Shot-noise Fano factor

Kamil Rajdl* and Petr Lansky

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlarska 2, 611 37 Brno, Czech Republic and Institute of Physiology, Academy of Sciences of the Czech Republic, Videnska 1083, 142 20 Prague 4, Czech Republic

(Received 15 June 2015; revised manuscript received 12 October 2015; published 24 November 2015)

A variability measure of the times of uniform events based on a shot-noise process is proposed and studied. The measure is inspired by the Fano factor, which we generalize by considering the time-weighted influence of the events given by a shot-noise response function. The sequence of events is assumed to be an equilibrium renewal process, and based on this assumption we present formulas describing the behavior of the variability measure. The formulas are derived for a general response function, restricted only by some natural conditions, but the main focus is given to the shot noise with exponential decrease. The proposed measure is analyzed and compared with the Fano factor.

DOI: 10.1103/PhysRevE.92.052135 PACS number(s): 02.50.Ey, 05.45.Tp

I. INTRODUCTION

From many experiments or observations, data arise in the form of sequences of times of occurrence of identical events (e.g., radioactive particles emission, accidents, earthquakes, neuronal spikes, arrivals of customers or cars, etc.). The numbers of events in time intervals of a certain length are often examined when analyzing these sequences. For example, the very frequently used variability measure, the Fano factor, is defined as the variance to the mean ratio of numbers of events that occurred up to a time t. The Fano factor was first defined and used in [1] for a slightly different purpose—as a measure of fluctuations of the produced number of ion pairs in a volume of gas. Nevertheless, since then it has been commonly used as a variability measure in many situations, where discrete identical events occur in time. For example, the Fano factor is useful in analysis of natural disasters (earthquakes, fires) [2–4], neural spike trains [5–7], chemical reactions [8,9], or many other phenomena [10–13].

The Fano factor implicitly assumes that all the events have the same effect independently of the times of their occurrence, which may not always be valid. Sometimes, it is more convenient to give a greater influence to the more recent events or to distribute their weights depending on time in some other way. This can be done by employing a function representing the dependence of the weights on time and summing the values of weights corresponding to the single events.

As an example of a scientific field where weighting the events has a transparent empirical reason, let us mention the analysis of neural spike trains. In such an analysis, the Fano factor calculated based on events that occurred in a time window is often supposed to represent the variability that a neuron "registers" at the time corresponding to the end of the window. Nevertheless, a standard property of neurons is that they constantly lose information about the received spikes (the accumulated membrane potential spontaneously decreases). It seems natural, therefore, to improve the Fano factor so that the recent spikes have a larger influence. Most of the standard neuronal models (leaky integrator) describe the "forgetting" using an exponential decay [14], thus a natural choice for the weighting function is a decreasing exponential.

We believe that weighting the events can be convenient in any situation in which the Fano factor is used to measure the instantaneous variability of a series of events, mainly when some "effect" of the events decreases in time. There might not always be a rigorous theoretical justification resulting from the character of the sequences of events, but intuitively the events nearer the time at which the variability is studied should have a larger influence on the measure than the more distant events.

The proposed measure can be formally described as follows. Assuming m time intervals of a length t in which some identical events occur, the "counting" approach yields numbers of events $n_1,\dots,n_m$ in the individual intervals. On the other hand, the "weighting" approach assesses the occurrence of the events by values $s_1,\dots,s_m$. Every value s_i, $i=1,\dots,m$, is created by summing the weights of the events in every interval. We define the weights using a weighting function w so that the weight $w(t-x)$ is assigned to an event that occurred at a time $x>0$. A simple example of the weighting function can be a decreasing exponential function, yielding weights $e^{-(t-x)/\tau}$, where $\tau>0$ is a time constant (for illustration, see Fig. 1).

Using the terminology of filtering theory, the values of s_i arise as the times of the events filtered by a response function $w(x)$, thus as convolution of the (stochastic) counting process describing the events with the function $w(x)$. In this specific case, where the filtered process is created by discrete values, the resulting process is often called shot noise [15]. We define the proposed variability measure analogously to the Fano factor using the values s_i, thus as the variance-to-mean ratio of a shot-noise process. Shot noise has a wide range of applications in many fields [16–22], while it is mostly considered with an exponential distribution of events [23], i.e., the Poisson process. This assumption is too restrictive for our purposes, so we use a more general model, namely an equilibrium renewal process. Two situations are studied: the shot-noise process with an exponential response function, which is the most often used case, and the shot-noise process with a general response function. The renewal shot-noise process was also studied recently in [24,25].

