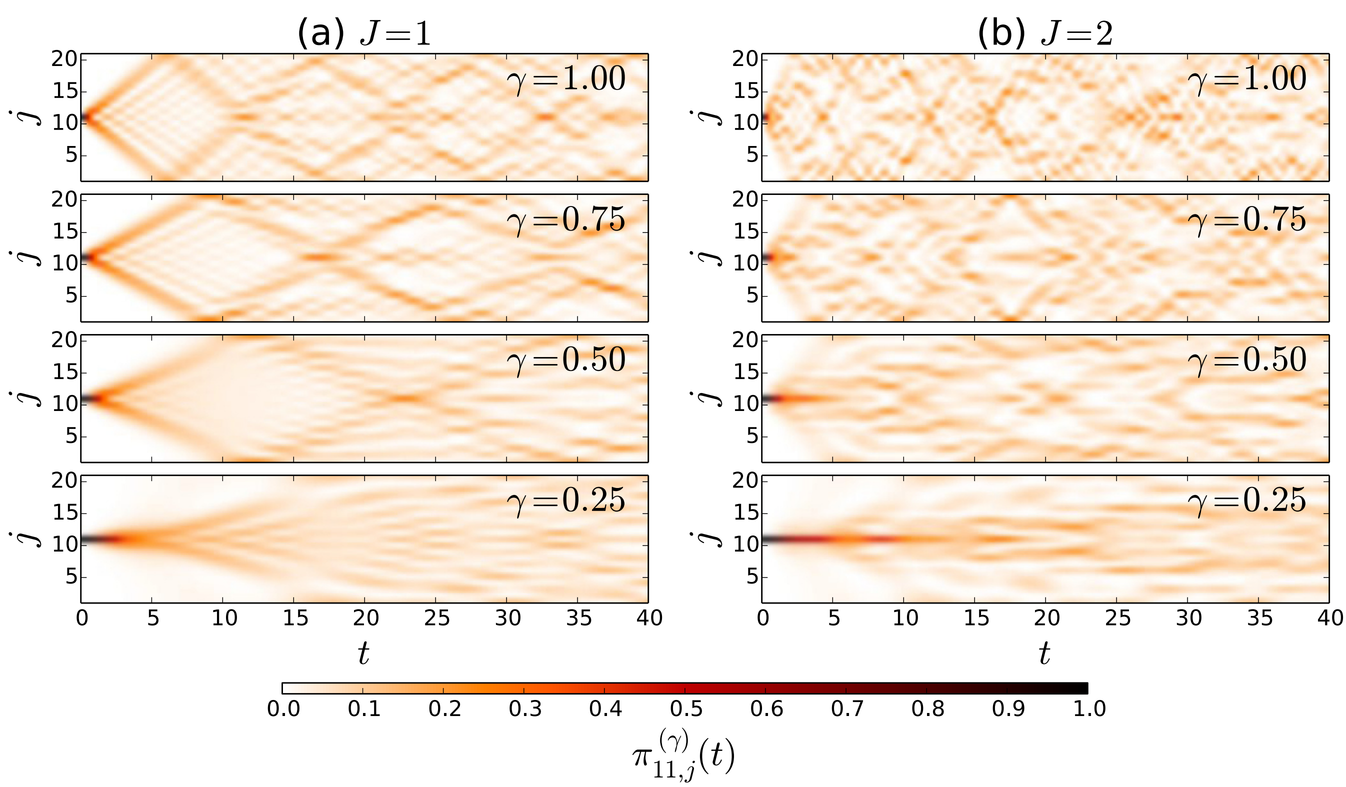

FIG. 2. (Color online) Transition probability $\pi_{11,j}^{(\gamma)}(t)$ to start in the state $|11\rangle$ at $t = 0$ and measure $|j\rangle$ at time t for different values of γ in interacting cycles with $N = 21$ nodes. (a) Ring, $J = 1$. (b) $J = 2$. The colored bar indicates the values of the probability $\pi_{11,j}^{(\gamma)}(t)$ determined by Eq. (17).

