General response formula and application to topological insulator in quantum open system

H. Z. Shen,[1,2] M. Qin,[1,2] X. Q. Shao,[1] and X. X. Yi[1,*]

[1]*Center for Quantum Sciences and School of Physics, Northeast Normal University, Changchun 130024, China*

[2]*School of Physics and Optoelectronic Technology Dalian University of Technology, Dalian 116024, China*

(Received 30 June 2015; revised manuscript received 12 October 2015; published 17 November 2015)

It is well-known that the quantum linear response theory is based on the first-order perturbation theory for a system in thermal equilibrium. Hence, this theory breaks down when the system is in a steady state far from thermal equilibrium and the response up to higher order in perturbation is not negligible. In this paper, we develop a nonlinear response theory for such quantum open system. We first formulate this theory in terms of general susceptibility, after which we apply it to the derivation of Hall conductance for open system at finite temperature. As an example, the Hall conductance of the two-band model is derived. Then we calculate the Hall conductance for a two-dimensional ferromagnetic electron gas and a two-dimensional lattice model. The calculations show that the transition points of topological phase are robust against the environment. Our results provide a promising platform for the coherent manipulation of the nonlinear response in quantum open system, which has potential applications for quantum information processing and statistical physics.

DOI: 10.1103/PhysRevE.92.052122 PACS number(s): 67.10.Fj, 03.65.Yz, 76.20.+q

I. INTRODUCTION

The Kubo Formula is an equation that expresses the linear response of an observable due to a time-dependent perturbation, which has been widely used in condensed matter physics [1–4] since it was first derived by Kubo in 1957 [5] and well developed in recent years [6–10]. This theory tells us how to calculate the response of an observable of a quantum system to a perturbation applied to the quantum system, a good example of which is the electric current as a response to an external electric field. When considering the response only to the first order of the perturbation, the theory is the so-called quantum linear response theory.

The Kubo's theory is only valid for a system in equilibrium and in the linear regime [11]. In recent years, linear response theory for open system (i.e., its steady state might not be a thermal equilibrium state) has attracted more and more attention in biophysics, nanophysics, and condensed matter physics. A linear response theory based on the master equation [12–16] and the hierarchical equation of motion [17,18] has been developed for open system. The key issue and difference of those approaches are how to get the reduced density matrix of the open system—the first approach obtains the density matrix by master equation, the second by the Hierarchical equation. Both approaches need to calculate the density matrix or the dynamics of the density matrix, which means we have to trace out the environment first, then calculate the response—a treatment on nonequal footing for the system and environment. It is worth addressing that, based on the reduced density matrix, a response theory for the open system was developed; we refer the readers to Ref. [19] for details.

Topological insulators (TIs) were theoretically predicted and have been experimentally discovered in Refs. [20–23]. TIs are a broad class of unconventional materials that are insulating in the interior but conduct along the edges. Over the past decades, topological insulators have attracted a great deal of interest due to their interesting features and possible applications in quantum computation [24,25]. Recently, efforts have also been made to investigate topological insulator for open system, e.g., density-matrix Chern insulators by thermal noise [26,27], zero-temperature Hall conductance subjected to decoherence [19], and topological order by dissipation [28]. These works motivate us to develop a response theory for open system of high order in perturbation and apply it to study topological insulators subjected to the environment.

Here we develop a quantum response theory for open system with a different approach, treating the system and the environment on equal footing. This approach does not need to get the reduced density matrix and the theory is beyond the linear response regime for open system [29,30]. Our treatment is not limited to a specific system. Rather, it is applicable to any strength of perturbation and open quantum system. In this sense, our theory of higher-order response [31,32] is applicable to open system subjected to the strong external field. We will apply this theory to topological insulators (TIs) [33] described by the two-band model and study the effect of the environment on the Hall conductance at both zero and finite temperature.

The structure of the paper is as follows: In Sec. II, we extend the linear response theory of closed system to open system and nonlinear case. In Sec. III, as an application of our theory, we derive the finite-temperature Hall conductance for open system and exemplify it into the two-band model. The Hall conductance for a two-dimensional ferromagnetic electron gas and a two-dimensional lattice model is also discussed in this section. Discussions and conclusions are given in Sec. IV.

II. NONLINEAR QUANTUM RESPONSE THEORY FOR QUANTUM OPEN SYSTEM

In this section, we first present a model to describe the system under study. Then we calculate the change of the expectation value of a system observable caused by the external perturbation, assuming the system and its environment are in thermal equilibrium without the perturbation. We define a nonlinear response (called susceptibility in later discussions) of the system to the perturbation up to arbitrary order and

*yixx@nenu.edu.cn

present a method to calculate the nth order susceptibility based on the generalized projection operator technique.

A. Model Hamiltonian

To begin with, we consider a quantum system described by a time-dependent Hamiltonian $H_S(t)$ coupled to the environment H_R in an external field $H_e(t)$. The total Hamiltonian $H_T(t)$ reads

$$H_T(t) = H_S(t) + H_R + H_{SR} + \varepsilon H_e(t), \tag{1}$$

where H_{SR} denotes the interaction between the system and the environment. ε stands for the perturbation parameter. The Hamiltonian of the system-field coupling reads

$$H_e(t) = -\sum_{\nu} f_\nu(t) C_\nu, \tag{2}$$

where C_ν denote system operators, $f_\nu(t)$ are coupling coefficients that depend on the time t, which can be interpreted as the external field in general. In later discussions, we set $H(t) = H_0(t) + H_{SR}$ with $H_0(t) = H_S(t) + H_R$.

What we are interested in is the response of the system to the time-varying external field $f_\nu(t)$, i.e., the change rates of expectation values of observables F_μ of the system with respect to the perturbations. The expectation values of F_μ are $\mathrm{Tr}_S\mathrm{Tr}_R[F_\mu \rho_T(t)]$, where $\rho_T(t)$ is the total density matrix of the system plus environment, determined by the Liouville equation,

$$\dot{\rho}_T(t) = -\frac{i}{\hbar}[H_T(t),\rho_T(t)] \equiv -\frac{i}{\hbar} L_T(t)\rho_T(t). \tag{3}$$

To calculate the change of the expectation of F_μ, we write the total density matrix into two parts,

$$\rho_T(t) = \rho_{T0}(t) + \rho_{Te}(t), \tag{4}$$

where ρ_{T0} is the total density matrix when $f_\nu(t) = 0$ with the initial condition $\rho_T(0) \equiv \rho_{T0}(0)$, i.e., the external field is applied at time $t = 0$. $\rho_{Te}(t)$ stands for the change in $\rho_T(t)$ due to the external field. Similarly, we split the Liouville operator as

$$L_T(t) = L_{T0}(t) + \varepsilon L_{Te}(t), \tag{5}$$

where $L_{T0}(t)\rho = [H_T(\varepsilon = 0),\rho]$, $L_{Te}(t)\rho = [H_e,\rho]$. Substituting Eqs. (4) and (5) into Eq. (3), we obtain

$$\rho_e(t) = -\frac{i}{\hbar}\varepsilon \mathrm{Tr}_R \int_{t_0}^{t} g(t,u) L_{Te}(u)\rho_T(u) du, \tag{6}$$

where the reduced system dynamics $\rho_e(t)$ was defined as $\rho_e(t) = \mathrm{Tr}_R[\rho_{Te}(t)]$, which denotes the change of the density matrix induced by the external field. Tr_R denotes a trace over the environment.

Equation (6) completely describes the influence of the external field on the system subjected to the environment. $g(t,u)$ is the chronological time-ordering operator, $g(t,u) = T_+ \exp[-\frac{i}{\hbar}\int_u^t d\tau L(\tau)]$, $(t > u)$, where $L(\tau)\rho = [H(\tau),\rho]$. The operator T_+ describes the chronological time ordering. It orders any product of superoperators such that the time argument increases from right to left.

B. Nonlinear response tensor for open system

In this section, we first define the response tensor, then we expend it in orders of the perturbation. Of course, the Kubo's linear response theory is the linear response of a closed system to the perturbation in first order of perturbation.

Noticing Eq. (4), we can rewrite Eq. (6) as

$$\rho_e(t) = -\varepsilon \mathrm{Tr}_R \int_{t_0}^{t} s(t,u)[\rho_{T0}(u) + \rho_{Te}(u)] du, \tag{7}$$

with

$$s(t,u) = \frac{i}{\hbar} g(t,u) L_{Te}(u). \tag{8}$$

Equation (7) is in fact an integral equation with respect to $\rho_{Te}(t)$. By iterating repeatedly on Eq. (7), we obtain

$$\rho_e(t) = -\varepsilon \mathrm{Tr}_R \int_{t_0}^{t} s(t,u) x(u) du, \tag{9}$$

where

$$x(u) = \rho_{T0}(u) + \sum_{n=1}^{\infty} (-\varepsilon)^n \int_{t_0}^{u} dt_1 \ldots \times \int_{t_0}^{t_{n-1}} dt_n s(u,t_1)\ldots s(t_{n-1},t_n)\rho_{T0}(t_n). \tag{10}$$

Now we can define the general nonlinear response tensor $\chi_{\mu\nu}(t,u)$ for the open system via the change of the expectation value of a system operator F_μ caused by the external field, $\langle F_\mu(t)\rangle_e = \mathrm{Tr}_S[\rho_e(t) F_\mu]$, substituting Eq. (9) into which, we obtain

$$\langle F_\mu(t)\rangle_e \equiv \sum_\nu \int_{t_0}^{t} du\, \chi_{\mu\nu}(t,u) f_\nu(u), \tag{11}$$

where the nonlinear response tensor $\chi_{\mu\nu}(t,u)$ is given by

$$\chi_{\mu\nu}(t,u) = \frac{i}{\hbar}\varepsilon \mathrm{Tr}_{SR}\{F_\mu g(t,u)[C_\nu, x(u)]\}, \tag{12}$$

in which $x(u)$ is given by Eq. (10). The commutation relation in Eq. (12) is defined by $[A,B] = AB - BA$ with the operators A and B. $\mathrm{Tr}_{SR} \equiv \mathrm{Tr}_S\mathrm{Tr}_R$ denotes traces over the system and environment, respectively.