Note that there are also other variability measures of sequences of uniform events such as the very often used coefficient of variation, measures based on entropy or Fisher information [26], or a measure reducing the influence of rate

*xrajdl@math.muni.cz

 ©2015 American Physical Society

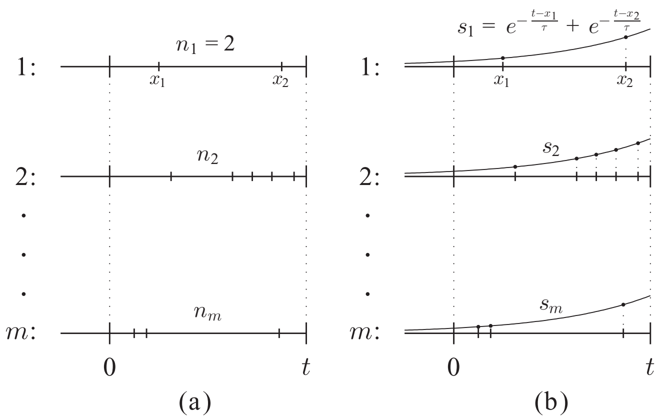

FIG. 1. Data construction based on events that occurred in m time intervals of a length $t > 0$. (a) The events are counted and their numbers $n_1, \dots, n_m$ obtained. (b) To every event a value is assigned using a weighting (exponential) function, and these values are summed. The individual sums are denoted as $s_1, \dots, s_m$.

changes [27]. Nevertheless, they all differ from the approach proposed in this article.

II. MODEL OF OCCURRENCE OF EVENTS

First, we specify the considered model of occurrence of events, and we present its properties. The time at which the observation of the events starts is denoted as zero, and the random times at which the events occur are denoted as $X_1, X_2, X_3, \dots$ [see Fig. 2(a)]. Next, we assume that this sequence can be described as an equilibrium renewal process. Thus, the lengths of intervals between the events [interevent intervals (IEIs)] are independent and identically distributed continuous random variables, denoted by T with the probability density function $f(t)$. The term "equilibrium" moreover specifies that the time zero is unrelated to the sequence of events. In such a situation, the time up to the first event (X_1), i.e., the forward recurrence time, generally has a different probability distribution from T.

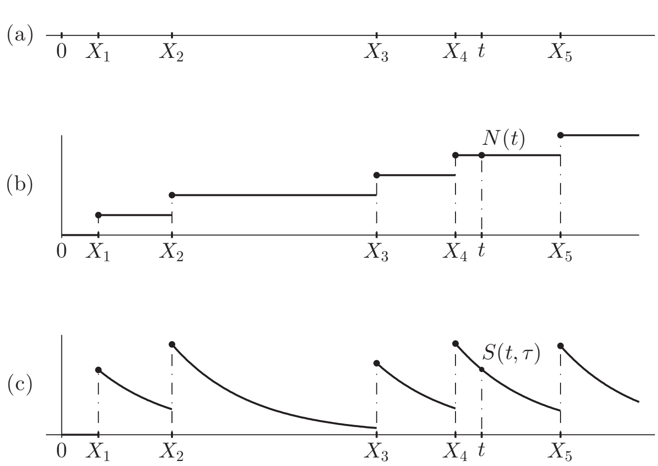

FIG. 2. (a) Illustration of times of events that occurred after time zero. (b) Illustration of the counting process $N(t)$. (c) Illustration of the shot-noise process $S(t,\tau)$ with an exponential response function.

The random process $N(t)$ describes the number of events in an interval $(0,t]$, $t > 0$, where time zero is "randomly positioned" with respect to the sequence of times [see Fig. 2(b)]. The process $N_o(t)$ representing the number of events in $(0,t]$ on the condition that an event occurred at time zero is called an ordinary renewal process. The time up to the first event has, in this situation, the same probability distribution as the IEIs.

Next, we denote the intensity of the renewal process at time t as

$$\mu(t) = \frac{dE(N(t))}{dt}, \quad \mu_o(t) = \frac{dE(N_o(t))}{dt}. \tag{1}$$

Then, since [28]

$$E(N(t)) = \frac{t}{E(T)}, \tag{2}$$

it holds that

$$\mu(t) = \mu = \frac{1}{E(T)}, \tag{3}$$

which is also the mean number of events in an interval of unit length. The function $\mu_o(t)$ cannot be generally expressed so easily. One possibility is the relationship [28]

$$\mu_o(t) = \sum_{k=1}^{\infty} f_k(t), \tag{4}$$

where $f_k(t)$, $k = 1,2,\dots$, is the probability density of the sum of k random variables with density $f(t)$. Using the Laplace transform ($\mathcal{L}$), one can rewrite Eq. (4) in the form

$$\mathcal{L}\{\mu_o\}(s) = \frac{\mathcal{L}\{f\}(s)}{1 - \mathcal{L}\{f\}(s)}. \tag{5}$$

Note that

$$\lim_{t\to\infty} \mu_o(t) = \mu \tag{6}$$

and that the only renewal process for which $\mu_o(t) = \mu$ for all $t > 0$ is the Poisson process (the renewal process with exponentially distributed T).