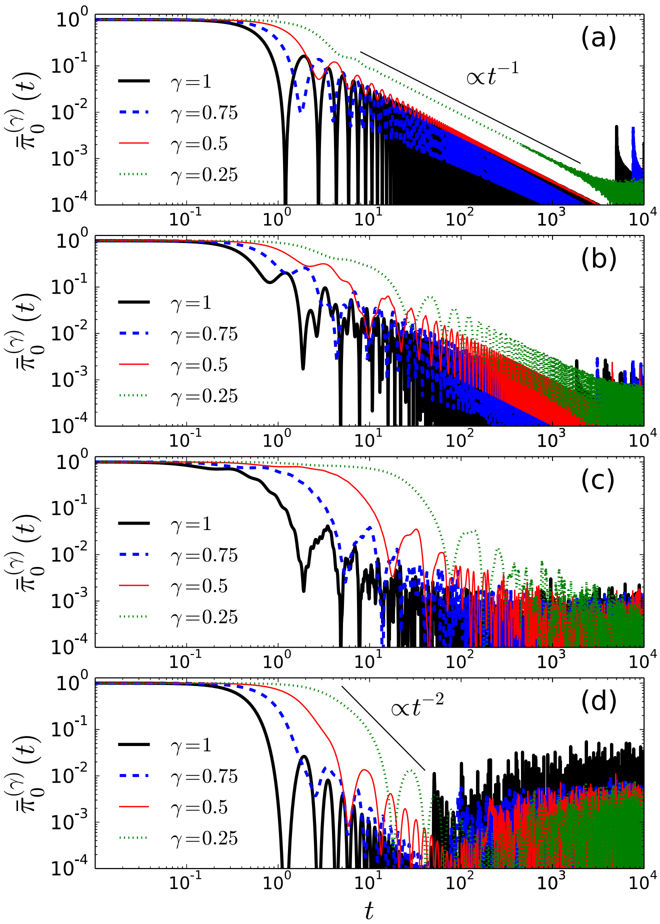

FIG. 3. (Color online) Average probability of return $\bar{\pi}_0^{(\gamma)}(t)$ as a function of time t for circulant networks with $N = 10^4$ and different values of γ. The results are obtained by the analytical expression (18) and the eigenvalues of the corresponding Laplacian matrix for interacting cycles with eigenvalues given by Eq. (19). (a) A ring, $J = 1$; the solid line denotes the behavior $\bar{\pi}_0^{(\gamma)}(t) \sim t^{-1}$. (b) $J = 2$. (c) $J = 10$. (d) 2D square lattice with dimensions 100×100 and periodic boundary conditions.

where $\alpha_n^{(\gamma)}(t)$ denotes the probability amplitude of transition to a distance $n = 0,1,2,\ldots$, from the initial node. In terms of this quantity we have $\pi_n^{(\gamma)}(t) = |\alpha_n^{(\gamma)}(t)|^2$ for the respective probability of transition.

The integral in Eq. (22) allows an exact calculation of the transition amplitudes. There are some specific cases with closed expressions for this integral. For example, for the value $\gamma = 1$, the dynamics of a normal quantum walk on a ring is described. For this case

$$\alpha_n^{(1)}(t) = \frac{e^{-2it}}{2\pi}\int_0^{2\pi} e^{i(n\theta+2t\cos\theta)}d\theta = i^n e^{-2it}\, J_n(2t),$$

where $J_n(x)$ denotes the Bessel function of the first kind [51]. Therefore,

$$\pi_n^{(1)}(t) = |J_n(2t)|^2. \qquad (23)$$

This is the well-known result for the transition probability for normal quantum transport on an infinite ring [6].