C. Series expansion of the nonlinear response tensor for open system

In the following, we will derive a general nonlinear response function, which can be expanded up to arbitrary order in the strength of the external field. The nonlinear susceptibility is calculated by the generalized projection operator method with the external field here.

Assuming that H_e was turned on at $t_0 \to -\infty$, we then are interested in the general response at $t = 0$. Equation (12) follows

$$\chi_{\mu\nu}(0,-t) = \frac{i}{\hbar}\varepsilon \mathrm{Tr}_{SR}\{F_\mu e^{-\frac{i}{\hbar}Lt}[C_\nu, x(-t)]\}. \tag{13}$$

In the following, we will restrict ourself to consider a time-independent system. The formalism can be easily generalized to time-dependent system. Suppose the whole system is in an equilibrium state $\rho_{T0}(u) \equiv \rho_{eq}$ at temperature T. The main task of the general nonlinear response theory is to calculate the susceptibility:

$$\chi_{\mu\nu}(\omega) = \int_0^\infty dt e^{i\omega t}\chi_{\mu\nu}(0,-t) = \frac{i}{\hbar}\varepsilon\int_0^\infty dt e^{i\omega t}\mathrm{Tr}_{SR}\{F_\mu e^{-\frac{i}{\hbar}Lt}[C_\nu,x(-t)]\}. \quad (14)$$

Defining $q_\nu(t) = \mathrm{Tr}_R e^{-\frac{i}{\hbar}Lt}[C_\nu,x(-t)]$, we rewrite the susceptibility as

$$\chi_{\mu\nu}(\omega) = \frac{i}{\hbar}\varepsilon\int_0^\infty dt e^{i\omega t}\mathrm{Tr}_S[F_\mu q_\nu(t)]. \quad (15)$$

With a modified Laplace transformation $g(\omega) = \int_0^\infty dt e^{i\omega t} g(t)$ [29,34], where $g(t)$ is a general function about the time t, then the nonlinear susceptibility $\chi_{\mu\nu}(\omega)$ is given by

$$\chi_{\mu\nu}(\omega) = \frac{i}{\hbar}\varepsilon\mathrm{Tr}_S[F_\mu q_\nu(\omega)]. \quad (16)$$

According to the formula above, we find that the task to obtain the nonlinear susceptibility is reduced to calculating the quantity $q_\nu(\omega)$. A compact formula can be given when we consider a realistic example. In the following we will present an expansion for the susceptibility in orders of the perturbation. With this expansion, one can have the susceptibility up to any order in perturbation if required. Substituting Eq. (10) into Eq. (16), we obtain

$$\chi_{\mu\nu}(\omega) = \varepsilon\chi^{(1)}_{\mu\nu}(\omega) + \sum_{n=2}^{\infty}\varepsilon^n\chi^{(n)}_{\mu\nu}(\omega), \quad (17)$$

where $\chi^{(n)}_{\mu\nu}(\omega) = \frac{i}{\hbar}\mathrm{Tr}_S[F_\mu q^{(n)}_\nu(\omega)]$ denotes the nth nonlinear susceptibility with

$$q^{(n)}_\nu(\omega) = \int_0^\infty dt\exp(i\omega t)q^{(n)}_\nu(t), \qquad q^{(1)}_\nu(t) = \mathrm{Tr}_R e^{-\frac{i}{\hbar}Lt}[C_\nu,\rho_{eq}], \quad (18)$$

and

$$q^{(n)}_\nu(t) = (-1)^{n+1}\mathrm{Tr}_R\left\{e^{-\frac{i}{\hbar}Lt}\left[C_\nu,\int_{-\infty}^{-t}dt_1\cdots\int_{-\infty}^{t_{n-2}}dt_{n-1} \cdot s(-t,t_1)\cdots s(t_{n-2},t_{n-1})\rho_{eq}\right]\right\}, (n\geqslant 2). \quad (19)$$

Here, the first term in Eq. (17) represents the linear response, which returns to Kubo's linear response theory when the system is closed, while the others are nonlinear responses of the open system to the external field. We expand the quantum nonlinear response function $\chi_{\mu\nu}(\omega)$ as a sum of n-order nonlinear susceptibility $\chi^{(n)}_{\mu\nu}(\omega)$ (n is arbitrary). This provides us with a natural link between theory and experiment [35,36]. The external field and environment enter the system through the multitime convolutions in Eq. (19). When $q^{(n)}_\nu(t)$ is calculated in terms of the superoperator $s(t,u)$ for the external field with thermal equilibrium as the initial state, we need to perform the calculation for a given external field.

Equation (17) suggests that in order to get the nonlinear susceptibility, we have to calculate $q_\nu(\omega)$ through $q_\nu(t)$. The time evolution of $q_\nu(t)$ is given by (see Appendix B)

$$\dot{q}_\nu(t) = -\frac{i}{\hbar}[H_S,q_\nu(t)] + \int_{t_0}^{t}c(t-\tau)q_\nu(\tau)d\tau + K_\nu(t), \quad (20)$$

where the kernel $c(t)$ and the inhomogeneous term $K_\nu(t)$ are given by $c(t) = -\mathrm{Tr}_R[Le^{-\frac{i}{\hbar}QLt}QL\rho_R/\hbar^2]$ and

$$K_\nu(t) = -\frac{i}{\hbar}\mathrm{Tr}_R\left[Le^{-\frac{i}{\hbar}QLt}Q\Lambda_\nu(0)\right] + \mathrm{Tr}_R e^{-\frac{i}{\hbar}Lt}\dot{\Lambda}_\nu(t) - \frac{i}{\hbar}\mathrm{Tr}_R\left[L\int_0^t d\tau e^{-\frac{i}{\hbar}QL(t-\tau)}Qe^{-\frac{i}{\hbar}L\tau}\dot{\Lambda}_\nu(\tau)\right], \quad (21)$$

respectively. $\Lambda_\nu(t) = [C_\nu,x(-t)]$, and $x(t)$ is given by Eq. (10). The operator Q in Eq. (20) is defined by $Q = 1 - P$ with $P(\ldots) = \rho_R\otimes\mathrm{Tr}_R(\ldots)$, which are projection operators satisfying $P^2 = P$ and $Q^2 = Q$.

In Appendix C we use an example, a single-mode cavity system coupled to the environment in a time-dependent external field, to illustrate the nonlinear response when the external field is not weak. In addition, we show that it is difficult to derive analytically the nonlinear response of a general open system to a perturbation at finite temperatures. Considering the fact that two-dimensional topological insulators described by the two-band model has been experimentally observed [20–23], we will focus on the linear response theory for an open system in the following.

We assume that the system and environment are initialized in their thermal equilibrium, $\rho_{T0}(0) = \rho_{\mathrm{eq}} = e^{-\beta H}/\mathrm{Tr}_{SR}[e^{-\beta H}]$. To calculate the linear response, we take the first term in Eq. (10) as $x(u) \equiv \rho_{\mathrm{eq}}$. Furthermore, we take $\rho_{\mathrm{eq}} = \rho_S\otimes\rho_R + \mathcal{O}(H_{SR})$ for simplicity, where ρ_S and ρ_R are the density operators for the system and environment, respectively. The term $K_\nu(t)$ in Eq. (20) vanishes since $Q[C_\nu,\rho_{\mathrm{eq}}] = Q[C_\nu,\rho_S]\rho_R = 0$. After performing the modified Laplace transformation for Eq. (20) and considering its expansion to the second order in the system-environment coupling H_{SR} [30], we have

$$-\frac{i}{\hbar}L_S q_\nu(\omega) + [i\omega + c(\omega)]q_\nu(\omega) = -q_\nu(0), \quad (22)$$

where the superoperator

$$c(\omega) = \frac{-1}{\hbar^2}\int_0^\infty dt\langle L_{SR}\mathcal{H}^I_{SR}(t)\rangle_R\exp[i(\omega - L_S/\hbar)t], \quad (23)$$

with these definitions $\mathcal{H}^I_{SR}(t) = \exp[-iL_0 t/\hbar]H_{SR}$, $L_{SR}\rho = [H_{SR},\rho], L_0\rho = [H_0,\rho]$, and $L_S\rho = [H_S,\rho]$. Here and hereafter, the perturbation parameter ε is absorbed in C_ν. Equations (17) and (22) are of vital importance in calculating the response function. Equation (17) is the nonlinear response of the open system to the external field, and Eq. (22) is the key element to derive the linear response.

So far, we have presented a derivation for the susceptibility of an open quantum system. In the next section, we will apply the analytical expression to the Hall conductance based on the two-band model.