The Fano factor is defined as

$$F(t) = \frac{\mathrm{var}(N(t))}{E(N(t))}, \tag{7}$$

and sometimes the term "Fano factor" also denotes the limit

$$F = \lim_{t\to\infty} F(t). \tag{8}$$

Note that it holds [28] that

$$\lim_{t\to 0} F(t) = 1, \tag{9}$$

$$\lim_{t\to\infty} F(t) = C_v^2, \tag{10}$$

where C_v, the coefficient of variation, is defined as

$$C_v = \frac{\sqrt{\mathrm{var}(T)}}{E(T)}. \tag{11}$$

For a specific t and the known probability distribution of T, $F(t)$ has to be mostly calculated numerically, e.g., using the formula [29]

$$F(t) = \frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1+\mathcal{L}\{f\}(s)}{s^2[1-\mathcal{L}\{f\}(s)]}\right\}(t) - \frac{t}{E(T)}. \quad (12)$$

On the other hand, empirical values are evaluated using estimates of quantities in Eq. (7). As was mentioned, next we generalize the Fano factor assuming the shot-noise process instead of $N(t)$.

III. SHOT-NOISE PROCESS WITH AN EXPONENTIAL RESPONSE FUNCTION

The shot-noise process is a random process wherein the value is increased when an event occurs but otherwise decreases in time—the events are being "forgotten." The forgetting is given by a response function that represents the time effect of one event on the process [23]. Let us note that the function does not need to be (monotonously) decreasing, however that is the most natural case. First, we assume that this response function is exponentially decreasing [see Fig. 2(c) for an illustration]. The definition of such a process is

$$S_E(t,\tau) = \sum_{i=1}^{N(t)} e^{-\frac{t-X_i}{\tau}}, \quad (13)$$

where $\tau > 0$ is the time constant. Equation (13) represents the process $S_E(t,\tau)$ from the discrete-time point of view. Using the Dirac delta function, $\delta(x)$, it is possible to rewrite it into a continuous-time form,

$$S_E(t,\tau) = \int_0^t u(s)e^{-\frac{t-s}{\tau}}\,ds, \quad (14)$$

where

$$u(t) = \sum_{i=1}^{\infty}\delta(t-X_i). \quad (15)$$

This form is often used in filtering theory, however in this paper we prefer to represent it by the discrete formula (13), which is more intuitive for our purposes.

In analogy to the Fano factor, we define

$$G_E(t,\tau) = \frac{\mathrm{var}(S_E(t,\tau))}{E(S_E(t,\tau))}, \quad (16)$$

$$G_E(\tau) = \lim_{t\to\infty} G_E(t,\tau). \quad (17)$$

To be able to study this measure, first we need the mean and variance of $S_E(t,\tau)$. For the mean, there are very simple formulas (see the Appendix),

$$E(S_E(t,\tau)) = \mu\tau(1-e^{-t/\tau}), \quad (18)$$

$$E(S_E(\tau)) = \lim_{t\to\infty} E(S_E(t,\tau)) = \mu\tau. \quad (19)$$

Thus, taking into account Eq. (3), we do not need more information about T than its mean value. On the other hand, the variance of $S_E(t,\tau)$ depends fully on the distribution of T (see the Appendix),

$$\mathrm{var}(S_E(t,\tau)) = \mu\int_0^t\int_0^t e^{-\frac{x+y}{\tau}}[\mu_o(|y-x|)-\mu]\,dx\,dy + \frac{1}{2}\mu\tau(1-e^{-2t/\tau}). \quad (20)$$

This formula shows that the variance is highly dependent on the difference of $\mu_o(x)$ and μ; the smaller the difference is, the closer the variance is to $\mu\tau(1-e^{-2t/\tau})/2$, which is the variance of shot noise with Poisson-distributed events.