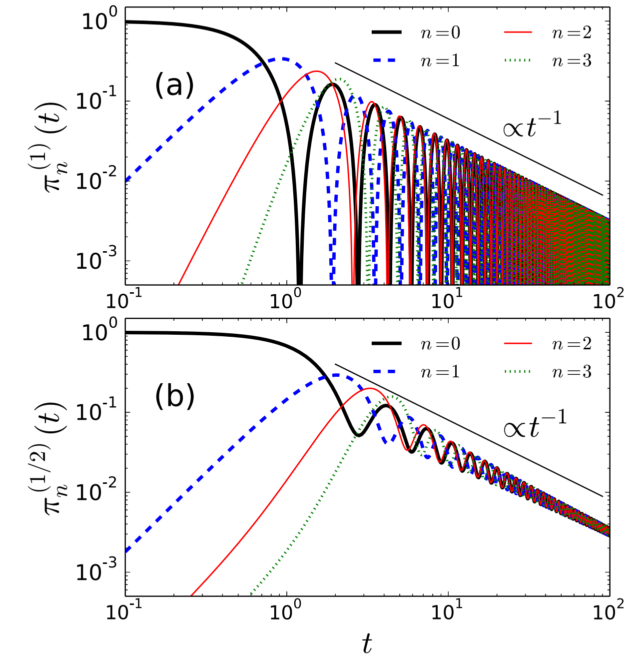

FIG. 4. (Color online) Transition probability $\pi_n^{(\gamma)}(t)$ on an infinite ring as a function of t for different values of the distance n. The results are obtained from the analytical expressions given by Eqs. (23) and (28). (a) Local dynamics determined by $\gamma = 1$. (b) Fractional dynamics based on Lévy flights with $\gamma = 1/2$.

On the other hand, the case with $\gamma = 1/2$ combines Lévy flights and the quantum dynamics on an infinite ring; for this case we have

$$\alpha_n^{(1/2)}(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(2-2\cos\theta)^{1/2}t} e^{in\theta}d\theta = \frac{1}{\pi}\int_0^{\pi} e^{i(2n\theta-2t\sin\theta)}d\theta. \qquad (24)$$

In terms of the Anger function [51,54],

$$\mathbf{J}_\nu(z) = \frac{1}{\pi}\int_0^{\pi}\cos(\nu\theta - z\sin\theta)\,d\theta, \qquad (25)$$

and the Weber function [51,54],

$$\mathbf{E}_\nu(z) = \frac{1}{\pi}\int_0^{\pi}\sin(\nu\theta - z\sin\theta)\,d\theta, \qquad (26)$$

Eq. (24) takes the form

$$\alpha_n^{(1/2)}(t) = \mathbf{J}_{2n}(2t) + i\mathbf{E}_{2n}(2t). \qquad (27)$$

Therefore,

$$\pi_n^{(1/2)}(t) = |\mathbf{J}_{2n}(2t)|^2 + |\mathbf{E}_{2n}(2t)|^2, \qquad (28)$$

which gives an analytical expression for the transition probability to a distance n. In Fig. 4 we plot the probabilities for the normal quantum walk given by Eq. (23) and the fractional quantum dynamics on a ring with $\gamma = 1/2$ given by Eq. (28). In both cases it is observed that for $t \gg 1$ the probabilities decay with the same inverse power-law relation t^{-1}. In this way, it is observed that the long-time dynamics of the excitations on the infinite ring is similar, no matter if the transitions are restricted to nearest neighbors (in the case of $\gamma = 1$) or include the long-range jumps that emerge from the fractional formalism. A similar nonexponential power-law decay is well known in quantum mechanics [55,56].

VII. EFFICIENCY

Once we have introduced the fractional quantum transport on networks, the transition probabilities in Eq. (13), and the average probability of return in Eq. (14), we analyze now global quantities that describe this process in general networks. We start defining two quantities similar to χ_{ij} and $\bar{\chi}$ in Eqs. (3) and (4), introduced in the study of the normal quantum transport [6]. For the fractional case we define the long-time average as

$$\chi_{ij}^{(\gamma)} \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \pi_{ij}^{(\gamma)}(t)dt = \sum_{l,m=1}^{N}\delta_{\mu_l^\gamma,\mu_m^\gamma}\langle j|\Psi_l\rangle\langle\Psi_m|j\rangle\langle i|\Psi_m\rangle\langle\Psi_l|i\rangle\,, \tag{29}$$

a result that is obtained from Eq. (13), and the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-\mathrm{i}(E_n-E_m)t}dt = \delta_{E_n,E_m}.$$