III. APPLICATION TO HALL CONDUCTANCE IN THE TWO-BAND MODEL

In order to apply the response theory to the study of the spin-Hall effect in quantum open system, we first introduce a two-band model, which can describe quantum anomalous Hall insulators [37,38]. The quantum Hall insulator can be understood as a consequence of momentum-space topology [1], which is robust against local perturbations. In the following, we will derive the quantum anomalous Hall conductance for open system by applying the quantum response theory to open system in order to study the influence of the environment on quantum anomalous Hall conductance. The most general two-band Hamiltonian describing a 2D noninteracting system can be expressed in the form

$$H_S = \sum_{\vec{p}} H_S(\vec{p}), H_S(\vec{p}) = \varepsilon(\vec{p}) + \sum_{\alpha=x,y,z} d_\alpha(\vec{p})\sigma_\alpha, \quad (24)$$

where $\varepsilon(\vec{p}) = \vec{p}^2/2m^*$ is the kinetic energy with the band electron effective mass m^* and σ_α are the Pauli matrices. And $\vec{p} = (p_x, p_y)$ stands for the Bloch wave vector of the electron. The two bands may stand for different physical degrees of freedom depending on the context, e.g., d_α describes the spin-orbit coupling if they are the components of a spin-1/2 electron, when they are the orbital degrees of freedoms, d_α denotes the hybridization between bands.

The two-band system is an idealization but can be realized approximatively with ultracold atoms [39] using different techniques, e.g., with superlattices [40–42], with hole-related band as suggested by Raghu *et al.* in Ref. [43] and in spin-1/2 electrons with the spin-orbit coupling [3]. Apart from the possibility of experimental realization, the two-band model is also interesting as a simple model in condensed matter physics.

The eigenenergies of the two-band Hamiltonian Eq. (24) are, $E_m(\vec{p}) = \varepsilon(\vec{p}) + md(\vec{p})$, with $m = \pm$ and $d = \sqrt{d_x^2 + d_y^2 + d_z^2}$. The corresponding eigenstates take

$$|+(\vec{p})\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}, |-(\vec{p})\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}, \quad (25)$$

where $\cos\theta = d_z/d$ and $\tan\phi = d_y/d_x$ were defined.

The Hall conductance tensor $\sigma_{\mu\nu}$ can be calculated through the current density $J_\mu(t)$ in the μ direction ($\mu = x,y,z$), as a response to the external electric field $E_\nu(t) = \mathrm{Re}[E_\nu \exp(i\omega t)]$ in that direction, the current density reads

$$J_\mu(t) = \lim_{\omega\to 0} \mathrm{Re}[\sigma_{\mu\nu}(\omega)E_\nu \exp(i\omega t)], \quad (26)$$

where Einstein conversion of summation was used. Quite often our interest is the dc conductance, which is obtained by taking the limit $\omega \to 0$. Consider a total Hamiltonian in the momentum space [26],

$$H_T(t) = H_S + H_R + H_{SR} + H_e(t), \quad (27)$$

with $H_R = \sum_j \hbar\omega_j b_j^\dagger b_j$, $H_{SR} = \sum_j \hbar g_j \sigma_- b_j^\dagger + \mathrm{H.c.}$, and $H_e(t) = -\sum_\nu \mathrm{P}_\nu E_\nu(t)$. Here H_S is given by Eq. (24). $H_e(t)$ denotes the Hamiltonian of the system's coupling to the external electrical field with polarization $\mathrm{P}_\nu = \sum_j q_j r_\nu^j$ and particle charges q_j as well as position operators r_ν^j. H.c. stands for the Hermitian conjugate and $\sigma_- = (\sigma_x - i\sigma_y)/2$. The environment Hamiltonian H_R is introduced as a bosonic bath modeled by a set of oscillators denoted by annihilation operators b_j, which are coupled to the two-band system with coupling constants g_j. The system-environment interaction Hamiltonian is given by H_{SR}. Equation (27) denotes the Hamiltonian of a system composed of many particles with charges q_j and position operators $r_\nu^j = i\hbar\frac{\partial}{\partial p_\nu^j}$ subjected to the environmental noises H_R. Therefore, in the linear response theory, taking $C_\nu = \mathrm{P}_\nu$ and $F_\mu = J_\mu$ with Eq. (16), we can write the Hall conductance as [44]

$$\sigma_{\mu\nu}(\omega) = \frac{1}{i\hbar}\int_0^\infty dt\langle[\mathrm{P}_\nu(0), J_\mu(t)]\rangle e^{i\omega t} = \frac{-V}{\hbar\omega}\int_0^\infty dt\langle[J_\nu(0), J_\mu(t)]\rangle e^{i\omega t}, \quad (28)$$

where $J_\nu = \dot{\mathrm{P}}_\nu/V$, and V denotes the mode volume for system. Making use of the relations $J = ev/V$, where e is the electric charge and $v_\mu(t) = e^{\frac{i}{\hbar}H_S t} v_\mu e^{-\frac{i}{\hbar}H_S t}$ is the kinetic velocity operator. Collecting all these together, we obtain

$$\begin{aligned}\sigma_{\mu\nu}(\omega) &= \frac{-e^2}{\hbar\omega V}\int_0^\infty dt\langle[v_\nu(0), v_\mu(t)]\rangle e^{i\omega t} \\ &= \frac{e^2}{\hbar\omega V}\int_0^\infty dt \mathrm{Tr}_{SR}\{v_\mu e^{-\frac{i}{\hbar}Lt}[v_\nu, \rho_S]\}e^{i\omega t} + \mathcal{O}(H_{SR}) \\ &\simeq \frac{e^2}{\hbar\omega V}\mathrm{Tr}_S[v_\mu q_\nu(\omega)],\end{aligned} \quad (29)$$

where $q_\nu(\omega)$ is given by Eq. (22), therefore $q_\nu(0) = [v_\nu, \rho_S] + \mathcal{O}(H_{SR})$. Being aware of Eq. (27), taking the Lorentzian spectrum density $J(\omega) = \frac{\Gamma}{2\pi}\frac{\lambda^2}{\omega^2+\lambda^2}$ [45] into consideration, under the Markovian approximation, from Eq. (23) we obtain

$$c(\omega)q_\nu(\omega) = \Gamma[2\sigma_- q_\nu(\omega)\sigma_+ - \sigma_+\sigma_- q_\nu(\omega) - q_\nu(\omega)\sigma_+\sigma_-], \quad (30)$$

where Γ is the decay rate induced by the environment. The dissipator in Eq. (30) describes a decay of the system from the high band to the low band with conserved momenta. Here we did not consider the effect of the finite-temperature environment on the system due to the average photon number of the thermal reservoir $N(T) = 1/(e^{\hbar\Delta_0/k_B T} - 1) \to 0$, where k_B is the Boltzmann constant, and T the temperature, and the energy gap Δ_0 much larger than $k_B T$ in two-dimensional topological insulator. Substituting Eq. (30) into Eq. (22), and inserting the identity $\sum_m |m(\vec{p})\rangle\langle m(\vec{p})| = I$ into the result, we obtain

$$0 = S_{\nu,nm} + iq_{\nu,nm}[\hbar\omega - E_n(\vec{p}) + E_m(\vec{p})] + 2\Gamma\hbar\sum_{ij}(\sigma_-)_{ni}q_{\nu,ij}(\sigma_+)_{jm} - F_{\nu,nm} - F_{\nu,mn}^\dagger, \quad (31)$$

where the coefficients $F_{\nu,nm} = \hbar\Gamma\sum_{\vec{p},j} q_{\nu,nj}(\sigma_+\sigma_-)_{jm}$, $S_{\nu,nm} = \hbar[f_m(\vec{p}) - f_n(\vec{p})]v_{\nu,nm}$, in which $y_{mn} = \langle m(\vec{p})|y|n(\vec{p})\rangle$. Here we have applied the fact that $\rho_S = \sum_j |j(\vec{p})\rangle f_j(\vec{p})\langle j(\vec{p})|$, and $f_j(\vec{p}) = 1/\{\exp[\beta(E_j(\vec{p}) - \mu)] + 1\}$ is the Fermi-Dirac distribution function with μ being the chemical potential, and $\beta = 1/k_B T$. With Eq. (31), tedious but straightforward algebra yields ($m \neq n$)

$$q_{v,nm} = \frac{S_{v,mn}A_{nm} - \hbar\Gamma B_{nm}cos2\theta - S_{v,nm}(\hbar\omega + i\hbar\Gamma)\hbar\Gamma}{2\left[D_{nm} + 2(\hbar\omega)^3 + 8i(\hbar\omega)^2\hbar\Gamma - i\hbar\Gamma e_{nm}^2 cos2\theta\right]}, \tag{32}$$

where the coefficients take $e_{nm} = E_n(\vec{p}) - E_m(\vec{p})$, $A_{nm} = 4i(\hbar\omega)^2 - \hbar\omega(11\hbar\Gamma + 4ie_{nm}) - \hbar\Gamma(7i\hbar\Gamma - 6e_{nm})$, $B_{nm} = S_{v,nm}(\hbar\omega + i\hbar\Gamma) + S_{v,mn}(\hbar\omega + i\hbar\Gamma - 2e_{nm})$, and $D_{nm} = -2\hbar\omega[5(\hbar\Gamma)^2 + e_{nm}^2] - i\hbar\Gamma[4(\hbar\Gamma)^2 + 3e_{nm}^2]$.