Next, we explore the limit properties of $G_E(t,\tau)$ for $t\to 0$ and $t\to\infty$. It holds that (see the Appendix)

$$\lim_{t\to 0} G_E(t,\tau) = 1 \quad (21)$$

and

$$G_E(\tau) = \frac{1}{1-\mathcal{L}\{f\}(1/\tau)} - \mu\tau - \frac{1}{2}. \quad (22)$$

The function $G_E(\tau)$ represents $G_E(t,\tau)$ for the shot-noise process at steady state. We can see that with a known Laplace transform of $f(t)$, one can simply calculate $G_E(\tau)$, e.g., we obtain

$$G_E(\tau) = \frac{1}{1-\left(1+\frac{C_v^2}{\mu\tau}\right)^{-\frac{1}{C_v^2}}} - \mu\tau - \frac{1}{2} \quad (23)$$

for the gamma probability distribution of T, and

$$G_E(\tau) = \frac{1}{1-e^{\frac{1}{C_v^2}\left(1-\sqrt{1+2\frac{C_v^2}{\mu\tau}}\right)}} - \mu\tau - \frac{1}{2} \quad (24)$$

for the inverse Gaussian (IG) distribution of T. Both of these probability distributions contain two parameters, e.g., $\mu = 1/E(T)$ and C_v, as in the forms assumed by us. With an unknown distribution of T, formula (22) can be approximated for $\mu\tau\to\infty$ as (see the Appendix)

$$G_E(\tau) \approx \frac{\mu\tau}{1-\frac{1+C_v^2}{2\mu\tau}} - \mu\tau - \frac{1}{2}, \quad (25)$$

which yields

$$C_v^2 \approx \frac{4E(S_E(\tau))G_E(\tau) - 2G_E(\tau) - 1}{2G_E(\tau) + 2E(S_E(\tau)) + 1}. \quad (26)$$

Independently of IEI distribution, it also holds that

$$\lim_{C_v\to 0} G_E(\tau) = \frac{1}{1-e^{-\frac{1}{\mu\tau}}} - \mu\tau - \frac{1}{2}, \quad (27)$$

which expresses the behavior of $G_E(\tau)$ when IEIs are all equal, thus when $f(t) = \delta(t-1/\mu)$, where $\delta(x)$ is the Dirac delta function. We can see that even then $G_E(\tau)$ is not zero, which is caused by the equilibrium property of renewal processes we use for the modeling of the sequences of events.

Finally, we present the limit relationships of quantity (16) with respect to τ (see the Appendix),

$$\lim_{\tau\to 0} G_E(t,\tau) = \tfrac{1}{2}, \quad (28)$$

$$\lim_{\tau\to\infty} G_E(t,\tau) = F(t), \quad (29)$$

$$\lim_{\tau\to 0} G_E(\tau) = \tfrac{1}{2}, \quad (30)$$

$$\lim_{\tau\to\infty} G_E(\tau) = \tfrac{1}{2}C_v^2. \quad (31)$$

The second formula can be intuitively explained so that for $\tau\to\infty$ the value of the shot-noise process does not decrease and thus the process is equal to $N(t)$. Note that Eqs. (10), (29), and (31) imply that the order of the limits (with respect to τ and t) cannot be changed.

IV. SHOT-NOISE PROCESS WITH THE GENERAL RESPONSE FUNCTION

Although the decreasing exponential is the most important example of the shot-noise response function, sometimes a different type of decrease of the process is assumed, e.g., a decaying power law [30,31]. Thus, we will next consider the shot-noise process with the response function in the general form $w(x)$. Such a process can be defined as

$$S(t,\tau) = \sum_{i=1}^{N(t)} w\left(\frac{t - X_i}{\tau}\right), \tag{32}$$

where $\tau > 0$ is the scale parameter. The parameter τ is not usually part of the definition of the general shot-noise process, nevertheless it provides a universal way to change the rate of forgetting of the events (for a decreasing response function, the larger the τ is, the slower is the forgetting), which is suitable for our purposes. Note that process (13) is obtained for response function $w_E(x) = e^{-x}$, $x \in [0,\infty)$.

It is not necessary to restrict the form of $w(x)$, however we want it to fulfill some natural conditions. We assume that the functions $w(x)$ and $w'(x) = \mathrm{d}w(x)/\,dx$ are continuous and bounded functions defined on $[0, \infty)$ such that

$$\lim_{x\to\infty} w(x) = \lim_{x\to\infty} w'(x) = 0. \tag{33}$$

Moreover, $w(x)$ is non-negative and integrable. For the integrals, we use the notation

$$I_k(t) = \int_0^t w^k(x)\,dx, \quad k = 1,2 \tag{34}$$

and

$$I_k = \lim_{t\to\infty} I_k(t). \tag{35}$$

The stated conditions mainly ensure that $G(t,\tau)$ is not divergent in dependence on t and τ, which is reasonable behavior for a variability measure. However, note that they exclude some standard power-law response functions, which are, for example, unbounded in proximity of zero.