In a similar way, by using the average probability of return, we have

$$\bar{\chi}^{(\gamma)} \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \bar{\pi}_0^{(\gamma)}(t)dt = \frac{1}{N}\sum_{j,l,m=1}^{N}\delta_{\mu_l^\gamma,\mu_m^\gamma}|\langle j|\Psi_l\rangle|^2\,|\langle j|\Psi_m\rangle|^2. \tag{30}$$

Also, as a consequence of the Cauchy-Schwarz inequality, a lower bound exists for $\bar{\chi}^{(\gamma)}$ and is given by

$$\frac{1}{N^2}\sum_{l=1}^{N}\sum_{m=1}^{N}\delta_{\mu_l^\gamma,\mu_m^\gamma} \leqslant \bar{\chi}^{(\gamma)}. \tag{31}$$

The equality is obtained for circulant networks; in this case $|\langle j|\Psi_m\rangle|^2 = 1/N$. On the other hand, the maximum value that $\bar{\chi}^{(\gamma)}$ can take is obtained for a complete graph (fully connected network) for which $\mu_1 = 0$, $\mu_l = N$ for $l = 2,\dots,N$ [48]; therefore,

$$\bar{\chi}_{\mathrm{c}}^{(\gamma)} = \frac{1+(N-1)^2}{N^2}, \tag{32}$$

and for $N \gg 1$, we have $\bar{\chi}_{\mathrm{c}}^{(\gamma)} \approx 1$.

From the expressions in Eqs. (29) and (30), we establish that the introduction of the fractional quantum mechanics to define the fractional quantum transport on networks do not change the global quantities $\chi_{ij}^{(\gamma)}$ and $\bar{\chi}^{(\gamma)}$ due to the fact that taking values of γ in the interval $0 < \gamma \leqslant 1$ changes the values of the spectrum of $\mathbf{H}_\gamma$ but it does not change their degeneracy; as a consequence, the sums in Eqs. (29) and (30) are independent of γ. This suggests that for a network, the long-time dynamics is similar, no matter if the transitions are long range, like the process that emerges in the fractional case, or local, like in the normal quantum transport ($\gamma = 1$). This universality of the quantum transport on networks is in contrast to the classical counterpart studied in Refs. [34,35]. This is consistent with the result established by means of the analysis of $\bar{\pi}_0^{(\gamma)}(t)$ in Fig. 3.

On the other hand, in the limit $\gamma \to 0$ we have $\lim_{\gamma\to 0}\mu_2^\gamma,\dots,\lim_{\gamma\to 0}\mu_N^\gamma = 1$; therefore, the degeneracy of the energy spectrum changes and $\lim_{\gamma\to 0}\bar{\chi}^{(\gamma)} = \frac{(N-1)^2+1}{N^2}$. This is the value obtained for a complete graph in Eq. (32). In the following part we quantify how the fractional process approaches the behavior observed in the limit $\gamma \to 0$. In this case we introduce a quantity similar to $\chi_{ij}^{(\gamma)}$ in Eq. (29) but with a finite time. We define the short-time average

$$\eta_{ij}^{(\gamma)} \equiv \frac{1}{N}\int_0^N \pi_{ij}^{(\gamma)}(t)dt, \tag{33}$$

and an analog value based in $\bar{\chi}^{(\gamma)}$ defined in Eq. (30),

$$\eta^{(\gamma)} \equiv \frac{1}{N}\int_0^N \bar{\pi}_0^{(\gamma)}(t)dt\,. \tag{34}$$

Now, from Eq. (14) we have

$$\eta^{(\gamma)} = \sum_{j,l,m=1}^{N}\frac{\Theta\big[(\mu_l^\gamma-\mu_m^\gamma)N\big]}{N}|\langle j|\Psi_l\rangle|^2|\langle j|\Psi_m\rangle|^2\,, \tag{35}$$

where

$$\Theta(x) = \begin{cases}1 & \text{if} \quad x = 0,\\ \sin(x)/x & \text{otherwise.}\end{cases} \tag{36}$$

In this way, the function $\Theta(x)$ smooths the values given by the Kronecker deltas in the sum in Eq. (29), and in addition to the states with equal energy, little differences in the spectrum are included in the global quantity $\eta^{(\gamma)}$.