Note that the diagonal elements of $q_v(\omega)$ are not listed here, since they have no contribution to the Hall conductance. In the weak dissipation limit, $\Gamma \to 0$, we can expand $q_{v,nm}$ in powers of Γ. To first order in Γ, $q_{v,nm}$ can be written as

$$q_{v,nm} = q_{v,nm}^{(0)} + \hbar\Gamma q_{v,nm}^{(1)}, \tag{33}$$

where the zeroth- and first-order of $q_{v,nm}$ take

$$\begin{aligned} q_{v,nm}^{(0)} &= \frac{iS_{v,nm}}{\hbar\omega - e_{nm}}, \\ q_{v,nm}^{(1)} &= \frac{g(\theta)S_{v,nm}}{4(\hbar\omega - e_{nm})^2} + \frac{h(\theta)S_{v,mn}}{4\left(\hbar^2\omega^2 - e_{nm}^2\right)}, \end{aligned} \tag{34}$$

respectively, where $g(\theta) = 5 - \cos 2\theta$, $h(\theta) = 1 - \cos 2\theta$. Substituting Eq. (33) into the third equation of Eq. (29), we have

$$\sigma_{\mu v}(\omega) = \sigma_{\mu v}^{(0)}(\omega) + \hbar\Gamma\sigma_{\mu v}^{(1)}(\omega), \tag{35}$$

with the zeroth- and first-order Hall conductance at finite-temperature T,

$$\begin{aligned} \sigma_{\mu v}^{(0)}(\omega) &= \frac{e^2}{\hbar\omega V}\sum_{\vec{p},m\neq n}\langle m(\vec{p})|\upsilon_\mu|n(\vec{p})\rangle\left(\frac{iS_{v,nm}}{\hbar\omega - e_{nm}}\right), \\ \sigma_{\mu v}^{(1)}(\omega) &= \frac{e^2}{\hbar\omega V}\sum_{\vec{p},m\neq n}\langle m(\vec{p})|\upsilon_\mu|n(\vec{p})\rangle\left[\frac{g(\theta)S_{v,nm}}{4(\hbar\omega - e_{nm})^2}\right. \\ &\quad \left. + \frac{h(\theta)S_{v,mn}}{4\left(\hbar^2\omega^2 - e_{nm}^2\right)}\right]. \end{aligned} \tag{36}$$

One of the main results of our work is the construction of this object of the Hall conductance for quantum open system, which characterizes the topological structure of insulators in the presence of dissipation. Furthermore, by taking Eq. (35) into account, we can obtain the finite-temperature Hall conductance for the open system in the weak dissipation limit (i.e., $\Gamma \to 0$, see Appendix D),

$$\sigma_{\mu v} = \sigma_{\mu v}^{(0)} + \sigma_{\mu v}^{(1)} \equiv C\frac{e^2}{h}, \tag{37}$$

where

$$\sigma_{\mu v}^{(0)} = \frac{e^2}{2\hbar}\int \tau(\vec{k})\varepsilon_{\alpha\beta\gamma}\frac{d^2k}{(2\pi)^2},$$

$$\sigma_{\mu v}^{(1)} = \frac{\Gamma e^2}{2\hbar}\sum_{\alpha,\beta,\gamma}\int \tau(\vec{k})\left(\frac{\hbar}{4}g(\theta)\frac{d_\alpha d_\beta}{d^2 d_\gamma} + \frac{id}{4\omega}h(\theta)\frac{D_{\alpha\beta}}{d_\gamma}\right)\frac{d^2k}{(2\pi)^2}, \tag{38}$$

with $\tau(\vec{k}) = \frac{[f_+(k) - f_-(k)]}{d^3}\frac{\partial d_\alpha}{\partial k_\mu}\frac{\partial d_\beta}{\partial k_v}d_\gamma$, $D_{\alpha\beta} = \text{Im}[\langle +(\vec{p})|\sigma_\alpha| -(\vec{p})\rangle\langle +(\vec{p})|\sigma_\beta| -(\vec{p})\rangle]$. C in Eq. (37) defines the Chern number of the open system. Here we have used $\sum_{\vec{p}} \to \frac{V}{(2\pi\hbar)^2}\int dp_x dp_y$ and $\vec{p} = \hbar\vec{k}$. A nonvanishing Hall conductance for quantum open system witnesses a topological nontrivial phase presented in Eq. (37). For the case of a closed system, the steady state $q_v(\omega)$ is a pure Bloch state, i.e., $c(\omega) \to 0$ in Eq. (22) being equivalent to $\sigma_{\mu v}^{(1)} \to 0$. Thus we can recover the conventional topological invariant (Chern number) $\sigma_{\mu v}^{(0)}$ in Eq. (37) for the two-band model in closed system [3].

The Hall conductance for quantum open system $\sigma_{\mu v}$ in Eq. (37) has two different terms. The first one is a weighted integration of curvatures for two bands. This term has topological meaning on its own and it can distinguish between phases with or without topological structure. The second term represents a correction of the environment to the Hall conductance. Clearly, taking the environment into account, the Hall conductance may not be an integer. This will not be an obstacle to use the Hall conductance to detect topological properties of insulator states [46] for quantum open system, since the phase transition points are still there.

Note that the correction $\sigma_{\mu v}^{(1)}$ in general is complex, it contains a real part (the first term) and an imaginary part (the second term). In the following, we will present two examples, for the first of which, the correction to the Hall conductance from the system-environment coupling is real, while in the second, it is imaginary. The two examples together exemplify the effects of the environment on the Hall conductance.

Example 1. Consider a two-dimensional ferromagnetic electron gas in the presence of both Rashba and Dresselhaus spin-orbit couplings [4,47,48]. This system can be described by Hamiltonian Eq. (24) with

$$\begin{aligned} d_x &= \lambda_0\hbar k_y - \beta_0\hbar k_x, \\ d_y &= -\lambda_0\hbar k_x + \beta_0\hbar k_y, \\ d_z &= h_0, \end{aligned} \tag{39}$$

where k_x and k_y are the electron momentum, λ_0 and β_0 are the strengths of the Rashba and Dresselhauss spin-orbit couplings. h_0 is the material specific constant. We show that this case corresponds to the quantum anomalous Hall effect, which originates from the Rashba and Dresselhauss spin-orbit interactions instead of the external magnetic field. Using Eq. (37), we numerically calculate the quantum anomalous Hall conductance and plot the results as a function of β_0 in Fig. 1. Figure 1(a) shows a phase transition at $\beta_0 = \pm\lambda_0$. When $\beta_0^2 > \lambda_0^2$, the Chern number of the closed system at zero-temperature is 0.5, while for $\beta_0^2 < \lambda_0^2$, the Chern number is -0.5. This is in agreement with the analytical results of quantum anomalous Hall conductance of the closed system given by $\sigma_{xy}^{(0)}(T = 0) = \frac{1}{2}\text{sgn}(\beta_0^2 - \lambda_0^2)\text{sgn}(h_0)$ in units of $\frac{e^2}{h}$. For the open system, the plot shows that the phase transition can still emerge. This can be found in Fig. 1(c), the Chern number at zero-temperature is about 0.5 when $\beta_0^2 > \lambda_0^2$, while for $\beta_0^2 < \lambda_0^2$, the Chern number is about -0.5. A critical value $\beta_0 = \pm\lambda_0$ at which the quantum anomalous Hall conductance changes abruptly can be found. This suggests that the topological phase transition survives for open system.

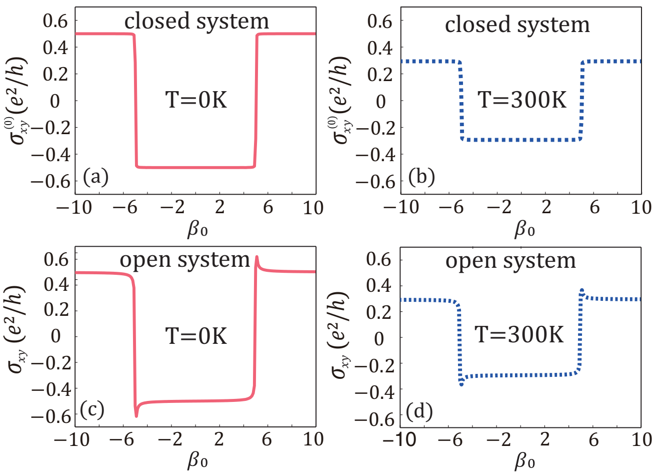

FIG. 1. (Color online) The zero-order quantum anomalous Hall conductance $\sigma_{xy}^{(0)}$ [(a) and (b)] (i.e., $\Gamma = 0$) and the total conductance σ_{xy} [(c) and (d)] as a function of β_0 (meVnm/$\hbar$). The quantum anomalous Hall conductance is exactly the conductance of the closed system. It is worth addressing that the imaginary part of the Hall conductance is zero in this case. Parameters chosen are $\Gamma = 0.2$ meV/$\hbar$, $\lambda_0 = 5$ meVnm/$\hbar$, $\mu = 1$ meV, $h_0 = 2$ meV, $m^* = 0.9\text{m}_e$, m_e is the mass of electron. The temperature $T = 0$ K for (a) and (c), and $T = 300$ K for (b) and (d).

We also find that at the critical point λ_0, σ_{xy} increases compared to $\sigma_{xy}^{(0)}$, while it decreases at $-\lambda_0$. Except at the two critical points, the effect of the environment on the conductance is almost zero. Thermal fluctuation diminishes the difference of the quantum anomalous Hall conductance between different phases, but it does not change the nature of the phase; see Figs. 1(b) and 1(d). The change in sign for quantum anomalous Hall conductance [49] can be observed experimentally in diluted magnetic GaAs quantum wells, where the Rashba and Dresselhaus couplings have usually the same order of magnitude. The Rashba coupling is tunable by controlling a gate field perpendicular to the electron gas. Thus there is no practical difficulty to achieve the situation of $\lambda_0 = \beta_0$. The gate field can be used to tune the direction of the anomalous Hall current. This effect can be identified as the mechanism for quantum anomalous Hall effect. It is worth addressing that the imaginary part of $\sigma_{xy}^{(1)}$ vanishes for this model, which is different from the example below.