Analogically to $G_E(t,\tau)$ we define the variability measure

$$G(t,\tau) = \frac{\mathrm{var}(S(t,\tau))}{E(S(t,\tau))}, \tag{36}$$

$$G(\tau) = \lim_{t\to\infty} G(t,\tau). \tag{37}$$

The mean and variance of $S(t,\tau)$ can be expressed as (see the Appendix)

$$E(S(t,\tau)) = \mu\tau I_1(t/\tau) \tag{38}$$

and

$$\mathrm{var}(S(t,\tau)) = \mu \int_0^t \int_0^t w\left(\frac{x}{\tau}\right) w\left(\frac{y}{\tau}\right)[\mu_o(|y-x|) - \mu] \times dx\,dy + \mu\tau I_2(t/\tau), \tag{39}$$

which is suitable to rewrite into the form

$$\mathrm{var}(S(t,\tau)) = 2\mu\tau^2 \int_0^{t/\tau} w(x) \int_0^x w(x-y)\mu_o(\tau y)\,dy\,dx - [\mu\tau I_1(t/\tau)]^2 + \mu\tau I_2(t/\tau). \tag{40}$$

The ratio of relationships (39) and (38) gives a general formula for $G(t,\tau)$. At the steady state ($t \to \infty$), it simplifies to the form

$$G(\tau) = 2\frac{\tau}{I_1} \int_0^\infty w(x) \int_0^x w(x-y)\mu_o(\tau y)\,dy\,dx - \mu\tau I_1 + \frac{I_2}{I_1}. \tag{41}$$

On the other hand, for $t \to 0$ (see the Appendix),

$$\lim_{t\to 0} G(t,\tau) = w(0). \tag{42}$$

Finally, the limit relationships with respect to τ are (see the Appendix)

$$\lim_{\tau\to 0} G(t,\tau) = \frac{I_2}{I_1}, \tag{43}$$

$$\lim_{\tau\to\infty} G(t,\tau) = w(0)F(t), \tag{44}$$

$$\lim_{\tau\to 0} G(\tau) = \frac{I_2}{I_1}, \tag{45}$$

$$\lim_{\tau\to\infty} G(\tau) = \frac{I_2}{I_1} C_v^2. \tag{46}$$

Relationships (45) and (46) resemble the well-known limit properties of the Fano factor (9) and (10). Because the Fano factor can be seen as a special case of $G(t,\tau)$, it is possible to obtain it also using formula (36). Let us consider the response function in the form

$$w_R(x) = \begin{cases} 1 & \text{for } x \in [0, 1], \\ 0 & \text{otherwise} \end{cases} \tag{47}$$

and denote $G(t,\tau)$ with this response function as $G_R(t,\tau)$. Then clearly $G_R(t,\tau) = F(\min\{t,\tau\})$, thus $G_R(\tau) = F(\tau)$, and formulas (45) and (46) are generalizations of formulas (9) and (10).

V. ILLUSTRATION OF THE BEHAVIOR OF $G_E(t,\tau)$ and $F(t)$

Figure 3 illustrates the dependence of $G_E(t,\tau)$ on t and the dependence of $G_E(\tau)$ on τ for gamma, IG, Weibull, and log-normal probability distributions of T and compared with the Fano factor. These distributions are probably the most often used distributions of interevent times in various processes. Let us mention, for example, that gamma, IG, and log-normal distributions are useful in the analysis of neural spike trains [32], and gamma and Weibull distributions are useful in the analysis of earthquakes or tsunamis [33,34]. All values were calculated based on formulas presented in this paper.

First, we compared $G_E(t,\tau)$ for $\tau = 1$ with $F(t)$. Both of these quantities start at the value 1, but with increasing t their behavior differs. Mainly $G_E(t,1)$ is lower than $F(t)$. According to Eq. (29), the value of $G_E(t,\tau)$ converges to $F(t)$ with increasing τ. Next, we compared $G_E(\tau)$ with $F(\tau)/2$. The purpose is to find out how much these values differ, because