In the particular case of circulant networks, the Eq. (35) takes the form

$$\eta^{(\gamma)} = \frac{1}{N^2}\sum_{l=1}^{N}\sum_{m=1}^{N}\Theta\big[(\mu_l^\gamma-\mu_m^\gamma)N\big]\,, \tag{37}$$

a result that gives additional information to the value $\bar{\chi}^{(\gamma)}$ in Eq. (30). As a consequence, in circulant networks we have a global measure that depends only on the Laplacian spectrum. In the limit $\gamma \to 0$, we have

$$\lim_{\gamma\to 0}\eta^{(\gamma)} = \frac{1}{N^2}\left[1+(N-1)^2+2(N-1)\frac{\sin(N)}{N}\right],$$

and for $N \gg 1$, $\eta^{(0)} \approx 1$. From the definition in Eq. (35), the global value $\eta^{(\gamma)}$ is the area under the curve $\bar{\pi}_0^{(\gamma)}(t)/N$ in the interval $0 \leqslant t \leqslant N$. Cases with $\eta^{(\gamma)} \approx 1$ describe efficiencies near those observed for the dynamics on a complete graph. In this way, comparing $\eta^{(\gamma)}$ for different structures allows us to identify networks for which low values of γ lead to an optimal efficiency of the transport, i.e., an efficiency similar to the transport on a complete graph.

In Fig. 5 we depict the values of the function Θ and the global quantity $\eta^{(\gamma)}$ determined by Eq. (37) for different interacting cycles with $N = 500$. In Figs. 5(a) and 5(b) we present the values of the function $|\Theta[(\mu_l^\gamma - \mu_m^\gamma)N]|$ for $l,m = 1,2,\dots N$. These are the values used to calculate the global quantity $\eta^{(\gamma)}$, whereas only the values with $\Theta = 1$ are included in the calculation of $\bar{\chi}^{(\gamma)}$. In comparison with the limit $\gamma \to 1$, significant changes are obtained for the fractional case