The quantum anomalous Hall effect (QAHE) in spin-orbit coupling (SOC) discussed in the previous case is important and interesting, not only because they are conceptually new but also because they are realizable in physics [50]. Recently, many potential compounds with nontrivial topological electronic states are proposed. The realizations of various topological electronic states have strongly promoted this field, and a lot of interesting experiments on their nontrivial topological properties are now becoming possible. The ferromagnetic proximity effect will open up a gap for the Dirac surface states on both surfaces of the TI, which gives rise to the QAHE [51], to sandwich a TI thin film with ferromagnetic insulators, dope the magnetic ions into the TI thin film and make it a ferromagnetic semiconductor [52–54]. The only one of these proposals for the QAHE has been realized experimentally by tuning the doping properly [55,56].

There have been several different types of proposals for the realization of the QAHE. Such as ferromagnetic Weyl semimetal (such as $HgCr_2Se_4$) at a certain film thickness [57], graphene, silicene, or other honeycomb lattice on top of magnetic insulators [58], toneycomb lattice by growing double-layer thin films of a transition metal oxide with perovskite structure [59], heavy metal on magnetic insulator substrate [60], strained epitaxial film with interface band inversion in EuO/GdN [61] or CdO/EuO [62].

The present prediction can be observed in the six-terminal device (for details, see Ref. [63]). The environment can be simulated by the use of Büttiker's virtual probes [64,65]. Now we discuss the experimental realization in our ferromagnetic electron gas in the presence of both Rashba and Dresselhaus spin-orbit couplings [Eq. (39)]. When $\beta_0 = 0$, Eq. (39) is reduced to Rashba spin-orbit coupling. And our model describes the low-energy physics in HgTe/CdTe quantum wells when $\lambda_0 = 0$ [20,52]. Thus, the experimental situation corresponds to the highest plateau at $e^2/2h$ in our model. In the presence of the dephasing, the quantum plateaus behave quite differently with a small fluctuation around the critical point (see Fig. 1). The plateau structure remains almost unchanged. This suggests that QAHE is insensitive to the environment. As we have shown, the phase transition remains in the open system. Thus, these quantum plateaus are observable in mesoscopic samples even with the environmental noise. Comparing with the experimental results [20,55,63], we conjecture that the small deviations between theoretical and experimental results may originate from the dephasing effect induced by the environment.

Example 2. As another example, we consider tight-binding electrons in a two-dimensional lattice described by the Hamiltonian [66,67]

$$H = -t_a \sum_{\langle i,j \rangle} {}_x c_j^\dagger c_i e^{i\theta_{ij}} - t_b \sum_{\langle i,j \rangle} {}_y c_j^\dagger c_i e^{i\theta_{ij}}, \quad (40)$$

where c_j is the fermion annihilation operator on the lattice site j, t_a, and t_b denote the hopping amplitudes along the x and y direction, respectively. If we restrict ourselves to two branches coupled by $|l|$th-order perturbation, the effective Hamiltonian then takes the form of Eq. (24) with $d_x = \delta \cos(k_y l)$, $d_y = \delta \sin(k_y l)$, and $d_z = 2t_a \cos(k_x + 2\pi \frac{p}{q} m_0)$, where p and q are integers. δ is proportional to (is the order of) $t_b^{|l|}$.

In Fig. 2, we show numerically the Hall conductance at zero-temperature for the open system as a function of t_a and δ, the Fermi energy is set in the gap. We find that the Hall conductance $\sigma_{xy}^{(0)}$ and Im[$\sigma_{\mu\nu}^{(1)}$] change its sign when t_a crosses zero. The topological phase transition at zero temperature survives in the open system. This can be observed by examining the Hall conductance, which changes from -0.5 and -0.1 to 0.5 and 0.1 when $m_0 = m_1$ [in units of e^2/h; see Figs. 2(a) and 2(c)]. Similar changes from 0.5 and 0.1 to -0.5 and -0.1 for $m_0 = m_2$ [see Figs. 2(b) and 2(d)] are found. We here address that the behavior of the imaginary part Im[$\sigma_{\mu\nu}^{(1)}$] of the Hall conductance for the open system is the same as $\sigma_{\mu\nu}^{(0)}$ for the closed system (see Figs. 2 and 3) except their amplitudes. This means that the topological properties of open

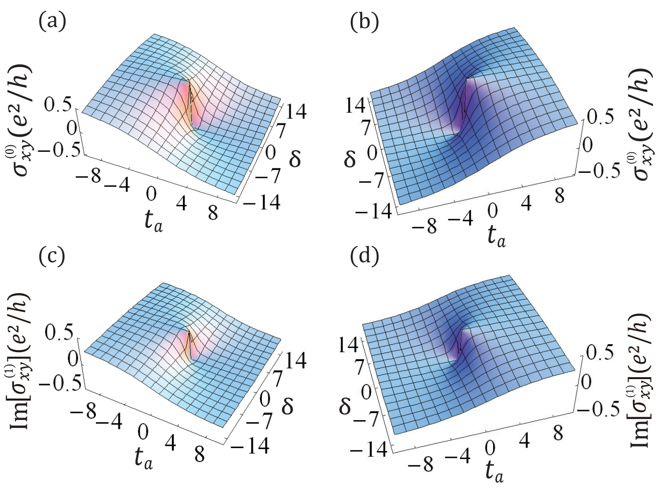

FIG. 2. (Color online) The zero-order Hall conductance $\sigma_{xy}^{(0)}$ [(a) and (b)] and the imaginary part $\text{Im}[\sigma_{\mu\nu}^{(1)}]$ [(c) and (d)] of the first-order Hall conductance as a function of t_a (meV) and δ (meV) at zero temperature. Note that the real part $\text{Re}[\sigma_{\mu\nu}^{(1)}]$ of the Hall conductance of the open system is zero in this case. Define two coefficients, $m_1 = 4n_1 + 1$ or $4n_1 + 4$ ($n_1 = 0, 1, 2, \ldots$), and $m_2 = 4n_2 + 2$ or $4n_2 + 3$ ($n_2 = 0, 1, 2, \ldots$). Parameters chosen are $\Gamma = 0.1$ mev/$\hbar$, $\omega = 0.2$ mev/$\hbar$, $p = 1, q = 4, l = 1$. $m_0 = m_1$, for (a) and (c). $m_0 = m_2$, for (b) and (d).

and closed system are the same to each other. This observation can be explained as follows. We analytically calculate the Hall conductance at zero temperature and obtain

$$\begin{aligned} \sigma_{\mu\nu}^{(0)} &= A_0 \frac{e^2}{2h} \text{sgn}[\delta_{m_0 m_1} - \delta_{m_0 m2}], \\ \sigma_{\mu\nu}^{(1)} &= i B_0 \sigma_{\mu\nu}^{(0)}, \end{aligned} \tag{41}$$

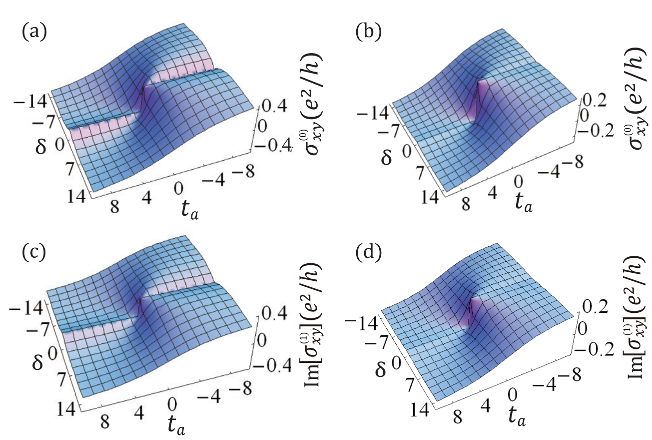

FIG. 3. (Color online) The Hall conductance of the closed system $\sigma_{xy}^{(0)}$ [(a) and (b)] and the imaginary part of the first order Hall conductance $\text{Im}[\sigma_{\mu\nu}^{(1)}]$ [(c) and (d)] of the open system. We plot them as a function of t_a (meV) and δ (meV) at finite temperature. We address that the real part $\text{Re}[\sigma_{\mu\nu}^{(1)}]$ of the Hall conductance of the open system is zero in this case. Parameters chosen are $\Gamma = 0.1$ mev/$\hbar$, $\omega = 0.2$ mev/$\hbar$, $p = 1, q = 4, l = 1, m_0 = m_1, T = 30$ K for (a) and (c). $T = 300$ K for (b) and (d).

where $A_0 = -\frac{l t_a}{\sqrt{M}}$, $B_0 = \frac{\Gamma(13M + 2\delta^2)}{12\omega M}$, and $M = 4t_a^2 + \delta^2 > 0$. Especially, we find

$$\begin{aligned} \sigma_{\mu\nu}^{(0)} &= -l \frac{e^2}{2h} \text{sgn}(t_a) \text{sgn}[\delta_{m_0 m_1} - \delta_{m_0 m2}], \\ \sigma_{\mu\nu}^{(1)} &= \frac{13 i \Gamma}{12\omega} \sigma_{\mu\nu}^{(0)}, \end{aligned} \tag{42}$$

when $\delta = 0$. This gives the phase transition points for the closed and open system at zero temperature with $\delta = 0$ [see Figs. 2(a) and 2(c), 2(b) and 2(d)]. In contrast to the system in the last example, the correction to the conductance due to the system-environment coupling is imaginary, which we will refer to as environment-induced reactance. This environment-induced reactance describes the energy exchange between the system and the environment, reminiscent of the non-Markovian effect. Moreover, from Eq. (41) we find that the reactance $\text{Im}[\sigma_{\mu\nu}^{(1)}]$ and $\sigma_{\mu\nu}^{(0)}$ have the same sign due to $B > 0$. At finite temperature, for example, $T = 30$ K [Figs. 2(a) and 2(c)] and 300 K [Figs. 2(b) and 2(d)] (see Fig. 3), the thermal effect diminishes the amplitude of the Hall conductance compared with the value at $T = 0$ K (Fig. 2), but it does not change the topological phase.