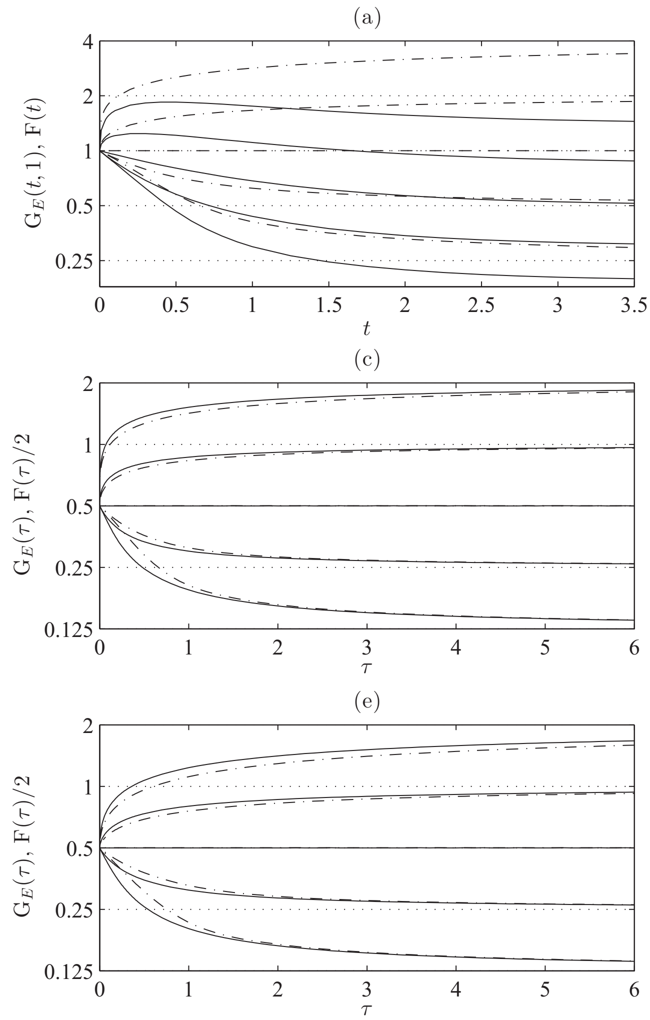

FIG. 3. Illustration of the behavior of $G_E(t,\tau)$ and $F(t)$ for gamma (a),(c), IG (b),(d), Weibull (e), and log-normal (f) probability distributions of IEIs with mean 1 and with various values of C_v^2 (values that correspond to the lines from bottom to top: 0.25, 0.5, 1, 2, 4). The vertical axes are logarithmically scaled. (a),(b) Solid lines: $G_E(t,\tau)$ in dependence on t for $\tau = 1$. Dash-dotted lines: $G(t)$ in dependence on t. (c)–(f) Solid lines: $G_E(\tau)$ in dependence on τ. Dash-dotted lines: $F(\tau)/2$ in dependence on τ. We can see that the behavior in dependence on t of the variability measures differs; however, at the steady state, $G_E(\tau)$ is approximately $G(\tau)/2$.

their limit values for $\tau \to 0$ and $\tau \to \infty$ indicate that $G_E(\tau)$ could be approximately $F(\tau)/2$. We can see that there are some differences, however they are vanishing with increasing τ, as expected.

VI. CONCLUSIONS AND DISCUSSION

The Fano factor, which is a frequently employed measure of the variability of sequences of times of events, was generalized by assuming the time-weighted influence of the events. Such an approach leads to the shot-noise process and variability measure $G(t,\tau)$ defined as its variance-to-mean ratio. Shot noise is almost always considered to be driven by the Poisson process. However, under this assumption, the value of $G(t,\tau)$ depends only on the arguments t and τ, which is clearly too restrictive. Therefore, shot noise with events described by an equilibrium renewal process was assumed. For this process, we presented the mean and variance, yielding formulas for $G(t,\tau)$. Moreover, the limit properties of $G(t,\tau)$ were derived and the behavior of the new variability measure was compared with the Fano factor.

Our results are mainly theoretical, but we also briefly describe some potential applications. They depend on whether we deal with data in the form of times of events or in the form of realizations of a shot-noise process. In the processing of times of events, we suggest $G(t,\tau)$ (its empirical version) as a measure of local variability. For such a purpose, the Fano factor is often used. Specifically, in the situation when we have multiple parallel records of times of events, the variability at time t can be measured using the Fano factor calculated based on events in the time interval $[t-\tau,t]$ for a $\tau > 0$. It corresponds to $G(t,\tau)$, where $w(x)$ is a rectangular function (47). However, if there is no specific reason for the rectangular function, a decreasing smooth function (mainly exponential) might be more suitable—it might reflect the importance of the time-dependent events more naturally. Our theoretical results can be then used to make inferences about the character of the data.

An advantage of $G_E(t,\tau)$ over $F(\tau)$ is also that the theoretical values of $G_E(t,\tau)$ can be more easily calculated. If the data records are long enough with respect to τ (values of the response function outside the interval $[0,t]$ are negligible), the behavior of $G_E(t,\tau)$ can surely be well approximated using $G_E(\tau)$, for which relatively simple formulas exist. Moreover, we have shown that $F(\tau) \approx 2G_E(\tau)$, which can be used to approximate the theoretical values of the Fano factor, mainly for gamma and IG distributions.