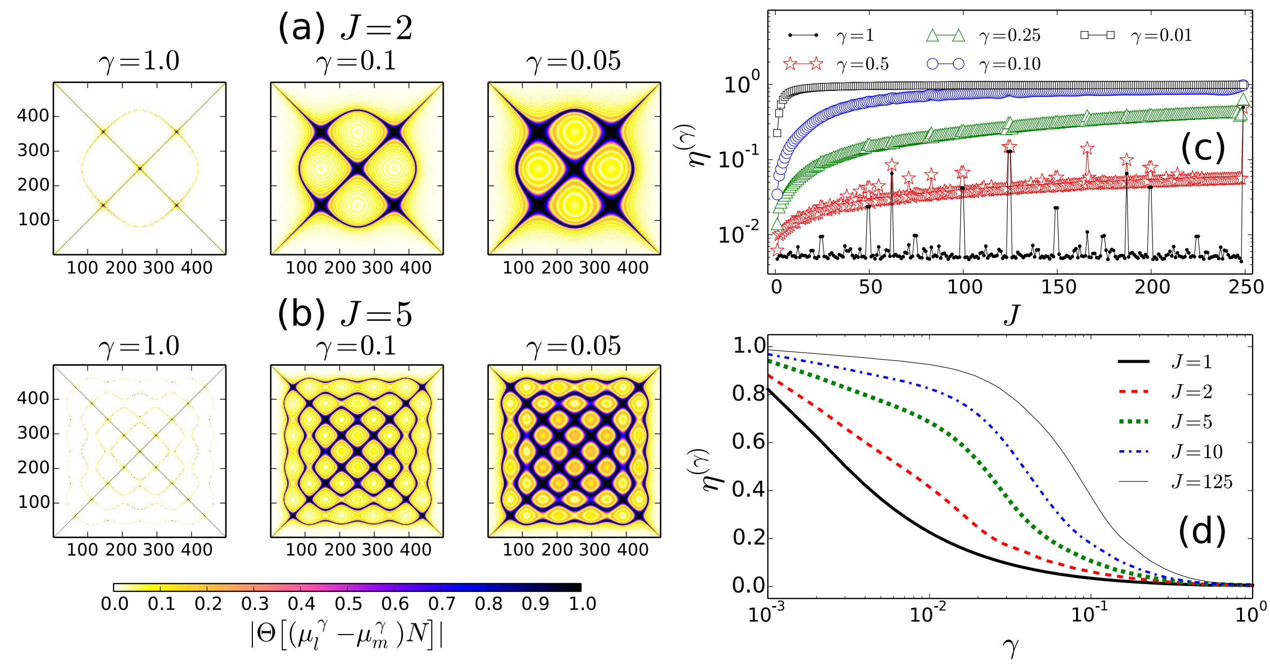

FIG. 5. (Color online) Analysis of the function Θ and the short-time average $\eta^{(\gamma)}$ for interacting cycles with $N=500$. Absolute value of the function $\Theta[(\mu_l^\gamma-\mu_m^\gamma)N]$ for (a) $J=2$, (b) $J=5$; the axes denote the values l and m. We calculate this function by using Eq. (36) and the spectrum in Eq. (19). It is observed how the changes of γ modify the values of the terms in Eq. (35). In (c) we show $\eta^{(\gamma)}$ in terms of J for different values of γ; we calculate the values using Eq. (37). In (d) we depict $\eta^{(\gamma)}$ as a function of γ for fixed values of J.

with $\gamma=0.1$ and $\gamma=0.05$. In Fig. 5(c) we plot the global quantity $\eta^{(\gamma)}$ as a function of J for all the interacting cycles with $N=500$. It is observed how by decreasing the values of γ, the values of $\eta^{(\gamma)}$ increase. In particular, $\gamma\leqslant 0.01$ gives the value $\eta^{(\gamma)}\approx 1$ obtained for a complete graph. In Fig. 5(d) we analyze $\eta^{(\gamma)}$ as a function of γ for interacting cycles with different values of J. It is deduced that in small-world networks ($J=125$) the transition to an optimal dynamics is achieved in the interval $0<\gamma\leqslant 0.01$; an equivalent efficiency in large-world networks requires lower values of γ. In this way, the analysis of the probability of return and the global quantity $\bar{\chi}^{(\gamma)}$ reveal the universality of the transport based on the fractional quantum mechanics on networks for $0<\gamma\leqslant 1$. On the other hand, with the study of the global quantity $\eta^{(\gamma)}$, in the limit $\gamma\to 0$ we observe two regimens, one where the dynamics is similar to the normal quantum walk and the other for which the efficiency of the process is similar to that observed in a complete graph.

VIII. CONCLUSIONS

In summary, in this paper we have used the matrix $\mathbf{L}^\gamma$ as the equivalent of the fractional Laplacian operator in order to define the fractional Schrödinger equation for networks. This fractional approach allows one to define continuous-time quantum walks that combine a dynamics with long-range displacements in the network, similar to Lévy flights, and the quantum superposition of states. Our general fractional formalism applies to any type of connected undirected networks, including regular, random, and complex networks, and can be implemented from the spectral properties of the Laplacian matrix. We explore this process by means of the transition probability, the average probability of return, and global quantities that describe the efficiency of this quantum process for any kind of network. Also, we explore in detail, with some analytical results, the fractional quantum dynamics in circulant networks; in particular, we obtain exact results for finite rings, interacting cycles, complete graphs, and infinite rings. In the case of an infinite ring, we found that the average probability of return to a node decays as a power law. Finally, our approach, applicable to any type of connected undirected networks, provides a general formalism that connects two important fields: fractional quantum mechanics and dynamical processes on networks.

ACKNOWLEDGMENTS

A.P.R. acknowledges support from the Secretaría de Ciencia, Tecnología e Innovación (SECITI), the Centro Latinoamericano de Física (CLAF), and the Consejo Nacional de Ciencia y Tecnología (CONACYT), México.

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