IV. DISCUSSION

The Kubo formula derived within the framework of the linear response theory applies only to the equilibrium system. In this paper, we have developed a quantum response theory for a system far from thermal equilibrium (referred to as open system) beyond the linear regime. A general nonlinear susceptibility for the open system is derived. This theory provides us with a formalism to extend the notion of the finite-temperature Hall conductance form equilibrium to nonequilibrium systems. This comes into play when decoherence effects on the Hall conductance in open system are studied. We exemplify the theory in a two-band model that describes topological insulators, the results show that the environment slightly affects the topological phase transition at finite-temperature, and the decoherence effect on the Hall conductance can be controlled in the two-band model. This observation makes the TIs immune to the influences of the environment, which supports its application in quantum computation. Although the analysis has been restricted to the two-band models, we believe that the general response theory could be extended to higher dimensions and nonlinear Hall conductance for many sorts of topological insulators.

ACKNOWLEDGMENTS

We thank Prof. D. Culcer, Prof. Shun-Qing Shen, and Prof. Zheng-Yuan Xue for valuable discussions. This work is supported by National Natural Science Foundation of China (NSFC) under Grants No. 11175032, No. 61475033, and No. 11204028.

APPENDIX A: THE VALIDITY OF THE APPROXIMATION $q_\nu(0) \simeq [C_\nu,\rho_S]$

Here, we examine the validity of the approximating $q_\nu(0) = \mathrm{Tr}_R[C_\nu,\rho_{eq}] \simeq [C_\nu,\rho_S] + O(H_{SR})$ in Eq. (22). To this end, we employ the formula with two Hermitian operators A and B:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} e^{(\text{high-order terms for } B)}. \tag{A1}$$

Under weak system-environment interaction H_{SR}, we can expand ρ_{eq} in powers of H_{SR}. To the second-order in H_{SR} and setting $A = -\beta H_0$ and $B = -\beta H_{SR}$ in Eq. (A1), we obtain

$$q_\nu(0) = [C_\nu,\mathrm{Tr}_R\rho_{\mathrm{eq}}] \simeq [C_\nu,\rho_S] + \left[C_\nu,\rho_S^{(2)}\right], \tag{A2}$$

with

$$\begin{aligned}\rho_S^{(2)} = \rho_S[a_1(&H_S\sigma_+H_S\sigma_- + \sigma_+H_S\sigma_-H_S\\ &-H_S\sigma_+\sigma_-H_S - \sigma_+H_SH_S\sigma_-)\\ &+a_2\sigma_+H_S\sigma_- + a_3\sigma_+\sigma_-],\end{aligned} \tag{A3}$$

where the coefficients $a_1 = \frac{\Gamma\lambda\beta^4}{16}$, $a_2 = \frac{\Gamma\lambda\beta^3}{4}$, and $a_3 = \frac{\Gamma\lambda\beta^2}{4}$. Here we have used the Lorentzian spectral density $J(\omega) = \frac{\Gamma}{2\pi}\frac{\lambda^2}{\omega^2+\lambda^2}$ [45].

In order to check the validity of $q_\nu(0) \simeq [C_\nu,\rho_S]$ in Eq. (22), we numerically plot the Hall conductance for closed and open system in the tight-binding model Eq. (40) with different δ. In Fig. 4, we show a comparison of the results with two different orders in $q_\nu(0)$ given in Eq. (A2); the simulation is performed for the Hall conductance given by the third equation of Eq. (29) with Eq. (22). The blue-dashed line in Fig. 4 [closed system, Figs. 4(a) and 4(b); and open system, Figs. 4(c) and 4(d)] is for $q_\nu(0) = [C_\nu,\rho_S]$, which are in good agreement with the results obtained with $q_\nu(0) = [C_\nu,\rho_S] + [C_\nu,\rho_S^{(2)}]$. In addition, we find the higher-order terms in $q_\nu(0)$, which have no effect on the phase transition point and can be ignored with respect to the zero-order term in the two-band model.

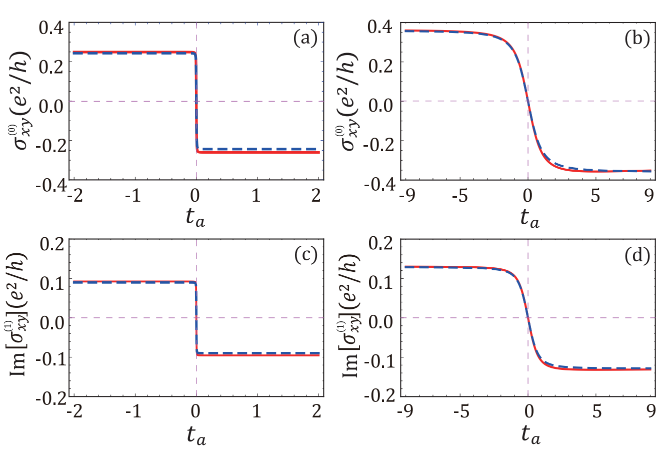

FIG. 4. (Color online) The Hall conductance for closed system [(a) and (b)] and open system [(c) and (d)] given by the third equation in Eq. (29). The purpose of this figure is to show the difference in $\sigma_{\mu\nu}$ caused by different order of $q_\nu(0)$ in Eq. (A2) by numerically solving Eq. (22). The red-solid and blue-dashed lines correspond to $q_\nu(0) = [C_\nu,\rho_S] + [C_\nu,\rho_S^{(2)}]$ and $q_\nu(0) = [C_\nu,\rho_S]$, respectively. Parameters chosen are $\omega = 0.2$ mev$/\hbar$, $\lambda = 25$mev$/\hbar$, $p = 1, q = 4, L = 1, m_0 = 1, T = 30K$, $\delta = 0.01$ mev, $\Gamma = 0$ for (a); $\delta = 2$ mev, $\Gamma = 0$ for (b); $\delta = 0.01$ mev, $\Gamma = 0.1$ mev$/\hbar$ for (c); and $\delta = 2$ mev, $\Gamma = 0.1$ mev$/\hbar$ for (d).

APPENDIX B: THE DERIVATION OF THE EQUATION FOR $\dot{q}_\mu(t)$

We apply the generalized projection operator method to derive the time derivative of $q_\nu(t)$. Note that $q_\nu(t) = \mathrm{Tr}_R e^{-\frac{i}{\hbar}Lt}[C_\nu,x(-t)]$ is not a density matrix since the operator C_ν does not commute with $x(-t)$ given by Eq. (10). To derive the equation, we define superoperators $P(\ldots) = \rho_R \otimes \mathrm{Tr}_R(\ldots)$ and $Q = 1 - P$, which are projection operators satisfying $P^2 = P$ and $Q^2 = Q$. By the standard procedure for deriving a master equation in Refs. [29,68,69], with two time-evolution operators defined by (the initial time is $t_0 = 0$)

$$\begin{aligned} W_\nu(t) &= Pe^{-\frac{i}{\hbar}Lt}\Lambda_\nu(t),\\ Y_\nu(t) &= Qe^{-\frac{i}{\hbar}Lt}\Lambda_\nu(t),\end{aligned} \tag{B1}$$

we have

$$\dot{W}_\nu(t) = -\frac{i}{\hbar}PLW_\nu(t) - \frac{i}{\hbar}PLY_\nu(t) + Pe^{-\frac{i}{\hbar}Lt}\dot{\Lambda}_\nu(t), \tag{B2}$$

and

$$\dot{Y}_\nu(t) = -\frac{i}{\hbar}QLY_\nu(t) - \frac{i}{\hbar}QLW_\nu(t) + Qe^{-\frac{i}{\hbar}Lt}\dot{\Lambda}_\nu(t). \tag{B3}$$

Solving Eq. (B3), we have

$$\begin{aligned} Y_\nu(t) = {}& e^{-\frac{i}{\hbar}QLt}Q\Lambda_\nu(0) - \frac{i}{\hbar}\int_0^t e^{-\frac{i}{\hbar}QL(t-\tau)}QLW_\nu(\tau)d\tau\\ &+ \int_0^t e^{-\frac{i}{\hbar}QL(t-\tau)}Qe^{-\frac{i}{\hbar}L\tau}\dot{\Lambda}_\nu(\tau)d\tau.\end{aligned} \tag{B4}$$

Substituting Eq. (B4) into Eq. (B2), we obtain

$$\begin{aligned}\dot{W}_\nu(t) = {}& -\frac{i}{\hbar}PLW_\nu(t) - \frac{i}{\hbar}PLe^{-\frac{i}{\hbar}QLt}Q\Lambda_\nu(0) - \frac{1}{\hbar^2}PL\\ &\cdot\int_0^t e^{-\frac{i}{\hbar}QL(t-\tau)}QLW_\nu(\tau)d\tau - \frac{i}{\hbar}PL\\ &\cdot\int_0^t e^{-\frac{i}{\hbar}QL(t-\tau)}Qe^{-\frac{i}{\hbar}L\tau}\dot{\Lambda}_\nu(\tau)d\tau + Pe^{-\frac{i}{\hbar}Lt}\dot{\Lambda}_\nu(t).\end{aligned} \tag{B5}$$

Finally, by applying $P(\ldots) = \rho_R \otimes \mathrm{Tr}_R(\ldots)$ into Eq. (B5), we arrive at Eq. (20).