The second area of application is in processing data that we assume to be values of a shot-noise process (with an exponential response function). Such data arise in many situations, as shot-noise realizations can be interpreted, for example, in traffic noise [35,36], river flows [37], or neural membrane potential [38]. The presented results describe some theoretical properties of such data and can be used, for example, to construct simple moment estimators of C_v of the underlying renewal process. To deduce the value of C_v with a known sample mean and variance of the shot noise, it is possible to use Eq. (26) directly or, while assuming gamma or IG distribution of IEIs, Eqs. (23) and (24). An assumption about the probability distribution might seem to be too restrictive, however there are two reasons why gamma distribution in particular is often suitable. First, it is relatively flexible and thus it could appropriately approximate a wide range of various distributions. The second reason is its connection with the Poisson process. The gamma distribution with $C_v = 1$ is exponential and the events create the Poisson process. Thus, C_v estimation using the gamma distribution assumption indicates whether the shot-noise events are Poissonian or less or more variable.

ACKNOWLEDGMENT

This work was supported by the Czech Science Foundation project 15-06991S.

APPENDIX

We focus mainly on the case with a general response function $w(x)$. The special formulas are their simple consequences. In the following, we assume that $w(x)$ satisfies the conditions stated in Sec. IV.

First, we derive formulas for the mean and variance of $S(t,\tau)$. Using results from [39], we simply obtain

$$E(S(t,\tau)) = \mu\tau \int_0^{t/\tau} w(x)\,dx \tag{A1}$$

and

$$\mathrm{var}(S(t,\tau)) = \mu \int_0^t \int_0^t w\left(\frac{x}{\tau}\right) w\left(\frac{y}{\tau}\right)[\mu_o(|y-x|) - \mu]\,dx\,dy + \mu\tau \int_0^{t/\tau} w^2(x)\,dx. \tag{A2}$$

Next, we derive the limit properties (45) and (46). Limit (45) can be deduced directly from relationship (41) considering Eq. (4). To derive limit (46), let us first denote

$$H_o(t) = E(N_o(t)) = \int_0^t \mu_o(x)\,dx. \tag{A3}$$

Then it holds [28] that

$$H_o(t) = \mu t + \tfrac{1}{2}\left(C_v^2 - 1\right) + g(t), \tag{A4}$$

where $g(t)$ is a bounded function satisfying

$$\lim_{t\to\infty} g(t) = 0. \tag{A5}$$

Using integration by parts, it is possible to derive

$$\int_0^x w(x-y)\mu_o(\tau y)\,dy = \frac{1}{\tau} w(0) H_o(\tau x) + \frac{1}{\tau}\int_0^x w'(x-y) H_o(\tau y)\,dy. \tag{A6}$$

Now, combining relationships (41), (A4), and (A6), one can obtain

$$G(\tau) = \frac{I_2}{I_1} C_v^2 + \frac{2}{I_1}\int_0^\infty w(x)\int_0^x w'(x-y) g(\tau y)\,dy\,dx + \frac{2w(0)}{I_1}\int_0^\infty w(x) g(\tau x)\,dx, \tag{A7}$$

which yields relationship (46).

Another relationship we derive is (22). Putting formula (4) and $w(x) = e^{-x}$ into (41) gives

$$G_E(\tau) = 2\tau \sum_{k=1}^\infty \int_0^\infty e^{-2x}\int_0^x e^y f_k(\tau y)\,dy\,dx - \mu\tau + \frac{1}{2}. \tag{A8}$$

Then, using integration by parts with $u(x) = \int_0^x e^y f_k(\tau y)\,dy$ and $v'(x) = e^{-2x}$, we obtain

$$\begin{aligned} G_E(\tau) &= \tau \sum_{k=1}^\infty \int_0^\infty e^{-x} f_k(\tau x)\,dx - \mu\tau + \frac{1}{2} \\ &= \sum_{k=0}^\infty \mathcal{L}^k\{f\}\left(\frac{1}{\tau}\right) - \mu\tau - \frac{1}{2} \\ &= \frac{1}{1 - \mathcal{L}\{f\}(1/\tau)} - \mu\tau - \frac{1}{2}. \end{aligned} \tag{A9}$$

Approximation (25) can be derived from relationship (A9) using a Taylor expansion of the exponential function in the Laplace transform.

It remains to prove formulas (42), (43), and (44). They are not too complicated, so we present only their rather intuitive explanations. The first formula can be explained using the fact that clearly $\lim_{t\to 0} S(t,\tau) = \lim_{t\to 0} w(0)N(t)$. Next, the second formula holds because for $\tau \to 0$ the process $S(t,\tau)$ converges to a steady state ("most of the response function" will be in the interval $[0,t]$) and thus the limit value is the same as the limit value of $G(\tau)$. Finally, Eq. (44) stems from the obvious relationship $\lim_{\tau\to\infty} S(t,\tau) = w(0)N(t)$.