APPENDIX C: QUANTUM RESPONSE BEYOND THE LINEAR REGIME

As mentioned before, the linear response theory is not valid when the external field is not weak enough. Here we present an example to show the difference between the linear response theory and the nonlinear response theory, which is also an illustration for the essential role of the nonlinear response theory. We exemplify the difference through $n_e(t)$, which is defined as the difference of photon number inside the cavity with and without the external field. The results show that when the coupling between the field and system is weak, the linear response is a good approximation; otherwise, nonlinear response should be taken into account.

Consider a single-mode cavity system with bare frequency ω_0 coupled to a non-Markovian reservoir, driven by an external laser with frequency ω_L. We assume the reservoir modeled by a set of harmonic oscillators is at a finite (ambient) temperature. In a rotating frame, the Hamiltonian of the total system reads

$$H_T = H + H_e, \tag{C1}$$

with

$$H = \hbar\Delta a^\dagger a + \sum_k \hbar\Omega_k b_k^\dagger b_k + \sum_k (\hbar g_k^* a b_k^\dagger + \hbar g_k b_k a^\dagger), \tag{C2}$$

and

$$H_e = \hbar\Omega(a + a^\dagger), \tag{C3}$$

where $\Delta = \omega_0 - \omega_L$ and $\Omega_k = \omega_k - \omega_L$. Ω is the strength of the external field, a is the cavity annihilation operator, and b_k and g_k are the reservoir annihilation operator and coupling constant. In the following we will solve the exact non-Markovian dynamics in Heisenberg picture.

Suppose the system and the environment is initially uncorrelated—the reservoir modeled by Hamiltonian $H_R = \sum_k \hbar\omega_k b_k^\dagger b_k$ is in a thermal equilibrium state, while the system is in a coherent state. The initial state of the total system is

$$\rho_T(0) = |\alpha\rangle\langle\alpha| \otimes \rho_R(0),\, \rho_R(0) = \frac{e^{-\beta H_R}}{\mathrm{Tr}e^{-\beta H_R}}, \tag{C4}$$

where $|\alpha\rangle$ can be obtained by defining it as an eigenstate of the annihilation operator a with an eigenvalue α, and $\beta = 1/\kappa_B T$.

Our task is to obtain exactly the response of the system to the external field and compare it with the nonlinear response. We use the word *exactly* to denote that the response is not a perturbative result—it is exact, including all orders in the external field. With the formal solution of $b_k(t) = e^{\frac{i}{\hbar}H_T t} b_k(0) e^{-\frac{i}{\hbar}H_T t}$, we rewrite the equation for $a(t)$,

$$\frac{d}{dt}a(t) = -i\Delta a(t) - \int_0^t a(\tau) f(t-\tau) d\tau - ib(t) - i\Omega(t), \tag{C5}$$

where $b(t) = \sum_k g_k b_k(0) e^{-i\Omega_k t}$. The memory kernel

$$f(\tau) = \sum_k |g_k|^2 e^{-i\Omega_k \tau} \equiv \int d\omega J(\omega) e^{-i(\omega-\omega_L)\tau}$$

characterizes the non-Markovian dynamics of the reservoir.

Because of the linearity of Eq. (C5), $a(t)$ can be expressed as $a(t) = u_1(t)a(0) + v_1(t)$, where $a(0)$ and $b_k(0)$ are the operators at the initial time. Here time-dependent coefficients $u_1(t)$ and $v_1(t)$ can be calculated by Eq. (C5),

$$\frac{d}{dt}u_1(t) = -i\Delta u_1(t) - \int_0^t u_1(\tau) f(t-\tau) d\tau, \tag{C6}$$

$$\frac{d}{dt}v_1(t) = -i\Delta v_1(t) - \int_0^t v_1(\tau) f(t-\tau) d\tau - ib(t) - i\Omega(t), \tag{C7}$$

with initial conditions $u_1(0) = 1$ and $v_1(0) = 0$. $v_1(t)$ can be given analytically by solving the inhomogeneous equation of Eq. (C7), it leads to $v_1(t) = -i\int_0^t [b(\tau) + \Omega(\tau)] u_1(t-\tau) d\tau$.

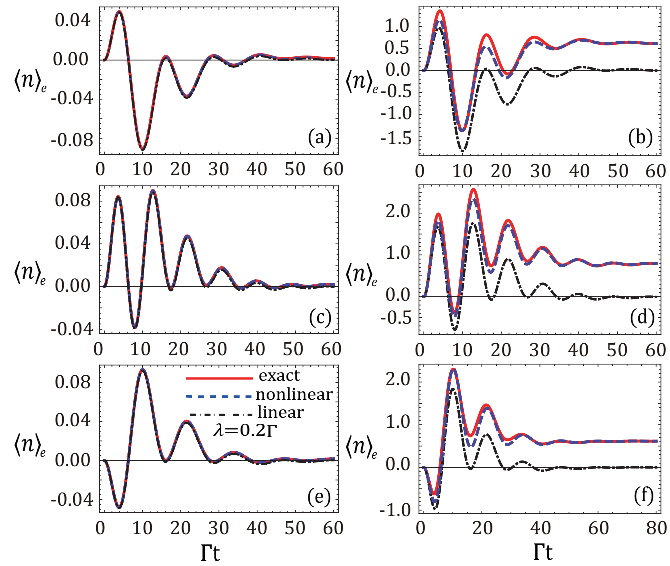

FIG. 5. (Color online) $n_e(t)$ as a function of time. This figure corresponds to non-Markovian case in nonlinear response with $\lambda = 0.2\Gamma$. The result of linear response is given in (a), (c), and (e), while (b), (d), and (f) are for two-order nonlinear response. The red solid line, blue dashed line, and black dash-dotted line denote the exact infinity-order response Eq. (C8), the linear response given by Eq. (11) containing only the first term of Eq. (10), and the nonlinear response obtained by Eq. (11) containing only the first two terms in Eq. (10), respectively. The parameters chosen are $\Omega = 0.01\Gamma, \alpha = 4, \Delta = 0.2\Gamma$ for (a); $\Delta = 0.4\Gamma$ for (c); and $\Delta = -0.2\Gamma$ for (e). $\Omega = 0.2\Gamma$, $\Delta = 0.2\Gamma$ for (b); $\Delta = 0.4\Gamma$ for (d); and $\Delta = -0.2\Gamma$ for (f).

Taking the state in Eq. (C4) into account, we can calculate the change of the expectation value of the cavity photon $n(t) = \mathrm{Tr}_S[a^\dagger(t)a(t)\rho(0)]$ in relevance to the external field Eq. (C3) as

$$n_e(t) = n_T(t) - n(t) = 2\mathrm{Re}[u_1^*(t)y(t)] + D(t), \tag{C8}$$

where $y(t) = -i\alpha^* D_1(t), D(t) = |D_1(t)|^2, D_1(t) = \Omega\int_0^t u_1(\tau)d\tau$.

We assume that the system coupled to a reservoir has a Lorentzian spectral density $J(\omega) = \frac{\Gamma}{2\pi}\frac{\lambda^2}{(\omega-\omega_0)^2+\lambda^2}$ [70,71]. In order to examine the validity of linear response, we plot the response of the average photon number to the external field in three regimes divided by linear, two-order nonlinear, and exact response in Figs. 5 and 6. In non-Markovian regime, e.g., $\lambda = 0.2\Gamma$, we can see that the results given by Eq. (11) under the first-order approximation (containing only the first term) in Eq. (10) are in good agreement with those obtained by the exact response Eq. (C8) when the interaction strength Ω is weak [see Figs. 5(a), 5(c), and 5(e)]. With the interaction strength Ω increasing [see Figs. 5(b), 5(d), and 5(f)], i.e., the dynamics of the Eq. (11) involving only the first- and second-order terms in Eq. (10) are in good agreement with those obtained by the exact response Eq. (C8), but the results obtained by the first-order approximation (linear regime) have serious deviations from the exact one, Eq. (C8). This difference comes

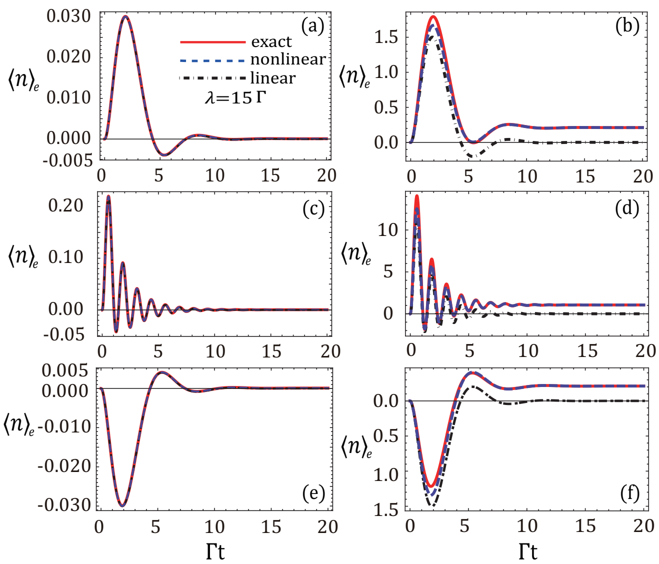

FIG. 6. (Color online) $n_e(t)$ as a function of time. This figure corresponds to Markovian case in nonlinear response with $\lambda = 15\Gamma$. Comparison of the linear response [(a), (c), and (e)] and two-order nonlinear response [(b), (d), and (f)] in the Markovian regime. The red solid line, blue dashed line, and black dash-dotted line denote the exact response Eq. (C8), the linear response obtained by Eq. (11) contains only the first term in Eq. (10), and the nonlinear response obtained by Eq. (11) contains only the first two terms in Eq. (10), respectively. The parameters chosen are $\alpha = 4$, $\Delta = \Gamma$, $\Omega = 0.01\Gamma$ for (a); $\Delta = 5\Gamma, \Omega = 0.1\Gamma$ for (c); and $\Delta = -1\Gamma, \Omega = 0.01\Gamma$ for (e). $\Delta = 1\Gamma, \Omega = 0.5\Gamma$ for (b); $\Delta = 5\Gamma, \Omega = 5\Gamma$ for (d); and $\Delta = -1\Gamma$, $\Omega = 0.5\Gamma$ for (f).

from the nonlinear terms, which are ignored in linear response theory.

Examining the Markovian regime, e.g., $\lambda = 15\Gamma$, we find that the results given by the linear response [containing only the first term in Eq. (10)] are in good agreement with those obtained by the exact response Eq. (C8) when the the interaction Ω is weak [see Figs. 6(a), 6(c), and 6(e)]. When the the interaction strength Ω becomes strong [see Figs. 6(b), 6(d), and 6(f)], the dynamics given by Eq. (11) involving only the first- and second-order terms in Eq. (10) are in good agreement with those obtained by the exact expression Eq. (C8). However, the results obtained by the first-order approximation (linear regime) have serious deviations from those obtained by the exact expression Eq. (C8). The same observation can be found in the non-Markovian regime.

From the analysis above, we can conclude that the change of the mean photon number of the cavity mode to the external field can be treated exactly by the second-order nonlinear response for quantum open system, which is the correction to the first-order response when the strength of the driving field becomes strong. Although our results have been fixed to the second order, our conclusion is general. Namely, with the increase of the field strength, we need to consider the revised response of higher order. Therefore, the nonlinear response given by Eqs. (11) and (17) plays an important role, which is applicable to any strength of perturbation for quantum open system.

APPENDIX D: THE DERIVATION OF EQ. (37)

We first calculate the zeroth-order Hall conductance at finite-temperature by rewriting Eq. (36) as

$$\sigma^{(0)}_{\mu\nu}(\omega) = \frac{ie^2}{\omega V}\sum_{\vec p, m\neq n} f_n(\vec p)\bigg[\frac{\langle n(\vec p)|\upsilon_\mu|m(\vec p)\rangle\langle m(\vec p)|\upsilon_\nu|n(\vec p)\rangle}{\hbar\omega + e_{nm}} - \frac{\langle m(\vec p)|\upsilon_\mu|n(\vec p)\rangle\langle n(\vec p)|\upsilon_\nu|m(\vec p)\rangle}{\hbar\omega - e_{nm}}\bigg]. \tag{D1}$$

In the limit $|\omega/e_{nm}| \ll 1$,

$$\frac{1}{\hbar\omega \pm e_{nm}} = \frac{1}{e_{nm}}\left(\pm 1 - \frac{\hbar\omega}{e_{nm}}\right) + \mathcal{O}(\omega^2). \tag{D2}$$

Substituting Eq. (D2) into Eq. (D1), we have $\sigma^{(0)}_{\mu\nu}(\omega) = \sigma_1(\omega) + \sigma_2(\omega)$ with

$$\sigma_1(\omega) = \frac{ie^2}{\omega V}\sum_{\vec p, m\neq n} f_n(\vec p)[\langle n(\vec p)|\upsilon_\mu|m(\vec p)\rangle\langle m(\vec p)|\upsilon_\nu|n(\vec p)\rangle + \langle n(\vec p)|\upsilon_\nu|m(\vec p)\rangle\langle m(\vec p)|\upsilon_\mu|n(\vec p)\rangle]/e_{nm}, \tag{D3}$$

and

$$\sigma_2(\omega) = \frac{-i\hbar e^2}{V}\sum_{\vec p, m\neq n} f_n(\vec p)[\langle n(\vec p)|\upsilon_\mu|m(\vec p)\rangle\langle m(\vec p)|\upsilon_\nu|n(\vec p)\rangle - \langle n(\vec p)|\upsilon_\nu|m(\vec p)\rangle\langle m(\vec p)|\upsilon_\mu|n(\vec p)\rangle]/e^2_{nm}. \tag{D4}$$

Now we show that the first term $\sigma_1(\omega)$ vanishes. Setting $\mu = x$ and $\nu = y$, we have $\upsilon_x = \frac{i}{\hbar}[H_S,x]$, then

$$\langle n(\vec p)|\upsilon_x|m(\vec p)\rangle = \frac{i}{\hbar}\langle n(\vec p)|[H,x]|m(\vec p)\rangle = \frac{i}{\hbar}e_{nm}\langle n(\vec p)|x|m(\vec p)\rangle, \tag{D5}$$

and thus

$$[\langle n(\vec p)|\upsilon_x|m(\vec p)\rangle\langle m(\vec p)|\upsilon_y|n(\vec p)\rangle + \langle n(\vec p)|\upsilon_y|m(\vec p)\rangle \cdot\langle m(\vec p)|\upsilon_x|n(\vec p)\rangle] = \frac{i}{\hbar}e_{nm}[\langle n(\vec p)|x|m(\vec p)\rangle \cdot\langle m(\vec p)|\upsilon_y|n(\vec p)\rangle - \langle n(\vec p)|\upsilon_y|m(\vec p)\rangle\langle m(\vec p)|x|n(\vec p)\rangle]. \tag{D6}$$

The factors e_{nm} cancel each other, noticing $\sum_m |m(\vec p)\rangle\langle m(\vec p)| = \mathbb{I}$, we have

$$\sigma_1(\omega) = \frac{e^2}{\omega V\hbar}\sum_{\vec p, m\neq n} f_n(\vec p)\langle n(\vec p)|[\upsilon_y,x]|n(\vec p)\rangle \equiv 0, \tag{D7}$$

since the commutator $[x,\upsilon_y]$ vanishes, $\sigma^{(0)}_{\mu\nu} \equiv \sigma_2$. As for the second term in Eq. (D4), simple algebra yields

$$\sigma^{(0)}_{\mu\nu} = \frac{e^2\hbar}{V}\sum_{\vec p, m\neq n}\frac{f_{mn}\mathrm{Im}[\langle m(\vec p)|\upsilon_\mu|n(\vec p)\rangle\langle n(\vec p)|\upsilon_\nu|m(\vec p)\rangle]}{e^2_{nm}}, \tag{D8}$$

where $f_{mn} = [f_m(\vec p) - f_n(\vec p)]$. Therefore, the finite-temperature Hall conductance for open system is given by

$\sigma_{\mu\nu} = \sigma_{\mu\nu}^{(0)} + \hbar\Gamma\sigma_{\mu\nu}^{(1)}$. Following the same procedure, we have

$$\sigma_{\mu\nu}^{(1)} = \frac{e^2\hbar}{2V}\sum_{\vec{p},m\neq n} g(\theta) f_{mn}\mathrm{Re}[\langle m(\vec{p})|\upsilon_\mu|n(\vec{p})\rangle \cdot\langle n(\vec{p})|\upsilon_\nu|m(\vec{p})\rangle]/e_{nm}^3 + \frac{e^2}{4\omega V}\sum_{\vec{p},m\neq n} h(\theta) f_{mn} \cdot\langle m(\vec{p})|\upsilon_\mu|n(\vec{p})\rangle\langle m(\vec{p})|\upsilon_\nu|n(\vec{p})\rangle/e_{nm}^2. \tag{D9}$$

In the Heisenberg picture, the velocity operator ($\mu = x,y,z$) is defined as

$$\upsilon_\mu = \frac{i}{\hbar}[H_S(p), r_\mu] = \frac{p}{m^*} + \frac{\partial d_\alpha}{\partial p_\mu}\sigma_\alpha, \tag{D10}$$

in which the summations with respect to α automatically assumed in the Einstein summation convention. With Eq. (D10),

$$\langle m(\vec{p})|\upsilon_\mu|n(\vec{p})\rangle = \frac{\partial d_\alpha}{\partial p_\mu}\langle m(\vec{p})|\sigma_\alpha|n(\vec{p})\rangle, \tag{D11}$$

for $m \neq n$, it is easy to prove that in the two-band model,

$$\begin{aligned} \mathrm{Im}[\langle m(\vec{p})|\sigma_\alpha|n(\vec{p})\rangle\langle n(\vec{p})|\sigma_\beta|m(\vec{p})\rangle] &= m\varepsilon_{\alpha\beta\gamma}\frac{d_\gamma}{d}, \\ \mathrm{Re}[\langle m(\vec{p})|\sigma_\alpha|n(\vec{p})\rangle\langle n(\vec{p})|\sigma_\beta|m(\vec{p})\rangle] &= -\frac{d_\alpha d_\beta}{d^2}. \end{aligned} \tag{D12}$$

Collecting all together, we can obtain the finite-temperature Hall conductance Eq. (37) for open system.

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