[1] U. Fano, Phys. Rev. **72**, 26 (1947).
[2] A. Talbi, K. Nanjo, J. C. Zhuang, K. Satake, and M. Hamdache, Geophys. J. Int. **194**, 1823 (2013).
[3] L. Telesca and L. Toth, Fluct. Noise Lett. **09**, 157 (2010).
[4] L. Telesca, G. Amatulli, R. Lasaponara, M. Lovallo, and A. Santulli, Ecol. Model. **185**, 531 (2005).
[5] F. Farkhooi, M. F. Strube-Bloss, and M. P. Nawrot, Phys. Rev. E **79**, 021905 (2009).
[6] M. P. Nawrot, C. Boucsein, V. R. Molina, A. Riehle, A. Aertsen, and S. Rotter, J. Neurosci. Methods **169**, 374 (2008).
[7] T. Omi and S. Shinomoto, Neural Comput. **23**, 3125 (2011).
[8] S. Chaudhury, J. Phys. Chem. B **118**, 10405 (2014).
[9] A. C. Barato and U. Seifert, J. Phys. Chem. B **119**, 6555 (2015).
[10] C. Anteneodo, R. D. Malmgren, and D. R. Chialvo, Eur. Phys. J. B **75**, 389 (2010).
[11] Q. M. Pei, X. Zhan, L. J. Yang, C. Bao, W. Cao, A. B. Li, A. Rozi, and Y. Jia, Phys. Rev. E **89**, 032715 (2014).
[12] Y. Yuan, X. Zhuang, Z. Liu, and W. Huang, Chaos Solitons Fractals **45**, 838 (2012).
[13] M. Karsai, K. Kaski, A. L. Barabsi, and J. Kertsz, Sci. Rep. **2**, 397 (2012).
[14] W. Gerstner and W. M. Kistler, *Spiking Neuron Models* (Cambridge University Press, Cambridge, 2002).
[15] D. R. Cox and V. Isham, *Point Processes* (Chapman & Hall, London, 1980).
[16] D. Vere-Jones, J. Roy. Stat. Soc. B **32**, 1 (1970).
[17] C. Kluppelberg and T. Mikosch, Bernoulli **1**, 125 (1995).
[18] A. J. Lawrance and N. T. Kottegoda, J. R. Stat. Soc. A Sta. **140**, 1 (1977).
[19] P. A. W. Lewis, J. Roy. Stat. Soc. B **26**, 398 (1964).
[20] F. Muller-Hansen, F. Droste, and B. Lindner, Phys. Rev. E **91**, 022718 (2015).
[21] I. Rodriguez-Iturbe, D. R. Cox, and V. Isham, Proc. R. Soc. London **410**, 269 (1987).
[22] G. Samorodnitsky, Lect. Notes Stat. **114**, 332 (1996).
[23] A. Papoulis, *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, New York, 1991).
[24] A. Iksanov, Stoch. Proc. Appl. **123**, 1987 (2013).
[25] A. Iksanov, A. Marynych, and M. Meiners, Stoch. Proc. Appl. **124**, 2132 (2014).
[26] L. Kostal, P. Lansky, and O. Pokora, PLoS ONE **6**, e21998 (2011).
[27] S. Shinomoto, K. Miura, and S. Koyama, BioSystems **79**, 67 (2005).
[28] D. R. Cox, *Renewal Theory* (Methuen & Co., London, 1962).
[29] K. Rajdl and P. Lansky, Math. Biosci. Eng. **11**, 105 (2014).
[30] S. B. Lowen and M. C. Teich, IEEE Trans. Inf. Theor. **36**, 1302 (1990).
[31] A. P. Petropulu, J.-C. Pesquet, X. Yang, and J. Yin, IEEE Trans. Signal Process. **48**, 1883 (2000).
[32] S. Ditlevsen and P. Lansky, Neural Comput. **23**, 1944 (2011).
[33] S. Pasari and O. Dikshit, Earth Planets Space **67**, 129 (2015).
[34] E. L. Geist and T. Parsons, Geophys. Res. Lett. **35**, L02612 (2008).
[35] A. H. Marcus, J. Appl. Probab. **10**, 377 (1973).
[36] G. H. Weiss, Transport Res. **4**, 229 (1970).
[37] M. Lefebvre and F. Bensalma, Int. J. Eng. Math. **2014**, 1 (2014).
[38] P. L. Smith, J. Math. Psychol. **54**, 266 (2010).
[39] J. Rice, Adv. Appl. Probab. **9**, 553 (1977).